J. OPERATOR THEORY
20(1988), 311–327

LIFTING THEOREMS FOR NEST ALGEBRAS

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The dilation and model theory for contractions on a Hilbert space begins with the Sz.-Nagy dilation theorem which asserts that every contraction posseses a unitary dilation. This result characterises the contractive Hilbert space representations of the disc algebra $A(\mathbf{D})$ and moreover shows that such representations admit $*$-dilations to representations of $C(\mathbf{T})$ on a larger Hilbert space. For pairs of commuting contractions this result has two apparently different, but actually equivalen generalizations. Namely, the Sz.-Nagy—Foiaş lifting theorem and Ando's dilation theorem.

In this paper we obtain analogues of these results, where the disc algebra is replaced by an arbitrary nest algebra on a separable Hilbert space. The analogue of Sz.-Nagy's theorem for nest algebras was already obtained in [9].

Recently, Ball and Gohberg [4] have studied the contractive representations of upper triangular matrix algebras which have $*$-dilations to the enveloping full matrix algebras, and in this context they obtain a lifting theorem for a contraction commuting with the representation, thus obtaining an analogue of the Sz.-Nagy—Foiaş lifting theorem.

In Section 1, we prove the analogue of Ando's theorem for a finite dimensional nest algebra and a commuting contraction, which yields a new proof of the Ball-Gohberg result. In Section 3 we use the results of [9] to extend this result to arbitrary nest algebras on separable Hilbert spaces.

In Section 2, we prove the analogue of Ando's theorem where both contractions are replaced by commuting contractive representations of finite dimensional nest algebras. We then extend this result in Section 3 to arbitrary nest algebras on separable Hilbert spaces. In particular, we show that a pair of commuting σ-weakly continuous contractive representations of a pair of nest algebras admits a pair of commuting σ-weakly continuous $*$-dilations.

In Section 4 we use a lifting theorem to characterise the operator norm of abstract Hankel operators H_x associated with a nest algebra $\mathscr{A}$. We find that

$$\|H_x\| = \text{dist}(X, \mathscr{A}) = \sup_{E \in \text{Lat}\,\mathscr{A}} \|(I - E)XE\|$$

which is analogous to the Nehari theorem for classical Hankel operators, and which also includes the Arveson distance formula. This result was also obtained in [11].

The lifting theorems have fundamental implications for tensor products of various non-selfadjoint operator algebras. We discuss this and related matters in another paper.

McAsey and Muhly have observed in [6] that contractive representations of upper triangular matrix algebras are completely contractive, and so, by Arveson's dilation theorem, admit $*$-dilations. This was obtained by direct construction in [9], and here we pursue similar techniques together with the Sz.-Nagy—Foiaş lifting theorem to obtain generalised lifting and dilation theorems in the finite dimensional case. The extension to σ-weakly continuous contractive representations of nest algebras is obtained by exploiting the semi-discreteness property obtained in [9]. This property says that for the given nest algebra $\mathscr{A}$ on a separable Hilbert space there are finite dimensional nest algebras $\mathscr{A}_n$, completely contractive σ-weakly continuous maps $\varphi_n: \mathscr{A} \to \mathscr{A}_n$, and completely contractive homomorphisms $\psi_n: \mathscr{A}_n \to \mathscr{A}$, such that $\psi_n \circ \varphi_n(X)$ converges to X σ-weakly for each X in $\mathscr{A}$.

This paper is self-contained with the exception of the proofs of semi-discreteness and the following two well-known results.

THE SZ.-NAGY—FOIAŞ LIFTING THEOREM. *Let T be a contraction on a Hilbert space $\mathscr{H}$ with isometric dilation V on a Hilbert space $\mathscr{K} \supset \mathscr{H}$, and let X be an operator with $XT = TX$. Then there exists an operator Y on $\mathscr{K}$ commuting with V suc that $\|Y\| = \|X\|$ and $X = P_{\mathscr{H}} Y|\mathscr{H}$, where $P_{\mathscr{H}}$ is the orthogonal projection from $\mathscr{K}$ to $\mathscr{H}$.*

We usually consider the isometric dilation V on $\mathscr{K} = \mathscr{H} \oplus \mathscr{H} \oplus \ldots$ defined by $V(h_1, h_2, \ldots) = (Th_1, D_T h_1, h_2, \ldots)$, where $D_T = (I - T^*T)^{1/2}$. It is important to note that Y can be chosen in this case so that $Y^*\mathscr{H} \subset \mathscr{H}$, where $\mathscr{H}$ is identified with the first summand of $\mathscr{K}$. (See [15].) In particular, we have $T^n X^m = P_{\mathscr{H}} V^n X^m|\mathscr{H}$, for $n, m = 0, 1, 2, \ldots$.

THE ARVESON DILATION THEOREM. ([2]). *Let $\mathscr{A}$ be a unital subalgebra of the C^*-algebra $\mathscr{B}$, and let $\rho: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ be a contractive unital representation.*

Then the following conditions are equivalent:

(i) ρ *is completely contractive;*

(ii) *there is a unital* $*$*-representation* $\pi: \mathscr{B} \to \mathscr{L}(\mathscr{K})$ *on a Hilbert space* $\mathscr{K} \supset \mathscr{H}$ *such that* $\rho(A) = P_{\mathscr{H}}\pi(A)\,|\mathscr{H}$ *for all* A *in* $\mathscr{A}$.

Recall that a linear map φ from a space of operators $\mathscr{S}$ into $\mathscr{L}(\mathscr{H})$ is said to be *completely contractive* if the induced maps φ_n between the normed operator matrix spaces $M_n(\mathscr{S})$ and $M_n(\mathscr{L}(\mathscr{H}))$ are contractive for $n = 1, 2, \ldots$. The implication (ii) $\Rightarrow$ (i) is elementary, and the direction (i) $\Rightarrow$ (ii) is obtained in two stages. First the completely positive map $\tilde{\rho}$ defined on $\mathscr{A} + \mathscr{A}^*$ by $\tilde{\rho}(A_1 + A_2^*) = \rho(A_1) + \rho(A_2)^*$, is extended to a completely positive map φ from $\mathscr{B}$ to $\mathscr{L}(\mathscr{H})$, by an extension theorem of Arveson. Then φ is dilated to π by means of Stinespring's theorem [15]. In particular, if $\mathscr{B}$ and $\mathscr{H}$ are separable, the dilation space $\mathscr{K}$ can be assumed separable. Further details may be found in [2] and [8].

A *nest algebra* $\mathscr{A}$ on a Hilbert space $\mathscr{R}$ is an algebra of operators which leaves invariant the subspaces in a preassigned nest of subspaces. We always take $\mathscr{R}$ to be separable, and if $\mathscr{R}$ is finite dimensional we call $\mathscr{A}$ a *finite dimensional nest algebra.* General facts about nest algebras, and the density of compact and finite rank operators, may be found in the lecture notes [13], or the forthcoming book of Davidson [5].

We write $C(\mathbf{T})$ for the C^*-algebra of continuous complex valued functions on the unit circle, and write $A(\mathbf{D})$ for the disc algebra regarded as a closed subalgebra of $C(\mathbf{T})$.

1. LIFTING THEOREMS FOR FINITE DIMENSIONAL NEST ALGEBRAS

The lifting theorem of Ball and Gohberg [4] asserts that if an operator X commutes with a contractive representation ρ of a finite dimensional nest algebra $\mathscr{A}$ then there is a norm preserving lifting Y commuting with the $*$-dilation π of ρ. Theorem 1.2 below is a generalisation of this which obtains a lifting with much more structure, and can be viewed as an analogue of Ando's theorem that commuting contractions admit commuting unitary dilations. Recall that Ando's theorem and the Sz.-Nagy—Foiaş lifting theorem are essentially equivalent. The deduction of the lifting theorem from Ando's theorem is elementary, whilst the other direction is obtained by a somewhat non trivial two-stage argument. For details, see the discussion in Parrott [7] and our Remark 1.8 below.

The following result is a structured form of the Sz.-Nagy—Foiaş lifting theorem which will be used in the proofs of Theorems 1.2 and 2.1.

THEOREM 1.1. *Let* X_1, X_2 *and* T *be contractions on the Hilbert space* $\mathscr{H}$ *such that* $X_1T = TX_2$, *and such that with respect to the decomposition* $\mathscr{H} = \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_1$

(m times), we have representing operator matrices

$$X_i = \begin{bmatrix} 0 & X_{i,1} & & & & \\ & 0 & X_{i,2} & & & \\ & & 0 & \cdot & & \\ & & & \cdot & & \\ & & & & \cdot & X_{i,m-1} \\ & & & & & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} T_1 & & & & \\ & T_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & T_m \end{bmatrix},$$

for $i = 1, 2$, (where the unspecified entries are zero). Then there are isometric dilations $\tilde{X}_i$ on the Hilbert space $\tilde{\mathscr{H}} = \mathscr{H} \oplus \mathscr{H} \oplus \ldots$ of the form

$$\tilde{X}_i = \begin{bmatrix} 0 & \tilde{X}_{i,1} & & & & \\ & 0 & \tilde{X}_{i,2} & & & \\ & & 0 & \cdot & & \\ & & & \cdot & & \\ & & & & \cdot & \tilde{X}_{i,m-1} \\ U & & & & & 0 \end{bmatrix}, \quad i = 1, 2,$$

with respect to $\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_1 \oplus \ldots \oplus \tilde{\mathscr{H}}_1$ (m times), where $\tilde{\mathscr{H}}_1 = \mathscr{H}_1 \oplus \mathscr{H}_1 \oplus \ldots$, where U is the unilateral shift on $\tilde{\mathscr{H}}_1$, and there is a contaction $\tilde{T}$ on $\tilde{\mathscr{H}}$ of the form $\tilde{T} = \tilde{T}_1 \oplus \ldots \oplus \tilde{T}_n$, such that $\tilde{X}_1\tilde{T} = \tilde{T}\tilde{X}_2$, and

$$\tilde{T}_i = \begin{bmatrix} T_i & 0 \\ * & * \end{bmatrix}$$

with respect to the decomposition $\tilde{\mathscr{H}}_i = \mathscr{H}_i \oplus (\tilde{\mathscr{H}}_i \ominus \mathscr{H}_i)$, $1 \leqslant i \leqslant m$.

Proof. Define $\tilde{X}_{ij}$ on $\tilde{\mathscr{H}}_1$ by $\tilde{X}_{ji}(h_1, h_2, \ldots) = (X_{ij}h_1, D_{ij}h_1, h_2, \ldots)$, where $D_{ij} = (I - X_{ij}^* X_{ij})^{1/2}$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant m-1$ and observe that the associated operator $\tilde{X}_i$ is an isometric dilation of X_i. By the Sz.-Nagy—Foiaş lifting theorem there is a contraction $\hat{T}$ on $\hat{\mathscr{H}}$ of the form

$$\hat{T} = \begin{bmatrix} T & 0 \\ * & * \end{bmatrix}$$

with respect to $\hat{\mathscr{H}} = \mathscr{H} \oplus (\mathscr{H})^{\perp}$, such that $\tilde{X}_1\hat{T} = \hat{T}\tilde{X}_2$.

Let D be the diagonal operator $I \oplus wI \oplus \ldots \oplus w^{m-1}I$ on $\mathscr{H} = \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_1$ where w is the primitive m^{th} root of unity, and note that $D^*X_iD = wX_i$, $i = 1, 2$, and $D^*TD = T$. Also D has a natural extension $\tilde{D}$ on $\hat{\mathscr{H}}$ such that $\tilde{D}^*\tilde{X}_i\tilde{D} = w\tilde{X}_i$. Observe that $(\tilde{D}^*)^j\hat{T}\tilde{D}^j$ has compression equal to T and also intertwines $\tilde{X}_1$ and $\tilde{X}_2$. It follows that the operator $T = m^{-1}\sum_{j=1}^{m}(\tilde{D}^*)^j\hat{T}\tilde{D}^j$ has the required properties. ▨

The proof of the next theorem contains the basic construction used in [9] of $*$-dilations for contractive representations of finite dimensional nest algebras $\mathscr{A} \subseteq M_n$.

THEOREM 1.2. *Let ρ be a contractive representation of a finite dimensional nest algebra $\mathscr{A} \subseteq M_n$ on the Hilbert space $\mathscr{H}$, and let X be a contraction that commutes with $\rho(A)$, for all A in $\mathscr{A}$. Then there exists a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, a $*$-homomorphism $\pi\colon M_n \to \mathscr{L}(\mathscr{K})$, and a unitary operator U on $\mathscr{K}$ which commutes with $\pi(B)$ for all B in M_n, such that*

$$X^n\rho(A) = P_{\mathscr{H}}U^n\pi(A)\,|\,\mathscr{H},$$

for $n = 0, 1, 2, \ldots$, and A in $\mathscr{A}$.

Proof. We may assume that ρ is unital. We first consider the case where $\mathscr{A} = \mathscr{A}_u$ is the upper triangular matrix algebra in M_n.

For each i the operator $E_i = \rho(e_{i,i})$ is a self-adjoint projection, with range space $\mathscr{H}_i$ and $\mathscr{H} = \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$. The contraction $\rho(e_{i,j})$ has range contained in $\mathscr{H}_i$ and kernel containing $(\mathscr{H}_j)^{\perp}$, for $1 \leqslant i \leqslant j \leqslant n$. Let $T_{ij} = E_i\rho(e_{ij})E_j$ and we have $\rho((a_{ij})) = (a_{ij}T_{ij})$ as an operator matrix on $\mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$, for (a_{ij}) in $\mathscr{A}$, and $T_{ij} = T_{i,i+1} \ldots T_{j-1,j}$, $1 \leqslant i \leqslant j \leqslant n$. The representation ρ is determined by the contractions $T_i = T_{i,i+1}$ and we write $\rho = \rho_{\{T_i\}}$ to indicate such a representation. Since X commutes with ρ we see that $X = X_1 \oplus \ldots \oplus X_n$, a diagonal operator on $\mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$, and the $X_iT_i = T_iX_{i+1}$, for $1 \leqslant i \leqslant n-1$.

Without loss of generality we assume that $\mathscr{H}_i = \mathscr{H}_j$ for all $1 \leqslant i, j \leqslant n$. If this does not already hold then we can arrange it to be true for a trivial dilation of the pair ρ and X obtained by adding trivial summands.

Let $\hat{T}_i$ be the isometric dilation of the operator T_i acting on the space $\hat{\mathscr{H}}_i = \mathscr{H}_i \oplus \mathscr{H}_i \oplus \ldots$, given by $\hat{T}_i(h_1, h_2, \ldots) = (T_i h_1, C_i h_1, h_2, \ldots)$ where $C_i = (I - T_i^* T_i)^{1/2}$, $1 \leqslant i \leqslant n-1$. By Theorem 1.1 (with reversed notation) there exist contractions $\hat{X}_i$ on $\hat{\mathscr{H}}_i$ of the form

$$\begin{bmatrix} X_i & 0 \\ * & * \end{bmatrix}$$

with respect to the decomposition $\hat{\mathscr{H}}_i = \mathscr{H}_i \oplus (\hat{\mathscr{H}}_i \ominus \mathscr{H}_i)$, such that $\hat{X}_i \hat{T}_i = \hat{T}_i \hat{X}_{i+1}$, for $1 \leqslant i \leqslant n-1$.

These relations imply that $\hat{X}$ commutes with $\hat{\rho}(A) = \rho_{\{\hat{T}_i\}}(A)$ on $\hat{\mathscr{H}}$ and that $X^n \rho(A) = P_{\mathscr{H}} \hat{X}^n \hat{\rho}(A) | \mathscr{H}$ for all $n = 0, 1, 2, \ldots$, and A in $\mathscr{A}_u$. Here we identify $\mathscr{H}_i$ with $\mathscr{H}_i \oplus 0 \oplus 0 \ldots$ in $\hat{\mathscr{H}}_i$.

Now define an isometry W on $\hat{\mathscr{H}}$ by setting $W(\hat{h}_1, \ldots, \hat{h}_n) = (\hat{h}_1, \hat{T}_1 \hat{h}_2, \ldots \ldots, \hat{T}_1 \cdots \hat{T}_{n-1} \hat{h}_n)$ and define a $*$-homomorphism $\pi_0 \colon M_n \to \mathscr{L}(\hat{\mathscr{H}})$ via $\pi_0(e_{ij}) = \hat{E}_{ij}$, where e_{ij} are the canonical matrix units in M_n and $\hat{E}_{ij}$ are the canonical matrix units for $\hat{\mathscr{H}} = \hat{\mathscr{H}}_1 \oplus \ldots \oplus \hat{\mathscr{H}}_n$. (Recall that $\mathscr{H}_i = \mathscr{H}_j$ and so $\hat{\mathscr{H}}_i = \hat{\mathscr{H}}_j$.) Let $Y = \hat{X}_1 \oplus \ldots \oplus \hat{X}_1$.

We claim that $W^* Y^n \pi_0(A) W = \hat{X}^n \pi_0(A)$ for all $n = 0, 1, \ldots$, and A in $\mathscr{A}_u$. To see this, note that for $2 \leqslant i < j \leqslant n$, $W^* Y^n \hat{E}_{ij} W$ is the operator matrix which is 0 except for the (i, j)-th entry which is,

$$\hat{T}_{i-1}^* \ldots \hat{T}_1^* \hat{X}_1^n \hat{T}_1 \ldots \hat{T}_{j-1} = \hat{T}_{i-1}^* \ldots \hat{T}_1^* \hat{T}_1 \ldots \hat{T}_{i-1} \hat{X}_i^n \hat{T}_i \ldots \hat{T}_{j-1} = \hat{X}_i^n \hat{T}_{i,j}.$$

This last quantity is clearly the (i, j)-th entry of $\hat{X}^n \hat{\rho}(E_{ij})$, which is also 0 in its remaining entries. The calculation for other E_{ij} in $\mathscr{A}_u$ follows similarly.

Thus, for any h, k in $\mathscr{H}$, we have $\langle X^n \rho(A) h, k \rangle = \langle Y^n \pi_0(A) W h, W k \rangle$. If we identify $\mathscr{H}$ with $W\mathscr{H} \subseteq \hat{\mathscr{H}}$, then this last equation becomes $X^n \rho(A) = P_{\mathscr{H}} Y^n \pi_0(A) | \mathscr{H}$.

Finally, if we let U_1 be the unitary dilation of $\hat{X}_1$ on $\mathscr{K}_0$, $\hat{\mathscr{H}}_i \subseteq \mathscr{K}_0$, set $U = U_1 \oplus \ldots \oplus U_1$ on $\mathscr{K} = \mathscr{K}_0 \oplus \ldots \oplus \mathscr{K}_0$ (n copies), and let $\pi \colon M_n \to \mathscr{L}(\mathscr{K})$ be the obvious representation, we then obtain the desired result, for the case that $\mathscr{A}$ is the algebra of upper triangular matrices. Note that $\mathscr{H}_i$ is contained in the i-th copy of $\mathscr{K}_0$.

The case of a general nest subalgebra $\mathscr{A}$ of M_n is deduced by first restricting ρ to the upper triangulars $\mathscr{A}_u$, applying the above result to obtain (π, U), and observing that the desired relations also hold for all A in $\mathscr{A}$ as well as just in $\mathscr{A}_u$. To see this it will be sufficient to let $i < j$ such that $e_{ji} \in \mathscr{A}$ and show that $X^n \rho(e_{ji}) = P_{\mathscr{H}} U^n \pi(e_{ji}) | \mathscr{H}$.

Let $W_i: \mathcal{H}_i \to \mathcal{K}_0$ be the isometric inclusion obtained above, so that $W: \mathcal{H} \to \mathcal{K}$ defined by $W(h_1, \ldots, h_n) = (W_1 h_1, \ldots, W_n h_n)$ satisfies $X^n \rho(A) = W^* U^n \pi(A) W$ for A in $\mathcal{A}_n$. In terms of operator matrices this says that,

$$X_i^n T_{ij} = W_i^* U_1^n W_j,$$

for $n = 0, 1, 2, \ldots,$ and $1 \leqslant i \leqslant j \leqslant n$, with $T_{ii} = I_{\mathcal{H}_i}$. Since $\rho(E_{ij})\rho(E_{ji}) = \rho(E_{ii})$, we have that $T_{ij}T_{ji} = I_{\mathcal{H}_i}$. Hence, $W_i^* W_j W_j^* W_i = W_i^* W_i$ and so $W_j W_j^* W_i = W_i$. Thus, $X_j^n T_{ji} = (W_j^* U_1^n W_i)(W_j^* W_i) = W_j^* U_1^n W_i$, and so the operator matrix $X^n \rho(E_{ji})$ is equal to $W^* U^n \pi(E_{ji}) W$. After again identifying $\mathcal{H}$ with $W\mathcal{H}$, we obtain the desired result. ▨

What we have really shown in the above proof, is that the relations $X_i T_i = T_i X_{i+1}$, $i = 1, \ldots, n-1$, have a representation $(U_1, W_1, \ldots, W_n)$, where U_1 is unitary and the W_i are isometries, such that $X_i^n T_i = W_i^* U_1^n W_{i+1}$, and $W_i W_i^* W_{i+1} = W_{i+1}$. The initial relations determine a representation $\rho_{\{T_i\}}$ and commuting contraction X, while the latter clearly yield the dilation.

Let $\mathcal{A} \subseteq M_n$ be a nest algebra and let $M_n(C(\mathbf{T}))$ denote the algebra of $n \times n$ matrices with entries from $C(\mathbf{T})$. We identify $\mathcal{A} \otimes A(\mathbf{D})$ with the subalgebra of $M_n(C(\mathbf{T}))$ consisting of those matrices of functions (f_{ij}) such that f_{ij} belongs to $A(\mathbf{D})$ and $f_{ij} = 0$ if e_{ij} does not belong to $\mathcal{A}$. The next corollary is an immediate consequence of the last theorem and the complete contractivity of compression mappings and $*$-representations. By Arveson's dilation theorem it is in fact equivalent to Theorem 1.2.

COROLLARY 1.3. *Let $\mathcal{A}$ be a finite dimensional nest algebra and let $\rho_1: \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\rho_2: A(\mathbf{D}) \to \mathcal{L}(\mathcal{H})$ be commuting contractive representations. Then the representation $\rho_1 \otimes \rho_2$ of $\mathcal{A} \otimes A(\mathbf{D})$ defined by $\rho_1 \otimes \rho_2((f_{ij})) = \sum_{i,j} \rho_1(f_{ij})\rho_2(e_{ij})$ is completely contractive.*

COROLLARY 1.4. (Ball and Gohberg). *Let $\mathcal{A}$ be a finite dimensional nest algebra with enveloping matrix algebra M_n, let $(\rho, \mathcal{H})$ be a representation of $\mathcal{A}$ with a contractive M_n-dilation $(\pi, \mathcal{K})$, and let X be an operator on $\mathcal{H}$ such that $X\rho(A) = \rho(A)X$, for all operators A in $\mathcal{A}$. Then there exists an operator Y on $\mathcal{K}$ such that $\|Y\| = \|X\|$, $Y\pi(A) = \pi(A)Y$ for all A in M_n, and $X = P_{\mathcal{H}} Y \mid \mathcal{H}$.*

Proof. Let $\mathcal{M}$ be the minimal reducing subspaces for $\pi(M_n)$ which contains the subspace $\mathcal{H}$. Then the associated restriction representation is a minimal M_n-dilation of $(\rho, \mathcal{H})$, and is unique up to the usual notion of unitary equivalence of dilations.

Without loss, let X be a contraction, and let $(\pi_1, \mathcal{K}_1)$ and U in $\mathcal{L}(\mathcal{K}_1)$ be the commuting dilations of $(\rho, \mathcal{H})$ and X provided by Theorem 1.1. If $\mathcal{M}_1$ is the minimal reducing subspace for $\pi_1(M_n)$ containing $\mathcal{H}$, then $(\pi, \mathcal{M})$ and $(\pi_1, \mathcal{M}_1)$ are unitarily equivalent dilations, and so we may identify them. Define $Y_0 = P_{\mathcal{M}} U \mid \mathcal{M}$ and note that Y_0 commutes with the operators $\pi(A) \mid \mathcal{M}$. Let $Y = X_0 \oplus 0$ on $\mathcal{M} \oplus \mathcal{M}^{\perp} = \mathcal{H}$ and we are finished. ∎

REMARK 1.5. The intertwining version of the lifting theorem concerns an operator X satisfying $X\rho_1(A) = \rho_2(A)X$ for all A in $\mathcal{A}$, where ρ_1 and ρ_2 are contractive representations of the nest algebra $\mathcal{A}$. The existence of an intertwining extension for dilations π_1, π_2 of ρ_1, ρ_2 follows easily from the theorem above and the familiar observation that the contractive representation $\rho = \rho_2 \oplus \rho_1$ commutes with the operator

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

1.6. Ball and Gohberg provide two proofs of the lifting theorem above both of which are quite different from ours. The most elaborate of these, which also yields information about *all* the commuting liftings, makes use of the Krein space approach to the analysis of invariant subspaces for representations of nest algebras ([3], [4]). The other arugument uses a dual extremal formulation and a use of the Hahn-Banach theorem. This latter argument is analogous to Sarason's proof of his early version of the lifting theorem for contractions related to the unilateral shift [14].

1.7. A different proof of Theorem 1.2 can be given that is similar to arguments used to deduce Ando's theorem from the Sz.-Nagy—Foiaş lifting theorem as discussed by Parrott [7]. Here the lifting theorem is used to obtain a dilation $\tilde{T}$ of T commuting with the isometric dilation $\tilde{X}$ of the contraction X. At this point it must be observed that the pair $\tilde{T}$, $\tilde{X}$ provide a commuting power dilation of the commuting pair T, X. Next an extension $\hat{T}$ of $\tilde{T}$ is constructed using the unitary dilation $\hat{X}$ of $\tilde{X}$, so that $\hat{T}$ and $\hat{X}$ provide a commuting power dilation for the pair T, X. In fact $\hat{T}$ is essentially the strong limit of the sequence $(\hat{X}^*)^n\tilde{T}\hat{X}^n$. In this way the dilation problem is reduced to the case of a commuting pair where one of the contractions is unitary, and there are direct methods to treat this.

Suppose now that we have as before, contractions on a common Hilbert space satisfying the relations $X_iT_i = T_iX_{i+1}$, $i = 1, \ldots, n-1$, and hence a representation ρ of the upper triangular $n \times n$ matrix algebra commuting with the contraction $X_1 \oplus \ldots \oplus X_n$. Let $\tilde{X}$ and $\hat{X}$ be the natural isometric and unitary dilations respectively for X, with summands on a common dilation space. Then, using the lifting theorem, we can obtain dilations $\tilde{T}_i$ of T_i, satisfying the dilated relations, and hence a representation $\tilde{\rho}$ of ρ such that $\tilde{\rho}$ and $\tilde{X}$ are a commuting dilating pair for ρ and X. As in the last paragraph we next construct the norm preserving extension $\hat{T}_i$ of $\tilde{T}_i$ as the strong limit of the sequence $(\hat{X}_i^*)^n \tilde{T}_i \hat{X}_{i+1}^n$, to obtain a representation $\hat{\rho}$ such that $\hat{\rho}$, X form a commuting dilating pair for $\hat{\rho}$, $\hat{X}$. Once more we have reduced to the case where X is a unitary contraction and various direct methods can be used for this case. One such method is indicated in the next remark.

1.8. For doubly commuting contractions Ando's theorem has a more elementary proof. Similarly, if both X and X^* commute with the representation ρ in the statement of Theorem 1.2, then we can provide more elementary arguments. A useful result in this context is the lifting theorem of Arverson for the commutant of the range of a completely positive mapping (see [2] and [8, p. 162]): if π is a unital $*$-representation of a C^*-algebra B, on the Hilbert space $\mathscr{K}$, and if $P: \mathscr{K} \to \mathscr{H}$ is an orthogonal projection, then there is a $*$-isomorphism from the commutant $\{P\pi(\mathscr{B})P\}'$ onto $\{\pi(B)\}' \cap \{P\}'$. Using this principle we can obtain a dilation π_1 of ρ commuting with X_1 and X_1^*, where X_1 is a dilation of X. Applying the principle again, for the C^*-algebra generated by the unitary dilation U of X_1, we obtain a representation π commuting with U, with the required properties.

2. COMMUTING CONTRACTIVE REPRESENTATIONS OF FINITE DIMENSIONAL NEST ALGEBRAS

We now turn to the proof of an Ando-type dilation theorem for a pair of commuting contractive representations of finite dimensional nest algebras.

THEOREM 2.1. *Let ρ_1 and ρ_2 be contractive unital representations of the finite dimensional nest algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, on the common Hilbert space $\mathscr{H}$, such that $\rho_1(A_1)\rho_2(A_2) = \rho_2(A_2)\rho_1(A_1)$ for all A_i in $\mathscr{A}_i$, $i = 1, 2$. Then there exist unital $*$-representations π_1, π_2 of the enveloping matrix algebras $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, on a Hilbert space $\mathscr{K} \supset \mathscr{H}$, such that*

(i) $\rho_1(A_1)\rho_2(A_2) = P_{\mathscr{H}}\pi_1(A_1)\pi_2(A_2)|\mathscr{H}$,

(ii) $\pi_1(B_1)\pi_2(B_2) = \pi_2(B_2)\pi_1(B_1)$,

for all A_i in $\mathscr{A}_i$ and B_i in $\mathscr{B}_i$, $i = 1, 2$.

Proof. Assume first that $\mathscr{A}_1$ and $\mathscr{A}_2$ are the algebras of upper triangular $n \times n$ and $m \times m$ matrices, respectively, spanned by the matrix units e_{ij}, $1 \leqslant i \leqslant j \leqslant n$, and f_{ij}, $1 \leqslant i \leqslant j \leqslant m$, respectively. Let $\mathscr{H}_i = \rho_1(e_{ii})\mathscr{H}$, $1 \leqslant i \leqslant n$, and let $\mathscr{H}_{i,j} = \rho_2(f_j)\mathscr{H}_i$ for $1 \leqslant j \leqslant m$. Without loss we may assume that $\mathscr{H}_{i,j} = \mathscr{H}_{1,1}$ for all i, j. With respect to the decomposition $\mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$ the operators $T = \rho_1(e_{1,2} + \ldots + e_{n-1,n})$ and $X = \rho_2(f_{1,2} + \ldots + f_{m-1,m})$ have representing operator matrices

$$T = \begin{bmatrix} 0 & T_1 & & & & \\ & 0 & T_2 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & \cdot & 0 & T_{n-1} \\ & & & & & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} X_1 & & & & \\ & X_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & X_n \end{bmatrix},$$

and with respect to $\mathscr{H}_i = \mathscr{H}_{i,1} \oplus \ldots \oplus \mathscr{H}_{i,m}$ we have

$$T_i = \begin{bmatrix} T_{i,1} & & & & \\ & T_{i,2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & T_{i,m} \end{bmatrix}$$

for $1 \leqslant i \leqslant n-1$, and,

$$X_i = \begin{bmatrix} 0 & X_{i,1} & & & \\ & 0 & X_{i,2} & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & X_{i,m-1} \\ & & & & 0 \end{bmatrix}$$

for $1 \leqslant i \leqslant n$. Note that X commutes with T if and only if $X_{i,j}T_{i,j+1} = T_{i,j}X_{i+1,j}$, for $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant m-1$. Conversely if we have operators satisfying these relations then the operators $T_1, \ldots, T_{n-1}$ determine a representation of $\mathscr{A}_1$ commuting with the representation $\rho = \rho_2^1 \oplus \ldots \oplus \rho_2^n$ of $\mathscr{A}_2$ on $\mathscr{H}$, determined by the representations ρ_2^i of $\mathscr{A}_2$ on $\mathscr{H}_i$ associated with the contractions $X_{i,1}, \ldots, X_{i,m-1}$, for $1 \leqslant i \leqslant n$.

By Theorem 1.1, and its proof, if $\tilde{X}_{i,j}$ is the usual isometric dilation of X_{ij} on $\tilde{\mathscr{H}}_{i,j} = \mathscr{H}_{i,j} \oplus \mathscr{H}_{i,j} \oplus \ldots$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m-1$, then there are dilations $\tilde{T}_i = \tilde{T}_{i,1} \oplus \ldots \oplus \tilde{T}_{i,m}$ of T_i, $1 \leqslant i \leqslant n-1$, such that $\tilde{X}_{i,j}\tilde{T}_{i,j+1} = \tilde{T}_{i,j}\tilde{X}_{i+1,j}$. Hence we obtain associated commuting contractive representations $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Moreover, in view of the special form of the operators $\tilde{T}_{i,j}$, products of the operators $\tilde{T}_{ij}$ dilate the corresponding of the operators T_{ij} and hence $\rho_1(A_1)\rho_2(A_2) = P_{\mathscr{H}}\tilde{\rho}_1(A_1)\tilde{\rho}_2(A_2)|\mathscr{H}$, for A_i in $\mathscr{A}_i$, $i = 1, 2$.

Exchanging the roles ρ_1 and ρ_2 in the argument above we may assume that $\tilde{\rho}_1$ is the dilation of ρ_1 obtained by the canonical isometric dilations $\hat{T}_1, \ldots, \hat{T}_{n-1}$ of $T_1, \ldots, T_{n-1}$, and that $\tilde{\rho}_2$ is a contractive commuting dilation such that this pair $\tilde{\rho}_1$, $\tilde{\rho}_2$ dilate the pair ρ_1, ρ_2. Write $\hat{X}_1 \oplus \ldots \oplus \hat{X}_n$ for the dilation $\tilde{\rho}_2(f_{1,2} + \ldots + f_{n-1,n})$ of X.

As in the proof of Theorem 1.2, define the isometry W on $\tilde{\mathscr{H}}$ by $W(\hat{h}_1, \ldots, \hat{h}_n) = (\hat{h}_1, \hat{T}_1 h_2, \ldots, \hat{T}_1 \cdots \hat{T}_{n-1} h_n)$, and define the $*$-isomorphism $\sigma_1 \colon M_n \to \mathscr{L}(\tilde{\mathscr{H}})$ by $\sigma_1(e_{ij}) = \tilde{E}_{ij}$, where $\hat{E}_{ij}$ is the partial isometry identifying the j^{th} and i^{th} summands of $\tilde{\mathscr{H}}$. Let $Y = \hat{X}_1 \oplus \ldots \oplus \hat{X}_1$ (n times), and observe that, as before, for A in $\mathscr{A}_1$,

$$X^n \rho_1(A) = P_{\mathscr{H}} Y^n \sigma_1(A) \,|\, \mathscr{H},$$

where we have identified $\mathscr{H}$ with $W\mathscr{H} \subset \hat{\mathscr{H}}$. Using Y we can construct a contractive unital representation $\tau = \tau_1 \oplus \ldots \oplus \tau_1$ (n times) of $\mathscr{A}_2$, which commutes with σ_1, and satisfies

$$\rho_2(A_2)\rho_1(A_1) = P_{\mathscr{H}}\tau(A_2)\sigma_1(A_1) \mid \mathscr{H}$$

for A_i in $\mathscr{A}_i$, $i = 1, 2$. We have now reduced to the case where one of the representations is an inflation and this can be dealt with in a very explicit way. Let $\pi_2^1 \colon M_m \to \mathscr{L}(\mathscr{K}_1)$, $\mathscr{K}_1 \supset \mathscr{H}_1$ be the canonical $*$-dilation of τ_1. Let $\pi_2 = \pi_2^1 \oplus \ldots \oplus \pi_2^1$ (n times) on $\mathscr{K} = \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_1$ and let $\pi_1 \colon M_n \to \mathscr{L}(\mathscr{K})$ be the obvious representation, which dilates σ_1 and commutes with π_2. Then π_1 and π_2 give the desired dilation of ρ_1 and ρ_2.

The case of general finite dimensional nest algebras, $\mathscr{A}_1$, $\mathscr{A}_2$ is now derived by first restricting ρ_1 and ρ_2 to the upper triangular subalgebras $\mathscr{A}_{1,\mathrm{u}}$, $\mathscr{A}_{2,\mathrm{u}}$ respectively and obtaining the dilating commuting pair π_1, π_2 for the restrictions of ρ_1 and ρ_2. The argument in the final paragraph of the proof of Theorem 1.1 already shows that π_1 and π_2 necessarily have the dilation properties for $\mathscr{A}_1$, $\mathscr{A}_2$.

3. DILATION AND LIFTING THEOREMS

We now generalise the results of the last two sections to general nest algebras acting on a separable Hilbert space $\mathscr{R}$. Our method is to use the semidiscretness of nest algebras to obtain the complete contractivity of a representation of a spatial tensor product algebra associated with the given representations.

It was shown in [9] that a nest algebra $\mathscr{A}$ on a separable Hilbert space is semidiscrete in the sense that there are finite dimensional nest algebras $\mathscr{A}_1, \mathscr{A}_2, \ldots$, completely contractive σ-weakly continuous maps $\varphi_n : \mathscr{A} \to \mathscr{A}_n$, and completely isometric σ-weakly continuous homomorphism $\psi_n : \mathscr{A}_n \to \mathscr{A}$, such that $\psi_n \circ \varphi_n(A) \to A$ σ-weakly for all A in $\mathscr{A}$. Moreover. we can arrange that $\mathrm{dist}(K, \psi_n(A_n)) \to 0$ for each compact operator K in $\mathscr{A}$, and we shall need this extra detail in the proofs below.

THEOREM 3.1. *Let $\mathscr{A}$ be a nest algebra on a separable Hilbert space $\mathscr{R}$, let ρ be a σ-weakly continuous contractive representation of $\mathscr{A}$ on $\mathscr{H}$, and let X be a contraction on $\mathscr{H}$ that commutes with $\rho(\mathscr{A})$. Then there is an inflation $\pi : \mathscr{L}(\mathscr{R}) \to \mathscr{L}(\mathscr{R} \oplus \mathscr{R} \oplus \ldots)$ given by $\pi(A) = A \oplus A \oplus \ldots$, with at most countably many copies, a unitary U that commutes with $\pi(\mathscr{A})$, and an isometry $V : \mathscr{H} \to \mathscr{R} \oplus \mathscr{R} \oplus \oplus \ldots$, such that*

$$X^n \rho(A) = V^* U^n \pi(A) V$$

for all $n = 0, 1, 2, \ldots$, and A in $\mathscr{A}$.

Proof. Let $\mathscr{B}_1$ denote the C^*-algebra generated by the compact operators and the identity and let $\mathscr{A}_1 = \mathscr{A} \cap \mathscr{B}_1$. We regard $\mathscr{A}_1 \otimes A(\mathbf{D})$ as a subalgebra of the C^*-algebra $\mathscr{B}_1 \otimes C(\mathbf{T})$.

Let $\mathscr{C}_1, \mathscr{C}_2, \ldots$ be subalgebras of $\mathscr{A}$ which are completely isometric images of finite dimensional nest algebras, and satisfy $\mathrm{dist}(K, \mathscr{C}_n) \to 0$ for every compact operator K in $\mathscr{A}$. Clearly, $\mathrm{dist}(A, \mathscr{C}_n) \to 0$ for every A in $\mathscr{A}_1$.

By Corollary 1.3, X and $\rho \mid \mathscr{C}_n$ gives rise to a completely contractive representation of $\mathscr{C}_n \otimes A(\mathbf{D})$. From this it follows that X and $\rho \mid \mathscr{A}_1$ gives rise to a completely contractive respresentation of the algebra $\mathscr{A}_1 \otimes A(\mathbf{D})$.

Hence, there exists a separable Hilbert space $\mathscr{K}$, a $*$-homomorphism $\pi : \mathscr{B}_1 \to \mathscr{L}(\mathscr{K})$, a unitary U on $\mathscr{K}$ which commutes with $\pi(\mathscr{B}_1)$, and an isometry $V : \mathscr{H} \to \mathscr{K}$ such that

$$X^n \rho(A) = V^* U^n \pi(A) V,$$

$n = 0, 1, 2, \ldots$, and A in $\mathscr{A}_1$.

The $*$-homomorphism π decomposes as $\pi_1 \oplus \pi_0$ on $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_0$ with π_1 faithful on the compacts and π_0 zero on the compacts. Relative to this decomposition $U = U_1 \oplus U_0$ with U_i in the commutant of $\pi_i(\mathscr{B}_1)$, $i = 1, 2$.

Now using the σ-weak continuity of ρ, and choosing a sequence K_n of compacts in $\mathscr{A}_1$ which converges σ-weakly to the identity (see [13] or [5]), we see that in fact, $V\mathscr{H} \subseteq \mathscr{K}_1$ and $X''\rho(A) = V^* U_1'' \pi_1(A) V$ for A in $\mathscr{A}_1$. Note that π_1 is, up to unitary equivalence, a countable direct sum of the identity representation. Hence, π_1 is σ-weakly continuous, and since $\mathscr{A}_1$ is σ-weakly dense in $\mathscr{A}$ the remainder of the proof follows. ▨

The following corollary generalises the Ball-Gohberg theorem and is obtained easily from Theorem 3.1 and elementary arguments.

COROLLARY 3.2. *Let $\mathscr{A}$ be a nest algebra on $\mathscr{R}$, let ρ be a σ-weakly continuous contractive representation of $\mathscr{A}$ on $\mathscr{H}$, with σ-weakly continuous $\mathscr{L}(\mathscr{R})$-dilation π on $\mathscr{K} \supset \mathscr{H}$, and let X be an operator commuting with the range of ρ. Then there exists an operator Y on $\mathscr{K}$ which commutes with the range of π and satisfies $\|Y\| = \|X\|$, $X = P_{\mathscr{H}} Y | \mathscr{H}$.*

THEOREM 3.3. *Let $\mathscr{A}_1$, $\mathscr{A}_2$ be nest algebras on separable Hilbert spaces $\mathscr{R}_1$, $\mathscr{R}_2$. Let ρ_1, ρ_2 be σ-weakly continuous representations of $\mathscr{A}_1$ and $\mathscr{A}_2$ on the separable Hilbert space $\mathscr{H}$, such that $\rho_1(A_1)\rho_2(A_2) = \rho_2(A_2)\rho_1(A_1)$ for all A_i in $\mathscr{A}_i$, $i = 1, 2$. Then there exist σ-weakly continuous $*$-isomorphisms π_1, π_2 of $\mathscr{L}(\mathscr{R}_1)$ and $\mathscr{L}(\mathscr{R}_2)$ on a separable Hilbert space $\mathscr{K} \supseteq \mathscr{H}$, such that*

(i) $\rho_1(A_1)\rho_2(A_2) = P_{\mathscr{H}}\pi_1(A_1)\pi_2(A_2) | \mathscr{H}$,

(ii) $\pi_1(B_1)\pi_2(B_2) = \pi_2(B_2)\pi_1(B_1)$,

for all A_i in $\mathscr{A}_i$, B_i in $\mathscr{L}(\mathscr{R}_i)$, $i = 1, 2$.

Proof. Let $\mathscr{C}_1^{(i)}, \mathscr{C}_2^{(i)} \ldots, i = 1, 2$, be subalgebras of $\mathscr{A}_i$ which are completely isometric images of finite dimensional nest algebras, and which satisfy dist$(\mathscr{K}_i, \mathscr{C}_n^{(i)}) \to 0$ for every compact operator K_i in $\mathscr{A}_i$. Let $\mathscr{A}_i^1$ be the C^*-algebra generated by the compact operators in $\mathscr{A}_i$ together with the identity operator.

By Theorem 2.1 the representation $\rho_1 \otimes \rho_2$ restricted to $\mathscr{C}_n^{(1)} \otimes \mathscr{C}_n^{(2)}$ is completely contractive. From this it follows that $\rho_1 \otimes \rho_2$ is completely contractive on the operator algebra $\mathscr{A}_1^1 \otimes \mathscr{A}_2^1 \subset \mathscr{L}(\mathscr{R}_1 \otimes \mathscr{R}_2)$. Hence there exists a separable Hilbert space $\mathscr{K} \supset \mathscr{H}$ and a $*$-isomorphism π of $\mathscr{B}_1 \otimes \mathscr{B}_2$ (where B_i is the C^*-algebra generated by the identity and compacts on $\mathscr{R}_i$) which dilates $\rho_1 \otimes \rho_2$. As in the proof of Theorem 3.1 π decomposes as $\pi_1 \oplus \pi_0$ on $\mathscr{K}_1 \oplus \mathscr{K}_0$ with π_1 faithful on the compacts and π_0 zero on the compacts. Using the σ-weak continuity of $\rho_1 \otimes \rho_2$ and choosing sequences of compact operators K_n^i in $\mathscr{A}_i$, which converge σ-weakly to the identity, we see that $\mathscr{H} \subset \mathscr{K}_1$, and that the restriction representations $\pi | \mathscr{B}_1$ and $\pi | \mathscr{B}_2$ provide the desired commuting dilations of ρ_1 and ρ_2. ▨

4. GENERALISED HANKEL OPERATORS

It is well known that Nehari's theorem for Hankel operators on the Hardy space H^2 is a simple consequence of the Sz.-Nagy—Foiaş lifting theorem. Ball and Gohberg obtained an analogous Nehari theorem in the triangular matrix context, where triangular truncation replaces the Riesz projection. More general Nehari type theorems were also obtained independently in [11], [12], for general nest algebras and for nest subalgebras of semi-finite factors, the main tools there being generalized Riesz factorisation, and Arveson's distance formula. Here we note how such results and Arveson's distance formula follow from the lifting theorem, Theorem 3.1.

To prove these results, it will be useful to consider *anti-representations*, i.e. multiplication reversing representations. A general principle says that every dilation theorem about representations has a corresponding statement for anti-representations and we wish to point out why this is so. Let $\mathscr{A}$ be a subalgebra of the C^*-algebra $\mathscr{B}$ and suppose that $\rho:\mathscr{A}\to\mathscr{L}(\mathscr{H})$ is a contractive anti-representatoion and we wish to know if ρ dilates to a $*$-anti-homomorphism $\pi:\mathscr{B}\to\mathscr{L}(\mathscr{K})$, $\mathscr{H}\subset\mathscr{K}$. We call this an *anti-dilation*. If we let $\mathscr{B}_{\text{op}}$ denote $\mathscr{B}$ with multiplication reversed then $\mathscr{B}_{\text{op}}$ is a C^*-algebra and π is a $*$-homomorphism on $\mathscr{B}_{\text{op}}$. Moreover, ρ is a representation of the subalgebra $\mathscr{A}_{\text{op}}$. Thus, by Arveson's theorem it is enough to know that ρ is completely contractive on $\mathscr{A}_{\text{op}}$. We must be careful though because the norms on $M_n(\mathscr{A}_{\text{op}})$ are inherited from $M_n(\mathscr{B}_{\text{op}})$. We use $\|(b_{ij})\|_{\text{op}}$ to denote the norm of (b_{ij}) in $M(\mathscr{B}_{\text{op}})$. We leave it to the reader to check that $\|(b_{ij})\|_{\text{op}} = \|(b_{ij})^{\text{t}}\|$, where t denotes the transpose. Thus, to see that an anti-homomorphism has an anti-dilation one needs to verify that

$$\|(\rho(a_{ij}))\| \leqslant \|(a_{ij})\|_{\text{op}} = \|(a_{ij})^{\text{t}}\|.$$

Now if $\mathscr{A}$ is a nest algebra and $\rho:\mathscr{A}\to\mathscr{L}(\mathscr{H})$ is a contractive anti-representation, consider $\tilde{\rho}:\mathscr{A}\to\mathscr{L}(\mathscr{H})_{\text{op}}$, $\tilde{\rho}(a)=\rho(a)$. This is a contractive representation and so completely contractive. Thus, $\|(a_{ij})\| \geqslant \|(\tilde{\rho}(a_{ij}))\| = \|(\rho(a_{ij}))\|_{\text{op}} = \|(\rho(a_{ij}))^{\text{t}}\|$ from which it follows that ρ has an anti-dilation. Hence, we have that every contractive anti-representation of a nest algebra has an anti-dilation.

Similar arguments yield "anti'' versions of our other theorems concerning nest algebras and we use these freely in what follows.

Let $\mathscr{E}$ be a complete nest of projections on a separable Hilbert space $\mathscr{R}$, with nest algebra $\mathscr{A}$. Let $\mathscr{C}_2$ be the Hilbert space of Hilbert-Schmidt operators on $\mathscr{R}$ and let $H^2(\mathscr{E}) = \mathscr{C}_2\cap\mathscr{A}$ be the upper triangular subspace, with orthogonal projection $\mathscr{P}:\mathscr{C}_2\to H^2(\mathscr{E})$. For X in $\mathscr{L}(\mathscr{R})$ define the generalised multiplication operator L_X on $\mathscr{C}_2$ and the generalised Hankel operator $H_X: H^2(\mathscr{E})\to(H^2(\mathscr{E}))^{\perp}$, by

$$L_X T = XT, \quad T\in\mathscr{C}_2,$$

$$H_X A = \mathscr{P}^{\perp} L_X | H^2(\mathscr{E}).$$

THEOREM 4.1. *Let $X \in \mathscr{L}(\mathscr{R})$. Then there exists an operator $Y \in \mathscr{L}(\mathscr{R})$ such that $H_X = H_Y$ and $\|Y\| = \|H_X\|$. Moreover,*

$$\|H_X\| = \operatorname{dist}(X, \mathscr{A}) = \sup_{E \in \mathscr{E}} \|(I - E)XE\|.$$

Proof. Let ρ_1, ρ_2 be the σ-weakly continuous contractive unital anti-representations of $\mathscr{A}$ on $H^2(\mathscr{E})$ and $(H^2(\mathscr{E}))^\perp$ given by

$$\rho_1(A) = R_A \,|\, H^2(\mathscr{E}),$$

$$\rho_2(A) = \mathscr{P}^\perp R_A \,|\, (H^2(\mathscr{E}))^\perp,$$

where R_A is the right multiplication operator on $\mathscr{C}_2$, $R_A T = TA$. Then, for A_1 in $H^2(\mathscr{E})$ and A in $\mathscr{A}$ we have

$$\rho_2(A) H_X A_1 = \rho_2(A) \mathscr{P}^\perp (XA_1) =$$

$$= \mathscr{P}^\perp((\mathscr{P}^\perp(XA_1))A) = \mathscr{P}^\perp((\mathscr{P}^\perp(XA_1) + \mathscr{P}(XA_1))A) =$$

$$= \mathscr{P}^\perp((XA_1)A) = \mathscr{P}^\perp(X(A_1A)) = H_X \rho_1(A) A_1.$$

By the intertwining version of the anti-representation version of Theorem 3.1, there is a operator $\tilde{Y}$ on $\mathscr{C}_2$ such that

(i) $\|\tilde{Y}\| = \|H_X\|$,

(ii) $\pi_2(B)\tilde{Y} = \tilde{Y}\pi_1(B)$, $B \in \mathscr{L}(\mathscr{R})$,

(iii) $H_X = \mathscr{P}^\perp \tilde{Y} \,|\, H^2(\mathscr{E})$,

where π_1 and π_2 are the $*$-anti-isomorphisms of $\mathscr{L}(\mathscr{R})$ on $\mathscr{C}_2$ given by $\pi_i(B) = R_B$, and which are $\mathscr{R}(\mathscr{L})$-dilations of ρ_1, ρ_2.

Condition (ii) implies that $\tilde{Y} = L_Y$ for some operator Y in $\mathscr{L}(\mathscr{R})$ with $\|X\| = \|\tilde{Y}\|$, and so the first part of the theorem follows. Note that if $H_X = H_Y$ then $A = X - Y$ belongs to $\mathscr{A}$, and so $\operatorname{dist}(X, \mathscr{A}) \leqslant \|Y\| = \|H_X\|$. The inequality $\|H_X\| \leqslant \operatorname{dist}(X, \mathscr{A})$ is elementary, and so the first equality holds. It remains only to show that

$$\|H_X\| = \sup_{E \in \mathscr{E}} \|(I - E)XE\|.$$

Note that if $Q = E - E_-$ is an atom of $\mathscr{E}$ then $\mathscr{C}_2 Q$ is a reducing subspace for L_X and

$$H_X \,|\, H^2(\mathscr{E})Q = H_X \,|\, E\mathscr{C}_2 Q = L_{E^\perp XE} \,|\, \mathscr{C}_2 Q.$$

If $\mathscr{C}$ is purely atomic then $\mathscr{C}_2 = \oplus\mathscr{C}_2 Q$, where the direct sum is taken over all atoms, and so $\|H_X\| = \sup\|L_{E^\perp EX}|\mathscr{C}_2 Q\| = \sup\|E^\perp XE\|$, as desired.

In a general nest it is easy to see that if $F < E$ then $\|H_X\| \geqslant \|(I-E)XF\|$, by considering the subspace $F\mathscr{C}_2(E-F)$ of $H^2(\mathscr{E})$. Thus if $E_- = E$ we have $\|H_X\| \geqslant \|(I-E)XE\|$. Our earlier reasoning gives this inequality when E is an atom $(E \neq E_-)$ and so it follows that we need only show that $\|H_X\|$ is dominated by $\sup_{E\in\mathscr{E}}\|(I-E)XE\|$. Choose $A \in H^2(\mathscr{E})$ and $B \in (H^2(\mathscr{E}))^\perp$ of unit norm so that $\langle XA, B\rangle \geqslant \|H_X\| - \varepsilon$. There is a finite nest $\mathscr{E}_n \subset \mathscr{E}$ so that $\|\mathscr{P}_n^\perp B - B\|_{\mathscr{C}_2} < \varepsilon\|X\|$ where $\mathscr{P}_n$ is the truncation operator for $H^2(\mathscr{E}_n)$. Let $B_1 = \mathscr{P}_n^\perp B$ and note that $H^2(\mathscr{E}_n) \supset H^2(\mathscr{E})$. Then, using the formula in the finite (purely atomic) case, we have

$$\max_{E\in\mathscr{E}_n}\|(I-E)XE\| = \|\mathscr{P}_n^\perp L_X \mathscr{P}_n\| \geqslant$$

$$\geqslant |\langle XA, \mathscr{P}_n^\perp B\rangle| \geqslant |\langle XA, B\rangle| - \varepsilon \geqslant \|H_X\| - 2\varepsilon,$$

and so

$$\sup_{E\in\mathscr{E}}\|(I-E)XE\| \geqslant \|H_X\|$$

as desired. ▨

The first author was supported in part by an NSF grant.
The second author was supported by a D.G. Bourgin visiting scholarship at the University of Houston, by a Fullbright grant, and by the Science and Engineering Research Council.

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Received August 19, 1987.