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A DUALITY FOR AN ACTION OF A COUNTABLE AMENABLE GROUP ON A HYPERFINITE II_1-FACTOR

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1. INTRODUCTION

In the theory of automorphism groups of operator algebras, pioneering works about the Galois type duality correspondence were done in [18], [19], [20] and [21] by M. Nakamura and Z. Takeda. Their works modelled the theory of classical simple algebras. Meanwhile, great advancements in the theory of operator algebras were accomplished by A. Connes' thesis [4] and by M. Takesaki's duality theorem [25], which gave remarkable structure theories of factors of type III. Furthermore, the Galois type correspondence was studied in A. Connes and M. Takesaki [8] as an application of the relative commutant theorem. In [23], J. E. Roberts gave a characterization of crossed product with a group dual, in which several useful methods were given for investigating automorphism groups. A duality theory for compact abelian automorphism groups was shown in [2] by H. Araki and A. Kishimoto. In [1], H. Araki, D. Kastler, M. Takesaki and H. Haag gave an extension of it to compact automorphism groups in general. In their work, Roberts' idea of "a Hilbert space in a von Neumann algebra" played a substantial role in proving an analogy of Tannaka's duality theorem for a compact automorphism group. The same type of duality theories for a compact action on a C^*-algebra appeared in [16] and [3]. On the other hand, remarkable classifications of automorphism groups of a hyperfinite II_1-factor were originated by A. Connes [5] and [6], by further expanding the line of ideas of D. McDuff [17]. These two Connes' works were extended to the classifications of the action of finite groups and countable amenable groups by V. F. R. Jones [12] and A. Ocneanu [22]. Through Takesaki's duality theorem, these classifications were applied to classifications of compact abelian automorphism groups on a hyperfinite II_1-factor ([13] and [15]).

The purpose of this note is to give a certain aspect of the duality theory of a hyperfinite II_1-factor R and its automorphism group under some condition about the asymptotic behaviour; if an automorphism θ satisfies the *asymptotic fixed point* *property*, i.e.

$$\lim_{n\to\infty} \|\theta(x_n) - x_n\|_2 = 0$$

for every $(x_n) \in \ell^\infty(\mathrm{N}; R)$ such that $\lim_{n\to\infty} \|\alpha_g(x_n) - x_n\|_2 = 0$ for all $g \in G$, then θ must be of the form

$$\theta = \alpha_k, \quad k \in G,$$

when the group G acting on is a countable amenable group.

We are motivated to introduce the asymptotic fixed point property by the inspiring results of U. Haagerup [10]. This asymptotic duality theory holds good only for amenable groups. In fact it is false for Bernoulli shifts of non-amenable discrete groups.

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2. NOTATIONS AND PRELIMINARIES

Let G be a countable discrete amenable group and R be a hyperfinite II_1-factor. Let (R, G, α) be a W^*-covariant system where α is an outer action of G on R (α_g are outer automorphisms for $g \neq e$) throughout this paper. For $x \in R$, define the norm $\|x\|_2$ by

$$\|x\|_2 = \tau(x^*x)^{1/2}$$

where τ is the faithful normal tracial state on R. Let ω be a free ultrafilter on N and let I_ω (resp. I_∞) of ℓ^∞ (N; R) be the ideal defined by

$$\{(x_n) \in \ell^\infty(\mathrm{N}; R) \,;\, \lim_{n\to\omega} \|x_n\|_2 = 0 \ (\text{resp. } \lim_{n\to\infty} \|x_n\|_2 = 0)\}.$$

The quotient algebras $\ell^\infty(\mathrm{N}; R)/I_\omega$ and $\ell^\infty(\mathrm{N}; R)/I_\infty$ by I_ω and I_∞ are denoted by $R(\omega)$ and $R(\infty)$ respectively. Then $R(\omega)$ becomes a II_1-factor with a unique trace $\tau_\omega((x_n)) = \lim_{n\to\omega} \tau(x_n)$ for $(x_n) \in R(\omega)$ and the relative commutant $R_\omega = R' \cap R(\omega)$ is again a II_1-factor. Let $\mathrm{Aut}(R)$ (resp. $\mathrm{Int}(R)$) be the automorphism group (resp. inner automorphism group) of R. All $\theta \in \mathrm{Aut}(R)$ can be extended to an automorphism (denoted by the same symbol θ without confusion) of $R(\infty)$, $R(\omega)$ or R_ω by

$$\theta(x) = \theta((x_n)) = (\theta(x_n))$$

for $x = (x_n) \in R(\infty)$, $R(\omega)$ or R_ω. Therefore we can define fixed point algebras for the action α of G:

$$R(\infty)^G = \{(x_n) \in R(\infty) ; \lim_{n\to\infty} \|\alpha_g(x_n) - x_n\|_2 = 0, \quad \text{for } g \in G\},$$

and similarly $R(\omega)^G$ and R_ω^G. For further informations of ultraproduct algebras, we refer to [5], [17] and [22].

Let (M, H, β) be a W^*-covariant system with a discrete group H. We define one-cocycle $Z_\beta^1(H; U(M))$ and coboundary $B_\beta^1(H; U(M))$ by

(2.1) $\quad Z_\beta^1(H; U(M)) = \{u \; ; u_h \in U(M) \text{ with } u_g\beta_g(u_h) = u_{gh} \text{ for } g, h \in H\}$

(2.2) $\quad B_\beta^1(H; U(M)) = \{u \in Z_\beta^1(H; U(M)) \; ; u_h = v^*\beta_h(v) \text{ for some } v \in U(M)\}$

where $U(M)$ is the unitary group of M. If $Z_\beta^1(H; U(M)) = B_\beta^1(H; U(M))$, then we say that the first cohomology vanishes for (M, H, β). A. Ocneanu showed in [22], Proposition 7.2 that the first cohomology vanishes for the W^*-covariant system (R_ω, G, α). His proof is valid even for the W^*-covariant system $(R(\omega), G, \alpha)$. We note that when the equality $u_g\beta_g(u_h) = u_{gh}$ in (2.1) is replaced by $\beta_g(u_h)u_g = u_{gh}$, we should change $u_h = v^*\beta_h(v)$ in (2.2) as $\beta_h(v)v^* = u_h$.

3. THE RELATIVE COMMUTANT OF ASYMPTOTIC FIXED POINT ALGEBRAS

For $a \in R$, $\delta > 0$ and a finite subset F of G (denoted by $F \Subset G$), we define $C_G(a, \delta, F)$, $C_G(a)$ and B_G by

$$C_G(a, \delta, F) = \overline{\text{conv}}\{uau^* \; ; u \in U(R), \|\alpha_g(u) - u\|_2 < \delta, g \in F\}$$

$$C_G(a) = \bigcap \{C_G(a, \delta, F) \; ; \delta > 0, F \Subset G\}$$

$$B_G = \{x \in R \; ; \lim_{n\to\infty} \|u_n x u_n^* - x\|_2 = 0 \text{ for } u_n \text{ satisfying}$$

$$\lim_{n\to\infty} \|\alpha_g(u_n) - u_n\|_2 = 0, \quad g \in G\}$$

where $\overline{\text{conv}}$ means the closure of the convex hull in the σ-weak topology.

LEMMA 3.1. *An element $a \in R$ is contained in B_G if and only if $C_G(a) = \{a\}$.*

Proof. Suppose $a \in B_G$. Then for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $F_0 \Subset G$ such that

$$u \in U(R);\ \ \|\alpha_g(u) - u\|_2 < \delta(\varepsilon), \quad g \in F_0$$

implies

$$\|ua - au\|_2 < \varepsilon.$$

Take $b \in C_G(a)$ and we can find $u_i \in U(R), \lambda_i > 0, i = 1, 2, \ldots, n$ such that $\sum_i \lambda_i = 1$, $\|\alpha_g(u_i) - u_i\|_2 < \delta(\varepsilon)$ for $g \in F_0$ with $\|b - \sum_i \lambda_i u_i a u_i^*\|_2$ sufficiently small. Then we have

$$\|b - a\|_2 \leqslant \|b - \sum_i \lambda_i u_i a u_i^*\|_2 + \|\sum_i \lambda_i u_i a u_i^* - a\|_2 \leqslant$$

$$\leqslant \|b - \sum_i \lambda_i u_i a u_i^*\|_2 + \sum_i \lambda_i \|u_i a u_i^* - a\|_2 < \varepsilon.$$

Since ε is arbitrary, we obtain $a = b$.

Conversely suppose $a \in B_G$. There exists a sequence $\{u_n\} \subset U(R)$ with

$$\lim_{n\to\infty} \|\alpha_g(u_n) - u_n)\|_2 = 0 \quad \text{for } g \in G$$

$$\|u_n a u_n^* - a\|_2 > c > 0 \quad \text{for some positive constant } c.$$

Take an accumulation point b of $\{u_n a u_n^*;\ n = 1, 2, \ldots\}$ in the σ-weak topology. Since $b \in C_G(a)$, we have $b = a$. Hence we have (if necessary, by taking a subsequence)

$$\lim_{n\to\infty} \|u_n a u_n^* - a\|_2 =$$

$$= \lim_{n\to\infty} 2(\|a^2\|_2 - \mathrm{Re}\,\tau(u_n a^* u_n^* a)) =$$

$$= 2(\|a\|_2^2 - \mathrm{Re}\,\tau(b^* a)) = 2(\|a\|_2^2 - \mathrm{Re}\,\tau(a^* a)) = 0,$$

which is a contradiction.

LEMMA 3.2. *For all $a \in R$, we have*

$$C_G(a, \delta, F) \cap \mathbf{C}1 \neq \emptyset \textit{ for all } \delta > 0,\ F \Subset G \textit{ if and only if } B_G = \mathbf{C}1.$$

Proof. Suppose $B_G = \mathbf{C}1$. The function

$$C_G(a) \ni x \to \|x\|_2 = \sup\{\tau(b^*a);\ b \in R,\ \|b\|_2 = 1\}$$

is σ-weakly lower semi-continuous on the σ-weakly compact convex set $C_G(a)$. Therefore there exists a unique minimal element $d \in C_G(a)$, i.e. $\|d\|_2 \leqslant \|b\|_2$ for all $b \in C_G(a)$. Since

$$C_G(d, \delta, F) \subset C_G(a, \delta + \gamma, F)$$

for all $\gamma > 0$, we have $C_G(d) \subseteq C_G(a)$. For all $b \in C_G(d)$, take unitaries u_i and positive numbers λ_i with $\sum_i \lambda_i = 1$ such that $\left\|\sum_i \lambda_i u_i^* d u_i - b\right\|_2$ is sufficiently small, and $\|\alpha_g(u_i) - u_i\|_2$ are sufficiently small for $g \in F \Subset G$. Since we have

$$\|b\|_2 \leqslant \left\|\sum_i \lambda_i u_i d u_i^*\right\|_2 + \left\|b - \sum_i \lambda_i u_i d u_i^*\right\|_2 \leqslant$$

$$\leqslant \|d\|_2 + \left\|b - \sum_i \lambda_i u_i d u_i^*\right\|_2,$$

we obtain $\|b\|_2 \leqslant \|d\|_2$. By the minimality of d, we get $b = d$, and $C_G(d) = \{d\}$. By Lemma 3.1 and $B_G = \mathbf{C}1$, the element d is scalar. Since $C_G(a, \delta, F) \cap \mathbf{C}1 \supset$ $\supset C_G(d) \cap \mathbf{C}1 \ni d$ for all $\delta > 0$ and F, we conclude that for each $a \in R$, $C_G(a, \delta, F) \cap \mathbf{C}1 \neq \emptyset$ for $\delta > 0$ and $F \Subset G$.

Conversely, take $a \in B_G$. Then for all $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $F_0 \Subset G$ such that $\|\alpha_g(u) - u\|_2 < \delta(\varepsilon)$, $g \in F_0$ implies $\|uau^* - a\|_2 < \varepsilon$. We take $\lambda_i \geqslant 0$ with $\sum_i \lambda_i = 1$, $u_i \in U(R)$ and $\lambda(\varepsilon) \in \mathbf{C}$ such that

$$\left\|\sum_i \lambda_i u_i a u_i^* - \lambda(\varepsilon)1\right\|_2 < \varepsilon$$

$$\|\alpha_g(u_i) - u_i\|_2 < \delta(\varepsilon) \quad \text{for all } g \in F_0.$$

Since $\left\|\sum_i \lambda_i u_i a u_i^* - a\right\|_2 \leqslant \sum_i \lambda_i \|u_i a u_i^* - a\|_2 < \varepsilon$, we have $\|a - \lambda(\varepsilon)1\|_2 < \varepsilon$. Then the operator a is scalar.

PROPOSITION 3.3. *The relative commutant in R of the fixed point algebra $R(\infty)^G$ is trivial*, i.e. $B_G = \mathbf{C}1$.

Proof. By Lemma 3.2, we have only to show $C_G(a, \delta, F) \cap \mathbf{C}1 \neq \emptyset$ for all $\delta > 0$, $F \Subset G$. Let β be an outer action of G on R with an ergodic automorphism γ commuting with β (for example, let β_g be a product action $\otimes \alpha_g$ on $\underset{n \in \mathbf{Z}}{\otimes} R_n \cong R$ where $R_n = R$) and γ be a shift on $\underset{n \in \mathbf{Z}}{\otimes} R_n$). By [22] 1.2 and 1.4, for $\delta > 0$, $F \Subset G$, there exists a one-cocycle v_g for α with

$$\|v_g - 1\|_2 < \delta/3 \quad \text{for } g \in F$$

$$\operatorname{Ad} v_g \cdot \alpha_g = \Phi \beta_g \Phi^{-1} \quad \text{for } g \in G,$$

for $\Phi \in \operatorname{Aut}(R)$. As the closure $\overline{\operatorname{Int}(R)} = \operatorname{Aut}(R)$, we can find $V \in R(\omega)$ with $\operatorname{Ad} V(x) = \gamma(x)$ for $x \in R$. Since

$$\operatorname{Ad} V^* \beta_g(V)(x) = \gamma^{-1} \beta_g \gamma \beta_g^{-1}(x) = x \quad x \in R,$$

we have $V^* \beta_g(V) \in R_\omega$, moreover $V^* \beta_g(V) \in Z^1_\beta(G; U(R_\omega))$. By the first cohomology vanishing theorem, we can take a unitary $W \in R_\omega$ with $V^* \beta_g(V) = W^* \beta_g(W)$ for $g \in G$. Then we obtain $\beta_g(VW^*) = VW^*$ for $g \in G$. Let (v_n) be a representative of VW^*. For $g \in F$ and $m \in \mathbf{Z}$, we have

$$\|\alpha_g(\Phi(v_n^m)) - \Phi(v_n^m)\|_2 \leqslant$$

$$\leqslant \|\operatorname{Ad} v_g \alpha_g(\Phi(v_n^m)) - \alpha_g(\Phi(v_n^m))\|_2 + \|\Phi(\beta_g(v_n^m)) - \Phi(v_n^m)\|_2 \leqslant$$

$$\leqslant 2\|v_g - 1\|_2 + \|\beta_g(v_n^m) - v_n^m\|_2 .$$

Hence we obtain $\Phi(v_n^m \Phi^{-1}(a) v_n^{m*}) = \Phi(v_n^m) a \Phi(v_n^{m*}) \in C_G(a, \delta, F)$ for $a \in R$ and sufficiently large n. Then we get $\Phi \gamma^m \Phi^{-1}(a) \in C_G(a, \delta, F)$. Hence there exists a unique element d in $\overline{\operatorname{conv}}\{\Phi \gamma^m \Phi^{-1}(a);\ m \in \mathbf{Z})$ such that $\|d\|_2 \leqslant \|b\|_2$ for any b in $\overline{\operatorname{conv}}\{\Phi \gamma^m \Phi^{-1}(a);\ m \in \mathbf{Z}\}$. Since γ is an ergodic automorphism, the operator d must be scalar. Thus we obtain $C_G(a, \delta, F) \cap \mathbf{C}1 \neq \emptyset$ for all $\delta > 0$ and $F \Subset G$.

4. A DUALITY THEOREM FOR COUNTABLE DISCRETE AMENABLE GROUPS

Let (R, G, α) be as above and θ be an automorphism of R satisfying the asymptotic fixed point property throughout the rest of this paper:

$$\lim_{n \to \infty} \|\theta(x_n) - x_n\|_2 = 0 \tag{4.1}$$

for every $(x_n) \in \ell^\infty(\mathrm{N}; R)$ with $\lim_{n\to\infty} \|\alpha_g(x_n) - x_n\|_2 = 0$ for all $g \in G$. Since the following lemma is easily established, we omit its proof.

LEMMA 4.1. *$R(\omega)^G$ (resp. R_ω^G) is contained in $R(\omega)^0$ (resp. R_ω^0).*

Since our proof of the main Theorem 4.8 relies on P. Eymard's and K. Saito's duality theorems ([9], [24]), we have to prepare some notations and definitions about representations of G. A pair $\{\pi, H_\pi\}$ is called a representation in $R(\omega)^G$ (resp. R_ω^G) if H_π is a separable Hilbert subspace of $L^2(R(\omega); \tau_\omega)$ (resp. $L^2(R_\omega; \tau_\omega)$) and π is a non-degenerate unitary representation of G on H_π with $\pi(g) \in R(\omega)^G$ (resp. R_ω^G) for all $g \in G$. A representation $\{\lambda, H_\lambda\}$ in R_ω^G is called regular if $\tau_\omega(\lambda(g)) = \delta_{e,g}$ where δ is Kronecker's delta and e is the unit of G and H_λ is the closed linear spant of $\{\lambda(g)\ ;\ g \in G\}$. Since R_ω^G is a II_1-von Neumann algebra (in fact, R_ω^G is known to be a II_1-factor), by [22], Lemma 8.3, R_ω^G has a hyperfinite subfactor. Hence, by [22], Proposition 4.4, we can construct a regular representation λ in R_ω^G.

Let $B(G)$ be the Fourier-Stieltjes algebra of G (see [9]). We define a subsoe $B_\alpha(G)$ by

$$B_\alpha(G) = \{\tau_\omega(a^*\pi(\cdot)b) \in B(G);\ \pi \text{ runs through all representations in } R(\omega)^G \text{ and all } a, b \in H_\pi\}.$$

It is remarked that since a representation π in $R(\omega)^G$ is a one-cocycle for α, we can find a unitary v in $R(\omega)$ with $v^*\alpha_g(v) = \pi(g)$ for $g \in G$ by the first cohomology vanishing theorem.

LEMMA 4.2. *Let π be a representation in $R(\omega)^G$. For two unitaries $v_i \in R(\omega)$ $(i = 1,2)$ with $v_i\alpha_g(v_i^*) = \pi(g)$ for all $g \in G$, we have*

$$\tau_\omega(a^*\theta(v_1^*)\alpha_g(v_1)b) = \tau_\omega(a^*\theta(v_2^*)\alpha_g(v_2)b)$$

for all $g \in G$ and $a, b \in R(\omega)$.

Proof. Since $\alpha_g(v_1v_2^*) = v_1\pi(g)\pi(g)^*v_2^* = v_1v_2^*$ for all $g \in G$, we obtain $\theta(v_1v_2^*) = v_1v_2^*$ by the assumption for θ. Therefore,

$$\begin{aligned}\tau_\omega(a^*\theta(v_1^*)\alpha_g(v_1)b) &= \tau_\omega(a^*\theta((v_1v_2^*v_2)^*)\alpha_g(v_1v_2^*v_2)b) = \\ &= \tau_\omega(a^*\theta(v_2^*)\theta(v_1v_2^*)^*\alpha_g(v_1v_2^*)\alpha_g(v_2)b) = \\ &= \tau_\omega(a^*\theta(v_2^*)(v_1v_2^*)^*(v_1v_2^*)\alpha_g(v_2)b) = \tau_\omega(a^*\theta(v_2^*)\alpha_g(v_2)b).\end{aligned}$$

Lemma 4.2 means that the value of $\tau_\omega(a^*\theta(v^*)\alpha_g(v)b)$ is independent of the choice of the unitary v with $v^*\alpha_g(v) = \pi(g)$ for $g \in G$. We want to prove further that $\tau_\omega(a^*\theta(v^*)\alpha_g(v)b)$ depends only on $\tau_\omega(a^*\pi(\cdot)b) \in B(G)$, or it is independent on the particular choice of π and $a, b \in H_\pi$. For this purpose, we show the following lemma, which plays a decisive role in our work.

LEMMA 4.3. *For a representation* π *in* $R(\omega)^G$ *and* $v \in R(\omega)$ *with* $v^*\alpha_g(v) = \pi(g)$ $(g \in G)$, *there exists a unique operator* d *in the double commutant* $\pi(G)''$ *of* $\{\pi(g) ; g \in G\} \subseteq R(\omega)^G$ *such that*

$$\tau_\omega(a^*\theta(v^*)\alpha_g(v)b) = \tau_\omega(a^*d^*v^*\alpha_g(v)b)$$

for all $a, b \in R(\omega)$.

Proof. Let $(\varepsilon_n, G_n, (K_i^n)_i, (L_{i,j}^n)_{i,j}, k^n, (l_g^n)_g)_n$ be a paving structure for G. (See [22], Propositions 3.4 and 3.5, and we use the same notations as in [22].) By non-abelian Rohlin theorem ([22], 6.1), there exists a partition of unity $(E_{i,k}^n)_{i=1,2,\dots,N(n),\, k\in K_i^n} \in R_\omega$ such that

$$\sum_{i=1}^{N(n)} |K_i^n|^{-1} \sum_{k,l\in K_i^n} |\alpha_{kl^{-1}}(E_{i,l}^n) - E_{i,k}^n|_\tau < 5\varepsilon_n^{1/2}$$

$$[E_{i,k}^n, \alpha_g(E_{j,l}^n)] = 0 \quad \text{for all } g, i, j, k, l$$

where $|\ \ |$ means the cardinality and $|x|_\tau = \tau_\omega(|x|)$ for $x \in R(\omega)$. Let N, F and $\mathscr{B}$ (which appear in [22], 5.3 — fast reindexation trick) be W^*-subalgebras of $R(\omega)$ and an automorphism group generated by $(E_{i,k}^n, \theta(E_{i,k}^n), \alpha_g(E_{i,k}^n); i, k \in K_i^n, g \in G\}$, $\{a, b, \pi(g) ; g \in G\}$ $\{\alpha_g, \theta ; g \in G\}$ respectively. Let Φ stand for the normal injective $*$-homomorphism $\Phi\colon N \to R(\omega)$ in [22], 5.3 and define $F_{i,k}^n$ by $F_{i,k}^n = \Phi(E_{i,k}^n)$. Then, according to [22], Proposition 7.2, we get unitaries $v^{(n)} \in R(\omega)$ by

$$v^{(n)} = \sum_i \sum_{k\in K_i^n} \pi(g)^* F_{i,k}^n$$

(4.2)

$$\tilde{v}_g^{(n)} = v^{(n)}\pi(g)^* \alpha_g(v^{(n)*}).$$

Then we have

$$|\tilde{v}_g^{(n)} - 1|_\tau < 32\varepsilon_n^{1/2} \quad \text{for } g \in G_n.$$

For $a, b \in R(\omega)$, we have

$$\tau_\omega(a^*\theta(v^{(n)})^*\alpha_g(v^{(n)})b) =$$

$$= \tau_\omega(a^*(\sum \sum_k \theta(F_{i,k}^n)\pi(k))\alpha_g(\sum \sum \pi(h)^* F^n_{,h})b) =$$

(4.3)

$$= \sum_{i,j,k,h} \tau_\omega(a^*\theta(F_{i,k}^n)\pi(k)\pi(h)^*\alpha_g(F_{j,h}^n)b) =$$

$$= \sum_{i,j,k,h} \tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)\alpha_g(E_{j,h}^n)),$$

where the last equality follows from [22], 5.3. For $h \in K_j^n \cap g^{-1}K_j^n$, we have

$$\Big|\sum_{i,k\in K_i^n}\sum_{j,h\in K_j^n}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)(\alpha_g(E_{j,h}^n)-E_{j,gh}^n))\Big| \leqslant$$

$$\leqslant \sum_{j,h}|\tau_\omega\{(\sum_{i,k}\tau_\omega(a^*\pi(kh^{-1})b)\theta(E_{i,k}^n))(\alpha_g(E_{j,h}^n)-E_{j,gh}^n)\}| \leqslant$$

$$\leqslant \sum_{j,h}\|\sum_{i,k}\tau_\omega(a^*\pi(kh^{-1})b)\theta(E_{i,k}^n)\|\,|\alpha_g(E_{j,h}^n)-E_{j,gh}^n|_\tau.$$

Since $\{\theta(E_{i,k}^n)\}_{i,k}$ is a partition of unity, we have

$$\Big\|\sum_{i,k}\tau_\omega(a^*\pi(kh^{-1})b)\theta(E_{i.k}^n)\Big\| \leqslant \|a\|_\tau\|b\|_\tau \tag{4.4}$$

where $\|a\|_\tau = \tau_\omega(a^*a)^{1/2}$. By [22], Corollary 6.1(4), we have, for $h \in K_j^n \cap$ $\cap\, g^{-1}K_j^n$,

$$\Big|\sum_{i,k\in K_i^n}\sum_{j,h\in K_j^n}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)(\alpha_g(E_{j,h}^n)-E_{j,gh}^n))\Big| \leqslant$$

$$\leqslant \|a\|_\tau\|b\|_\tau\sum_{j,h}|\alpha_s(E_{j,h}^n)-E_{j,gh}^n|_\tau \leqslant$$

$$\leqslant 10\varepsilon_n^{1/2}\|a\|_\tau\|b\|_\tau.$$

Since

$$\sum_{i,k}\sum_{j,h\in K_j^n\cap g^{-1}K_j^n}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)E_{j,gh}^n) =$$

$$= \sum_{i,k}\sum_{j,h\in K_j^n\cap gK_j^n}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,gh}^n),$$

we obtain

$$\Big|\sum_{i,k}\sum_{j,h}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)\alpha_g(E_{j,h}^n)) -$$

$$-\sum_{i,k}\sum_{j,h}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,h}^n)\Big| \leqslant$$

$$\leqslant\Big|\sum_{i,k}\sum_{j,h\notin K_j^n\cap g^{-1}K_j^n}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)\alpha_g(E_{j,h}^n))\Big| +$$

$$+\Big|\sum_{i,k}\sum_{j,k\notin K_j^n\cap gK_j^n}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,h}^n)\Big| +$$

$$+\,10\varepsilon_n^{1/2}\|a\|_\tau\|b\|_\tau.$$

By the same calculation as (4.4), we have

$$\Big|\sum_{i,k}\sum_{j,h\notin K_j^n\cap g^{-1}K_j^n}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)\alpha_g(E_{j,h}^n))\Big| \leqslant$$

$$\leqslant \|a\|_\tau\|b\|_\tau \sum_{j,h\notin K_j^n\cap g^{-1}K_j^n}\tau_\omega(E_{j,h}^n) \leqslant$$

$$\leqslant \|a\|_\tau\|b\|_\tau(\varepsilon_n + 5\varepsilon_n^{1/2})$$

where the last inequality follows from [22], Corollary 6.1(5) and (ε_n, G_n)-invariance of K_j^n ($|g^{-1}K_j^n \ominus K_j^n| \leqslant \varepsilon_n|K_j^n|$ for $g\in G_n$). Similarly, we obtain

$$\Big|\sum_{i,k}\sum_{j,h\notin K_j^n\cap gK_j^n}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,h}^n)\Big| \leqslant$$

$$\leqslant \|a\|_\tau\|b\|_\tau(\varepsilon_n + 5\varepsilon_n^{1/2}).$$

Therefore,

$$\Big|\sum_{i,k}\sum_{j,h}\tau_\omega(a^*\pi(kh^{-1})b)\tau_\omega(\theta(E_{i,k}^n)\alpha_g(E_{j,h}^n)) -$$

$$- \sum_{i,k}\sum_{j,h}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,h}^n)\Big| \leqslant \tag{4.5}$$

$$\leqslant \|a\|_\tau\|b\|_\tau C_n\varepsilon_n^{1/2},$$

where C_n is some constant (independent of a, b) and $g\in G_n$. Since $\{K_i^n\}$ can be chosen mutually disjoint by [22], 3.5, let $\tilde{\mu}_n$ be a probability measure on $G\times G$ obtained by putting $\tilde{\mu}_n((k, h)) = (\theta(E_{i,k}^n)E_{j,h}^n)$. Define a probability measure μ_n on G by $\mu_n(E) = \int_{E\times G} \mathrm{d}\tilde{\mu}_n(hk, k)$ for $E \Subset G$. Then,

$$\sum_{i,k}\sum_{j,h}\tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E_{i,k}^n)E_{j,h}^n) =$$

$$= \int_{G\times G}\tau_\omega(a^*\pi(hk^{-1})^*\pi(g)b)\,\mathrm{d}\tilde{\mu}_n(h, k) =$$

$$= \int_G \tau_\omega(a^*\pi(h^*)\pi(g)b)\,\mathrm{d}\mu_n(h) = \tag{4.6}$$

$$= \tau_\omega(a^*\pi(\mu_n)^*\pi(g)b),$$

where $\pi(\mu_n) = \int_G \pi(h)\, d\mu_n(h)$. Since $\|\pi(\mu_n)\| \leqslant \int_G d\mu_n(h) = 1$, if necessary by taking a subsequence, we may assume that $\pi(\mu_n)$ converges to some operator $d \in \pi(G)''$ in the σ-weak topology. By (4.5) we have

$$\lim_{n\to\omega} \sum_{i,k} \sum_{j,h} \tau_\omega(a^*\pi(kh^{-1})\pi(g)b)\tau_\omega(\theta(E^n_{i,k})E^n_{j,h}) =$$

$$= \tau_\omega(a^*d^*\pi(g)b).$$

The C^*-subalgebra C of $\ell^\infty(\mathbf{N}, R(\omega))$ and the automorphism group A, which appear in [22], 5.5 (index selection trick), are generated by $\{\{a\}_n, \{b\}_n, \{F^n_{i,k}\}_n, \{\pi(g)\}_n; i, k \in K^n_i, g \in G\} \subset L^\infty(\mathbf{N}, R(\omega))$ and $\{\alpha_g, \theta; g \in G\}$ respectively. Put $v = \Phi(\{v^{(n)}\})$ where Φ is the same homomorphism as in [22], 5.5. By (4.2), (4.3), (4.5), (4.6), (4.7), and the property (1) in [22], 5.5, we have

$$\tau_\omega(a^*\theta(v^*)\alpha_g(v)b) = \lim_{n\to\omega} \tau_\omega(a^*\theta(v^{(n)})^*\alpha_g(v^{(n)})b) =$$

$$= \tau_\omega(a^*d^*\pi(g)b) = \tau_\omega(a^*d^*v^*\alpha_g(v)b)$$

for $v^*\alpha_g(v) = \pi(g)$ for $g \in G$. We remark that the operator d depends only on the partition of unity $\{E^n_{i,k}\}$. Therefore, we complete the proof of

$$\tau_\omega(a^*\theta(v^*)\alpha_g(v)b) = \tau_\omega(a^*d^*v^*\alpha_g(v)b)$$

for all $a, b \in R(\omega)$. The uniqueness of the operator d is clear.

The following methods are essentially borrowed from [1] and [11]. Let Θ be a map of $B_\alpha(G)$ into $B(G)$ defined by

$$\Theta(\psi)(g) = \tau_\omega(a^*\theta(v^*)\alpha_g(v)b)$$

for $\psi \in B_\alpha(G) \equiv \{\psi(g) = \tau_\omega(a^*\pi(g)b)$; some representation π in $R(\omega)^G$, $a, b \in H_\pi\}$ and some unitary v with $v^*\alpha_g(v) = \pi(g)$.

LEMMA 4.4. *$B_\alpha(G)$ is a subspace of $B(G)$ and the map Θ is well-defined, surjective, isometric and linear of $B_\alpha(G)$ onto $B_\alpha(G)$.*

Proof. It is noted that $B_\alpha(G)$ is closed under scalar multiplication. Let $\psi_i \in B_\alpha(G)$ with $\psi_i(g) = \tau_\omega(a_i^*\pi_i(g)b_i)$ for some representation π_i in $R(\omega)^G$ and a_i, $b_i \in H_{\pi_i}$ $(i = 1, 2)$. Since the fixed point algebra R^G_ω is a II_1-factor, we can take a projection $e \in R^G_\omega$ with $\tau_\omega(e) = 1/2$. By the fast reindexation trick, we may assume that the projection e commutes with $\pi_i(g)$, $g \in G$ and all elements in H_i $(i = 1, 2)$. Hence we can construct a new representation by

$$\rho(g) = \pi_1(g)e + \pi_2(g)(1-e), \quad H_\rho = eH_{\pi_1} + (1-e)H_{\pi_2},$$

(4.8)

$$a_+ = ea_1 + (1-e)a_2, \quad b_\pm = eb_1 \pm (1-e)b_2.$$

Then

$$\tau_\omega(a_+^*\rho(g)b_+) = \tau_\omega(e)\tau_\omega(a_1^*\pi_1(g)b_1) + \tau_\omega(1-e)\tau_\omega(a_2^*\pi_2(g)b_2)$$

$$=(1/2)\tau_\omega(a_1^*\pi_1(g)b_1) + (1/2)\tau_\omega(a_2^*\pi_2(g)b_2),$$

where the first equality follows from [22], 5.3. Therefore the sum $\psi_1 + \psi_2 \in B_\alpha(G)$. Then $B_\alpha(G)$ is a subspace of $B(G)$. To see that Θ is well-defined, by Lemma 4.2, we have only to prove that, if

$$\tau_\omega(a_1^*\pi_1(g)b_1) = \tau_\omega(a_2^*\pi_2(g)b_2),$$

then

$$\tau_\omega(a_1^*\theta(v_1^*)\alpha_g(v_1)b_1) = \tau_\omega(a_2^*\theta(v_2^*)\alpha_g(v_2)b_2)$$

for $g \in G$ and some unitaries v_i with $v_i^*\alpha_g(v_i) = \pi_i(g)$ $(i = 1, 2)$. Set a new unitary w by $w = ev_1 + (1-e)v_2$ with $w^*\alpha_g(w) = \rho(g)$ for $g \in G$. We have

$$\tau_\omega(a_+^*\rho(g)b_-) = (1/2)\tau_\omega(a_1^*\pi_1(g)b_1) - (1/2)\tau_\omega(a_2^*\pi_2(g)b_2) = 0,$$

for all $g \in G$, hence

$$\tau_\omega(a_+^*\theta(w^*)\alpha_g(w)b_-) = \tau_\omega(a_+^*d^*\rho(g)b_-) = 0,$$

where $d \in \rho(G)''$ by Lemma 4.3. Since

$$\tau_\omega(a_+^*\theta(w^*)\alpha_g(w)b_-) =$$

$$= (1/2)\tau_\omega(a_1^*\theta(v_1^*)\alpha_g(v_1)b_1) - (1/2)\tau_\omega(a_2^*\theta(v_2^*)\alpha_g(v_2)b_2),$$

we conclude

$$\tau_\omega(a_1^*\theta(v_1^*)\alpha_g(v_1)b_1) = \tau_\omega(a_2^*\theta(v_2^*)\alpha_g(v_2)b_2)$$

for all $g \in G$. We show that Θ is surjective and linear. Since

$$\tau_\omega(a_+^*\rho(g)b_+) = (1/2)\tau_\omega(a_1^*\pi_1(g)b_1) + (1/2)\tau_\omega(a_2^*\pi_2(g)b_2),$$

$$\tau_\omega(a_+^*\theta(w^*)\alpha_g(w)b_+) =$$

$$= (1/2)\tau_\omega(a_1^*\theta(v_1^*)\alpha_g(v_1^*)b_1) + (1/2)\tau_\omega(a_2^*\theta(v_2^*)\alpha_g(v_2)b_2),$$

the map Θ is linear of $B_\alpha(G)$ into $B_\alpha(G)$. Since $v^*\theta(v)$ is a unitary operator in $\pi(G)''$ by Lemma 4.3, the map Θ is isometric. Since $\tau_\omega((d^*a)^*\theta(v^*)\alpha_g(v)b) = \tau_\omega(a^*\pi(g)b)$, we conclude that Θ is a surjective linear map of $B_\alpha(G)$ onto $B_\alpha(G)$.

The subspace $B_\alpha(G)$ is not in general a subalgebra of $B(G)$. It follows from the existence of the regular representation in R_ω^G that $B_\alpha(G)$ contains the Fourier algebra $A(G)$ of G. (See [9] for the definition of $A(G)$.) Therefore we restrict the map Θ on $A(G)$.

LEMMA 4.5. *The restriction of Θ on $A(G)$ is a surjective and multiplicative linear map of $A(G)$ onto $A(G)$.*

Proof. We have only to prove the multiplicativity of Θ (the surjectivity of Θ can be proved as in Lemma 4.4). Let λ be a regular representation in R_ω^G. For $\psi_i \in A(G)$, we can take $a_i, b_i \in H_\lambda$ with $\psi_i(g) = \tau_\omega(a_i^*\lambda(g)b_i)$ for $g \in G$ $(i = 1,2)$. By [22], 5.3, we take a homomorphism Φ satisfying

$$[\lambda(g), \Phi(\lambda(g))] = 0 \quad \text{and} \quad [v, \Phi(v)] = 0$$

for some unitary $v \in R_\omega^G$ with $v^*\alpha_g(v) = \lambda(g)$ and

$$\psi_1(g)\psi_2(g) = \tau_\omega(a_1^*\lambda(g)b_1)\tau_\omega(a_2^*\lambda(g)b_2) =$$

$$= \tau_\omega(a_1^*\Phi(a_2^*)\lambda(g)\Phi(\lambda(g))b_1\Phi(b_2)))$$

for $a_i, b_i \in H_\lambda$ $(i = 1,2)$. Moreover we can choose Φ to satisfy

$$\lambda(g)\Phi(\lambda(g)) = (v\Phi(v))^*\alpha_g(v\Phi(v))$$

and

$$\Theta(\psi_1\psi_2)(g) = \tau_\omega\{(a_1\Phi(a_2))^*\theta(v\Phi(v))^*\alpha_g(v\Phi(v))(b_1\Phi(b_2))\} =$$

$$= \tau_\omega(a_1^*\theta(v^*)\alpha_g(v)b_1)\tau_\omega(a_2^*\theta(v^*)\alpha_g(v)b_2).$$

Therefore we have

$$\Theta(\psi_1\psi_2) = \Theta(\psi_1)\Theta(\psi_2).$$

PROPOSITION 4.6. *For a representation π in $R(\omega)^G$ and a unitary $v \in R(\omega)$ with $v^*\alpha_g(v) = \pi(g)$, $\theta(v)$ must be of the form*

$$\theta(v) = v\pi(k)$$

for some $k \in G$ and k depends only on θ.

Proof. It follows from Lemma 4.5 and [9], Théorème 3.34 that there exists $k \in G$ with $\Theta(\psi)(e) = \psi(k^{-1})$ for all $\psi \in A(G)$ where e is the unit of G. For a representation π in $R(\omega)^G$, applying the fast reindexation trick as in Lemma 4.5, we construct a new representation $\{\rho(g) = \pi(g)\Phi(\lambda(g)), H_\rho = H_\pi\Phi(H_\lambda)\}$ where λ is a regurar representation in R_ω^G and Φ is a homomorphism. This ρ is a tensor product epresentation of π and λ. Since ρ is quasi-equivalent to a regular representation of G, then $\tau_\omega(\xi^*\rho(\cdot)\eta) \in A(G)$ for $\xi, \eta \in H_\rho$. Let $w \in U(R_\omega)$ with $w^*\alpha_g(w) = \lambda(g)$ $(g \in G)$. By [22], 5.3 (3) (4), we can choose Φ to meet the following computations. For $a, b \in H_\pi$, $a', b' \in H_\lambda$,

$$\tau_\omega(a^*\Phi(a')\theta(v\Phi(w))^*\alpha_c(v\Phi(w))bV(b')) =$$

$$= \tau_\omega(a^*\theta(v)^*vb)\tau_\omega(a'^*\theta(w)^*wb') = \tag{4.9}$$

$$= \tau_\omega(a^*\theta(v)^*vb)\tau_\omega(a'^*\lambda(k^{-1})b')$$

where the last equality follows from $\tau_\omega(a'w^*\alpha_g(w)b') \in A(G)$ and Lemma 4.5 Since $\tau_\omega((a\Phi(a'))^*\rho(g)b\Phi(b')) \in A(G)$, we have

$$\Theta(\tau_\omega((a\Phi(a'))^*\rho(\cdot)b\Phi(b')))(e) =$$

$$= \tau_\omega((a\Phi(a'))^*\rho(k^{-1})b\Phi(b')) = \tag{4.10}$$

$$= \tau_\omega(a^*\pi(k^{-1})b)\tau_\omega(a'^*\lambda(k^{-1})b').$$

Therefore by (4.9) and (4.10), we obtain

$$\tau_\omega(a^*\theta(v)^*vb) = \tau_\omega(a^*\pi(k^{-1})b)$$

for all $a, b \in H_\pi$. Since $\theta(v)^*v \in \pi(G)''$, we conclude $\theta(v)^*v = \pi(k^{-1})$ for some $k \in G$.

We remark that $\alpha_k^{-1}\theta$ satisfies the asymptotic fixed point property (4.1). We put an automorphism $\bar{\theta}$ of R:

$$\bar{\theta} = \alpha_k^{-1}\theta,$$

then $\bar{\theta}$ satisfies $\bar{\theta}(v) = v$ for $v \in U(R(\omega))$ with $v^*\alpha_g(v) = \pi(g)$ and satisfies the asymptotic fixed point property. Thus, to show the main Theorem 4.8, we have only to prove that $\bar{\theta}$ is a trivial automorphism of R.

PROPOSITION 4.7. *The automorphism $\bar{\theta}$ commutes with G.*

Proof. By [22], Lemma 9.2 there exists a unitary representation V_g in $R(\omega)$ satisfying the restriction $\mathrm{Ad}\, V_g \mid R$ of $\mathrm{Ad}\, V_g$ on R is equal to α_g

$$\alpha_g(V_h) = V_{ghg^{-1}} \quad \text{for all } g,\ h \in G.$$

We note that V is a one-cocycle by $\alpha_g(V_h)V_g = V_{ghg^{-1}}V_g = V_{gh}$. By the first cohomology vanishing theorem for $(R(\omega),\ G,\ \alpha)$, there exists a unitary $v \in R(\omega)$ such that $\alpha_g(v)v^* = V_g$. Then we have

$$\alpha_h(v^* V_g v) = \alpha_h(v^*)\alpha_h(V_g)\alpha_h(v) = v^* V_h^* V_{hgh^{-1}} V_h v = v^* V_g v$$

for all $h \in G$. Since $v^* V_g v$ is a representation in $R(\omega)^G$, we can take again a unitary $w \in R(\omega)$ with $w^*\alpha_g(w) = v^* V_g v$. By the assumption of $\bar{\theta}$, we get $\bar{\theta}(w) = w$. Since

$$\alpha_g(wv^*) = \alpha_g(w)\alpha_g(v^*) = w(v^* V_g v)v^* V_g^* = wv^*$$

for $g \in G$, we have

$$\bar{\theta}(v) = \bar{\theta}((wv^*)^* w) = \bar{\theta}((wv^*)^*)\bar{\theta}(w) = (wv^*)^* w = v.$$

Since

$$\bar{\theta}(v^* V_g v) = v^* V_g v, \quad \bar{\theta}(v^*)\bar{\theta}(V_g)\bar{\theta}(v) = v^*\bar{\theta}(V_g)v,$$

we get $V_g = \bar{\theta}(V_g)$ for all $g \in G$. Therefore we conclude

$$\alpha_g = \mathrm{Ad}\, V_g|R = \mathrm{Ad}\, \bar{\theta}(V_g)|R = \bar{\theta}\alpha_g\bar{\theta}^{-1}.$$

We have already proved lemmas which are necessary to get the following main theorem.

THEOREM 4.8. *Let α be an outer action of a countable discrete amenable group G on a hyperfinite II_1-factor R. Suppose that an automorphism θ of R satisfies the asymptotic fixed point property* (4.1). *Then θ must be of the form*

$$\theta = \alpha_k$$

for some $k \in G$.

Proof. Let k be as in Proposition 4.6 and let $\bar{\theta}$ be as above. When the automorphism $\bar{\theta}$ is periodic, let p be the minimal positive integer such that $\bar{\theta}^p$ is trivial and let $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$. When $\bar{\theta}$ is aperiodic, the integer p is taken as zero and $\mathbf{Z}_p = \mathbf{Z}$. By Proposition 4.7, we can define an action β of the amenable group $\mathbf{Z}_p \times G$ by

$$\beta:\ \mathbf{Z}_p \times G \ni (n,\ g) \to \bar{\theta}^n \alpha_g.$$

Suppose first that $\bar{\theta}^n\alpha_g$ is inner for some $n \in \mathbf{Z}_p$, $g \in G$. For a regular representation λ in R_ω^G, we take a unitary $v \in R_\omega$ with $v^*\alpha_g(v) = \lambda(g)$ $(g \in G)$. Then, since $\bar{\theta}^n\alpha_g$ is inner, we have $\bar{\theta}^n\alpha_g(v) = v$. But

$$\bar{\theta}^n\alpha_g(v) = \bar{\theta}^n(v\lambda(g)) = \bar{\theta}^n(v)\bar{\theta}^n(\lambda(g)) = v\lambda(g).$$

Therefore $\lambda(g) = 1$ and g must be the unit $e \in G$. Then $\bar{\theta}^n$ is inner and the element n is the unit of $\mathbf{Z}_p$ by Proposition 3.3. We thus conclude that the action β is outer. By [22], Corollary 1.4, β is cocycle conjugate to a tensor product action $\bar{\theta}\otimes\alpha$ on $R\otimes R$:

$$(\bar{\theta}\otimes\alpha)_{(n,g)} = \bar{\theta}^n\otimes\alpha_g \quad (n, g) \in \mathbf{Z}_p\times G,$$

that is

$$\psi^{-1}(\bar{\theta}\otimes\alpha)_{(n,g)}\psi = \operatorname{Ad} u(n,g)\beta_{(n,g)}$$

for some isomorphism ψ of R onto $R\otimes R$ and one-cocycle $u(n, g)$ for the action β. Since every inner automorphism acts trivially on R_ω, we obtain

$$\psi(R_\omega^G) = (R\otimes R)_\omega^{\iota\otimes G}, \quad \psi(R_\omega^{\bar{\theta}}) = (R\otimes R)_\omega^{\bar{\theta}\otimes\iota},$$

where ι is a trivial automorphism. Therefore we get

$$R_\omega\otimes \mathrm{C} \subset (R\otimes R)_\omega^{\iota\otimes G} \subset (R\otimes R)_\omega^{\bar{\theta}\otimes\iota},$$

which implies $R_\omega \subset R_\omega^{\bar{\theta}}$. By [6], Theorem 3.2, $\bar{\theta}$ is inner. Since the unitary u, which implements $\bar{\theta}$, is in B_G, by Proposition 3.3, we conclude that $\theta = \alpha_k$ for some $k \in G$.

Finally we discuss an example given by V.F.R Jones [14]. Let H be a countable non-amenable discrete group. Consider

$$R = \bigotimes_{n\in H} M_2(\mathrm{C}), \quad \beta \text{ Bernoulli shift on } R.$$

Take $x = (x_n) \in R(\infty)^H$ and suppose $\inf_n \|x_n - \tau(x_n)\|_2 > c > 0$ (if necessary, by taking a subsequence of $\{x_n\}$). Set

$$y_n = (x_n - \tau(x_n))/\|x_n - \tau(x_n)\|_2,$$

then y_n are unit vectors orthogonal to $1 \in L^2(R, \tau)$ and satisfy

$$\lim_{n\to\infty}\|\beta_h(y_n) - y_n\|_2 = 0$$

for all $h \in H$. Hence the restriction of the unitary representation β of H on the ortho-complement of $1 \in L^2(R, \tau)$ contains weakly a trivial representation. By [14], Lemma 1 and the non-amenability of H, this is a contradiction. Thus we see that we should always have

$$\lim_{n\to\infty} \|x_n - \tau(x_n)\|_2 = 0.$$

Hence all automorphisms of R satisfy the property (4.1) for the Bernoulli shift β This means that the asymptotic duality theory is false in the case of Bernoulli shift by a countable non-amenable group. On the other hand, when β is an ergodic action of a discrete group with property T (see [7]), we have

$$\lim_{n\to\infty} \|x_n - \tau(x_n)\|_2 = 0$$

for all bounded sequences (x_n) with $\lim_{n\to\infty} \|\beta_h(x_n) - x_n\|_2 = 0$ for $h \in H$. Therefore the asymptotic duality theory is false for an ergodic action of a countable group with property T as well.

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