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COCYCLES AND FACTORIZATION IN ANALYTIC OPERATOR ALGEBRAS

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1. INTRODUCTION

Let M be a σ-finite von Neumann algebra and $\mathscr{S} \subseteq M$ a (not necessarily selfadjoint) subalgebra of M. An invertible operator T in M is said to have *factorization with respect to* $\mathscr{S}$ if we can write $T = UA$ where $U \in M$ is a unitary operator and both A and A^{-1} lie in $\mathscr{S}$.

The factorization problem with respect to non selfadjoint operator algebras was studied by several authors. (See [1], [2], [4], [7], [8], [11], [12], [13], [14]).

This problem is known to be related to the problem of similarity among non selfadjoint algebras.

In the present paper we study factorization with respect to analytic subalgebras associated with a σ-weakly continuous action α of a locally compact abelian group G on M. Let Γ be the dual group of G and Σ be a positive semigroup (such that $\Sigma \cap (-\Sigma) = \{0\}$ and Σ is the closure of its interior). Then the *analytic subalgebra* of M associated with α and Σ is

$$M^{\alpha}(\Sigma) = \{a \in M : \mathrm{sp}_{\alpha}(a) \subseteq \Sigma\}$$

where $\mathrm{sp}_{\alpha}(\cdot)$ is Arveson's spectrum.

It is known that every nest subalgebra of a von Neumann algebra M is of the form $M^{\alpha}(\mathbf{R}_{+})$ where α is a σ-weakly continuous inner action of $\mathbf{R}$ on M. In particular, if $M = B(H)$, we obtain all the nest algebras (for which the factorization problem was extensively studied).

In order to make the statements of the main results clearer we shall assume, for the rest of this section, that Σ totally orders Γ; i.e. $\Sigma \cup (-\Sigma) = \Gamma$. Otherwise, a condition called admissibility would have to be imposed on T (see Section 2).

In Section 2 we associate with every invertible operator T in M an α-cocycle $a(T)$ and we show (Theorem 2.3) that T can be factored with respect to $M^{\alpha}(\Sigma)$ if and only if $a(T)$ is trivial (i.e., $a(T) \simeq 1$).

We use it to show that if M is G-finite relative to α (see Definition 2.5(2)) and T in M is invertible then T can be factored if and only if M is G-finite relative to α^T (where $\alpha_t^T(x) = a(T)(t)\alpha_t(x)a(T)(t)^*$ for x in M).

This extends a result of D. Pitts [11] for nest subalgebras.

We also show (Theorem 2.10) that M can be embedded in a larger algebra (namely in $M \otimes B(L^2(G))$) with an action of G that agrees with α on (the image of) M (namely $\alpha \otimes \mathrm{ad}\,\lambda$ where λ is the regular representation) and such that, in this larger algebra, every invertible operator in (the image of) M can be factored.

In Section 3 we study the relation $a(T) \sim 1$ (i.e., $a(T)$ is quasi-equivalent to 1) for an invertible operator T in M. We show (Theorem 3.5) that $a(T) \sim 1$ if and only if $T = \sum u_i A_i$ where A_i are outer operators (to be defined in Section 3) in $M^\alpha(\Sigma)$ and u_i are partial isometries in M such that $u_i^* u_i$ is the range projection of A_i, $\sum u_i u_i^* = I$ and the central support of $\bigvee u_i^* u_i$, in M^α $(= \{a \in M : \alpha_t(a) = a, t \in G\})$, is I. We call this property a *weak factorization*.

We show an example of an operator that has a weak factorization but not a factorization (Example 3.4). However, if both M and M^{α^T} are properly infinite (or both are abelian) then (Lemma 3.8) weak factorization implies factorization.

2. FACTORIZATION

Let M be a σ-finite von Neumann algebra and let G be a locally compact abelian group with the dual group Γ. Let $\alpha = \{\alpha_t : t \in G\}$ be a representation of G) as a group of $*$-automorphisms of the von Neumann algebra M such that $t \mapsto \alpha_t(a$ is σ-weakly continuous for every a in M. (We call it an *action* of G on M.)

Let $\Sigma \subseteq \Gamma$ be a positive semigroup; i.e.

(i) $\Sigma + \Sigma \subseteq \Sigma$;

(ii) $\Sigma \cap (-\Sigma) = \{0\}$; and

(iii) Σ is the closure of its interior.

The *analytic subalgebra* associated with (α, Σ), $M^\alpha(\Sigma)$, is defined by

$$M^\alpha(\Sigma) = \{a \in M : \mathrm{sp}_\alpha(a) \subseteq \Sigma\}$$

where $\mathrm{sp}_\alpha(\cdot)$ is Arveson's spectrum. (For details see [9]).

We can assume, by choosing an appropriate representation, that there is a strongly continuous unitary group $\{W_t : t \in G\}$ that implements α; i.e. $\alpha_t(a) = W_t a W_t^*$. Using Stone's Theorem we find a unique projection-valued measure $P(\cdot)$ on Γ such that

$$W_t = \int_\Gamma \langle t, q \rangle \, \mathrm{d}P(q), \quad t \in G.$$

Given an invertible operator T in M and q in Γ we let Q_q be the projection onto the subspace $[TP(q + \Sigma)(H)]$ (where $[B]$ is the closed linear subspace spanned by B and H is the Hilbert space on which M acts).

Given a projection-valued measure $P(\cdot)$ we say that the invertible operator T in M is *(P, Σ)-admissible* if there is a projection-valued measure $Q(\cdot)$, on the Borel subsets of Γ, such that for q in Γ, $Q(q + \Sigma) = Q_q$.

Clearly if Γ is totally ordered (i.e. if $\Sigma \cup (-\Sigma) = \Gamma$) then every invertible operator in M is (P, Σ)-admissible.

Note also that if the measure $Q(\cdot)$ exists it is unique (see [10, 8.3.10].

Suppose now that T is (P, Σ)-admissible and $Q(q + \Sigma) = Q_q$ as above. Then we can define a strongly continuous unitary group

$$U_t = \int_\Gamma \langle t, q \rangle \, dQ(q), \quad t \in G,$$

and a group of $*$-automorphisms of $B(H)$,

$$\gamma_t(S) = U_t S U_t^*, \quad t \in G, \ S \in B(H).$$

We can also define a group of $*$-automorphisms of M' by setting

$$\beta_t(S) = W_t S W_t^*, \quad S \in M', \ t \in G.$$

Note that if $\mathrm{sp}_\beta(S) \subseteq q + \Sigma$ for some q in Γ, then (by [9, Theorem 2.13]) $SP(p + \Sigma)(H) \subseteq P(p + q + \Sigma)(H)$ for every p in Γ. Since S is in M' we have $SQ_p(H) \subseteq Q_{p+q}(H)$ for all p in Γ and therefore, using [9, Corollary 2.14],

$$\mathrm{sp}_\gamma(S) \subseteq q + \Sigma.$$

We can now conclude from [9, Corollary 2.11] (with Φ being the inclusion map of M' into $B(H)$) that

$$W_t S W_t^* = \beta_t(S) = U_t S U_t^*, \quad S \in M', \ t \in G.$$

Hence, for every t in G, $U_t W_t^* \in M$ and $U_t M U_t^* = M$.

We now define $\alpha_t^T(S) = U_t S U_t^*$ for S in M and

$$a(T)(t) = U_t W_t^* \in M.$$

Then α^T is an action of G on M and $a(T)$ is a unitary α-cocycle since for s and t in G,

$$a(T)(ts) = U_{ts} W_{ts}^* = U_s W_s^* W_s U_t W_t^* W_s^* = a(T)(s) \alpha_s(a(T)(t));$$

and

$$a(T)(s^{-1}) = U_s^* W_s = \alpha_s^{-1}(a(T)(s)^*).$$

LEMMA 2.1. *Suppose T is invertible and is (P, Σ)-admissible. Then α^T is the unique action of G on M satisfying*

$$M^{\alpha^T}(q + \Sigma) = TM^{\alpha}(q + \Sigma)T^{-1}, \quad q \in \Gamma.$$

Proof. Let $Q(\cdot)$ be the projection-valued measure satisfying $Q(q + \Sigma)(H) = [TP(q + \Sigma)(H)]$, $q \in \Gamma$. Then an operator S in M is in $M^{\alpha^T}(q + \Sigma)$ if and only if

$$SQ(p + \Sigma) = Q(p + q + \Sigma)SQ(p + \Sigma), \quad p \in \Gamma,$$

(by [9, Corollary 2.14]). But this holds if and only if $T^{-1}ST$ maps $P(p + \Sigma)(H)$ into $P(p + q + \Sigma)(H)$ for every p in Γ; hence if and only if $T^{-1}ST \in M^{\alpha}(q + \Sigma)$.

The uniqueness follows from Corollary 2.11 of [9] (with Φ = identity). ∎

The lemma shows that α^T does not depend on the choice of the implementing unitary group $\{W_t\}$.

Following [15, § 20] we denote by $Z_\alpha(G, U(M))$ the set of unitary α-cocycles of G in M.

For a cocycle a in $Z_\alpha(G, U(M))$ the equation

$$({}_a\alpha)_t(x) = a(t)\alpha_t(x)a(t)^*, \quad x \in M$$

defines an action of G on M whose fixed point algebra is denoted by M^a $(= \{x \in M : ({}_a\alpha)_t(x) = x,\ t \in G\})$.

Given a and b in $Z_\alpha(G, U(M))$ we say that a and b are *equivalent* and write $a \simeq b$ if there is a unitary operator u in $U(M)$ such that $a(s) = u^*b(s)\alpha_s(u)$ for all s in G.

Let F_2 be the type I_2 factor with the system of matrix units $\{e_{i,j} : 1 \leqslant i, j \leqslant 2\}$. Write $\tilde{M} = M \otimes F_2$. Then $\alpha \otimes \mathrm{id}$ is an action of G on $\tilde{M}$. For a and b in $Z_\alpha(G, U(M))$ we define $c = c(a, b) : G \to \tilde{M}$ by

$$c(s) = a(s) \otimes e_{11} + b(s) \otimes e_{22}, \quad s \in G,$$

and

$$\mathscr{I}(a, b) = \{x \in M : xb(s) = a(s)\alpha_s(x),\ s \in G\}.$$

We then have

PROPOSITION 2.2. [15, Proposition 20.2].

(1) *$c = c(a, b)$ is in $Z_{\alpha\otimes \mathrm{id}}(G, U(\tilde{M}))$.*

(2) *$I \otimes e_{11}$ and $I \otimes e_{22}$ are in $\tilde{M}^c$.*

(3) *a and b are equivalent if and only if $I \otimes e_{11}$ and $I \otimes e_{22}$ are equivalent projections in $\tilde{M}^c$.*

(4) $\mathscr{I}(a, a) = M^a$.

(5) $\mathscr{I}(a, b) = \mathscr{I}(b, a)^*$.

(6) $\mathscr{I}(a, w)\mathscr{I}(w, b) \subseteq \mathscr{I}(a, b)$ *for* w *in* $Z_\alpha(G, U(M))$.

(7) *If* x *is in* $\mathscr{I}(a, b)$ *with polar decomposition* $x = v|x|$ *then* $|x| \in \mathscr{I}(b, b)$ *and* v *is in* $\mathscr{I}(a, b)$.

(8) $\tilde{M}^c = \mathscr{I}(a, a) \otimes e_{11} + \mathscr{I}(a, b) \otimes e_{12} + \mathscr{I}(b, a) \otimes e_{21} + \mathscr{I}(b, b) \otimes e_{22}$.

THEOREM 2.3. (1) *Suppose* T *and* S *are invertible operators in* M *that are* (P, Σ)*-admissible. Then* $a(T) \simeq a(S)$ *if and only if* $T = USA$ *where* U *is in* $U(M)$ *(the unitary operators in* M*) and both* A *and* A^{-1} *lie in* $M^\alpha(\Sigma)$.

(2) *For an invertible operator* T *in* M, T *can be factored with respect to* $M^\alpha(\Sigma)$ *if and only if* T *is* (P, Σ)*-admissible and* $a(T) \simeq 1$ *(where* 1 *is the trivial cocycle* $1(s) = I$, $s \in G$*).*

Proof. (1) Suppose first that $T = USA$ where $U \in U(M)$ and $A \in M^\alpha(\Sigma) \cap \cap M^\alpha(\Sigma)^{-1}$. Then for q in Γ,

$$[AP(q + \Sigma)(H)] = P(q + \Sigma)(H)$$

and therefore,

$$[TP(q + \Sigma)(H)] = U[SP(q + \Sigma)(H)].$$

Since both T and S are (P, Σ)-admissible, there are unique projection-valued measures $Q(\cdot)$ and $F(\cdot)$, such that

$$Q(q + \Sigma)(H) = SP(q + \Sigma)(H)$$

and

$$F(q + \Sigma)(H) = TP(q + \Sigma)(H).$$

Hence $F(\cdot) = UQ(\cdot)U^*$. It follows immediately that

$$a(T)(t) = Ua(S)(t)\alpha_t(U^*), \quad t \in G.$$

For the other direction, if $a(T)(t) = Ua(S)(t)\alpha_t(U^*)$, $t \in G$, then $F(\cdot) = = UQ(\cdot)U^*$ where $F(\cdot)$ and $Q(\cdot)$ are as above. Hence, for every q in Γ,

$$[TP(q + \Sigma)(H)] = [USP(q + \Sigma)(H)]$$

and consequently,

$$S^{-1}U^*TP(q + \Sigma)(H) \subseteq P(q + \Sigma)(H).$$

This implies that $S^{-1}U^*T$ lies in $M^\alpha(\Sigma) \cap M^\alpha(\Sigma)^{-1}$.

(2) follows easily from (1) as $a(I) = 1$. ▨

The following corollary now follows immediately from Theorem 23.12 in [15].

COROLLARY 2.4. *Let M be a properly infinite semifinite von Neumann algebra, G be $\mathbf{R}$ and τ be a normal semifinite trace on M such that $\tau \circ \alpha_t = e^{-t}\tau$, $t \in \mathbf{R}$. Then every invertible operator T in M can be factored with respect to $M^\alpha(\mathbf{R}_+)$.* ∎

DEFINITION 2.5. (1) An *expectation* from a von Neumann algebra M onto a von Neumann subalgebra N is a σ-weakly continuous linear map Φ from M onto N such that $\|\Phi\| = 1$ and $\Phi \circ \Phi = \Phi$.

(2) Let β be an action of G on M with a fixed point algebra M^β. Then M is called *G-finite relative to β* if there is a faithful expectation Φ from M onto M^β such that

$$\Phi \circ \beta_t = \Phi, \quad t \in G.$$

THEOREM 2.6. *Suppose M is G-finite relative to α and T is an invertible operator in M. Then T can be factored with respect to $M^\alpha(\Sigma)$ if and only if T is (P, Σ)-admissible and M is G-finite with respect to α^T.*

Proof. Suppose first that T can be factored. Then T is (P, Σ)-admissible and $a(T)(t) = U\alpha_t(U^*)$ (see the proof of Theorem 2.3). Then for S in M,

$$\alpha_t^T(S) = a(T)(t)\alpha_t(S)a(T)^*(t) = U\alpha_t(U^*SU)U^*, \quad t \in G.$$

Hence $M^{\alpha^T} = UM^\alpha U^*$ and $\Psi(S) = U\Phi(U^*SU)U^*$ defines a faithful expectation onto M^{α^T} (where Φ is a faithful expectation onto M^α). If $\Phi \circ \alpha_t = \Phi$, $t \in G$, then $\Psi \circ \alpha_t^T = \Psi$, $t \in G$. Hence M is G-finite relative to α^T.

Assume now that T is (P, Σ)-admissible and M is G-finite relative to both α and α^T. Let c be $c(1, a(T))$ (in $\tilde{M} = M \otimes F_2$) and let $\tilde{\alpha}$ be ${}_c\alpha$. Then

$$\tilde{\alpha}_t(x \otimes e_{11}) = \alpha_t(x) \otimes e_{11} \quad \text{and} \quad \tilde{\alpha}_t(x \otimes e_{22}) = \alpha_t^T(x) \otimes e_{22}$$

for t in G. For S in $\tilde{M}^{\tilde{\alpha}}$ $(= \tilde{M}^c)$ with $S \geqslant 0$, $S \neq 0$, write $S = \sum S_{ij} \otimes e_{ij}$. Then either $S_{11} \neq 0$ or $S_{22} \neq 0$. Assume that $S_{11} \neq 0$. Then, by [6, Proposition 1], there is a normal state σ on M that is α-invariant and $\sigma(S_{11}) \neq 0$. Write $\tilde{\sigma}(\sum x_{ij} \otimes e_{ij}) = \sigma(x_{11})$. Then $\tilde{\sigma}$ is a normal state on $\tilde{M}$ that is $\tilde{\alpha}$-invariant and satisfies $\tilde{\sigma}(S) \neq 0$. The case where $S_{22} \neq 0$ can be dealt with in a similar way. We can now use [6, Proposition 1] and conclude that $\tilde{M}$ is G-finite relative to $\tilde{\alpha}$. There is therefore a faithful expectation $\tilde{\Phi}$ from $\tilde{M}$ onto $\tilde{M}^c$ that is $\tilde{\alpha}_t$-invariant for every t in G.

Now let $\tilde{T} \in \tilde{M}$ be $T \otimes e_{21} + T^{-1} \otimes e_{12}$. Then $\tilde{T}^2 = I$. Let $Q(\cdot)$ be the projection-valued measure associated with T; i.e. $Q(q + \Sigma)(H) = [TP(q + \Sigma)(H)]$,

$q \in \Gamma$. We now write $\tilde{P}(\cdot)$ for the projection-valued measure $P(\cdot) \otimes e_{11} + Q(\cdot) \otimes e_{22}$. Then, for S in $\tilde{M}$ and t in G,

$$\tilde{\alpha}_t(S) = \tilde{W}_t S \tilde{W}_t^*,$$

where

$$W_t = \int_\Gamma \langle t, q \rangle \, d\tilde{P}(q).$$

We have, for q in Γ,

$$\tilde{T}\tilde{P}(q + \Sigma) = TP(q + \Sigma) \otimes e_{21} + T^{-1}Q(q + \Sigma) \otimes e_{21} =$$

$$= Q(q + \Sigma)TP(q + \Sigma) \otimes e_{21} + P(q + \Sigma)T^{-1}Q(q + \Sigma) \otimes e_{12} =$$

$$= \tilde{P}(q + \Sigma)\tilde{T}\tilde{P}(q + \Sigma).$$

Hence $\tilde{T}$ is in $\tilde{M}^{\tilde{\alpha}}(\Sigma)$. Since

$$\tilde{T} = (I \otimes e_{11})\tilde{T}(I \otimes e_{22}) + (I \otimes e_{22})\tilde{T}(I \otimes e_{11})$$

and both $I \otimes e_{11}$ and $I \otimes e_{22}$ lie in $\tilde{M}^c = \tilde{\Phi}(\tilde{M})$, we have,

$$\tilde{\Phi}(\tilde{T}) = (I \otimes e_{11})\tilde{\Phi}(\tilde{T})(I \otimes e_{22}) + (I \otimes e_{22})\tilde{\Phi}(\tilde{T})(I \otimes e_{11}).$$

Hence $\tilde{\Phi}(\tilde{T}) = T_0 \otimes e_{12} + T_1 \otimes e_{21}$ for some T_0, T_1 in M. By [9, Theorem 3.8 the expectation $\tilde{\Phi}$ is multiplicative on $\tilde{M}^{\tilde{\alpha}}(\Sigma)$. Hence $\tilde{\Phi}(\tilde{T})^2 = I$ and therefore $T_1 = T_0^{-1}$. By Proposition 2.2 (8), T_0 lies in $\mathscr{I}(1, a(T))$. Let $T_0 = U|T_0|$ be its polar decomposition (and U is unitary since T_0 is invertible). Then, by Proposition 2.2(7), U is in $\mathscr{I}(1, a(T))$. Hence $Ua(T)(t) = \alpha_t(U)$, $t \in G$. Thus $a(T) \simeq 1$ and, using Theorem 2.3, this completes the proof. ▨

When Γ is totally ordered then every invertible operator is (P, Σ)-admissible. We therefore have

COROLLARY 2.7. *Suppose Γ is totally ordered (*i.e. $\Sigma \cup (-\Sigma) = \Gamma$*), M is G-finite relative to α and T is an invertible operator in M. Then T has a factorization with respect to $M^\alpha(\Sigma)$ if and only if M is G-finite relative to α^T.*

For inner actions of **R** this result was proved by D. Pitts [11].

COROLLARY 2.8. *Suppose Γ is totally ordered and M has a faithful normal semifinite trace τ such that*

(i) $\tau \circ \alpha_t = \tau$, $t \in G$; and

(ii) τ restricted to M^α is semifinite.

Then every invertible operator T in M such that τ restricted to M^{α^T} is semifinite can be factored with respect to $M^\alpha(\Sigma)$.

In particular, if τ is finite and $\tau\circ\alpha_t = \tau$ for every t in G then every invertible operator can be factored.

Proof. Suppose τ is a faithful normal trace satisfying (i) and (ii) and $T\in M$ is an invertible operator such that the restriction of τ to M^{α^T} is semifinite. Then, by [1, 6.1.3(4)], there is a unique faithful normal expectation Φ (resp. Ψ) from M onto M^α (resp. M^{α^T}) such that for every A in M^α satisfying $\tau(|A|)<\infty$ and for every B in M^{α^T} satisfying $\tau(|B|)<\infty$ we have

$$\tau(SA) = \tau(\Phi(S)A)$$

and

$$\tau(SB) = \tau(\Psi(S)B)$$

for every S in M. But then $\tau(\alpha_t(S)A) = (\tau\circ\alpha_t)(SA) = \tau(SA) = \tau(\Phi(S)A)$. Hence, by the uniqueness of Φ, Φ is α-invariant. Similarly Ψ is α^T-invariant (note that $\tau\circ\alpha_t^T = \tau$ since $\alpha_t^T(S) = a(T)(t)\alpha_t(S)a(T)(t)^*$ and $a(T)(t)$ is in $U(M)$). Now use Corollary 2.7. ∎

For the case where τ is finite this result was proved by Arveson in [1, Theorem 4.2.1]. (In fact, Arveson proved it for every finite maximal subdiagonal algebra, not only those that are analytic subalgebras.)

When G is compact M is G-finite relative to any action of G. We therefore have the following.

COROLLARY 2.9. *Suppose G is compact and T is an invertible operator in M. Then T can be factored with respect to $M^\alpha(\Sigma)$ if and only if T is (P,Σ)-admissible.*

In particular, if G is compact and if Γ is totally ordered, then every invertible operator can be factored with respect to $M^\alpha(\Sigma)$.

THEOREM 2.10. *Let α, $\{W_t\}$, $P(\cdot)$ and Σ be as before. Let T be an invertible operator in M that is (P,Σ)-admissible. Then the operator $T\otimes I$, in $M\otimes B(L^2(G))$, can be factored with respect to $M\otimes B(L^2(G))^{\alpha\otimes \mathrm{ad}\,\lambda}(\Sigma)$ where λ is the regular representation of G on $L^2(G)$;* i.e. $(\lambda_t f)(s) = f(t-s)$, *and* $(\mathrm{ad}\,\lambda_t)(S) = \lambda_t S\lambda_t^*$ *for S in* $B(L^2(G))$.

Proof. By [15, 20.4(2)] the cocycles $a(T)\otimes\lambda$ and $1\otimes\lambda$ (in $Z_{\alpha\otimes\mathrm{id}}(G, M\otimes B(L^2(G)))$) are equivalent. Hence there is a unitary operator U in $M\otimes B(L^2(G))$ such that

$$(a(T)\otimes\lambda)(t) = U(1\otimes\lambda)(t)(\alpha_t\otimes \mathrm{id})(U^*) =$$

$$= U(I\otimes\lambda_t)(W_t\otimes I)U^*(W_t^*\otimes I),\quad t\in G.$$

Hence

$$(a(T) \otimes 1)(t) = (a(T) \otimes \lambda)(t)(I \otimes \lambda_t^*) =$$

$$= U(W_t \otimes \lambda_t)U^*(W_t^* \otimes \lambda_t^*) = U(\alpha \otimes \mathrm{ad}\,\lambda)_t(U^*), \quad t \in G.$$

Thus $a(T) \otimes 1 \simeq 1$ (as $\alpha \otimes \mathrm{ad}\,\lambda$-cocycles). It is easy to check that $a(T \otimes I) = a(T) \otimes 1$ and the result now follows from Theorem 2.3. ▨

3. WEAK FACTORIZATION

Let a and b be two α-cocycles in $Z_\alpha(G, U(M))$ and let $\tilde{M}$, $\tilde{\alpha}$, $c(a, b)$ and $\mathscr{I}(a, b)$ be as in the discussion preceding Proposition 2.2. Following [3] we say that a and b are quasi-equivalent (and write $a \sim b$) if $I \otimes e_{11}$ and $I \otimes e_{22}$ have the same central support in $\tilde{M}^c$ (the fixed point algebra of $\tilde{\alpha}$).

LEMMA 3.1. *Write*

$$e(a, b) = \sup\{u^*u : u \text{ is a partial isometry in } \mathscr{I}(a, b)\}.$$

Then

(1) *$e(a, b)$ is in $Z(M^b)$, the center of M^b, and $e(b, a)$ is in $Z(M^a)$.*

(2) *$a \sim b$ if and only if $e(a, b) = e(b, a) = I$.*

Proof. (1) As $\mathscr{I}(a, b)^*\mathscr{I}(a, b) = \mathscr{I}(b, a)\mathscr{I}(a, b) \subseteq \mathscr{I}(b, b)$, $e(a, b)$ lies in $\mathscr{I}(b, b) = M^b$. If v is a unitary operator in M^b then

$$ve(a, b)v^* = \sup\{vu^*uv^* : u \text{ is a partial isometry in } \mathscr{I}(a, b)\}.$$

Since uv^* is also a partial isometry in $\mathscr{I}(a, b)$ (if u is), then $ve(a, b)v^* \leqslant e(a, b)$ for every unitary operator v in M^b. Hence $e(a, b)$ is in $Z(M^b)$.

(2) For every partial isometry u in $\mathscr{I}(a, b)$, let $\tilde{u}$ be $u \otimes e_{12} \in \tilde{M}^c$. Then $\tilde{u}^*(I \otimes e_{11})\tilde{u} = u^*u \otimes e_{22}$. Hence, if $e(a, b)$ is I, the central support (in $\tilde{M}^c$) of $I \otimes e_{11}$ is $I \otimes e_{11} + I \otimes e_{22}$. Similarly for $I \otimes e_{22}$.

Now assume that $a \sim b$. Write z for $I - e(b, a)$ and assume $z \neq 0$. Since the central support of $z \otimes e_{11}$ is contained in the central support of $I \otimes e_{22}$, there are non zero projections $z_0 \in M^a$ and $z_1 \in M^b$ such that $z_0 \leqslant z$ and $z_0 \otimes e_{11}$ is equivalent to $z_1 \otimes e_{22}$ in $\tilde{M}^c$. Hence there is some partial isometry $\tilde{u}$ in $\tilde{M}^c$ such that $\tilde{u}\tilde{u}^* = z_0 \otimes e_{11}$ and $\tilde{u}^*\tilde{u} = z_1 \otimes e_{22}$. But then $\tilde{u}$ has the form $\tilde{u} = u \otimes e_{12}$ for some u in $\mathscr{I}(a, b)$ and $uu^* = z_0$, $u^*u = z_1$. Therefore $z_0 = uu^* \leqslant e(b, a)$. Since $z_0 \leqslant z = I - e(b, a)$, we get a contradiction. ▨

LEMMA 3.2. (1) *Suppose u is a partial isometry in $\mathscr{I}(a,b)$ and p is a projection in M^a such that $uu^*p \neq 0$. Then there is a non zero partial isometry v in $\mathscr{I}(a,b)$ such that $vv^* \leqslant p$ and $v^*v \leqslant u^*u$.*

(2) *There is a partial isometry w in $\mathscr{I}(a,b)$ such that*

(i) *whenever u is a partial isometry in $\mathscr{I}(a,b)$ and $u^*w=0$ then $u^*u \leqslant w^*w$; and*

(ii) *the central support of w^*w in M^b is $e(a,b)$.*

Proof. (1) By the comparison theorem there is a projection z in $Z(M^a)$ and partial isometries v_1, v_2 in M^a such that $v_1^*v_1 = zp$, $v_1v_1^* \leqslant zuu^*$, $v_2^*v_2 = (I-z)uu^*$, and $v_2v_2^* \leqslant (I-z)p$. Now, if $v_1 \neq 0$, let v be v_1^*zu and, if $v_2 \neq 0$, let v be $v_2(I-z)u$ This completes the proof of part (1).

(2) We can use Zorn's lemma to find a maximal family of partial isometries $\{u_\gamma\}$ in $\mathscr{I}(a,b)$ such that $\{u_\gamma u_\gamma^*\}$ is an orthogonal family of projections and so is $\{u_\gamma^* u_\gamma\}$. We let w be $\sum u_\gamma$.

Let u be a partial isometry in $\mathscr{I}(a,b)$ such that $u^*w = 0$. If $u^*u(I-w^*w) \neq 0$ then, using part (1) (with u^* in place of u), we find a partial isometry v in $\mathscr{I}(a,b)$ such that $v^*v \leqslant I-w^*w$ and $vv^* \leqslant uu^* \leqslant I-ww^*$. But this contradicts the maximality of $\{u_\gamma\}$. Hence $u^*u \leqslant w^*w$.

For part (ii), assume that z is a central projection in M^b such that $w^*wz = 0$. Then we can find a non zero partial isometry u in $\mathscr{I}(a,b)$ such that $u^*u \leqslant z$ provided $z \leqslant e(a,b)$. The maximality property of w now implies that $uu^*ww^* \neq 0$ and, thus, we can use part (1) to get a partial isometry v in $\mathscr{I}(a,b)$ satisfying $vv^* \leqslant ww^*$ and $v^*v \leqslant u^*u \leqslant z$. Then w^*v is a non zero partial isometry in M^b with initial projection smaller than z and final projection smaller than $I-z$. Since z is central this is a contradiction. ∎

LEMMA 3.3. *For a and b in $Z_\alpha(G, U(M))$ there is a family $\{u_i : i \geqslant 1\}$ of partial isometries in $\mathscr{I}(a,b)$ such that*

(i) *$\{u_iu_i^* : i \geqslant 1\}$ is an orthogonal family of projections with sum $e(b,a)$.*

(ii) *For every i, $u_i^*u_i \leqslant u_1^*u_1$.*

(iii) *The central support of $u_1^*u_1$, in M^b, is $e(a,b)$.*

Proof. Let w be as in Lemma 3.2(2) and let $\{u_\gamma\}$ be a maximal family of partial isometries in $\mathscr{I}(a,b)$, containing w, whose final projections are pairwise orthogonal. Write f for $e(a,b) - \sum u_\gamma u_\gamma^*$. If $f \neq 0$ then there is a non zero partial isometry u in $\mathscr{I}(a,b)$ such that $uu^*f \neq 0$. Using Lemma 3.2(1) there is a non zero partial isometry v in $\mathscr{I}(a,b)$ such that $vv^* \leqslant f$. This contradicts the maximality of $\{u_\gamma\}$. Thus $f=0$. Since M is σ-finite $\{u_\gamma\}$ is at most countable and we are done. ∎

If $a(T)$ is a cocycle associated with an invertible operator T in M and $a(T)$ is equivalent to 1 then it is quasi-equivalent to 1. The converse is false as the following example shows.

EXAMPLE 3.4. Let H be $L^2[0, 1]$, P_t be the projection onto $L^2[0, t]$, $0 \leq t \leq 1$, and Q_t be the projection onto $L^2[0, t/2] \oplus L^2[1/2, 1/2 + t/2]$. Write

$$W_s = \int_0^1 e^{its} dP_t \quad \text{and} \quad U_s = \int_0^1 e^{its} dQ_t, \quad s \in \mathbf{R},$$

and $\alpha_s = \operatorname{ad} W_s$, $\beta_s = \operatorname{ad} U_s$ (on $M = B(H)$). By [7] there is some invertible operator T in $B(H)$ such that $[TP_t(H)] = Q_t(H)$, $t \in [0, 1]$. Hence $a(T)(s) = U_s W_s^*$. Since $\{P_t\}'$ is an abelian algebra but $\{Q_t\}'$ is not, T cannot be factored with respect to $B(H)^\alpha(\mathbf{R}_+)$ (this is the nest algebra associated with the nest $\{P_t\}$). Therefore $a(T)$ is not equivalent to 1.

Define isometries u_1 and u_2 on H as follows:

$$(u_1 f)(t) = \begin{cases} 2f(2t) & \text{if } 0 \leq t < 1/2 \\ 0 & \text{if } 1/2 \leq t \end{cases}$$

$$(u_2 f)(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1/2 \\ 2f(2t-1) & \text{if } 1/2 \leq t \end{cases}$$

for f in $L^2[0, 1]$.

Then $u_1^* u_1 = u_2^* u_2 = I$ and $u_1 u_1^* + u_2 u_2^* = I$. Also, for every $0 \leq t \leq 1$ and $i = 1, 2$, $u_i P_t = Q_t u_i$. Hence

$$u_i W_s = U_s u_i, \quad s \in \mathbf{R}, \; i = 1, 2.$$

We therefore have

$$u_i = U_s u_i W_s^* = U_s W_s^* \alpha_s(u_i) = a(T)\alpha_s(u_i).$$

It follows that u_i lie in $\mathscr{I}(a(T), 1)$ and u_i^* lie in $\mathscr{I}(1, a(T))$ for $i = 1, 2$. Therefore $e(a(T), 1) = e(1, a(T)) = I$ and, by Lemma 3.1, $a(T)$ is quasi-equivalent to 1.

Let α, Σ, W and $P(\cdot)$ be as in Section 2. An operator A in M is said to be *outer* if

(i) the range projection of A is in M^α; and

(ii) for every q in Γ, $[AP(q + \Sigma)(H)] = [A(H)] \cap P(q + \Sigma)(H)$.

(Note that this definition is consistent with Arveson's [2].) It follows from the definition that every outer operator is in $M^\alpha(\Sigma)$. Also, if A is invertible, then A is outer if and only if both A and A^{-1} lie in $M^\alpha(\Sigma)$.

An invertible operator T in M will be said to have a *weak factorization* with respect to $M^\alpha(\Sigma)$ if there are partial isometries $\{u_i : i \geq 1\}$ in M and outer operators $\{A_i : i \geq 1\}$ such that the following are satisfied.

(1) $u_i^* u_i$ is the range projection of A_i, for every i.

(2) $\{u_i u_i^* : i \geqslant 1\}$ is a family of orthogonal projections with sum I.

(3) The central support of $\bigvee u_i^* u_i$ in M^α is I.

(4) $T = \sum u_i A_i$ (where the convergence is in the σ-weak topology).

THEOREM 3.5. *Let T be an invertible operator in M. Then T has a weak factorization with respect to $M^\alpha(\Sigma)$ if and only if T is (P, Σ)-admissible and $a(T) \sim 1$.*

Proof. Suppose first that T has a weak factorization; i.e., $T = \sum u_i A_i$ where A_i are outer operators with range projections $u_i^* u_i$, $\sum u_i u_i^* = I$ and the central support of $\bigvee u_i^* u_i$ in M^α is I. Then, for every q in Γ,

$$(Q_q(H) =) [TP(q + \Sigma)(H)] = \sum\nolimits^{\oplus} u_i [A_i P(q + \Sigma)(H)] = \sum\nolimits^{\oplus} u_i P(q + \Sigma)(H).$$

Hence $Q_q = \sum u_i P(q + \Sigma) u_i^*$. Define the projection-valued measure $Q(\cdot) = \sum u_i P(\cdot) u_i^*$. Then $Q_q = Q(q + \Sigma)$, so T is (P, Σ)-admissible. Write

$$U_t = \int \langle t, q \rangle \, \mathrm{d}Q(q), \quad t \in G.$$

Also, we have for $j \geqslant 1$,

$$u_j^* Q(\cdot) = P(\cdot) u_j^*$$

(as $u_j^* u_i = 0$ if $i \neq j$). Hence $u_j^* U_t = W_t u_j^*$ and

$$u_j^* a(T) = \alpha_t(u_j^*).$$

Therefore $u_j^* \in \mathscr{I}(1, a(T))$ and $u_j \in \mathscr{I}(a(T), 1)$. Since $\sum u_j u_j^* = I$, $e(1, a(T)) = I$ and, since the central support of $\bigvee u_j^* u_j$ is I, $e(a(T), 1) = I$. Thus $a(T) \sim 1$.

For the other direction, suppose T is (P, Σ)-admissible and $a(T) \sim 1$. Then $e(1, a(T)) = e(a(T), 1) = I$ and, using Lemma 3.3, there are partial isometries $\{u_i : i \geqslant 1\}$ in $\mathscr{I}(a(T), 1)$ such that $\{u_i u_i^*\}$ is an orthogonal family of projections with sum I, $u_i^* u_i \leqslant u_1^* u_1$ for every i, and the central support of $u_1^* u_1$ in M^α is I.

Write $A_i = u_i^* T$. Then $\sum u_i u_i^* T = \sum u_i A_i$. Also, $[A_i(H)] = [u_i^* T(H)] = u_i^* u_i(H)$. For q in Γ and $i \geqslant 1$, $[A_i P(q + \Sigma)(H)] = [u_i^* TP(q + \Sigma)(H)] = u_i^* Q(q + \Sigma)(H)$ where $Q(q + \Sigma)(H) = [TP(q + \Sigma)(H)]$.

Since $u_i^* \in \mathscr{I}(1, a(T))$ we have,

$$u_i^* a(T)(t) = \alpha_t(u_i^*) = W_t u_i^* W_t^*, \quad t \in G.$$

We write $a(T)(t) = U_t W_t^*$ where $U_t = \int \langle t, q \rangle \mathrm{d}Q(q)$, and get,

$$u_i^* U_t = W_t u_i^*, \quad t \in G.$$

Hence

$$u_i^* Q(q + \Sigma) = P(q + \Sigma) u_i^*, \quad q \in \Gamma.$$

We now have,

$$[A_i P(q + \Sigma)(H)] = [u_i^* Q(q + \Sigma)(H)] = [P(q + \Sigma) u_i^*(H)] =$$

$$= [P(q + \Sigma) u_i^* u_i(H)] = [P(q + \Sigma)(H)] \cap [A_i(H)].$$

Hence A_i is outer for every i. ▨

LEMMA 3.6. *Let $T = \sum u_i A_i$ be a weak factorization of an invertible operator T in M. Then for every i and every y in H there is some z in H such that*

$$A_i z = A_i y \quad \textit{and} \quad A_j z = 0 \quad \textit{for every } j \neq i.$$

Proof. Let z be $T^{-1} u_i A_i y$. Then $A_i z = u_i^* T z = u_i^* u_i A_i y = A_i y$; and $A_j z = u_j^* T z = u_j^* u_i A_i y = 0$ if $i \neq j$. ▨

A set $\{A_i\}$ of outer operators will be called *independent* if for every i and every y in H there is some z in H such that $A_i z = A_i y$ and $A_j z = 0$ for $i \neq j$.

PROPOSITION 3.7. *Let $T \geqslant 0$ be an invertible operator in M. Then the following properties are equivalent.*

(1) *$T^{1/2}$ is (P, Σ)-admissible and $a(T^{1/2}) \sim 1$.*

(2) *$T = \sum A_i^* A_i$ for an independent set of outer operators $\{A_i\}$ such that the central support of $\bigvee p_i$ in M^α is I (where p_i is the range projection of A_i).*

Proof. If (1) holds then $T^{1/2}$ has a weak factorization and

$$T = (T^{1/2})^* T^{1/2} = (\sum u_i A_i)^* (\sum u_j A_j) = \sum A_i^* u_i^* u_i A_i = \sum A_i^* A_i.$$

Lemma 3.6 shows that $\{A_i\}$ is independent. Hence (2) holds.

Now assume that (2) is satisfied. Since $T = \sum A_i^* A_i$, we have $\|T^{1/2}x\|^2 = \sum \|A_i x\|^2$ for every x in H. Define u_i^* by $u_i^* T^{1/2} x = A_i x$ for x in H. Since $\|T^{1/2}x\| \geqslant \|A_i x\|$ and $T(H) = H$, u_i^* is well defined and $\|u_i^*\| \leqslant 1$.

Write $K = \sum^\oplus H_i$ where $H_i = H$ and define V from H into K by $V T^{1/2} x = \sum^\oplus A_i x$. Then V is an isometry since

$$\|T^{1/2}x\|^2 = \sum \|A_i x\|^2.$$

From the independence of $\{A_i\}$ it follows that $V(H)$ contains all finite sums $\sum^\oplus A_i h_i$, $h_i \in H$. As V is an isometry its range is closed; hence $V(H) = \sum^\oplus A_i(H)$. Let F_i be the projection in $B(K)$ with range H_i. Then $F_i V T^{1/2} x = A_i x$. But then (see the definition of u_i^* above) $u_i^* = F_i V$ (identifying H with H_i) and F_i commutes with the range projection of V. Hence u_i^* is a partial isometry in M with final space $A_i(H)$. Also, for every i, $\|A_i x\| = \|u_i^* T^{1/2} x\| = \|u_i u_i^* T^{1/2} x\|$. But then $\|T^{1/2} x\|^2 = \sum \|u_i u_i^* T^{1/2} x\|^2$. Hence for every y in H,

$$\|y\|^2 = \sum \|u_i u_i^* y\|^2.$$

It follows that $\sum u_i u_i^* = I$. Hence $T = \sum u_i u_i^* T = \sum u_i A_i$. Using Theorem 3.5, (1) follows. ▨

As we saw in Example 3.4, we might have an invertible operator T in M that has a weak factorization (equivalently $a(T) \sim 1$) but cannot be factored (i.e. $a(T) \not\simeq 1$). The following lemma shows that, in some cases, $a(T) \sim 1$ would imply $a(T) \simeq 1$.

LEMMA 3.8. *Suppose T has a weak factorization and either*

(1) M^α and M^{α^T} are both properly infinite, or

(2) M^α and M^{α^T} are both abelian.

Then T can be factored with respect to $M^\alpha(\Sigma)$.

Proof. If both are properly infinite the result follows from [15, 20.2(13)]. Suppose now that both M^α and M^{α^T} are abelian and $T = \sum u_i A_i$ as in the definition of weak factorization. Then, for $i \neq j$,

$$0 = u_i u_j^* u_i u_i^* u_j = u_i u_i^* u_i u_j^* u_j = u_i u_j^* u_j$$

(as $u_j^* u_i = 0$ and both $u_i u_j^*$ and $u_i u_i^*$ are in M^{α^T}). Hence $\{u_i^* u_i\}$ is an orthogonal family of projections in M^α. Since M^α is abelian and the central support of $\bigvee u_i^* u_i$ is I, we have $\sum u_i^* u_i = I$. We also have $\sum u_i u_i^* = I$. Hence $U = \sum u_i$ is a unitary operator in $\mathscr{I}(a(T), 1)$. Hence $a(T) \simeq 1$ and T can be factored. ▨

EXAMPLE 3.9. Let H be $L^2[0,1]$ and P_t, $0 \leqslant t \leqslant 1$, be the projection onto $L^2[0,t]$. Let $f:[0,1] \to [0,1]$ be an order isomorphism that does not preserve the Borel sets of measure 0 (see [5]). Then there is (by [7]) an invertible operator T in $B(H)$ such that $[TP_t(H)] = P_{f(t)}(H)$. However, there is no unitary operator U in $B(H)$ such that $UP_t(H) = P_{f(t)}(H)$. Hence T cannot be factored with respect to $B(H)^\alpha(\mathbf{R}_+)$ (where α is as in Example 3.4). Since $B(H) = \{P_t\}' = B(H)^{\alpha^T}$, both algebras are abelian and Lemma 3.8 shows that T does not have a weak factorization with respect to $B(H)^\alpha(\mathbf{R}_+)$. ▨

REFERENCES

1. ARVESON, W. B., Analyticity in operator algebras, *Amer. J. Math.*, **89**(1967), 578–642.
2. ARVESON, W. B., Interpolation problems in nest algebras, *J. Funct. Anal.*, **20**(1975), 203–233.
3. CONNES, A.; TAKESAKI, M., The flow of weights on a factor of type III, *Tôhoku Math. J.*, **29**(1977), 473–575.

4. DAUGHTRY, J., Factorization along several nests, preprint.
5. KADISON, R. V.; SINGER, I. M., Triangular operator algebras, *Amer. J. Math.*, **82**(1960), 227–259.
6. KOVÁCS, I.; SZÜCS, J., Ergodic type theorems in von Neumann algebras, *Acta Sci. Math. (Szeged)*, **27**(1966), 233–246.
7. LARSON, D. R., Nest algebras and similarity transformations, *Ann. of Math.*, **121**(1985), 409–427.
8. LARSON, D. R., Triangularity in operator algebras, Notes.
9. LOEBL, R. I.; MUHLY, P. S., Analyticity and flows in von Neumann algebras, *J. Funct. Anal.*, **29**(1978), 214–252.
10. PEDERSEN, G. K., *C^*-algebras and their automorphisms groups*, Academic Press, 1979.
11. PITTS, D., Thesis, University of California at Berkeley, 1986.
12. POWER, S., Factorisation in analytic operator algebras, preprint.
13. SOLEL, B., Analytic operator algebras (factorization and an expectation), *Trans. Amer. Math. Soc.*, **287**(1985), 799–817.
14. SOLEL, B., Factorization in operator aglgebras, *Proc. Amer. Math. Soc.*, to appear.
15. STRĂTILĂ, Ş., *Modular theory in operator algebras*, Abacus Press, Kent, England, 1981.

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