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ALGEBRAIC AND TOPOLOGICAL K-FUNCTORS OF COMMUTING n-TUPLE OF OPERATORS

R. N. LEVY

In the present paper we consider, from a cohomological point of view, commuting and essentially commuting n-tuples of linear bounded operators, acting on a Banach space. In Section 1, for an arbitrary commuting n-tuple T of operators we construct an element $\mathscr{K}(T)$ of the Grothendieck group of the category of all coherent sheaves on $\mathbf{C}^n \setminus \sigma_e(T)$, where $\sigma_e(T)$ is the joint essential spectrum of T. More precisely, $\mathscr{K}(T)$ is the alternated sum of the homology sheaves of the complex of holomorphic sections of the parametrized Koszul complex of T. On the other hand, for any essentially commuting n-tuple T we define a topological invariant $K(T) \in \mathrm{K}^0(\mathbf{C}^n \setminus \sigma_e(T))$. Proposition 1.4 shows that $K(T)$ can be considered as a generalization of $\mathrm{Ext}(T)$, defined by the Brown-Douglas-Fillmore theory in the case of essentially normal operators in Hilbert space. Further, we show that the elements $\mathscr{K}(T)$ and $K(T)$ are related by some natural transformation of functors. This implies that the essential spectrum of a commuting n-tuple T with $K(T) \neq 0$ must contain the boundary of a bounded complex set. This fact gives an obstruction to the lifting of an essentially commuting n-tuple, i.e. to representing an essentially commuting n-tuple as a compact perturbation of a commuting n-tuple. This obstruction depends on the complex geometry of the joint essential spectrum $\sigma_e(T)$. As another application of the results of Section 1 we obtain a characterization of all commuting n-tuples of essentially unitary operators in Hilbert space. Further, we give a short proof of Boutet de Monvel's formula for the index of a Toeplitz operator on a strongly pseudoconvex manifold.

In Section 3 we prove the functoriality of $\mathscr{K}(T)$ and $K(T)$ under suitable functional calculus of operators. This allows to obtain corresponding index theorems and shows that $K(T)$, resp. $\mathscr{K}(T)$, determines the index of all matrix functions of T. By generalizing the arguments, used in the proof of the functoriality of $\mathscr{K}(T)$, we obtain in Proposition 3.3 further information about the structure of the joint Fredholm spectrum of a commuting n-tuple. In Section 4 we have collected the technical results, necessary for the first tree sections (most of them are „folklore-type" or "well-known").

The main results of this paper were announced in the short communication [17]. In the mean time there appeared two papers [5] and [23] treating related subjects. The paper [5] contains the treatment of the particular case when the Fredholm spectrum is an 1-dimensional subset of $\mathbf{C}^n$. In [23], Theorem 3.1, there is a proof of the functoriality of the index of a commutative n-tuple T, or, equivalently, of the functoriality of the component of $\mathscr{K}(T)$ of (maximal) dimension n. Our results are also related to the works of several other authors, for instance [22], [8], [9], [10].

Throughout the paper the system $X_\bullet(z) = \{X_i, d_i(z)\}$ will be called a continuous (holomorphic) complex on the domain U, if X_i are Banach spaces and $d_i(z)\colon X_i \to X_{i+1}$ are continuous (holomorphic) operator-valued functions on U with $d_{i+1}(z) \circ d_i(z) \equiv 0$. Denote by $\mathscr{C}X$ ($\mathscr{O}X$) the sheaf of germs of continuous (holomorphic) X-valued functions on U. If $X_\bullet(z)$ is a continuous (holomorphic) complex on U, then the complex of sheaves $\mathscr{C}X_\bullet(z) = \{\mathscr{C}X_i, d_i(z)\}$ (resp. $\mathscr{O}X_\bullet(z) = \{\mathscr{O}X_i, d_i(z)\}$) will be called a complex of continuous (holomorphic) sections of $X_\bullet(z)$.

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1. MAIN DEFINITIONS

In his fundamental papers [25] and [26] J. L. Taylor introduced the notion of parametrized Koszul complex (we shall denote it by $K_\bullet(T, z)$) of a commuting n-tuple T of bounded linear operators acting on a Banach space. We recall the inductive definition of this complex. In the case $n = 1$ the Koszul complex of the single operator T acting on the space X is simply $0 \to X \xrightarrow{T-zI} X \to 0$. Let $T = (T_1, \ldots, T_{n-1}, T_n)$ be a commuting n-tuple, $z = (z_1, \ldots, z_{n-1}, z_n) = (z', z_n)$ be a point of $\mathbf{C}^n$, and $K_\bullet(T', z')$ be the parametrized Koszul complex for the $n-1$-tuple $T' = (T_1, \ldots, T_{n-1})$ in the point $z' \in \mathbf{C}^{n-1}$. Then the operator $T_n - z_n I$ defines an endomorphism of the complex $K_\bullet(T', z')$. The cone of this morphism is by definition the parametrized Koszul complex of the n-tuple T at the point z. In this paper it will be denoted by $K_\bullet(T, z) = \{K_i, d_i(z)\}_{i=0}^n$. It is easy to see that the i-th term K_i of this complex is equal to the direct sum of $\binom{n}{i}$ copies of the Banach space X. Note that all the differentials $d_i(z)\colon K_i \to K_{i+1}$ of the complex $K_\bullet(T, z)$ are linear and hence they are holomorphic functions of the variable $z \in \mathbf{C}^n$.

The *Taylor spectrum* $\sigma(T)$ of the commuting n-tuple T is by definition the set of all points $z \in \mathbf{C}^n$ such that the complex $K_\bullet(T, z)$ is not exact (see [25]). The *essential Taylor spectrum* $\sigma_e(T)$ of the commuting n-tuple T (see [22], [8], [10]) is the set of all points z such that the complex $K_{\bullet}(T, z)$ is not Fredholm, i.e. it has at least one infinite dimensional homology group. Finally, define the *Fredholm spectrum* $\sigma_F(T)$ of T as the complement of $\sigma_e(T)$ in $\sigma(T)$.

The complex $\mathcal{O}K_{\bullet}(T, z)$ of holomorphic sections of $K_{\bullet}(T, z)$ is a complex of sheaves on $\mathbf{C}^n$. Denote its homology sheaves by $\mathcal{H}_i(T)$, $i = 0, \ldots, n$. One can see from Lemma 4.2 that the union of supports of all $\mathcal{H}_i(T)$ coincides with $\sigma(T)$. It follows immediately from Corollary 4.4 that:

PROPOSITION 1.1. *All the sheaves $\mathcal{H}_i(T)$ are coherent on $\mathbf{C}^n \setminus \sigma_e(T)$. The Fredholm spectrum $\sigma_F(T)$ is a bounded complex subset of $\mathbf{C}^n \setminus \sigma_e(T)$.*

This assertion can be considered as analogous to Gleason's theorem on the holomorphic structure of the set of finitely generated maximal ideals of a commutative Banach algebra.

Let M be a complex manifold. Recall that the algebraic K-group $\mathrm{K}_0^{\mathrm{alg}}(M)$ is defined as the Grothendieck group of the category of all coherent sheaves on M. More precisely, the generators of $\mathrm{K}_0^{\mathrm{alg}}(M)$ are the classes $[\mathcal{F}]$ of coherent sheaves $\mathcal{F}$ on M, and the relations are determined by the short exact sequences $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, corresponding to the relations $[\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}]$.

DEFINITION. For a given commutative n-tuple T of operators we denote by $\mathcal{K}(T)$ the element of the group $\mathrm{K}_0^{\mathrm{alg}}(\mathbf{C}^n \setminus \sigma_e(T))$, determined by:

$$\mathcal{K}(T) = \sum_{i=0}^{n} (-1)^i [\mathcal{H}_i(T)].$$

From Lemma 4.6 it follows:

PROPOSITION 1.2. *Suppose that the commutative n-tuple T' is a finite-dimensional perturbation of the commutative n-tuple T. Then $\mathcal{K}(T') = \mathcal{K}(T)$.*

In fact, if $d_i(z)$, $d_i'(z)$ are the differentials of the Koszul complexes of T and T', respectively, then it is easy to see that $d_i(z) - d_i'(z)$ is a finite dimensional operator, not depending on the parameter z. By applying 4.7 and summing over all i, we obtain the invariance of $\mathcal{K}(T)$.

In the case of a single operator T the element $\mathcal{K}(T)$ has the form $\mathcal{K}(T) = \sum n_j \cdot 1_{U_j}$ where 1_{U_j} are one-dimensional trivial bundles on the bounded components U_j of $\mathbf{C} \setminus \sigma_e(T)$, and each n_j is equal to the index of $T - zI$ for $z \in U_j$. (It is easy to check that the sheaves with zero-dimensional support do not contribute to $\mathcal{K}(T)$.) For any essentially normal operator T the collection $\{n_j, U_j\}$, called the "spectral picture of T", plays an important role in the Brown-Douglas-Fillmore theory. So, for a commutative n-tuple T the element $\mathcal{K}(T)$ can be considered as higher-dimensional analogous of the spectral picture of an operator. We shall see below that for any commutative n-tuple T of essentially normal operators in Hilbert space the element $\mathscr{K}(T)$ determines T up to unitary equivalence modulo compact operators.

The element $\mathscr{K}(T)$ can be described more explicitely if it has a Jordan-Hölder decomposition, i.e. if it can be represented in the form:

$$\mathscr{K}(T) = \sum n_j \cdot [\mathcal{O}_{M_j}]$$

where n_j are integers, M_j are irreducible complex subsets of $\sigma_F(T)$, and $\mathcal{O}_M$ denotes the structure sheaf of the complex set M. In any case the above decomposition for the homology sheaves and their alternated sum holds locally on $\sigma_F(T)$.

EXAMPLE. Denote by S the right unilateral shift acting on a separable Hilbert space and consider the couple $T = (S^4, S^6)$. Then it is easy to see that $\mathscr{K}(T) = 2 \cdot [\mathcal{O}_M]$, where M is defined by $M = \{(z, w) \in \mathbf{C}^2 : z^2 = w^2, |z| < 1\}$. This can be shown directly or appllying the theorem of functoriality proved in Section 3.

The element $\mathscr{K}(T)$ can be defined in a more general situation. Namely, suppose that all commutators $[T_i, T_j]$ are finite dimensional operators. Then in the same way as above one can define the Koszul system $K_\bullet(T, z) = \{K_i, d_i(z)\}$. Now this system is not a complex, but it is easy to see that the composition $d_{i+1}(z) \circ d_i(z)$ of two consecutive differentials is a finite dimensional operator. Therefore for any i and z the space $\operatorname{im} d_i(z)/\ker d_{i-1}(z) \cap \operatorname{im} d_i(z)$ is finite dimensional. Define, as above, the essential Taylor spectrum of T as the set of all $z \in \mathbf{C}^n$ such that at least for one i the space $\ker d_i(z)/\operatorname{im} d_{i-1}(z) \cap \ker d_i(z)$ is infinite dimensional. For two subsheaves $\mathscr{F}$, $\mathscr{G}$ of a given sheaf we shall denote by $\mathscr{F} \triangle \mathscr{G}$ their symmetric difference in the Grothendieck group: $\mathscr{F} \triangle \mathscr{G} = \mathscr{F}/\mathscr{F} \cap \mathscr{G} - \mathscr{G}/\mathscr{F} \cap \mathscr{G}$. Then for an n-tuple T, commuting modulo finite dimensional operators, we define:

$$\mathscr{K}(T) = \sum_{i=0}^{n} (-1)^i \ker d_i(z) \triangle \operatorname{im} d_{i-1}(z)$$

where $d_i(z)$ are considered as morphisms of sheaves.

The only difference between this case and the commutative one is that in this case the support of $\mathscr{K}(T)$ may be an unbounded subset of $\mathbf{C}^n$. For this reason, it seems apropriate to modify slightly the definition of $\mathscr{K}(T)$. Note that the parametrized Koszul complex $K_\bullet(T, z)$ may be extended from the affine space $\mathbf{C}^n$ to the projective space $\mathbf{CP}^n$; indeed, if $w_0, \ldots, w_n$ are the projective coordinates in $\mathbf{C}^n$, then near infinity one can consider the Koszul complex of the operators $w_0 T_1 - w_1 I, \ldots, w_0 T_n - w_n I$. Thus, we consider $\mathscr{K}(T)$ as an element of $\mathrm{K}_0^{\mathrm{alg}}(\mathbf{CP}^n \setminus \sigma_e(T))$. All the theorems concerning $\mathscr{K}(T)$, proved in the present paper for the commutative case are still valid for the case of finite dimensional commutators. We omit the proofs, which may be obtained from the proofs in the commutative case by small modifications.

Now we shall define the corresponding topological invariant. Let the n-tuple $T = (T_1, \ldots, T_n)$ of linear bounded operators, acting on a Banach space, be essentially commutative, i.e. all the commutators $[T_i, T_j]$ be compact. Then the Koszul system $K_\bullet(T, z)$ of T is an essential complex, i.e. the composition of any two consecutive differentials is a compact operator. The essential Taylor spectrum $\sigma_e(T)$ of T is the complement in $\mathbf{C}^n$ of the set of all z such that the essential complex $K_\bullet(T, z)$ is essential Fredholm in the sense of the definition given in Section 4 (see [22] and [8]). When T is commutative, we must compare this definition of $\sigma_e(T)$ with the definition given above. If T acts in a Hilbert space, it is easy to see that these definitions are equivalent. In the case of Banach space, the essential spectrum, given by the second definition, is in general larger than that given by the first definition. Indeed, it may happen that at some point z the homologies of the complex $K_\bullet(T, z)$ are finite dimensional, but the kernel of some differential of this complex does not have a direct complement. In the rest of the paper, treating commutative n-tuples, we shall assume the first definition, and while treating essentially commutative n-tuples we assume the second one.

By applying the construction of Section 4 to the essential Fredholm complex $K_\bullet(T, z)$, we obtain an element of the topological K-group $\mathrm{K}^0(\mathbf{C}^n \setminus \sigma_e(T))$, which will be denoted by $K(T)$. (In the case of a commutative n-tuple we may apply directly Lemma 4.5.) The definition of $K(T)$ is an obvious generalization of the notion of index of essentially commuting n-tuples (cf. [22]). In some particular cases $K(T)$ is completely determined by the index of the essential Fredholm complex $K_\bullet(T, z)$ for some values of z. For instance, if $\mathbf{B}$ and $\mathbf{S}$ are the unit ball and the unit sphere in $\mathbf{C}^n$, and T is an essentially commutative n-tuple with $\sigma_e(T) = \mathbf{S}$, then $K(T)$ is equal to $k_\bullet[1_{\mathbf{B}}]$, where $1_{\mathbf{B}}$ is the trivial one dimensional bundle on $\mathbf{B}$, and k is the index of $K_\bullet(T, z)$ at an arbitrary point $z \in \mathbf{B}$. However, in general, the index of $K_\bullet(T, z)$ may be identically zero (this is always satisfied if $\mathbf{C}^n \setminus \sigma_e(T)$ is connected), but $K(T)$ may be nonzero. Below we prove that $K(T)$ is the index class of T, i.e. $K(T)$ determines in a natural way the index of any matrix-function of the operators of T in a suitable functional calculus.

The invariance of $K(T)$ under small or compact perturbations of T follows immediately from Lemma 4.10. More precisely:

PROPOSITION 1.3. *Let T be an essentially commutative n-tuple of operators. If T is a compact perturbation of T, then $\sigma_e(T') = \sigma_e(T)$ and $K(T') = K(T)$. If F is a closed subset of $\mathbf{C}^n \setminus \sigma_e(T)$ and the essentially commutative n-tuple T' is sufficiently close to T in the operator norm topology, then $\sigma_e(T') \cap F = \emptyset$ and the restrictions of $K(T')$ and $K(T)$ to F coincide.*

Suppose that X is a Hilbert space and the essentially commutative n-tuple T acting on X consists of essentially normal operators, i.e. all the commutators $T_i, T_j^*]$ are compact. Then there exists a unique $*$-homomorphism $f \to f(T)$ from the algebra $C(\sigma_e(T))$ of continuous functions on $\sigma_e(T)$ to the Calkin algebra $\mathfrak{A}(X) = L(X)/K(X)$, such that $z_i(T)$ is the class of T_i, for all i, $1 \leqslant i \leqslant n$. Hence the n-tuple T determines an element of the Brown-Douglas-Fillmore group $\mathrm{Ext}(\sigma_e(T))$, which will be denoted by $\mathrm{Ext}(T)$. In [15] the Alexander duality homomorphism $D\colon \mathrm{Ext}(F) \to \mathrm{K}^0(\mathbf{C}^n \setminus F)$ for a compact subset $F \subset \mathbf{C}^n$ is described.

PROPOSITION 1.4. *For any essentially commuting n-tuple T of essentially normal operators acting on Hilbert space, the element $K(T)$ is Alexander dual to* $\mathrm{Ext}(T)$, *i.e.* $K(T) = D(\mathrm{Ext}(T))$.

COROLLARY. *If $\sigma_e(T)$ is homeomorphic to a finite* CW-*complex, then $K(T)$ determines T up to a* BDF-*equivalence.*

Note that in [1] M. F. Atiyah describes the dual of $\mathrm{Ext}(T)$ by using a Clifford algebra. One can prove that our construction is equivalent to his. However, since in [1] there is no proof, we prefer to give a direct proof of the duality between $K(T)$ and $\mathrm{Ext}(T)$.

Proof of 1.4. Recall the construction from [15]. Let F be a compact subset of $\mathbf{C}^n$, and $\xi \in \mathrm{Ext}(F)$. Then for any metric space M the element ξ determines a homomorphism $\gamma_M(\xi) : \mathrm{K}^1(F \times M) \to \mathrm{K}^0(M)$. Let $U = \mathbf{C}^n \setminus F$, and denote by i the natural embedding $i : F \times U \to (\mathbf{C}^n \times \mathbf{C}^n) \setminus \Delta$, where Δ is the diagonal in $\mathbf{C}^n \times \mathbf{C}^n$. Denote by μ the generator of the group $\mathrm{K}^1(\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)$. Then the Alexander dual of ξ is equal to the image of the element $i^*(\mu) \in \mathrm{K}^1(F \times U)$ under the homomorphism $\gamma_U(\xi)$.

Consider the case $F = \sigma_e(T)$, $\xi = \mathrm{Ext}(T)$. Let M be a metric space and $G(z, m)$, $z \in F$, $m \in M$, be an invertible continuous matrix-valued function on $F \times M$. Then the image of the class of $G(z, m)$ under the homomorphism $\gamma_M(\xi)$ coincides with the element of $\mathrm{K}^0(M)$, determined by the continuous vector-function $G(\tilde{T}, m)$ on M with values in the Calkin algebra. Denote by $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ the coordinates on the first and the second copies of $\mathbf{C}^n$ in $(\mathbf{C}^n \times \mathbf{C}^n) \setminus \Delta$. Then the generator of the group $\mathrm{K}^1(\mathbf{C}^n \times \mathbf{C}^n \setminus \Delta)$ can be represented by the Koszul complex of the functions $z_1 - w_1, \ldots, z_n - w_n$, which is exact on $(\mathbf{C}^n \times \mathbf{C}^n) \setminus \Delta$. Applying $\gamma_U(\xi)$, i.e. replacing z_i by T_i, we obtain a representation of the Alexander dual of the element Ext T by the Koszul complex of the operators $T_1 - w_1 I, \ldots, T_n - w_n I$ for $w \in U = \mathbf{C}^n \setminus \sigma_e(T)$.

Now we shall consider a more general situation. Let $T = (T_1, \ldots, T_n)$ be an essentially commuting n-tuple of operators acting on the Banach space X, and let $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_n)$ be the corresponding n-tuple of elements in the Calkin algebra $\mathfrak{A}(X)$. Let A be a commutative subalgebra of $\mathfrak{A}(X)$ containing $\tilde{T}$. Suppose for simplicity that A is semisimple and that the Gelfand transforms of the elements $\tilde{T}_1, \ldots, \tilde{T}_n$ separate the points of the spectrum of A, and therefore the spectrum of A can be identified with a compact subset F of $\mathbf{C}^n$ containing $\sigma_e(T)$. If $f(z)$ is a function on F belonging to A, then we shall denote by $f(\tilde{T})$ the corresponding element of $\mathfrak{A}(X)$, and by $f(T)$ — an arbitrary operator representing it. We shall show that the functional calculus $f(z) \to f(T)$ determines an element ξ of the group $K_1(F)$, i.e. a natural transformation of functors $\gamma_M(\xi) : K^1(F \times M) \to K^0(M)$. Indeed, fix the compact M and denote by A_M the Banach algebra consisting of all continuous functions $f(z, m)$ on $F \times M$ such that $f(z, m_0) \in A$ for any fixed $m_0 \in M$. The corresponding norm is $\|f(z, m)\|_{A_M} = \sup_{m \in M} \|f(z, m)\|_A$. The Novodvorski theorem [20] shows that any element τ of the group $K^1(F \times M)$ can be represented by an invertible matrix $G(z, m)$ with entries in A_M, and that this representation is unique up to homotopy. Then the element $\gamma_M(\xi)\tau \in K^0(M)$ can be defined as the class of the $\mathfrak{A}(X)$-valued function $G(\tilde{T}, m)$, depending continuosly on the parameter $m \in M$. Using the construction quoted in the proof of 1.4, one can see that the restriction of $K(T)$ on $\mathbf{C}^n \setminus F$ coincides with the Alexander dual of ξ. This implies:

PROPOSITION 1.5. *Suppose that $L_\bullet(z) = \{L_i, d_i(z)\}$ is a bounded exact complex of free finite dimensional A-modules. Then $L_\bullet(T) = \{L_i \otimes X, d_i(T)\}$ is an essential Fredholm complex of Banach spaces, and its Euler characteristic (*i.e. *the Euler characteristic of any equivalent complex — see 4.9) is given by the formula*

$$\chi(L_\bullet(T)) = \langle [L_\bullet(z)], K(T) \rangle$$

where $[L_\bullet(z)]$ is the class of $L_\bullet(z)$ in the group $K^1(F)$, and $\langle \cdot , \cdot \rangle$ denotes the natural pairing between $K^1(F)$ and $K^0(\mathbf{C}^n \setminus F)$.

Proof. If $S_i(z) : L_i \to L_{i-1}$ are homomorphisms of A-modules such that $S_{i+1}(z) \circ d_i(z) + d_{i-1}(z) \circ S_i(z) = \mathrm{Id}$, then $S_{i+1}(T) \circ d_i(T) + d_{i-1}(T) \circ S_i(T) = \mathrm{Id} +$ + compact operator, and therefore the essential complex $L_\bullet(T)$ is Fredholm. Further we have

$$\chi(L_\bullet(T)) = \gamma_{\mathrm{pt}}(\xi)[L_\bullet(z)] = \langle [L_\bullet(z)], \xi \rangle = \langle [L_\bullet(z)], K(T) \rangle.$$

Now we are going to describe the connection between $\mathcal{K}(T)$ and $K(T)$ for a commutative n-tuple T. We shall use the natural morphism $\alpha_M : K_0^{\mathrm{alg}}(M) \to$ $\to K^0(M)$ for a complex manifold M, constructed in the Atiyah-Hirzebruch work [2]. Briefly, the action of α_M can be described as follows: Let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_M$-modules. Denote by $M_{\mathbf{R}}$ the underlying real-analytic manifold of M. Then $\mathcal{L}$ determines a coherent sheaf $\mathcal{L}_{\mathbf{R}}$ of modules over the sheaf $\mathcal{R}_M$ of germs of real-analytic functions on M. The manifold $M_{\mathbf{R}}$ can be embedded as a real-analytic submanifold in some complex Stein manifold $\tilde{M}$ ($\tilde{M}$ is in fact a neighborhood of $M_{\mathbf{R}}$ in its complexification — see [12]), and the sheaf $\mathcal{L}_{\mathbf{R}}$ can be extended up to a uniquely determined coherent sheaf $\tilde{\mathcal{L}}$ on M. For any compact subset F

of M one can choose a locally free resolution $R_\bullet \to \tilde{\mathscr{L}} \to 0$ of the sheaf $\tilde{\mathscr{L}}$, defined in some neighborhood of F in $\tilde{M}$. The class of the corresponding complex of vector bundles on F does not depend on the choice of the resolution and determine an element of the group $K^0(F)$. Then, taking the limit over an exhausting sequence of compact subsets of M, one obtains the element $\alpha_M(\mathscr{L}) \in K^0(M)$.

Let T be a commutative n-tuple of operators and $U = \mathbf{C}^n \setminus \sigma_e(T)$.

PROPOSITION 1.6. *The image of $\mathscr{K}(T)$ under the homomorphism α_U coincides with $K(T)$.*

COROLLARY. *If for the commutative n-tuples T and T' we have $\sigma_e(T) = \sigma_e(T')$ and $\mathscr{K}(T) = \mathscr{K}(T')$, then $K(T) = K(T')$.*

Proof of 1.6. Denote by $\mathscr{H}_i(T)$ the homology sheaves of the complex $\mathscr{O}K_\bullet(T, z)$ on U. Then the corresponding real-analytic sheaves $\mathscr{H}_i(T)_{\mathbf{R}}$ coincides with the homology sheaves of the complex $\mathscr{R}K_\bullet(T, z)$ of real-analytic sections of $K_\bullet(T, z)$. Let $\tilde{U}$ be a Stein neighborhood of $U_{\mathbf{R}}$ in its complexification, and let $\tilde{K}_\bullet$ be the canonical extensions of $K_\bullet(T, z)$ (considered as a real-analytic complex on $U_{\mathbf{R}}$) on $\tilde{U}$. Let F be a compact subset of U, and $H_\bullet$ be a holomorphic complex of vector bundles, quasi-isomorphic to $K_\bullet$ on some neighborhood of F in $\tilde{U}$. Then the restriction of $H_\bullet$ to F is a real-analytic complex of vector bundles, quasi--isomorphic to $K_\bullet(T, z)$, and therefore represents the restriction of $K(T)$ to F. On the other hand, it is easy to see that the homology sheaves of the complex $\mathscr{R}H_\bullet$ of real-analytic sections of $H_\bullet$ on F coincide with $\mathscr{H}_i(T)_{\mathbf{R}}$ and therefore the complex $H_\bullet$ represents the restriction of $\alpha_U(\mathscr{K}(T))$ to F. The proof is complete.

In a particular case the construction of α_U can be simplified. Let $\mathscr{L}$ be a coherent sheaf supported on $\sigma_F(T)$, and suppose that there exists an open set V such that $\sigma_F(T) \subset V \subset \mathbf{C}^n \setminus \sigma_e(T)$, and the sheaf $\mathscr{L}$ has a bounded locally free resolution $R_\bullet \to \mathscr{L} \to 0$ on V. Then the corresponding complex of holomorphic vector bundles determines an element $[R_\bullet]$ of the group $K^0(V, V \setminus \sigma_e(T))$. It is easy to see that $[R_\bullet]$ coincides with $\alpha_V(\mathscr{L})$. By using the excision homomorphism, one can consider $[R_\bullet]$ as an element of $K^0(\mathbf{C}^n \setminus \sigma_e(T), \mathbf{C}^n \setminus \sigma(T))$ and therefore as an element of $K^0(\mathbf{C}^n \setminus \sigma_e(T))$. Since the morphism α is compatible with the excision homomorphism, then in this group the element $[R_\bullet]$ coincides with $\alpha_U(\mathscr{L})$ again. If under the above conditions we have $\mathscr{K}(T) = [\mathscr{L}]$ then the image of $[R_\bullet]$ in $K^0(\mathbf{C}^n \setminus \sigma_e(T))$ will coincide with $K(T)$.

The invariant $K(T)$ determines the corresponding element in cohomology. For any essentially commuting n-tuple T denote $\operatorname{ch}(T) = \operatorname{ch}(K(T)) \in \bigoplus_{i=0}^{n} H^{2i}(U)$, where $U = \mathbf{C}^n \setminus \sigma_e(T)$ and $H^i(U)$ is the i-th cohomology group of U with coefficients in $\mathbf{C}$. On the other hand, in the paper [21] one defines the Chern character

$\mathrm{ch}(\mathscr{L})$ in Hodge cohomology for any coherent sheaf $\mathscr{L}$ on a complex manifold. Hence we have a homomorphism $\mathrm{ch} : \mathrm{K}_0^{\mathrm{alg}}(U) \to \bigoplus_{i=0}^{n} H^{i,i}(U)$, where $H^{p,q}(U)$ is the Hodge decomposition of $H^{p+q}(U)$. It is easy to see that the construction o Chern character in [21] agrees with the Atiyah-Hirzebruch construction, and therefore the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{K}_0^{\mathrm{alg}}(U) & \xrightarrow{\alpha_U} & \mathrm{K}^0(U) \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ \oplus H^{i,i}(U) & \longrightarrow & \oplus H^{2i}(U) \end{array}$$

(the morphism in the bottom row is the embedding, determined by the Hodge decomposition). Hence we obtain:

PROPOSITION 1.7. *If T is a commutative n-tuple, then* $\mathrm{ch}(T)$ *belongs to* $\bigoplus_{i=0}^{n} H^{i,i}(U)$.

Since $K(T)$ is invariant under compact perturbations, Proposition 1.7 gives us an obstruction for a given essentially commutative n-tuple T to be a compact perturbation of a commutative n-tuple. Another obstruction will be given in the next section.

One can construct a Chern character $\mathrm{ch}_\bullet(T)$ with values in the homology groups. Since $\mathrm{ch}(T)$ is defined as an element of the group $\oplus H^{2i}(U, U \setminus \sigma_{\mathrm{F}}(T))$, then its Poincaré dual is an element of the group $H_{2i}(\sigma_{\mathrm{F}}(T)^+)$, where $\sigma_{\mathrm{F}}(T)^+$ is the one-point compactification of $\sigma_{\mathrm{F}}(T)$; this element will be denoted by $\mathrm{ch}_\bullet(T)$. More explicitely, the linear functional

$$\omega \to \int_U \omega \wedge \mathrm{ch}(T)$$

defined on the space of all differential forms ω with compact support in U, is a closed current supported in $\sigma_{\mathrm{F}}(T)$ and represents $\mathrm{ch}_\bullet(T)$. The above construction can be localized. Fix an open subset V of $U = \mathbf{C}^n \setminus \sigma_e(T)$, and let $M = V \cap \sigma_{\mathrm{F}}(T)$. Then in the same way as above one defines the local invariants $\mathscr{K}_V(T) \in \mathrm{K}_0^{\mathrm{alg}}(M)$, $K_V(T) \in \mathrm{K}^0(V, V \setminus M)$, $\mathrm{ch}_V(T) \in \oplus H^{2i}(V, V \setminus M)$ and $\mathrm{ch}_{\bullet V}(T) \in \oplus H_{2i}(M^+)$. Since the Chern character commutes with restrictions, then $\mathrm{ch}_V(T)$ and $\mathrm{ch}_{\bullet V}(T)$ coincide with the corresponding restrictions of $\mathrm{ch}(T)$ and $\mathrm{ch}_\bullet(T)$, respectively. The Chern character of an n-tuple T of commuting operators can be calculated directly from its Koszul complex. For the local Chern characters this is easy: one can take a holomorphic complex of finite-dimensional spaces, locally quasi-isomorphic to

$K_\bullet(T, z)$, and calculate its Chern character by the use of the methods, developed in [21]. The global construction is more complicated and will be described elsewhere. Here we shall compute top-dimensional component of $\mathrm{ch}_\bullet(T)$. Let M be an irreducible complex set, and let $\mathscr{L}$ be a coherent sheaf on M. Then there exists an open Zariski subset M^0 of M such that the restriction of $\mathscr{L}$ on M^0 is locally free. Denote by $\dim \mathscr{L}$ the dimension of the restriction of $\mathscr{L}$ to M^0; since M is irreducible, then M^0 is connected and the dimension does not depend on the point. Now suppose that the Fredholm spectrum $\sigma_F(T)$ of the commuting n-tuple T is a complex set of dimension not exceeding p, and let $\{M_\alpha\}$ be the family of its irreducible subsets of dimension p. Denote $k_{\alpha,i} = \dim \mathscr{H}_i(T)|M_\alpha$, and let $k_\alpha = \sum_i (-1)^i k_{\alpha,i}$.

PROPOSITION 1.8. *Under the above conditions we have:*

$$\mathrm{ch}_\bullet(T) = \sum_\alpha k_\alpha[M_\alpha] + \textit{terms of dimension} \leqslant 2p-2,$$

where $[M_\alpha]$ *denotes the fundamental cycle of* M_α *in the group* $H_{2p}(\sigma_F(T)_+)$.

Proof. For any α denote by M_0^α an open Zariski subset, dense in the regular part of M_α, such that all the sheaves $\mathscr{H}_i(T)$ are locally free on M_α^0. Then $\sigma_F(T)\setminus\bigcup_\alpha M_\alpha^0$ is a complex set of complex dimension $\leqslant p-1$. Let V be an arbitrary open ball in $\mathbf{C}^n\setminus\sigma_e(T)$ such that $M_V = V\cap\sigma_F(T)$ is contained in some M_α^0. Then standard arguments show that in the group $\oplus H_{2i}(M_V^+)$ we have $\mathrm{ch}_{\bullet V}(T) = k_\alpha[M_V]$. Denote by ξ the element $\mathrm{ch}_\bullet(T) - \sum_\alpha k_\alpha[M_\alpha]$ of the group $\oplus H_{2i}(\sigma_F(T)^+)$. Then for any ball V satisfying the above condition the restriction of ξ is equal to zero in the group $\oplus H_{2i}(M_V^+)$. This means that the component of dimension $2p$ of ξ is equal to zero. Since $\mathrm{ch}_\bullet(T)$ contains only components of even dimension $\leqslant 2p$, this proves the proposition.

Now we shall give an explicit formula for the regular case. Let N be a p-dimensional complex manifold in $\mathbf{C}^n$, M — a precompact domain in N, E — an analytic vector bundle defined on some neighborhood of $\overline{M}$ in N (we shall denote by the same symbol the corresponding locally free sheaf of $\mathcal{O}_N$-modules). Let T be a commutative n-tuple of operators with $\sigma(T) = \overline{M}$, $\sigma_F(T) = M$, and $\mathscr{H}(T) = i_* E$, where i is the embedding of M in $\mathbf{C}^n$. Then $\mathrm{ch}(T)$ (in fact, its Alexander dual in $\oplus H_{2i}(\mathrm{b}M)$) can be computed by a standard procedure. Let V be a sufficiently small neighborhood of M in $\mathbf{C}^n\setminus\mathrm{b}M$. Then the couple (M, V) may be identified with the zero section of the normal bundle of M and some neighborhood of it in the total space of the normal bundle respectively. Denote by π the projection of the normal bundle on M, and by $i_! E$ — the Koszul-Thom complex of the normal bundle, tensorized by $\pi^* E$. Then $i_! E$ is a locally free resolution of the sheaf $i_* E$ on V and determines an element of the group $\mathrm{K}^0(V, V \setminus M)$. Applying the excision homomorphism to this element, we will obtain the element $K(T)$ in $\mathrm{K}^0(\mathbf{C}^n \setminus \mathrm{b}M)$ (see Proposition 1.6 and the remark following it). The Riemann-Roch theorem (see [14], § 24.5) asserts that

$$\mathrm{ch}(i_! E) = i_!(\mathrm{ch}(E) \cup \mathrm{Td}(\mathrm{T}M))$$

where in the right side $i_! \colon \bigoplus_k H^{2k}(M) \to \bigoplus_k H^{2k+2(n-p)}(V, V \setminus M)$ denotes the Thom-Gysin homomorphism in cohomology. By using the excision homomorphism, one can consider both sides of this equality as elements of $\bigoplus H^{2k}(\mathbf{C}^n \setminus \mathrm{b}M, \mathbf{C}^n \setminus \overline{M})$. or simply of $H^{2k}(\mathbf{C}^n \setminus \mathrm{b}M)$. Denote by 1_M the element of $H^0(N \setminus \mathrm{b}M)$, corresponding to the characteristic function of M, and by $[\mathrm{b}M] \in H_{2p-1}(\mathrm{b}M)$ — its Alexander dual on N. If $\mathrm{b}M$ is smooth, then $[\mathrm{b}M]$ is its fundamental cycle. The element $i_!(1_M)$ is in Alexander duality with $[\mathrm{b}M]$ on $\mathbf{C}^n$. One sees that:

$$\mathrm{ch}(T) = \mathrm{ch}(i_! E) = i_!(1_M \vee \mathrm{ch}(E) \vee \mathrm{Td}(\mathrm{T}M)).$$

Since E can be extended over $\mathrm{b}M$, one can consider the restrictions of $\mathrm{ch}(E)$ and $\mathrm{Td}(\mathrm{T}M)$ on $\mathrm{b}M$ (we shall denote them by the same symbols). Finally, we obtain:

PROPOSITION 1.9. *Under the above conditions* $\mathrm{ch}(T)$ *is the Alexander dual of the element* $(\mathrm{ch}(E) \cup \mathrm{Td}(\mathrm{T}M)) \cap [\mathrm{b}M]$ *of* $\bigoplus H^{2i}(\mathrm{b}M)$.

In the next section we give an application of this result.

2. APPLICATIONS

In the present section we give some immediate consequences of the results of the preceding one. At first, we construct an obstruction for lifting essentially commuting tuples of operators. Let $T = (T_1, \ldots, T_n)$ be an essentially commuting n-tuple of operators acting on a Banach space. We will say that T *can be lifted to a commuting n-tuple* if there exists a commuting n-tuple $T' = (T_1', \ldots, T_n')$ such that $T_i' - T_i$ are compact operators for $i = 1, \ldots, n$. Since the element $K(T)$ is invariant under compact perturbations and since for a commuting n-tuple T' the element $K(T')$ comes from an element of the group $\mathrm{K}^0(\mathbf{C}^n \setminus \sigma_e(T), \mathbf{C}^n \setminus \sigma(T))$, we obtain:

PROPOSITION 2.1. *Suppose that the essentially commuting n-tuple T can be lifted, and $K(T) \neq 0$. Then $\sigma_e(T)$ contains the boundary of some bounded complex subspace of* $\mathbf{C}^n \setminus \sigma_e(T)$.

Indeed, in this case the complex space $\sigma_F(T)$ can not be empty or zero dimensional.

This assertion can be made more precise. Let F be a compact CW-complex. Denote by F_p the p-skeleton of F and by $i_p\colon F_p \to F$ the natural embedding. Then we have a filtration of the group $K_1(F)$ by the subgroups $K_1^p(F) = \operatorname{im} i_{p*}$. For any $\xi \in K_1(F)$ denote $\deg\xi = \min\{p : \xi \in K_1^p(F)\}$ (this degree is always odd). Note that if $\mathrm{ch} : K_1(F) \to \oplus\, H_i(F)$ is the Chern character, then $\deg\xi \geqslant \deg \mathrm{ch}(\xi)$. Let T be an essentially commuting n-tuple, and suppose that $\sigma_e(T)$ is homeomorphic to a finite CW-complex. Denote the Alexander dual of the element $K(T)$ by $DK(T) \in \in K_1(\sigma_e(T))$.

PROPOSITION 2.1′. *Suppose that the n-tuple T, satisfying the above conditions, can be lifted to a commutative n-tuple, and* $\deg DK(T) = 2p - 1$. *Then $\sigma_e(T)$ contains the boundary of some bounded complex set of complex dimension greater or equal to p.*

Proof. Suppose that T is a compact perturbation of the commuting n-tuple T. Denote by V an open neighborhood of $\sigma_e(T)$ with smooth boundary such that $\sigma_e(T)$ is a homotopical retract of $\bar{V}$, and bV intersects transversally the regular part of $\sigma_F(T)$. Denote by k the maximal complex dimension of $\sigma_F(T)$. Then the set $M = bV \cap \sigma_F(T)$ is a CW-complex of dimension $2k - 1$. Denote by r the embedding of $\mathbf{C}^n \setminus V$ in $\mathbf{C}^n \setminus \sigma_e(T)$. Then $r^*K(T)$ belongs to the group $K^0(\mathbf{C}^n \setminus V, \mathbf{C}^n \setminus (\bar{V} \cup \sigma_F(T)))$. This means that the Alexander dual of $r^*K(T)$ belongs to the image of $K_1(M)$ in $K_1(\bar{V})$ under the homomorphism of embedding, and therefore belongs to $K_1^{2k-1}(\bar{V})$. Since $\sigma_e(T)$ is a homotopical retract of V and the filtration is invariant under a homotopical equivalence, then $DK(T)$ belongs to $K_1^{2k-1}(\sigma_e(T))$. Therefore, $k \geqslant p$, and $\sigma_F(T)$ is the desired complex subspace of $\mathbf{C}^n \setminus \sigma_e(T)$.

Since $K(T)$ is the index class of T (see Proposition 1.5), then the degree of $DK(T)$ can be estimated from below if we know the indices of the functions of T (for instance, if there exists a function of T with nonzero index, then $K(T) \neq 0$). The proposition above shows that if the n-tuple T with non-trivial index class lifts, then the essential spectrum must satisfy some complex-geometric conditions in $\mathbf{C}^n$. This condition can be easily formulated in the particular case when the essential spectrum is an union of submanifolds of $\mathbf{C}^n$. Recall that the complex tangent bundle $T^c(M)$ of the submanifold M is defined by the formula $T^c(M) = = T(M) \cap i_* T(M)$.

COROLLARY 2.2. *Suppose that $\sigma_e(T)$ is a finite union of submanifolds of $\mathbf{C}^n$ with complex tangent bundles of complex dimension less or equal to $p - 2$, and let* $\deg DK(T) \geqslant 2p - 1$. *Then the n-tuple T can not be lifted.*

Proof. Suppose that T lifts to a commutative n-tuple T', and denote by A the complex set $\sigma_F(T')$. From Proposition 2.1 one can see that the complex dimension of A is greater or equal to p. Applying the theorem of Chirka [6], we obtain that the closure $\bar{A}$ of A is a complex subset of $\mathbf{C}^n$. Since A is bounded, we obtain a contradiction.

As application, consider the question of lifting one-dimensional Toeplitz operators. Denote by $\mathbf{S}$ the unit circle, considered as a boundary of the unit disk $\mathbf{D}$ in $\mathbf{C}^n$, and by P — the orthogonal projection from $L^2 = L^2(\mathbf{S})$ onto the Hardy space H^2. For any function $f \in C(\mathbf{S})$ we denote by T_f the Toeplitz operator on H^2, $T_f = P \circ M_f$, where M_f is the operator of multiplication by f in L^2. If f belongs to the algebra $A(\mathbf{D})$ of continuous functions on $\mathbf{S}$, having a holomorphic continuation on $\mathbf{D}$, then T_f is simply the multiplication by f. For a given n-tuple of continuous functions $f = (f_1, \ldots, f_n)$, one may ask whether the n-tuple of operators $T_f = (T_{f_1}, \ldots, T_{f_n})$ lifts. Obviously it does if all f_i belong to $A(\mathbf{D})$. The next proposition shows that under some regularity conditions the general case reduces to the above by a change of the parameter.

PROPOSITION 2.3. *Let $f = (f_1, \ldots, f_n)$ be an n-tuple of continuous functions on $\mathbf{S}$ and suppose that f is a monomorphic map of $\mathbf{S}$ in $\mathbf{C}^n$ and the image Γ_f of f is a 2-smooth curve in $\mathbf{C}^n$. Then the n-tuple of operators T_f lifts if and only if there exists a continuous map $h : \mathbf{S} \to \mathbf{S}$, and functions $g_1, \ldots, g_n \in A(\mathbf{D})$ such that $f_i = g_i \circ h$, $i = 1, \ldots, n$.*

Proof. We prove first the sufficiency. Let T_h be the Toeplitz operator with symbol h. Then $\|T_h\| = 1$ and we have an $A(\mathbf{D})$-functional calculus for this operator. The operators $g_1(T_h), \ldots, g_n(T_h)$ commute and are a compact perturbation of $T_{f_1}, \ldots, T_{f_n}$ respectively, i.e. T_f lifts.

Suppose now that T_f lifts. The essential spectrum of T_f coincides with Γ_f, and the Alexander dual of $K(T_f)$ coincides with the fundamental cycle of Γ_f. Therefore, by Proposition 2.1, there exists a bounded (one-dimensional) complex subset $M \subset \mathbf{C}^n \setminus \Gamma_f$ with $\mathrm{b}M = \Gamma_f$. Using the Harvey-Lowson theorem [13], we obtain the existence of a closed subset E of Γ_f of zero linear measure, such that on $\Gamma_f \setminus E$ the couple (M, Γ_f) is an 1-smooth regular manifold with boundary. Since Γ_f is connected, then M is irreducible, and there exists a proper map $g : \mathbf{D} \to \mathbf{C}^n$, $g = (g_1, \ldots, g_n)$, whose image coincide with M. Theorem 33 of [7] shows that this map can be extended up to an 1-smooth map from $\bar{\mathbf{D}}$ to $\mathbf{C}^n$. The standard monotonicity argument shows that g induces an one to-one map from the boundary $\mathbf{S}$ of $\mathbf{D}$ to Γ_f. Denote $h = (g \mid S)^{-1} \circ f$; then h is continuous and $f_i = g_i \circ h$, $i = 1, \ldots, n$. The proposition is proved.

In particular, one can see that the $n + 1$-tuple $(T_{f_1}, \ldots, T_{f_n}, T_z)$ lifts iff it is commutative, i.e. iff all f_i belong to $A(\mathbf{D})$.

Combining our approach with the Brown-Douglas-Fillmore theory, one can obtain classification theorems for some classes of commuting tuples of essentially normal operators in Hilbert space.

PROPOSITION 2.4. *Suppose that $T = (T_1, \ldots, T_n)$ is a commuting n-tuple of essentially selfadjoint operators,* i.e. *all $T_i - T_i^*$ are compact. Then $K(T) = 0$. Moreover, if $\sigma_e(T)$ is a finite* CW-*complex, then T is a compact perturbation of a commuting n-tuple of normal operators.*

Proof. Since all T_i are essentially selfadjoint, then its essential spectrum is real, and therefore the joint essential spectrum $\sigma_e(T)$ lies in $\mathbf{R}^n$. Since there are no bounded complex subsets of $\mathbf{C}^n \setminus \mathbf{R}^n$, then, using 2.1, we obtain that $K(T) = 0$. The last assertion follows from 1.4.

Consider the case of a commuting n-tuple of essentially unitary operators, i.e. of an n-tuple $T = (T_1, \ldots, T_n)$ such that for all i the operators $I - T_i^* T_i$ and $I - T_i T_i^*$ are compact. First we give some examples of such n-tuples. Let $f = (f_1, \ldots, f_n)$ be an n-tuple of functions from $A(\mathbf{D})$ such that $|f_i(z)| = 1$ for all i and z, $|z| = 1$ (note that in this case $f_i(z)$ can be extended in a neighborhood of $\mathbf{D}$ and can be represented as a finite product of factors of the type $\theta(z - a)(1 - z\bar{a})^{-1}$ with $|\theta| = 1$, $|a| < 1$). Then $T_f = (T_{f_1}, \ldots, T_{f_n})$ and $T = (T_{\bar{f}_1}, \ldots, T_{\bar{f}_n})$ are commuting n-tuples of essentially unitary operators.

For a given compact set $F \subset \mathbf{C}^n$ we denote by D_F the n-tuple of commuting diagonal operators with joint essential spectrum equal to F.

PROPOSITION 2.5. *Suppose that T is a commuting n-tuple of essentially unitary operators and $\sigma_e(T)$ is a finite* CW-*complex. Then there exist two finite families $\{f^\alpha\}, \{g^\beta\}, f^\alpha = (f_1^\alpha, \ldots, f_n^\alpha), g^\beta = (g_1^\beta, \ldots, g_n^\beta)$ of the n-tuples of functions of the type considered above, such that T is unitarily equivalent to a compact perturbation of the n-tuple*

$$T' = (\bigoplus_\alpha T_{f^\alpha}) \oplus (\bigoplus_\beta T_{\bar{g}^\beta}) \oplus D_{\sigma_e(T)} .$$

Proof. Since all the T_i are essentially unitary, then $\sigma_e(T)$ is a subset of $\mathbf{T}^n = \{z = (z_1, \ldots, z_n) : |z_i| = 1, i = 1, \ldots, n\}$, and it is easy to see that the dimension of $\sigma_F(T)$ is less or equal to one. Indeed, if A is a component of $\sigma_F(T)$ of dimension 2, then recalling the fact that $\mathbf{T}^n$ has no complex tangent vectors, and using the theorem from [7], we obtain that $\bar{A}$ is a complex subset of $\mathbf{C}^n$, which is a contradiction. It is easy to see that the one-dimensional part of $\sigma_F(T)$ is contained in the closed unit n-disk $\bar{\mathbf{D}}^n$. Denote $W = \{(z_1, \ldots, z_n) : |z_i| > 1/2, i = 1, \ldots, n\}$. Then any irreducible component of $\sigma_F(T)$ has non-empty intersection with $\bar{\mathbf{D}}^n \setminus W$. Since $\sigma_e(T) \cap (\mathbf{D}^n \setminus W) = \emptyset$, then this implies that $\sigma_F(T)$ has finitely many irreducible one-dimensional components. Therefore, the element $\mathcal{K}(T)$ has the Jordan-Hölder decomposition of the form:

$$\mathcal{K}(T) = \sum_\alpha m_\alpha [M_\alpha] - \sum_\beta n_\beta [N_\beta]$$

where M_α, N_β are irreducible one-dimensional complex sets, and m_α, n_β are natural numbers. Since $\mathbf{T}^n$ is real-analytic, then, by using another theorem of Chirka (see [7], Section 1), one can see that M_α, N_β can be extended on some neighborhood of $\mathbf{D}^n$. Therefore there exist holomorphic maps f^α, g^β from the unit disk $\mathbf{D}$ in $\mathbf{C}$ into $\mathbf{C}^n$, which can be extended in a neighborhood of $\overline{\mathbf{D}}$, such that $f^\alpha(\mathbf{D}) = M_\alpha$, $g^\beta(\mathbf{D}) = N_\beta$. Let T' be the n-tuple defined above. Then we have $\mathcal{K}(T') = \mathcal{K}(T)$ and therefore $\mathrm{Ext}(T') = \mathrm{Ext}(T)$, which completes the proof.

The situation simplifies if one considers the class of T in the group $\mathrm{Ext}(\mathbf{T}^n)$. Recall that two n-tuples T, T' generate the same element of $\mathrm{Ext}(\mathbf{T}^n)$ iff $T \oplus D_{\mathbf{T}^n}$ is unitarily equivalent to a compact perturbation of $T' \oplus D_{\mathbf{T}^n}$. If T is a commuting n-tuple of essentially unitary operators, then $\sigma_F(T)$ is one-dimensional, and therefore the dual of $K(T)$ belongs to the group $K_1^1(\mathbf{T}^n) = \mathbf{Z}^n$. Then $\mathrm{Ext}(T)$ is determined by an n-tuple $\bar{k} = (k_1, \ldots, k_n)$ of integers. Denote by V the unilateral right shift (i.e., T_z), and let $V^{\bar{k}} = (V^{k_1}, \ldots, V^{k_n})$ if all $k_i \geqslant 0$.

Proposition 2.5'. *The n-tuple T is equivalent in* $\mathrm{Ext}(\mathbf{T}^n)$ *to an n-tuple of the type* $V^{\bar{m}} \oplus (V^*)^{\bar{p}}$, *where* $\bar{m}$ *and* $\bar{p}$ *are n-tuples of nonnegative integers.*

Indeed, take the n-tuples $\bar{m}, \bar{p}$ of nonnegative integers such that $\bar{k} = \bar{m} - \bar{p}$. Then it is easy to check that the n-tuple $V^m \oplus (V^*)^p$ generates the same element of $\mathrm{Ext}(\mathbf{T}^n)$ as T.

As another application of the results of Section 1 we shall give a simple proof of the index theorem of Boutet de Monvel for Toeplitz operators. The original proof of Boutet de Monvel in [4] was based on the Atiyah-Singer index theorem (and is in some sense equivalent to this theorem). It should be noted that recently Boutet de Monvel and Malgrange gave an independent proof of the index theorem for a more general class of operators. Now, let N be a complex--analytic submanifold of $\mathbf{C}^n$, M be a precompact strongly pseudoconvex domain in N, and E be a complex-analytic vector bundle on N. Denote by $L^2(E)$, $H^2(E)$ the space of all square-integrable, resp. holomorphic and square-integrable, sections of E on M, and by P — the ortogonal projection of $L^2(E)$ onto $H^2(E)$. The Toeplitz operator with symbol $f \in C(\overline{M})$ is the operator $T_f = P \circ M_f$. It is proved in [4] that the Toeplitz operators form an essentially commuting algebra, and the spectrum of its factor-algebra by the ideal of compact operators coincide with $\mathrm{b}M$. Denote by T_i, $i = 1, \ldots, n$, the operator of multiplication by the coordinate function z_i, and let $T = (T_1, \ldots, T_n)$. Then the algebra of Toeplitz operators is generated by the operators T_i, T_i^*, $i = 1, \ldots, n$; therefore, $\sigma_e(T) = \mathrm{b}M$ and $\mathrm{Ext}(T)$ coincides with the element of $\mathrm{Ext}(\mathrm{b}M)$, determined by the algebra of Toeplitz operators. Lemma 4.11 asserts that $\mathcal{K}(T) = i_* E$, where i is the embedding of M in $\mathbf{C}^n \setminus \mathrm{b}M$. Using Propositions 1.4 and 1.9, one can see that $\mathrm{Ext}(T) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM)) \cap$

$\cap$ [bM]. Therefore, for any continuous invertible matrix-valued function G on bM we obtain the Boutet de Monvel index formula:

$$\operatorname{ind}(T_G) = \langle G, \operatorname{Ext} T\rangle = \langle \operatorname{ch}(G) \cup \operatorname{ch}(E) \cup \operatorname{Td}(TM), \mathrm{b}M\rangle.$$

3. FUNCTORIALITY OF $K(T)$ AND $\mathscr{K}(T)$

In this section we prove the functoriality of $K(T)$ and $\mathscr{K}(T)$ under a suitable functional calculus, and some related results concerning the structure of $\mathscr{K}(T)$ for commutative n-tuples.

Let $T = (T_1, \ldots, T_n)$ be an essentially commuting n-tuple of operators acting on the Banach space X, and $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_n)$ be the corresponding n-tuple of elements of the Calkin algebra $\mathfrak{A}(X)$. Suppose that we are in the conditions of 1.5, i.e. we have a commutative Banach algebra $A \subset C(\sigma_e(T))$, and an A-functional calculus $f(z) \to f(\tilde{T})$ for $\tilde{T}$. If $f(z) = (f_1(z), \ldots, f_k(z))$ is a k-tuple of functions from A, then we shall denote by $f(T)$ an arbitrary k-tuple of operators representing $f(\tilde{T})$.

If $f\colon \sigma_e(T) \to \mathbf{C}^k$ is a continuous mapping, then there exists a natural morphism $f_* : \mathrm{K}^0(\mathbf{C}^n \setminus \sigma_e(T)) \to \mathrm{K}^0(\mathbf{C}^k \setminus f(\sigma_e(T)))$. If $\sigma_e(T)$ is a CW-complex, then this morphism coincides with the Alexander dual of the morphism $f_* : \mathrm{K}_1(\sigma_e(T)) \to \mathrm{K}_1(f(\sigma_e(T)))$; in the general case the construction of f_* is described in the proof below.

PROPOSITION 3.1. *If $f(z)$ is a k-tuple of functions from A, then $\sigma_e(f(T)) = f(\sigma_e(T))$ and $K(f(T)) = f_* K(T)$.*

Proof. The mapping f can be represented in a standard way as a composition of an embedding and a projection, and it is sufficient to prove the assertion in these two particular cases.

a) Let $f(z) = (f_1(z), \ldots, f_k(z))$ be a k-tuple of functions from A, and define the map $F(z) : \sigma_e(T) \to \mathbf{C}^{n+k}$ by the formula $F(z) = (z_1, \ldots, z_n, f_1(z), \ldots, f_k(z))$. Denote by $M \subset \mathbf{C}^n$ the spectrum of A. Then $F(z)$ can be extended canonicaly on M. Using Lemma 4.8, one can see that $\sigma_e(F(T)) = \sigma(F(\tilde{T})) \subset \sigma_A(F(T))$ and therefore $\sigma_e(F(T))$ is contained in the set $F(M)$. On the the other hand, applying for the projection $q\colon \mathbf{C}^{n+k} \to \mathbf{C}^n$ the projection property of the Taylor spectrum, we obtain $q(\sigma_e(F(T))) = \sigma(\tilde{T}) = \sigma_e(T)$. Therefore, we have $\sigma_e(F(T)) = F(\sigma_e(T))$. By applying the projection of $\mathbf{C}^{n+k}$ onto $\mathbf{C}^n$, we obtain $\sigma_e(F(T)) = f(\sigma_e(T))$.

We shall prove the second equality. Let $\tilde{f}_1(z), \ldots, \tilde{f}_k(z)$ be an arbitrary continuous extensions of $f_1(z), \ldots, f_k(z)$ on the whole $\mathbf{C}^n$. Denote by $K^0_.(z, w)$ the Koszul complex of the operators $T_1 - z_1 I, \ldots, T_n - z_n I, (f_1(z) - w_1)I, \ldots, (f_k(z) - w_k)I$. It is easy to see that $K^0_.(z, w)$ is Fredholm on $\mathbf{C}^{n+k} \setminus F(\sigma_e(T))$. We shall show that $K^0_.(z, w)$ represents the element $F_* K(T)$. Indeed, if $H.$ is a continuous complex of vector bundles on $\mathbf{C}^n \setminus \sigma_e(T)$, quasi-isomorphic to a compact perturbation of $K.(T, z)$.

then some compact perturbation of $K^0_\bullet(z, w)$ is quasi-isomorphic on $\mathbf{C}^{n+k}\setminus\sigma_e(F(T))$ to the image of the class of $H_\bullet$ under the Thom-Gysin homomorphism. On the other hand, the element $K(F(T))$ is represented by the Koszul complex of the operators $T_1 - z_1I, \ldots, T_n - z_nI, f_1(T) - w_1I, \ldots, f_k(T) - w_kI$. Denote it by $K^1_\bullet(z, w)$. We shall show that $K^0_\bullet(z, w)$ and $K^1_\bullet(z, w)$ are homotopic in the space of continuous essential Fredholm complexes on $\mathbf{C}^{n+k}\setminus\sigma_e(F(T))$. For $u \in \sigma_e(T)$, $z \in \mathbf{C}^n$, $0 \leqslant t \leqslant 1$, $1 \leqslant i \leqslant k$, denote $f_i(u, z, t) = (1-t)f_i(z) + tf_i(u)$, and let $K^t_\bullet(z, w)$ be the Koszul complex of the operators $T_1 - z_1I, \ldots, T_n - z_nI, f_1(T, z, t) - w_1I, \ldots, f_k(T, z, t) - w_kI$. We shall show that the essential complex $K^t_\bullet(z, w)$ is Fredholm off $\sigma_e(F(T))$. Indeed, if $z \notin \sigma_e(T)$, then $K^t_\bullet(z, w)$ is Fredholm for any w. Fix $z^0 \in \sigma_e(T)$. Then, using the assertion proved above, one can see that the joint essential spectrum of the operators $T_1, \ldots, T_n, f_1(T, z^0, t), \ldots, f_k(T, z^0, t)$ coincides with the set $\{(z, w) : z \in \sigma_e(T),\ w = (1-t)f(z^0) + tf(z)\}$ and does not contain the points (z^0, w) with $w \neq f(z^0)$. Hence, $K^t_\bullet(z, w)$ defines a homotopy between $K^0_\bullet(z, w)$ and $K^1_\bullet(z, w)$ and the functoriality of $K(T)$ with respect to the map $F(z)$ is proved.

b) Let $p: \mathbf{C}^{n+k} \to \mathbf{C}^n$ denote the projection on the last k coordinates, and let $T = (T_1, \ldots, T_n)$, $S = (S_1, \ldots, S_k)$, $T' = (T, S)$ be essentially commuting tuples of operators. Then $S = p(T')$. We shall prove that $K(S) = p_*K(T)$. Denote by S' the $n+k$-tuple $(0, S)$. Then the complex $K_\bullet(S', z, w)$ is equal to the image of the complex $K_\bullet(S, w)$ under the Thom-Gysin homomorphism, corresponding to the embedding $w \to (0, w)$ of $\mathbf{C}^k$ in $\mathbf{C}^{n+k}$. Therefore the morphism p_* maps the class of $K_\bullet(S', z, w)$ into the class of $K_\bullet(S, w)$. Let $\mathbf{B}$ be a ball in $\mathbf{C}^n$, centered in the origin and containing $\sigma_e(T)$, and let $\mathbf{B}' = \mathbf{B}\times\sigma_e(T)$. Then $K_\bullet(T', z, w)$ and $K_\bullet(S', z, w)$ determine the same element of the group $\mathrm{K}^0(\mathbf{C}^{n+k}, \mathbf{C}^{n+k}\setminus\mathbf{B}')$. Indeed, denote by $K_\bullet(tT, S, z, w)$ the Koszul complex of the $n+k$-tuple (tT, S), where $tT = (tT_1, \ldots, tT_n)$, $0 \leqslant t \leqslant 1$. Since $\sigma_e(tT, S) \subset \mathbf{B}'$, then $K_\bullet(tT, S, z, w)$ defines a Fredholm homotopy between $K_\bullet(T', z, w)$ and $K_\bullet(S', z, w)$ on $\mathbf{C}^{n+k}\setminus\mathbf{B}'$. This shows that the images of $K_\bullet(T', z, w)$ and $K_\bullet(S', z, w)$ under p_* coincide in the group $\mathrm{K}^0(\mathbf{C}^n\setminus\sigma_e(T))$.

Since $f = p \circ F$, then a) and b) prove the general case.

In the case of a commuting n-tuple T we shall show that a similiar property is valid for the element $\mathscr{K}(T)$.

PROPOSITION 3.2. *Let $f(z) = (f_1(z), \ldots, f_k(z))$ be a k-tuple of holomorphic functions defined in some neighborhood of the spectrum $\sigma(T)$ of the commuting n-tuple T. Then:*

a) $\sigma_e(f(T)) = f(\sigma_e(T))$;

b) *for each i we have $\mathscr{H}_i(f(T)) = f_*\mathscr{H}_{i+n-k}(T)$, therefore $\mathscr{K}(f(T)) = f_*\mathscr{K}(T)$.*

REMARK 1. Since the joint essential spectra of a commuting and an essentially commuting n-tuple are defined in different ways, the functoriality of $\sigma_e(T)$ in this case does not follow from 3.1. This functoriality was announced in [16] and independently proved in [9].

REMARK 2. It follows from the functoriality of $\sigma(T)$ and $\sigma_e(T)$ that $f(z)$ determines a holomorphic map from $\sigma_F(T)$ to $\sigma_F(f(T))$. Then any point of $\sigma_F(f(T))$ has a finite preimage, and the functor of direct image f_* is well-defined on the category of coherent sheaves. This functor is exact, and all its derived functors are equal to zero.

REMARK 3. Suppose that we choose a new coordinate system in $\mathbf{C}^n$ and take the corresponding system of generators in the linear span of $T_1, \ldots, T_n$. It follows immediately from the proposition that the homology sheaves $\mathscr{H}_i(T)$ and the homology spaces $H_i(T, z)$ remain invariant under such a change.

Proof. One follows the same argument as in the proof of 3.1. Denote $F(z) = (z, f(z))$, and $q(z, w) = z$. Then $\sigma(F(T)) = F(\sigma(T))$, and q defines an isomorphism between $\sigma(F(T))$ and $\sigma(T)$. Taking into account the functoriality under the projection q, which will be proved below, one can see that $\sigma_e(F(T)) = F(\sigma_e(T))$ and $\mathscr{H}_i(F(T)) = F_*\mathscr{H}_{i-k}(T)$.

The main point of the proof is to show the functoriality under the projections. Let $T = (T_1, \ldots, T_n)$, $S = (S_1, \ldots, S_k)$, $T' = (T, S)$ be commuting tuples of operators, and $p(z, w) = w$ be the projection from $\mathbf{C}^{n+k}$ to $\mathbf{C}^k$. Denote by $K_\bullet(T, z) = \{X_i, d_i(z)\}$ and $K_\bullet(T', z, w) = \{X_i', d_i'(z)\}$ the parametrized Koszul complexes of T and T'. Recall that the complex $K(T', z, w)$ is equal to the total complex of the bicomplex, whose i-th column coincides with the Koszul complex of the operators $S_1 - w_1 I, \ldots, S_k - w_k I$ in the space X_i $\Big($which is equal to the direct sum of $\binom{n}{i}$ copies of the space $X\Big)$, and the j-th row coincides with the Koszul complex of the operators $T_1 - z_1 I, \ldots, T_n - z_n I$ in the space X_j. The lower and the upper row of this bicomplex are equal to $K_\bullet(T, z)$, which determines an embedding of complexes $I_\bullet : K_\bullet(T, z) \to K_\bullet(T', z, w)$ of dergree k and a projection of complexes $P_\bullet : K_\bullet(T', z, w) \to K_\bullet(T, z)$ of degree 0. The morphisms of complexes $I_\bullet$ and $P_\bullet$ are natural; if T'' is a tuple of operators, $T \subset T'' \subset T'$, and $I_\bullet'$, $I_\bullet''$ are the embeddings of $K_\bullet(T, \cdot)$ in $K_\bullet(T'', \cdot)$ and of $K_\bullet(T'', \cdot)$ in $K_\bullet(T', \cdot)$, then $I_\bullet = I_\bullet'' \circ I_\bullet'$. The same property holds for the projection $P_\bullet$.

Fix a point $z^0 \in \mathbf{C}^n \setminus p(\sigma_e(T))$, a Stein neighborhood V of z^0 not intersecting $p(\sigma_e(T))$, and a polydisk $\mathbf{D}$ with polyradius $R = (R_1, \ldots, R_k)$ in $\mathbf{C}^k$ containing $\sigma(S)$. Let $x(z)$ be a section of $\mathscr{H}_i(T)$ over the open subset V' of V, represented by the X_i-valued holomorphic function $\tilde{x}(z)$, $d_i(z)\tilde{x}(z) \equiv 0$. Then $I_\bullet[\tilde{x}(z)]$ is a holomorphic X_{i+k}-valued function on $V \times \mathbf{D}$ whose image under $d'_{i+k}(z, w)$ is equal to zero, and we obtain a morphism of sheaves $I_i(z, D) : \mathscr{H}_i(T) \to p_*\mathscr{H}_{i+k}(T')$ on V. Denote by

$\mathcal{O}_w X_i'$ the sheaf of germs of X_i'-valued holomorphic functions on $\mathbf{D}$, and by $\mathcal{O}_w K_{\bullet}'(z) = \{\mathcal{O}_w X_i', d_i'(z, w)\}$, the complex of holomorphic sections of $K_{\bullet}(T', z, w)$ on $\{z\} \times \mathbf{D}$. Denote the i-th sheaf of homology of this complex by $\mathcal{H}_i^z(T')$. Put $\Gamma_w X_i' = \Gamma_{\mathbf{D}}(\mathcal{O}_w X_i')$. Then the Fréchet spaces $\Gamma_w X_i'$ form a complex $\Gamma_w K_{\bullet}'(z) = \Gamma_{\mathbf{D}}(\mathcal{O}_w K_{\bullet}'(z))$ with differentials holomorphically depending on $z \in V$. Then the complex of sheaves of its holomorphic sections $\mathcal{O}_z \Gamma_w K_{\bullet}'(z)$ coincides with $p_*(\mathcal{O}K_{\bullet}(T', z, w))$. Since $V \times \mathbf{D}$ is Stein, then it is easy to see that the sheaves of homology of the complex $\mathcal{O}_z \Gamma_w K_{\bullet}'(z)$ coincide with $p_* \mathcal{H}_i(T')$ and therefore are coherent (see Remark 2 above). Then a standard construction shows (see [24], I.4.15) that the complex of sheaves $\mathcal{O}_z \Gamma_w K_{\bullet}$ is perfect, i.e. it is quasi-isomorphic to a bounded complex of finite dimensional free $\mathcal{O}_z$-modules. This means that the complex of Fréchet spaces $\Gamma_w K_{\bullet}'(z)$ is $\mathcal{O}_z$-Fredholm in the sense of [18], and Propositions 1.3 and 1.4 from [18] imply that all its homology spaces $\Gamma_{\mathbf{D}} \mathcal{H}_i^z(T')$ are finite-dimensional for any $z \in V$. In the same way as above the embedding $I_{\bullet}$ defines a map of linear spaces $I_i(z^0, \mathbf{D})$: $: H_i(T, z^0) \to \Gamma_{\mathbf{D}} \mathcal{H}_{i+k}^{z^0}(T')$. We shall show that the maps $I_i(z, \mathbf{D})$ and $I_i(z^0, \mathbf{D})$ are isomorphisms.

We show that these maps are monomorphic by constructing their left inverse maps. Fix a sufficiently small positive ε, and denote $W_i = \{w : w \in \mathbf{D}, |w_i| > R_i - \varepsilon\}$, $i = 1, \ldots, k$, $W = \bigcap_{i=1}^{k} W_i$, $Z = \bigcap_{i=1}^{k} \mathbf{D} \setminus W_i$. Then the Fréchet spaces $H_Z^k(\mathbf{D}, \Gamma_w X_i') =$

$$= \Gamma_W(\mathcal{O}_w X_i') \Big/ \bigoplus_{j=1}^{k} \Gamma_{W_j}(\mathcal{O}_w X_i')$$ form a complex of Fréchet spaces $H_Z^k(\mathbf{D}, \Gamma_w K_{\bullet})$ with differentials, holomorphically depending on $z \in V$. It is well known that any $n+k$-tuple of operator-functions $g_1(z, w), \ldots, g_{n+k}(z, w)$ satisfying $(T_1 - z_1) \times g_1(z, w) + \ldots + (S_k - w_k) g_{n+k}(z, w) \equiv \mathrm{Id}_X$ determines a homotopy or the complex $K_{\bullet}(T', z, w)$, i.e. a family $G_{\bullet}(z, w) = \{G_i(z, w)\}$ of operators $G_i(z, w) : X_i' \to X_{i-1}'$ satisfying for any i the equality $G_{i+1}(z, w)\, d_i'(z, w) + d_{i-1}'(z, w) G_i(z, w) \equiv \mathrm{Id}$. On the domain $V \times W_j$ one can take $g_{n+j}(z, w) = (S_j - w_j)^{-1}$ and $g_m(z, w) = 0$ for $m \neq n + j$. Denote the corresponding homotopy of $K_{\bullet}(T', z, w)$ by $G_{\bullet}^j$. Let $y(w)$ be a X_i'-valued holomorphic function on $\mathbf{D}$, i.e. an element of $\Gamma_w X_i$. For any $(z, w) \in V \times W$ define $[r_i y](z, w) = G_{i-k+1}^k(z_j w) \circ \ldots \circ G_i^1(z, w) y(w)$. Then $r_{i+1} d_i - d_{i-k} r_i$ belongs to $\bigoplus_{j=1}^{k} \Gamma_{W_j}(\mathcal{O}_w X_{i-k+1}')$ and therefore $r_{\bullet}(z, w)$ defines a morphism of degree $-k$ from the complex $\Gamma_w K_{\bullet}(z)$ to the complex $H_Z^k(\mathbf{D}, \Gamma_w K_{\bullet}(z))$. Let $R_i' = R_i - \varepsilon/2$, $i = 1, \ldots, k$. We define a morphism of complexes $R_{\bullet}(z) : \Gamma_w K_{\circ}(z) \to K_{\bullet}(T, z)$ by the formula

$$R_i(z) y(w) = P_{i-k}(2\pi i)^{-k} \int_{|w_1| = R_1'} \ldots \int_{|w_k| = R_k'} [r_i y](z, w)\, \mathrm{d}w_1 \ldots \mathrm{d}w_k.$$

It is easy to see that $R_\bullet(z)$ commutes with the differentials and therefore induces the maps $R_i^{z^0}: \Gamma_{\mathbf{D}}\mathscr{H}_i^{z^0}(T') \to H_{i-k}(T, z^0)$ and $R_i(z): p_*\mathscr{H}_i(T') \to \mathscr{H}_{i-k}(T)$. We shall show that $R_{i+k}^{z^0} \circ I_i(z^0, \mathbf{D})$ is the identity in the space $H_i(T, z^0)$ and that $R_{i+k}(z) \circ I_i(z, \mathbf{D})$ is the identical morphism of the sheaf $\mathscr{H}_i(T)$ on V. Indeed, denote by $J_\bullet$ the right inverse to the morphism of complexes $P_\bullet$, i.e. the embedding of the complex $K_\bullet(T, z)$ in the upper row of the bicomplex, defining $K_\bullet(T', z, w)$ ($J_\bullet$ is not a morphism of complexes). Let $x \in \ker d_i(z^0)$ and $y(w) = I_i(z^0, \mathbf{D})x \in \Gamma_w X'_{i+k}$. Then as direct computation shows that $[r_{i+k}y](z, w) = (S_k - w_k)^{-1} \ldots (S_1 - w_1)^{-1}x$. Applying the Weil integral formula, we obtain the first equality, and the second can be proved in the same way. Therefore the spaces $H_i(T, z^0)$ are finite dimensional, and the functoriality of $\sigma_e(T)$ is proved.

We have to show that the map $I_\bullet(z, \mathbf{D})$ is an epimorphism. Since $I_\bullet$ is functorial in the sense, specified above, then it is sufficient to consider the case $k = 1$. We have $T' = (T_1, \ldots, T_n, S)$, where the operator S commutes with $T_1, \ldots, T_n$. Then S induces an endomorphism of the sheaf $p^*\mathscr{H}_i(T)$ which will be denoted by $\tilde{S}^i$. There is an exact sequence of sheaves on $V \times \mathbf{D}$:

$$0 \to \operatorname{coker}(\tilde{S}^{i-1} - wI) \to \mathscr{H}_i(T') \to \ker(\tilde{S}^i - wI) \to 0$$

where the first non-trivial arrow is induced by the embedding $I_\bullet$, and the second one — by the projection $P_\bullet$. In order to prove the epimorphicity of $I.(z, \mathbf{D})$ it is sufficient to show that $p_* \ker(\tilde{S}^i - wI) = 0$. Let V' be on open subset of V, $x(z, w)$ be a section of $\ker(\tilde{S}^i - wI)$ represented by the holomorphic X_i-valued function $\tilde{x}(z, w)$. Then there exists a holomorphic X_{i-1}-valued function $y(z, w)$ such that $(\tilde{S} - wI)\tilde{x}(z, w) = d_{i-1}(z)y(z, w)$. Put

$$u(z, w) = (2\pi i)^{-1} \int_{|s|=R'} (s - w)^{-1}(S - sI)^{-1}y(z, s)ds$$

where $|w| < R' < R$. It is easy to check that $d_{i-1}(z)u(z, w) = \tilde{x}(z, w)$, which shows that $x(z, w) = 0$. This proves the epimorphicity of $I_\bullet(z, w)$, and the proof of 3.2 is complete.

Note that the proof of the fact that $I_\bullet(z, \mathbf{D})$ and $I_\bullet(z^0, \mathbf{D})$ are isomorphisms is still valid if we do not assume that $p^{-1}(z^0) \cap \sigma_e(T') = \emptyset$; in this case the space $\Gamma_{\mathbf{D}}\mathscr{H}_i^{z^0}(T')$ must be replaced by its subspace, generated by the globally defined sections of $\ker d'_i(z^0, w)$, $w \in \mathbf{D}$ (in the case considered above they coincide), and analogously, the sheaf $p_*\mathscr{H}_i(T')$, by the i-th sheaf of homology of the complex of sheaves $p_*\mathcal{O}K_\bullet(T', z, w)$. Note also that for any j, $1 \leqslant j \leqslant k$, the isomorphism $I_\bullet(z^0, \mathbf{D})$

transfers the action of the operator S_j on $H_i(T, z^0)$ to the operator of multiplication by the coordinate w_j in the space $\Gamma_{\mathbf{D}}\mathscr{H}_i^{z^0}(T')$. Now, we shall prove a local version of Proposition 3.2. Fix a point $(z^0, w^0) \in \sigma_F(T')$, a sufficiently small Stein neighborhood V of z^0, and a polydisk $\tilde{\mathbf{D}}$, centered at w^0, such that $V \times b\tilde{\mathbf{D}} \cap \sigma(T') = \emptyset$ and $\{z^0\} \times \tilde{\mathbf{D}} \cap \sigma(T') = \{w^0\}$. Then the maps $I_\bullet(z^0, \tilde{\mathbf{D}})$, $I_\bullet(z, \tilde{\mathbf{D}})$ can be defined as above, and we shall give a construction for the maps $\tilde{R}_\bullet^{z^0}$, $\tilde{R}_\bullet(z)$, corresponding to the polydisk $\tilde{\mathbf{D}}$. Let $L_\bullet(z, w) = \{L_i, a_i(z, w)\}$ be a holomorphic complex of finite-dimensional spaces, quasi-isomorphic to $K_\bullet(T', z, w)$ on $V \times \tilde{\mathbf{D}}$, and $\varphi_\bullet(z, w): L_\bullet(z, w) \to K_\bullet(T', z, w)$ be the corresponding quasi-isomorphism. Then the complex $\Gamma_w L_\bullet(z)$ is a Fredholm complex of Fréchet spaces on V, quasi-isomorphic to the complex $\Gamma_w K'_\bullet(z)$. Denote by $\tilde{W}_i$, $\tilde{W}$, $\tilde{Z}$ the subspaces of $\tilde{\mathbf{D}}$, defined in the same way as W_i, W, Z above, and let $H^k_{\tilde{Z}}(\tilde{\mathbf{D}}, \Gamma_w L_i) = \Gamma_{\tilde{W}}(\mathcal{O}_w L_i) \Big/ \bigoplus_{j=1}^{k} \Gamma_{\tilde{W}_i}(\mathcal{O}_w L_i)$. For any j, $1 \leqslant j \leqslant k$, one can find a set of linear maps $\gamma_i^j(z, w): L_i \to L_{i-1}$, holomorphically depending on $(z, w) \in V \times \tilde{W}_j$, such that for any i we have

$$\gamma^j_{i+1}(z, w) \circ a_i(z, w) + a_{i-1}(z, w) \circ \gamma^j_i(z, w) = \mathrm{Id}_{L_i}.$$

Let $y(w)$ be an L_i-valued holomorphic function on $\tilde{\mathbf{D}}$. For any $(z, w) \in V \times \tilde{W}$ put $[\tilde{r}_i y](z, w) = \gamma^k_{i-k+1}(z, w) \circ \ldots \circ \gamma^1_i(z, w) y(w)$. Then $r_\bullet(z, w)$ is a morphism of degree $-k$ from the complex $\Gamma_w L_\bullet(z)$ to the complex $H^k_{\tilde{Z}}(\tilde{\mathbf{D}}, \Gamma_w L_\bullet(z))$. Now define the morphism $\tilde{R}_\bullet(z): \Gamma_w L_\bullet(z) \to K_\bullet(T, z)$ as follows:

$$\tilde{R}_i(z)\, y(w) = (2\pi i)^{-k} P_{i-k} \int\limits_{|w_1| = R'_1} \cdots \int\limits_{|w_k| = R'_k} \varphi_{i-k}(z, w) \circ \tilde{r}_i(z, w) y(w)\, dw.$$

This morphism induces the maps $\tilde{R}_i^{z^0}: \Gamma_{\tilde{\mathbf{D}}}\mathscr{H}_i^{z^0}(T') \to H_{i-k}(T, z^0)$ and $\tilde{R}_i(z): p_{*k}(\mathscr{H}_i T') \,|\, (V \times \tilde{\mathbf{D}}) \to \mathscr{H}_{i-k}(T)$. It is easy to see that the definition of $\tilde{r}_\bullet(z, w)$ does not depend up to a homotopy from the choice of $\gamma^j_\cdot$ z, w), and therefore the definitions of $\tilde{R}_\bullet^{z^0}$, $\tilde{R}_\bullet(z)$ are correct. Now the maps $G_i^{z^0} = \tilde{R}_{i+k}^{z^0} \circ I_i(z^0, \tilde{\mathbf{D}})$ and $G_i(z) = \tilde{R}_{i+k}(z) \circ I_i(z^0, \tilde{\mathbf{D}})$ may differ from the identity; we shall show that they are projections in the space $H_i(T, z^0)$ and in the sheaf $\mathscr{H}_i(T)$ respectively. Denote by $H'_i(T, z^0)$ the image of $G_i^{z^0}$, and by $\mathscr{H}'_i(T)$ — the image of $G_i(z)$. Then $H'_i(T, z^0)$ is a finite dimensional space, and $\mathscr{H}'_i(T)$ is a coherent sheaf. As we remarked above, the map $\tilde{R}_i(z)$ transfers the multiplication by the coordinate w_j to the action of the operator S_j:

therefore, $H_i'(T, z^0)$ is an invariant subspace of the operators $S_1, \ldots, S_k$, and the joint spectrum of the k-tuple S is contained in the polydisk $\tilde{\mathbf{D}}$. Shrinking $\tilde{\mathbf{D}}$ and V, we can see that this joint spectrum coincide with $\{w^0\}$. Now let $x \in H_i'(T, z^0)$. Then a direct computation shows that the element $\varphi_i(z, w) \circ \tilde{r}_{i+k}(z, w) \circ I_i(z, \tilde{D})x$ is homological to the elements $J_i(S_1 - w_1)^{-1} \ldots (S_k - w_k)^{-1}x$ in the complex $H^k_{\bar{Z}}(\tilde{\mathbf{D}}, \Gamma_w K_\bullet(z))$. Using again the Weil integral formula, we see that $G_i^{z^0}(x) = x$. The same arguments show that $G_i(z)$ is a projection also. Denote by $H_i''(T, z^0)$ and $\mathscr{H}_i''(T)$ the kernels of $G_i^{z^0}$ and $G_i(z)$ respectively. Take a polydisk $\mathbf{D}$ containing $\sigma(S)$; then the image of $H_i'(T, z^0)$ under the isomorphism $I_i(z^0, \mathbf{D})$ consists of sections of $\mathscr{H}^{z^0}_{i+k}(T')$ supported in $\{w^0\}$, and the image of $H_i''(T, z^0)$ consists of sections with support in $p^{-1}(z^0) \cap \cap \sigma(T') \setminus \{w^0\}$. In other words, the decomposition of $H_i(T, z^0)$, constructed above, is a spectral decomposition corresponding to the isolated point $\{w^0\}$ of the joint spectrum of S (since $H_i(T, z^0)$ is not a Banach space, this decomposition can not be used directly). Summarizing, we obtain

PROPOSITION 3.3. *Suppose that $T = (T_1, \ldots, T_n)$, $S = (S_1, \ldots, S_k)$ and $T' = (T, S)$ are commuting tuples of operators, the point $(z^0, w^0) \in \mathbf{C}^{n+k}$ belongs to $\sigma_{\mathrm{F}}(T')$, and the projection p of $\sigma_{\mathrm{F}}(T')$ to the coordinate subspace of the first n coordinates is proper on some neighborhood of (z^0, w^0). Then there exists a neighborhood V of z^0 such that for each i we have the decompositions*

$$H_i(T, z^0) = H_i'(T, z^0) \oplus H_i''(T, z^0), \quad \mathscr{H}_i(T) \,|\, V = \mathscr{H}_i'(T) \oplus \mathscr{H}_i''(T)$$

such that

a) *$H_i'(T, z^0)$ is finite-dimensional and $\mathscr{H}_i(T)$ is coherent. There exist on V a holomorphic complex $M_\bullet(z)$ of finite-dimensional spaces, and a homomorphism of complexes $\tau_\bullet(z) : M_\bullet(z) \to K_\bullet(T, z)$ such that $\tau_\bullet(z)$ induces an isomorphism between $H_i(M_\bullet(z^0))$ and $H_i'(T, z^0)$ and between $\mathscr{H}_i(M_\bullet(z))$ and $\mathscr{H}_i'(T)$.*

b) *The joint spectrum of S in $H_i(T, z^0)$ coincides with $\{w^0\}$, and the joint spectrum of S in $H_z''(T, z^0)$ does not contain w^0.*

c) *If $\mathbf{D}$ is a sufficienty small neighborhood of w^0, then $\mathscr{H}_i'(T) = p_* \mathscr{H}_{i+k}(T') \,|\, V \times D$ and $\mathscr{H}_{i+k}(T') = p^* \mathscr{H}_i'(T) \Big/ \bigoplus_{j=1}^{k} (S_j - w_j I) p^* \mathscr{H}_i'(T)$ on $V \times \mathbf{D}$.*

Indeed, under the conditions of the proposition one can perform the construction above. If $\psi_\bullet(z) : M_\bullet(z) \to \Gamma_w L_\bullet(z)$ is a quasi-isomorphism of the finite-dimensional complex $M_\bullet(z)$ in the complex $\Gamma_w L_\bullet(z)$, then we put $\tau_\bullet(z) = \tilde{R}_\bullet(z, w) \circ \psi_\bullet(z)$.

REMARK. It follows from b) that $H_i'(T, z^0)$ is the root space of $S_1 - w_1^0 I, \ldots$ $\ldots, S_k - w_k^0 I$ in $H_i(T, z^0)$, whence we obtain the formulae:

$$H_i'(T, z^0) = \bigcup_{p_1,\ldots,p_k \geq 0} \bigcap_{j=1}^{k} \ker(S_j - w_j^0 I)^{p_j} \,|\, H_i(T, z^0)$$

$$H_i''(T, z^0) = \bigcap_{p_1,\ldots,p_k \geq 0} \bigoplus_{j=1}^{k} (S_j - w_j^0)^{p_j} \,|\, H_i(T, z^0).$$

Now we shall show that the element $\mathscr{K}(T)$ determines the index of any holomorphic function of the n-tuple T. If $F_\bullet(z)$ is a finite complex of finite-dimensional free $\mathcal{O}$-modules, and z^0 is an isolated point of $\operatorname{supp} F_\bullet(z)$, then all the sheaves of homologies $\mathscr{H}_i(F_\bullet)$ of $F_\bullet$ are finite-dimensional at z^0 and we denote $\chi_{z^0}(F_\bullet) = \sum (-1)^i \dim_{z^0} \mathscr{H}_i(F_\bullet)$. Suppose that $F_\bullet(z)$ and $G_\bullet(z)$ are finite complexes of finite-dimensional free $\mathcal{O}$-modules, such that the set $B = \operatorname{supp} F_\bullet(z) \cap \operatorname{supp} G_\bullet(z)$ is finite, and let $z^0 \in B$. Then the intersection number $\langle F_\bullet, G_\bullet \rangle_{z^0}$ of $F_\bullet(z)$ and $G_\bullet(z)$ at z^0 is defined by the formula $\langle F_\bullet, G_\bullet \rangle_{z^0} = \chi_{z^0}(\operatorname{tot}(F_\bullet \otimes_{\mathcal{O}} G_\bullet))$, where $\operatorname{tot}(F_\bullet \otimes_{\mathcal{O}} G_\bullet)$ is the total complex of the bicomplex $F_\bullet \otimes_{\mathcal{O}} G_\bullet$. The global intersection number $\langle F_\bullet, G_\bullet \rangle$ is defined as the sum of the integers $\langle F_\bullet, G_\bullet \rangle_{z^0}$ over all the points $z^0 \in B$. It is easy to see that $\langle F_\bullet, G_\bullet \rangle$ depends only on the classes of $F_\bullet$ and $G_\bullet$ in the corresponding Grothendieck groups. If U is a Stein domain containing B, then $\langle F_\bullet, G_\bullet \rangle$ is equal to the Euler characteristic of the complex of Fréchet space $\Gamma_U(\operatorname{tot}(F_\bullet \otimes_{\mathcal{O}} G_\bullet))$.

PROPOSITION 3.4. *Suppose that $L_\bullet(z) = \{L_i, a_i(z)\}$ is a holomorphic complex of finite-dimensional spaces, defined on a polydisk $\mathbf{D}$ containing the joint spectrum $\sigma(T)$ of the commutative n-typle T, which is exact on $\sigma_e(T)$. Then the complex of Banach spaces $L(T) = \{L_i \otimes X, a_i(T)\}$ is Fredholm, and its Euler characteristic is equal to the intersection number $\langle \mathscr{K}(T), L_\bullet(z) \rangle$.*

Proof. We shall use a particular case of the quasi-isomorphism $R_\bullet(z)$ used in the proof of 3.2. In this case the map $R\colon \Gamma_{\mathbf{D}}(\mathcal{O}X) \to X$ will be defined by the formula

$$R[x(z)] = (2\pi i)^{-1} \int\limits_{|z_1| = R_1'} \cdots \int\limits_{|z_n| = R_n'} (T_1 - z_1)^{-1} \ldots (T_n - z_n)^{-1} x(z)\, dz$$

where $R_1, \ldots, R_n$ is the polyradius of $\mathbf{D}$, and $R_i' = R_i - \varepsilon$. Then it is easy to see that the complex $\Gamma_{\mathbf{D}}(\mathcal{O}K_\bullet(T, z)) \xrightarrow{R} X \to 0$ is exact. Indeed, this can be obtained from the proof of 3.2 with n replaced by 0, k replaced by n, k-tuple S replaced by the n-tuple T. Note that if $f(z)$ is a scalar holomorphic function on $\mathbf{D}$, then we have $R[f(z)x(z)] = f(T)R[x(z)]$. Therefore, if we take the operator of multiplication by $f(z)$ on the members of the complex $\Gamma_{\mathbf{D}}(\mathcal{O}K_\bullet(T, z))$, and the operator $f(T)$ on X, we obtain an endomorphism of the complex $\Gamma_{\mathbf{D}}(\mathcal{O}K_\bullet(T, z)) \to X \to 0$. This show that the complex $L(T)$ is quasi-isomorphic to the complex $\Gamma_{\mathbf{D}}(\text{tot}(\mathcal{O}L_\bullet(z) \otimes_{\mathcal{O}} \mathcal{O}K_\bullet(T, z)))$. Let $U \subset \mathbf{D}$ be a Stein neighborhood of the finite set $\text{supp}\, L_\bullet(z) \cap \sigma_{\mathrm{F}}(T)$, not intersecting $\sigma_e(T)$. Since the complex $\text{tot}(L_\bullet(z) \otimes K_\bullet(T, z))$ is exact on $\mathbf{D} \setminus U$, then the complex $\Gamma_U|\text{tot}(\mathcal{O}L_\bullet(z) \otimes_{\mathcal{O}} \mathcal{O}K_\bullet(T, z)))$ is quasi-isomorphic to $L_\bullet(T)$ again. On the other hand, if $M_\bullet(z)$ is a holomorphic complex of finite-dimensional spaces on U, quasi-isomorphic to $K_\bullet(T, z)$, then $\Gamma_U(\text{tot}(\mathcal{O}L_\bullet(z) \otimes_{\mathcal{O}} \mathcal{O}K_\bullet(T, z)))$ is quasi-isomorphic to the complex $\Gamma_U(\text{tot}(\mathcal{O}L_\bullet(z) \otimes_{\mathcal{O}} \mathcal{O}M_\bullet(z)))$. Then the Euler characteristic of this complex, i.e. the intersection number $\langle L_\bullet, M_\bullet\rangle$, coincides with the Euler characteristic of $L_\bullet(T)$. The proof is complete. Using standard techniques, one can prove this assertion for holomorphic complexes, defined in an arbitrary neighborhood of $\sigma(T)$.

Remark. The Marcus-Feldman theorem [19] states that if we have a square operator matrix, whose entries commute up to a trace class operator, then the index of the matrix is equal to the index of its determinant. In the commutative case this fact can be derived from 3.4. Indeed, any holomorphic $N \times N$ matrix $A(z)$ can be represented locally in an upper triangular form $\{a_{i,j}(z)\}$. Then it is easy to see that the sheaf $\mathcal{O}^N/A(z)\mathcal{O}^N$ is equivalent in the Grothendieck group to the direct sum $\bigoplus_{i=1}^{N} \mathcal{O}/a_{i,i}(z)\mathcal{O}$, which is in turn equivalent to $\mathcal{O}/a_{1,1}(z) \ldots a_{N,N}(z)\mathcal{O}$, and therefore the intersection numbers of these sheaves with $\mathscr{K}(T)$ coincide.

Let $T = (T_1, \ldots, T_n)$, $T' = (T_1, \ldots, T_n, S_1, \ldots, S_k)$ be commutative n-tuples of operators, $z^0 \in \sigma_{\mathrm{F}}(T)$, and let p be the projection of $\mathbf{C}^{n+k}$ onto the coordinate subspace of the first n coordinates. Suppose that the Jordan-Hölder decomposition of $\mathscr{K}(T')$ near $p^{-1}(z^0)$ is $\mathscr{K}(T') = \sum_j m_j[M_j] + \ldots$, where m_j are integers, M_j are irreducible components of $\sigma_{\mathrm{F}}(T')$ of dimension n, and the dots represent the summands of dimension $< n$.

Proposition 3.5. *Under the above conditions we have*

$$\chi(K_\bullet(T, z^0)) = \sum_j m_j \text{mult}_{p, z^0}(M_j).$$

Proof. Recall that the multiplicity $\text{mult}_{p, z^0}(M_j)$ is equal to the intersection number of M_j with the set $p^{-1}(z^0)$. Fix a holomorphic complex $M_\bullet(z)$ of finite-dimensional spaces, quasi-isomorphic to $K_\bullet(T, z)$ in a neighborhood of z^0. Then, by 3.2, $\mathscr{H}_i(M_\bullet(z)) = p_*\mathscr{H}_{i+k}(T')$. We have:

$$\chi(K_\bullet(T, z^0)) = \sum(-1)^i \dim H_i(M_\bullet(z^0)) = \sum(-1)^i\langle \mathscr{H}_i(M_\bullet), \{z^0\}\rangle =$$

$$= \sum(-1)^i\langle \mathscr{H}_{i+k}(T'), p^{-1}(z^0)\rangle = \langle \mathscr{K}(T'), p^{-1}(z^0)\rangle$$

which proves the assertion.

By using 3.3, one can obtain a local version of 3.5. Let $(z^0, w^0) \in \sigma_F(T')$. Suppose that the restriction of the projection p to $\sigma_F(T')$ is proper in some neighborhood of (z^0, w^0), and that near (z^0, w^0) the element $\mathscr{K}(T')$ has the same Jordan-Hölder decomposition as in 3.5. Take the decomposition $H_i(T, z^0) = H_i'(T, z^0) \oplus H_i''(T, z^0)$ corresponding to the point w^0, and denote $\chi'(K_\bullet(T, z^0)) = \sum(-1)^i \dim H_i'(T, z^0)$.

PROPOSITION 3.5'. *Under the above conditions we have*

$$\chi'(K_\bullet(T, z^0)) = \sum_j m_j \operatorname{mult}_{p,(z^0, w^0)}(M_j).$$

Proof. In this case the multiplicity $\operatorname{mult}_{p,(z^0, w^0)}(M_j)$ is equal to the local intersection number of M_j with $p^{-1}(z^0)$ at the point (z^0, w^0). Take the holomorphic complex $M_\bullet(z)$ as in 3.3 a), and let $V \subset \mathbf{C}^n$, $\mathbf{D} \subset \mathbf{C}^k$ be the same as in the proof of 3.3. Then we have $\mathscr{H}_i(M_\bullet) = p_*(\mathscr{H}_{i+k}(T') \mid V \times \mathbf{D})$. Replacing in the proof of 3.5 the global intersection numbers with the local intersection numbers at (z^0, w^0), we obtain a proof of 3.5'.

In the case $n = 1$ this assertion is proved under the more restrictive assumption that $z^0 \in \sigma_F(T)$ in [5], Theorem 5.11.

Proposition 3.3 implies that if the maximal dimension of $\sigma_F(T)$ near the point $z^0 \in \mathbf{C}^n$ is equal to m, then in some neighborhood of z^0 we have $\mathscr{H}_i(T) = 0$ for $i < n - m$. Indeed, one can change the coordinates in $\mathbf{C}^n$ in such a manner that $z_1, \ldots, z_m$ form a normal coordinate system for the m-dimensional part of $\sigma_F(T)$ near z^0, and then apply 3.3 c) with $n = m$, $k = n - m$. This allows us to obtain an explicit formula for $\mathscr{K}(T)$ in the case when $\sigma_F(T)$ is one-dimensional. Suppose that T is a commuting n-tuple of operators, $z^0 \in \sigma_F(T)$, and $\sigma_F(T)$ is one-dimensional near z^0. Changing the coordinates, we may suppose that the first coordinate z_1 is a normal coordinate system for $\sigma_F(T)$ near z^0. Let V be a sufficiently small neighborhood of z_1^0 in $\mathbf{C}$, and $V \times \mathbf{D}$ be a sufficiently small neighborhood of z^0. For any two sheaves $\mathscr{F}$, $\mathscr{G}$ on V (respectively, on $V \times \mathbf{D}$) we shall write $\mathscr{F} \sim \mathscr{G}$ iff $\mathscr{F}$ and $\mathscr{G}$ are isomorphic modulo sheaves supported in $\{z_1^0\}$ (respectively, in $\{z^0\}$). Denote by k and r the dimension of $H_0'(T_1)$ and $H_1'(T_1)$ respectively on $V \setminus \{z_1^0\}$. Then $\mathscr{H}_0'(T_1) \sim \mathcal{O}^k$, $\mathscr{H}_1'(T_1) \sim \mathcal{O}^r$. Denote by T^* the n-tuple $T_1^*, \ldots, T_n^*$, acting on X^*. Then $\sigma_F(T^*) = \sigma_F(T)$. It is easy to see that the decompositions of $H_i(T_1^*)$ and $\mathscr{H}_i(T_1^*)$, corresponding to the point $z^0 \in \sigma_F(T^*)$, agree with the decompositions of $H_i(T_1)$, $\mathscr{H}_i(T_1)$, $i = 0, 1$. Therefore we have $\mathscr{H}_0'(T_1^*) \sim \mathscr{H}_1'(T_1)$, $\mathscr{H}_1'(T_1^*) \sim \mathscr{H}_0'(T_1)$. Then on

$V \times \mathbf{D}$ we obtain

$$\mathscr{H}_{n-1}(T) = \mathscr{H}'_0(T_1) \Big/ \sum_{j=2}^{n} (T_j - z_j I)\mathscr{H}'_0(T_1) \sim$$

$$\sim \mathscr{H}'_1(T_1^*) \Big/ \sum_{j=2}^{n} (T_j - z_j I)\mathscr{H}'_1(T_1^*) =$$

$$= \mathcal{O}X^* \Big/ \sum_{j=1}^{n} (T_j^* - z_j I)\mathcal{O}X^*.$$

We have proved

PROPOSITION 3.6. Suppose that the Fredholm spectrum of the commuting n-tuple T of operators acting on the Banach space X is one-dimensional near the point $z^0 \in \mathbf{C}^n$. Then in some neighborhood of z^0 we have

$$\mathscr{K}(T) = \left[\mathcal{O}X \Big/ \sum_{j=1}^{n} (T_j - z_j I)\mathcal{O}X\right] - \left[\mathcal{O}X^* \Big/ \sum_{j=1}^{n} (T_j^* - z_j I)\mathcal{O}X^*\right].$$

A similar formula, using the ring of formal power series instead of the ring of germs of holmorphic functions, was obtained under more restrictive assumptions in [5], Theorem 5.13.

4. TECHNICAL LEMMAS

In this section all the complexes will be assumed to be bounded.

LEMMA 4.1. (Taylor, [26], Theorems 2.2 and 2.3). *Suppose that in the diagram* $X \xrightarrow{a(z)} Y \xrightarrow{b(z)} Z$, *$X$, Y, Z are Banach spaces and $a(z)$, $b(z)$ are continuous (holomorphic) operator-valued functions defined on the domain $U \subset \mathbf{C}^n$ such that $b(z) \circ a(z) \equiv 0$, and suppose that for $z^0 \in U$ we have* $\operatorname{im} a(z^0) = \ker b(z^0)$. *Then there exists a neighborhood V of z^0 such that*

a) *for any $z \in V$ we have* $\operatorname{im} a(z) = \ker b(z)$,

b) *for any continuous (holomorphic) Y-valued function $y(z)$ on V, satisfying $b(z)y(z) \equiv 0$, and for any element $x_0 \in X$, satisfying $a(z^0)x_0 = y(z^0)$, there exists a continuous (holomorphic) X-valued function $x(z)$ with $x(z^0) = x_0$ and $a(z)x(z) \equiv \equiv y(z)$.*

LEMMA 4.2. *Let $X_\bullet(z) = \{X_i, d_i(z)\}$ be a continuous (holomorphic) complex of Banach spaces on the domain U. Then $X_\bullet(z)$ is exact on U if and only if the complex $\mathscr{C}X_\bullet$ ($\mathcal{O}X_\bullet$) of sheaves of germs of its continuous (holomorphic) sections is exact.*

Proof. The "only if" part follows directly from 4.1. In order to prove the "if" part, fix $z^0 \in U$, and suppose that we have proved the exactness of $X_\bullet(z^0)$ in the stages $\geqslant i+1$. Let $x_0 \in \ker d_i(z^0)$ and let $x(z)$ be a continuous (holomorphic) X_i-valued function such that $x(z^0) = x_0$, $d_i(z)x(z) \equiv 0$. Then there exists a X_{i-1}-valued function $y(z)$ such that $d_{i-1}(z)y(z) \equiv x(z)$, and therefore $d_{i-1}(z^0)y(z^0) = x_0$.

LEMMA 4.3. *Let $X_\bullet(z) = \{X_i, d_i(z)\}$ be a continuous (holomorphic) complex of Banach spaces, defined in the domain U, and Fredholm in the point $z^0 \in U$. Then in a sufficiently small neighborhood of z^0 there exists a continuous (holomorphic) complex $L_\bullet(z) = \{L_i, a_i(z)\}$ of finite-dimensional spaces, and a continuous (holomorphic) quasi-isomorphism of complexes $\varphi_\bullet(z)\colon L_\bullet(z) \to X_\bullet(z)$.*

Proof. Since the complex $X_\bullet(z)$ is bounded, then we can assume that the spaces L_i and the homomorphisms $a_i(z)\colon L_i \to L_{i+1}$, $\varphi_i(z)\colon L_i \to X_i$ are constructed for $i > k$. We shall construct L_k, $a_k(z)$, $\varphi_k(z)$. Denote by $\tilde{X}_\bullet(z) = \{\tilde{X}_i, \tilde{d}_i(z)\}$, $i \geqslant k$, the cone of the morphism $\varphi_\bullet(z)$. The space $\ker \tilde{d}_k(z^0)/\operatorname{im} d_{k-1}(z^0)$ is finite-dimensional and one can choose elements $x_1, \ldots, x_s$ generating it. By 4.1 one can find continuous (holomorphic) X_k-valued functions $x_j(z), j = 1, \ldots, s$, defined in some neighborhood of z^0, such that $x_j(z^0) = x_j$ and $\tilde{d}_k(z)x_j(z) \equiv 0$. Put $L_k = \mathbf{C}^s$. Then the vector-functions $x_1(z), \ldots, x_s(z)$ define a continuous (holomorphic) map $x(z)\colon L_k \to \tilde{X}_k$. Recall that $\tilde{X}_k = X_k \oplus L_{k+1}$, and denote by $\varphi_k(z)$ and $a_k(z)$ the projections of $x(z)$ on the space X_k and L_{k+1} respectively. By using the equality $\tilde{d}_k(z)x(z) \equiv 0$, it is easy to check that all the necessary conditions are satisfied.

LEMMA 4.4. *Let $X_\bullet(z)$ be a holomorphic complex defined on the domain U. Then $X_\bullet(z)$ is Fredholm for all $z \in U$ iff all the sheaves of homologies of the complex $\mathcal{O}X_\bullet(z)$ of its holomorphic sections are coherent.*

Proof. It is easy to see that the quasi-isomorphism $\varphi_\bullet(z)$ constructed above defines an isomorphism between the complexes of sheaves $\mathcal{O}L_\bullet(z)$ and $\mathcal{O}X_\bullet(z)$ and therefore the homologies of $\mathcal{O}X_\bullet(z)$ are coherent. Conversely, suppose that all the sheaves $\mathscr{H}_i = \mathscr{H}_i(\mathcal{O}X_\bullet(z))$ are coherent. Then it is easy to compute

$$\sum_i (-1)^i \dim \mathscr{H}_i(X_\bullet(z)) = \sum_i (-1)^i \sum_{k \geqslant 0} \operatorname{Tor}_k(\mathscr{H}_{i+k}, e_z)$$

where e_z is the sheaf with stalk equal to $\mathbf{C}$ in the point z and zero elsewhere.

LEMMA 4.5. *Let F be a compact (compact and holomorphically convex) subset of $\mathbf{C}^n$, and let $X_\bullet(z)$ be a continuous (holomorphic) complex of Banach spaces, defined in a neighborhood of F and Fredholm on F. Then there exists a continuous (holomorphic) complex of (holomorphic) vector bundles $L_\bullet(z)$ on F, and a continuous (holomorphic) quasi-isomorphism $\varphi_\bullet(z)\colon L_\bullet(z) \to X_\bullet(z)$.*

Proof. Consider the holomorphic case. Since the complex $X_\bullet(z)$ is bounded, then one may suppose that $X_i = 0$ for $i < 0$. It is well known that if $\mathscr{L}$ is a coherent sheaf or a sheaf of the form $\mathscr{O}X$, X — a Banach space, then $H^i(F, \mathscr{L}) = 0$ for all $i > 0$. Since all the homology sheaves of the complex $\mathscr{O}X_\bullet(z)$ are coherent, then, decomposing it in short exact sequences, we obtain $H^i(F, \mathscr{L}) = 0$ for all $i > 0$ whenever $\mathscr{L}$ is one of the sheaves of the form $\ker d_k(z)$ or $\operatorname{im} d_k(z)$, $k \geqslant 0$. Suppose that the finite-dimensional vector spaces L_i and the homomorphisms $a_i(z)$, $\varphi_i(z)$ are constructed for all $i > k$, $k > 0$. Denote by $x_1, \ldots, x_s$ a generating system of global sections of the coherent sheaf $\ker \tilde{d}_k(z)/\operatorname{im} d_{k-1}(z)$ on F. Since $H^1(F, \operatorname{im} d_{k-1}(z)) = 0$, then this system can be lifted up to a system $x_1(z), \ldots, x_s(z)$ of sections of $\ker \tilde{d}_k(z)$ on F. Now the space L_k and the operators $a_k(z)$, $\varphi_k(z)$ can be constructed as in the proof of 4.3. In this way one obtains L_i, $a_i(z)$, $\varphi_i(z)$ for all $i > 0$. It is easy to see that the kernel of the operator $\tilde{d}_0(z)$ has constant finite dimension on F and therefore is a holomorphic vector bundle. Put $L_0 = \ker \tilde{d}_0(z)$ and denote by $a_0(z)$, $\varphi_0(z)$ the components of the natural embedding $\tilde{L}_0 \to \tilde{X}_0$. Thus the construction is complete.

In the continuous case all sheaves are soft, and the same proof holds without change.

LEMMA 4.6. *If $L_\bullet(z)$, $L'_\bullet(z)$ are complexes of finite-dimensional vector bundles, quasi-isomorphic to $X_\bullet(z)$ on U, then $L_\bullet(z)$ and $L'_\bullet(z)$ determine the same class in the group* $\mathrm{K}^0(F)$.

Proof. Let $\varphi_\bullet(z)\colon L_\bullet(z) \to X_\bullet(z)$, $\varphi'_\bullet(z)\colon L'_\bullet(z) \to X_\bullet(z)$ be continuous (holomorphic) quasi-isomorphisms. Then $\varphi_\bullet(z) \oplus \varphi_\bullet(z)\colon L_\bullet(z) \oplus L'_\bullet(z) \to X_\bullet(z)$ is a morphism of complexes. Denote its cone by $\tilde{X}_\bullet(z)$. Let $P_\bullet$, $P'_\bullet$ be the natural projections of $\tilde{X}_\bullet(z)$ onto $L_\bullet(z), L'_\bullet(z)$ respectively. The kernel of $P_\bullet$ coincides with the cone of $\varphi_\bullet(z)$ and therefore is an exact complex. This implies that $P_\bullet$ is a quasi-isomorphism, and the same is true for $P'_\bullet$. Choose a complex $M_\bullet(z)$ of vector bundles on F, quasi-isomorphic to $\tilde{X}_\bullet(z)$, and let $\tilde{\psi}_\bullet(z)\colon M_\bullet(z) \to \tilde{X}_\bullet(z)$ be the corresponding quasi-isomorphism. Then $\tilde{\psi}_i(z)$ can be represented in the form $(\psi_i(z), \psi'_i(z), S_i(z))$, where $\psi_i(z)\colon M_i \to L_i$, $\psi'_i(z)\colon M_i \to L'_i$, $S_i(z)\colon M_i \to X_{i-1}$ are linear maps. It is easy to see that $\psi_i(z) = P_i \circ \tilde{\psi}_i(z)$, $\psi'_i(z) = P'_i \circ \tilde{\psi}_i(z)$, and therefore $\psi_\bullet(z)$ and $\psi'_\bullet(z)$ are quasi-isomorphisms. We have see that both $L_\bullet(z)$ and $L'_\bullet(z)$ are quasi-isomorphic to $M_\bullet(z)$, which proves the lemma.

Let $\mathscr{F}$ and $\mathscr{G}$ be subsheaves of the sheaf $\mathscr{H}$, defined on the domain U. Denote by $\mathscr{F} \Delta \mathscr{G}$ the formal difference $\mathscr{F}/\mathscr{F} \cap \mathscr{G} - \mathscr{G}/\mathscr{F} \cap \mathscr{G}$ considered as an element of $\mathrm{K}_0^{\mathrm{alg}}(U)$.

LEMMA 4.7. *Let $A(z)\colon X \to Y$ be a holomorphic operator-valued function on the domain U, and let $\tilde{A}(z) = A(z) + K$, where K is a finite-dimensional operator,* *not depending on z. Then in the group* $K_0^{alg}(U)$ *we have*

$$\ker A(z)\,\Delta\,\ker\tilde{A}(z) + \operatorname{im} A(z)\,\Delta\,\operatorname{im}\tilde{A}(z) = 0.$$

Proof. In the case when X is finite-dimensional the assertion is obvious. In the general case, let $H = \ker K$, and denote by $\mathscr{L}$ the sheaf $\mathscr{L} = \operatorname{im} A(z)|H = \operatorname{im} A(z)|H$. Denote by $B(z)$, $\tilde{B}(z)$ the morphism from the sheaf $\mathcal{O}X/\mathcal{O}H = \mathcal{O}(X/H)$ to $\mathcal{O}Y/\mathscr{L}$, induced by $A(z)$, $\tilde{A}(z)$ respectively. Then

$$\ker A(z)\,\Delta\,\ker\tilde{A}(z) = \ker A(z)/\ker A(z)\cap\mathcal{O}H -$$

$$-\ker\tilde{A}(z)/\ker\tilde{A}(z)\cap\mathcal{O}H = \ker B(z) - \ker\tilde{B}(z) =$$

$$= \ker B(z)\,\Delta\,\ker\tilde{B}(z),$$

and in the same way $\operatorname{im} A(z)\,\Delta\,\operatorname{im}\tilde{A}(z) = \operatorname{im} B(z)\,\Delta\,\operatorname{im}\tilde{B}(z)$. Since X/H is finite-dimensional, the assertion follows.

DEFINITION. The system $X_\bullet(z) = \{X_i, d_i(z)\}$, where X_i are Banach spaces and $d_i(z): X_i \to X_{i+1}$ are continuous operator-valued functions, will be called an *essential complex*, if for all i and z the operators $d_{i+1}(z)\circ d_i(z)$ are compact. The essential complex $X_\bullet(z)$ will be called *Fredholm*, if for all i and z there exist operators $S_{i,z}: X_i \to X_{i+1}$ such that $S_{i+1,z}\circ d_i(z) + d_{i-1}(z)\circ S_{i,z}$ is equal in X_i to the identity plus a compact operator (see [22]). Two essential complexes $X_\bullet(z) = \{X_i, d_i(z)\}$ and $X'_\bullet(z) = \{X'_i, d'_i(z)\}$ will be called *equivalent*, if $X_i = X'_i$ and $d_i(z) - d'_i(z)$ are compact for all i and z.

In Section 1 we defined the essential joint spectrum $\sigma_e(T)$ of the essentially commuting n-tuple T as the complement in $\mathbf{C}^n$ of the set of all z such that the essential complex $K_\bullet(T, z)$ is an essential Fredholm complex.

LEMMA 4.8. *Let* $T = (T_1, \ldots, T_n)$ *be an essentially commuting n-tuple of operators acting on the Banach space X, and let* $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_n)$ *be the corresponding n-tuple of elements of the Calkin algebra* $\mathfrak{A}(X)$. *Then* $\sigma_e(T) = \sigma(\tilde{T})$, *where* $\sigma(\tilde{T})$ *denotes the Taylor spectrum of T considered as an n-tuple of operators of left multiplication on* $\mathfrak{A}(X)$.

Proof. Proposition 2.3 of [22] shows that the essential complex $X_\bullet = \{X_i, d_i\}$ is Fredholm iff for any k the complex $L(X_k, X_\bullet)/K(X_k, X_\bullet)$ is exact. Since the components of the Koszul complex $K_\bullet(T, z)$ are direct sum of finitely many copies of the space X, then one can see that the essential complex $K_\bullet(T, z)$ is Fredholm iff the complex $L(X, K_\bullet(T, z))/K(X, K_\bullet(T, z))$ is exact. On the other hand, it is easy to show that the latter complex is equal to the Koszul complex of $\tilde{T}$ in $\mathfrak{A}(X)$, which proves the lemma.

LEMMA 4.9. *For any continuous essential Fredholm complex $X_{\bullet}(z) = \{X_i, d_i(z)\}$, defined on the domain U, there exists on U a continuous Fredholm complex $X'_{\bullet}(z)$, equivalent to $X_{\bullet}(z)$.*

Proof. For given subspaces L and M of a Banach space we shall write $L \sim M$ iff both $L/L \cap M$ and $M/L \cap M$ are finite-dimensional. Similarly, for operators A and B we shall write $A \sim B$ iff $A - B$ has a finite-dimensional image. We are going to construct on U continuous operator-valued functions $d'_i(z): X_i \to X_{i+1}$, $S_i(z): X_i \to X_{i-1}$, such that for all i the operators $d_i(z) - d'_i(z)$ are compact, $d'_{i+1}(z) \circ d_i(z) \equiv 0$, and $S_{i+1}(z) \circ d'_i(z) + d_{i-1}(z) \circ S_i(z) \sim I$. Suppose that $d'_i(z)$, $S_{i+1}(z)$ are already constructed for $i > k$, and let us construct $d'_k(z)$, $S_{k+1}(z)$. Denote $P(z) = S_{k+2}(z) \circ d_{k+1}(z)$ and $Q(z) = I - P(z)$. Then $P^2(z) = S_{k+2}(z) \circ (I - S_{k+3}(z) \circ d'_{k+2}(z)) \circ d'_{k+1}(z) \sim P(z)$ and therefore $Q^2(z) \sim Q(z)$, $P(z) \circ Q(z) \sim 0$. Then it is easy to see that $\operatorname{im} P(z)$ is closed, $\operatorname{im} P(z) \sim \ker Q(z)$, and $\operatorname{im} P(z) \cap \ker d'_{k+1}(z)$ is finite-dimensional. Since $d'_{k+1}(z) \circ P(z) \sim d'_{k+1}(z)$, then we obtain that $\operatorname{im} P(z) \oplus \ker d'_{k+1}(z) \sim X_{k+1}$ and $\ker P(z) \sim \ker d'_{k+1}(z)$. Denote $\tilde{d}_k(z) = Q(z) \circ d_k(z)$. Then $d_k(z) - \tilde{d}_k(z) = P(z) \circ d_k(z)$ is compact. Since $\operatorname{im} Q(z) \sim \ker P(z) \sim \ker d'_{k+1}(z)$, then $d'_{k+1}(z) \circ \tilde{d}_k(z)$ is finite-dimensional. One can choose a continuously depending on z finite-dimensional operator $F(z): X_k \to X_{k+1}$ such that $d'_{k+1}(z) \circ F(z) = d'_{k+1}(z) \circ \tilde{d}_k(z)$. Then $d'_k(z) = \tilde{d}_k(z) - F(z)$ is the desired compact perturbation of $d_k(z)$.

It remains to construct $S_{k+1}(z)$. Fix the point $w \in U$ and the operators $S_{k+1,w}$ and $S_{k+2,w}$, such that the operator $S_{k+2,w} \circ d'_{k+1}(w) + d'_k(w) \circ S_{k+1,w} - I$ is compact. Put $\tilde{S}_{k+1}(z) = S_{k+1,w} \circ Q(z)$. Then, since $\operatorname{im} Q(z) \sim \ker d'_{k+1}(z)$, the operator

$$d'_k(w) \circ \tilde{S}_{k+1}(w) + S_{k+2}(w) \circ d'_{k+1}(w) - I =$$

$$= (d'_{k+1}(w) \circ S_{k+1,w} - I) \circ Q(w)$$

is compact. For all z sufficiently close to w the operator-function $H(z) = d'_k(z) \circ S_{k+1}(z) + S_{k+2}(z) \circ d'_{k+1}(z)$ can be represented in the form $H(z) = G(z) + K(z)$ where $G(z)$ and $K(z)$ continuously depend on z, $\|I - G(z)\| < 1/2$ and $K(z)$ is compact. Then there exists a continuous operator-function $L_{\bullet}(z): X_{k+1} \to X_{k+1}$, such that $L(z) \circ H(z) \sim H(z) \circ L(z) \sim I$. Since $P(z) \circ d_k(z)$ is finite-dimensional, then $P(z) \circ H(z) \sim P(z)$, and therefore $P(z) \sim P(z) \circ L(z)$. Put $S_{k+1}(z) = \tilde{S}_{k+1}(z) \circ L(z)$. Then $d'_k(z) \circ S_{k+1}(z) + S_{k+2}(z) \circ d_{k+1}(z) = H(z) \circ L(z) - P(z) L(z) + P(z) \sim I$. Now $S_{k+1}(z)$ is constructed in a neighborhood of w. Since w is arbitrary, then, using a suitable partition of unity, one can construct $S_{k+1}(z)$ on the whole U, which completes the proof.

DEFINITION A. Let $X_\bullet(z)$ be a continuous Fredholm complex of Banach spaces, defined on the domain U. Then for any compact $F \subset U$ there exists a continuous complex $L_\bullet(z)$ of vector bundles, quasi-isomorphic to $X_\bullet(z)$ in a neighborhood of F and determining an element of the group $\mathbf{K}^0(F)$. By 4.6 one can see that the elements, corresponding to the different F coincide, and therefore define a uniquely determined element of $\mathbf{K}^0(U)$, which will be denoted by $[X_\bullet(z)]$. If $X_\bullet(z)$ is a continuous essential Fredholm complex on U, we shall denote by $[X_\bullet(z)]$ the element $[X'_\bullet(z)]$, where $[X'_\bullet(z)]$ is an arbitrary Fredholm complex on U, equivalent to $X_\bullet(z)$.

DEFINITION B. Let $X_\bullet(z) = \{X_i, d_i(z)\}$ be a continuous essential Fredholm complex on the domain U, and $S_i(z) : X_i \to X_{i-1}$ be continuous operator-functions such that all the operators $\Delta_i(z) = S_{i+1}(z) \circ d_i(z) + d_{i-1}(z) \circ S_i(z)$ are equal to identity modulo compact operators. Denote $X_e = \bigoplus_i X_{2i}$, $X_0 = \bigoplus_i X_{2i+1}$. Let $D(z)\colon X_e \to X_0$ be the operator matrix, whose entries corresponding to the maps $X_{2i} \to X_{2i+1}$ are equal to $d_{2i}(z)$, those corresponding to $X_{2i} \to X_{2i-1}$ are equal to $S_{2i}(z)$, and all the others are zero (this construction is used, for instance, in [8]). Then $D(z)$ is Fredholm on U and determines an element of the group $\mathbf{K}^0(U)$, which will be denoted by $[X_\bullet(z)]$.

LEMMA 4.10. *The definitions* A *and* B *are correct and equivalent. The class* $[X_\bullet(z)]$ *is invariant under compact or small perturbations of the complex* $X_\bullet(z)$.

Proof. Let us show that $D(z)$ is Fredholm. Denote by $C_\bullet(z) : X_0 \to X_e$ the operator matrix, determined by the operators $d_{2i+1}(z)$, $S_{2i+1}(z)$, $i \in \mathbf{Z}$. Then all the entries of the matrices $C(z) \circ D(z)$ and $D(z) \circ C(z)$, lying above the diagonal, are of the type $d_{i+1}(z) \circ d_i(z)$, i.e. compact operators, and the diagonal entries are $\Delta_i(z)$. This means that $C(z) \circ D(z)$ and $D(z) \circ C(z)$ are Fredholm, and therefore similarly is $D(z)$. Further, note that Definition B does not depends on the choice of the homotopies $S_i(z)$. If $S'_i(z)$ is another family of homotopies of $X_\bullet(z)$, and $D'(z)$ is the corresponding operator matrix, then the linear homotopy between $S_i(z)$ and $S'_i(z)$ determines a homotopy between $D(z)$ and $D'(z)$ by Fredholm operators. Moreover, suppose that $S'_i(z)$ is a family of maps such that for all i we have

$$\| S'_{i+1}(z) \circ d_i(z) + d_{i-1}(z) \circ S_i(z) - I \|_c < 1/2,$$

where $\|T\|_c$ denotes the norm of the class of the operator T in the Calkin algebra. Then the same arguments show that $D(z)$ is homotopic to $D'(z)$ in the class of Fredholm operators. This implies the invariance of Definition B under small perturbation (the invariance under compact perturbations is obvious). Let F be a compact subset of U. Fix the operators of homotopy $S_i(z)$ for the complex $X_\bullet(z)$, and let $X'_\bullet(z) =$

$= \{X_i', d_i'(z)\}$ be another essential complex on U with $X_i' = X_i$, $i \in \mathbf{Z}$. Suppose that the norms of the differences $d_i(z) - d_i'(z)$ are sufficiently small on F, so that

$$\| S_{i+1}(z) \circ d_i'(z) + d_{i-1}'(z) \circ S_i(z) - I \|_c < 1/2$$

for any $i \in \mathbf{Z}$, $z \in F$. Then $X_i'(z)$ is Fredholm. Denote by $D'(z)$ the operator matrix, constructed starting from the operators $d_{2i}'(z)$, $S_{2i}(z)$, $i \in \mathbf{Z}$. Then the last argument shows that $D'(z)$ represents $[X_\bullet'(z)]$. On the other hand, it is easy to see that the linear homotopy between $D(z)$ and $D'(z)$ preserves the Fredholm property, and therefore $[X_\bullet(z)] = [X_\bullet'(z)]$.

Now we shall prove the equivalence between Definitions A and B. Note that if the essential Fredholm complex $X_\bullet'(z)$ is (geometrically) homotopic to $X_\bullet(z)$ in the space of essential Fredholm complexes, then both Definitions A and B give us $[X_\bullet'(z)] = [X_\bullet(z)]$. Further, if $X_\bullet'(z)$ is a direct sum of the complex $X_\bullet(z)$ and an exact complex, then $[X_\bullet'(z)] = [X_\bullet(z)]$ again. We shall use an induction on the length of the complex $X_\bullet(z)$. If $X_\bullet(z)$ has only two non-zero terms, then Definitions A and B obviously agree. Suppose that these definitions agree on the complexes of the length n. Let $X_\bullet(z) = \{X_i, d_i(z)\}$ be an essential complex of length $n+1$, $X_i = 0$ for $i < 0$ or $i > n$, and let $S_i(z)$ be homotopy operators for $X_\bullet(z)$. Denote $\tilde{X}_i = X_i$, $\tilde{d}_i(z) = d_i(z)$ for $i < n-2$, $\tilde{X}_{n-2} = X_{n-2} \oplus X_n$, $\tilde{X}_{n-1} = X_{n-1}$, $X_i = 0$ for $i > n-1$, and let $\tilde{d}_{n+2}(z)$ be defined as a sum of $d_{n-2}(z)$ and $S_n(z)$. Then it is easy to see that $\tilde{X}_\bullet(z) = \{\tilde{X}_i, \tilde{d}_i(z)\}$ is an essential Fredholm complex in U of length n. Denote by $U_\bullet(z)$ the complex in which $U_i = 0$ for $i \neq n-2$, $n-1$, and the $(n-2) \to (n-1)$ morphism is $X_n \xrightarrow{I} X_n$, and by $V_\bullet(z)$ — the complex in which $V_i = 0$ for $i \neq n-1, n$ and the $(n-1) \to (n)$ morphism is $X_n \xrightarrow{-I} X_n$. Then one can prove that the linear homotopy between $X_\bullet(z) \oplus U_\bullet(z)$ and $\tilde{X}_\bullet(z) \oplus V_\bullet(z)$ lies in the space of essential Fredholm complexes, and therefore $[X_\bullet(z)] = [\tilde{X}_\bullet(z)]$ in Definitions A and B. Since by the inductive assumption Definitions A and B coincide on $\tilde{X}_\bullet(z)$, then they coincide on $X_\bullet(z)$.

Let N be a Stein complex submanifold of $\mathbf{C}^n$, let M be a precompact strongly pseudoconvex domain in N, and let E be a holomorphic vector bundle on N. Denote by $\tilde{E}$ the locally free sheaf of $\mathcal{O}_M$-modules, corresponding to E, and by $H(M, E)$, respectively by $H^2(M, E)$ — the space of all, respectively all square-integrable, holomorphic sections of E on M. Let $T = (T_1, \ldots, T_n)$ be the n-tuple of the operators of multiplication by the coordinate functions $z_1, \ldots, z_s$ of $\mathbf{C}_n$, acting on the space $H^2(M, E)$, and let $K_\bullet(T, z)$ be the corresponding parametrized Koszul complex.

LEMMA 4.11. *On the domain $\mathbf{C}^n \setminus \mathrm{b}M$ the complex of spaces $K_\bullet(T, z)$ is Fredholm, and the complex of sheaves $\mathcal{O}K_\bullet(T, z)$ is quasi-isomorphic to the sheaf $i_* \tilde{E}$, where i denotes the embedding of M in $\mathbf{C}^n$.*

Proof. Choose a precompact strongly pseudoconvex domain M_1 such that $M_1 \Supset M$, and fix the point $z^0 \notin \overline{M}_1 \setminus M$. We shall prove that the morphism of restriction $r : H^2(M_1, E) \to H^2(M, E)$ induces a quasi-isomorphism between the parametrized Koszul complexes of T in the spaces $H^2(M_1, E)$ and $H^2(M, E)$ respectively at the point z^0. Let $\mathcal{U} = \{U_i\}_{i=0}^k$ be a finite covering of M_1 by strongly pseudoconvex domains such that $U_0 = M$, and $z^0 \notin U_i$ for $i > 0$. For a given subset $\alpha = (\alpha_1, \ldots, \alpha_s)$ of $\{0, 1, \ldots, k\}$ put $U_\alpha = U_{\alpha_1} \cap \ldots \cap U_{\alpha_s}$. Denote by $C^h_\cdot(M_1, \mathcal{U}, E)$ the alternated cochain complex of the spaces of square-integrable sections of E over the domains U, and let $C^h_0(M_1, \mathcal{U}, E) = H^2(M_1, E)$. Our first aim is to prove that the complex of Hilbert spaces $C^h_\cdot(M_1, \mathcal{U}, E)$ is exact. Denote by $\Omega^h_{0,q}(U, E)$ the space of all forms of bidegree $(0, q)$ on U with coefficients square-integrable E-valued functions. Then Hörmander's L^2-estimates for the operator $\bar{\partial}$ show that for any pseudoconvex domain U the $\bar{\partial}$-complex $0 \to H^2(U, E) \xrightarrow{J} \Omega^h_{0,\cdot}(U, E)$ is exact. Here J denotes the embedding of $H^2(U, E)$ in $L^2(U, E) = \Omega^h_{0,0}(U, E)$, and the other differentials are densely defined operators, determined by $\bar{\partial}$. Consider the bicomplex of Hilbert spaces $C^h_\cdot(M_1, \mathcal{U}, \Omega^h_{0,\cdot}(\cdot, E))$, consisting of the spaces of cochains with coefficients in $\Omega^h_{0,\cdot}(U_\alpha, E)$. Denote by $D^\infty(U)$ the space of all smooth functions f on U such that for any multiindex α, $|\alpha| \geqslant 0$, the function $\bar{\partial}^\alpha f$ is square-integrable on U. Note that if g is a smooth function on $\overline{U}$, then $gD^\infty(U) \subset D^\infty(U)$. If we denote by $C_\cdot(M_1, \mathcal{U}, E)$ the complex of cochains with coefficients in $D^\infty(U)$, then, using a partition of unity, one can prove that this complex is exact. Similarly, if we denote by $\Omega^\infty_{0,q}(U, E)$ the space of $(0, q)$-forms with coefficients in $D^\infty(U)$, then the complex $C_\cdot(M_1, \mathcal{U}, \Omega^\infty_{0,q}(\cdot, E))$ is exact. Fix an element $\xi \in C^h_k(M_1, \mathcal{U}, E)$ such that $\delta\xi = 0$. Applying the well-known procedure of diagram chasing, we obtain elements $\omega_i \in C_i(M_1, \mathcal{U}, \Omega_{0,k-i})$, $i = 0, \ldots, k-1$, such that $\delta\omega_i = \bar{\partial}\omega_{i+1}$ for $i = 0, \ldots, k-2$, $\delta\omega_{k-1} = J(\xi)$, $\bar{\partial}\omega_0 = 0$. Solving the corresponding $\bar{\partial}$-equations, we obtain elements $\alpha_i \in C^h_i(M_1, \mathcal{U}, \Omega^h_{0,k-i-1})$, $i = 0, \ldots, k-2$, such that $\bar{\partial}\alpha_0 = \omega_0$ and $\bar{\partial}\alpha_i = \omega_i - \delta\alpha_{i-1}$ for $i = 1, \ldots, k-2$. Then the element $\eta = \omega_{k-1} - \delta\alpha_{k-2}$ satisfies $\bar{\partial}\eta = 0$ and therefore belongs to $C^h_{k-1}(M_1, \mathcal{U}, E)$. On the other hand, it is easy to see that $\delta\eta = \xi$. The exactness of the complex $C^h_\cdot(M_1, \mathcal{U}, E)$ is proved.

Denote by $K_\cdot(C^h_\cdot(M_1, \mathcal{U}, E))$ the bicomplex, whose first differential is induced by the cochain map δ and whose second differential is equal to the Koszul complex of the n-tuple of operators $T - z^0 I$, acting on the spaces $C^h_p(M_1, \mathcal{U}, E)$. Let $KC^h_\cdot(M_1, \mathcal{U}, E)$ be its total complex. Since all the rows of $K_\cdot(C^h_\cdot(M_1, \mathcal{U}, E))$ are exact complexes, then the total complex is also exact. Denote by $K^1_\cdot$, $K_\cdot$ the Koszul complex of $T - z^0 I$ in the space $H^2(M_1, E)$, respectively $H^2(M, E)$, and by $\tilde{K}_\cdot$—the cone of the morphism of complexes $r_\cdot : K^1_\cdot \to K_\cdot$, induced by the restriction r. Let $P_\cdot$ be the natural projection from the complex $C^h_\cdot(M_1, \mathcal{U}, E)$ to the complex $0 \to H^2(M_1, E) \xrightarrow{r}$

$\xrightarrow{r} H^2(M,E) \to 0$. Then $P_\bullet$ induces a morphism of complexes $KP_\bullet : KC^h_\bullet(M_1, \mathcal{U}, E) \to \to \tilde{K}_\bullet$. The kernel of $KP_\bullet$ can be represented as a total complex of the bicomplex, whose columns are direct sums of the Koszul complexes of the n-tuple $T - z^0 I$ in the spaces $H^2(U_\alpha, E)$ for all the sets α, not containing 0. Since $z^0 \notin U_i$ for all $i > 0$, then all these complexes are exact, and therefore $KP_\bullet$ is a quasi-isomorphism. This implies that $\tilde{K}_\bullet$ is exact and $r_\bullet$ is a quasi-isomorphism of complexes.

Now let M_2 be a strongly pseudoconvex domain such that $z^0 \in M_2 \Subset M$. Denote by $r_1 : H^2(M_1, E) \to H(M, E)$, $r_2 : H(M,E) \to H^2(M,E)$, $r_3 : H^2(M,E) \to \to H(M_2, E)$ the restriction morphisms, and by $r^1_\bullet$, $r^2_\bullet$, $r^3_\bullet$ — the corresponding morphisms of Koszul complexes of the n-tuple $T - z^0 I$. We have proved that $r^2_\bullet r^1_\bullet$ is a quasi-isomorphism; on the other hand, it is easy to see that $r^3_\bullet r^2_\bullet$ is a quasi-isomorphism also. This implies that $r^2_\bullet$ is a quasi-isomorphism. The same arguments show that $r^2_\bullet$ induces a quasi-isomorphism of the corresponding complexes of sheaves of holomorphic sections near z^0. Since on the Fréchet space $H(M, E)$ the statements of the lemma are obviously satisfied, then they are also satisfied on the space $H^2(M, E)$.

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R. N. LEVY
Sofia State University,
Faculty of Mathematics,
Bd. A. Ivanov, 5,
Sofia 1126,
Bulgaria.

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