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THE MEASURE OF NON-COMPACTNESS OF A DISJOINTNESS PRESERVING OPERATOR

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1. INTRODUCTION

Let E be a Banach space and D a norm bounded subset of E. Then the *Kuratowksi measure of non-compactness* of D is defined as

$$\alpha(D) = \inf\left\{\lambda : D \subset \bigcup_{j=1}^{m} D_j,\ \operatorname{diam}(D_j) \leqslant \lambda\right\}$$

and the *Hausdorff measure of non-compactness* of D is defined as

$$\beta(D) = \inf\left\{r : D \subset \bigcup_{j=1}^{m} B(x_j, r),\ x_j \in E\right\},$$

where $B(x_j, r)$ denotes the ball in E with center x_j and radius r. If E and F are Banach spaces and $T\colon E \to F$ is a bounded linear operator, then one defines for T the corresponding measures of non-compactness

$$\alpha(T) = \inf\{k : \alpha(T(D)) \leqslant k\alpha(D) \text{ for all bounded } D \subset E\}$$

and

$$\beta(T) = \inf\{k : \beta(T(D)) \leqslant k\beta(D) \text{ for all bounded } D \subset E\} = \beta(T(B_E)),$$

where B_E denotes the unit ball in E. We recall some of the basic properties of $\alpha(T)$, respectively $\beta(T)$:

(1) $\frac{1}{2}\alpha(T) \leqslant \beta(T) \leqslant 2\alpha(T)$,

(2) $\alpha(T^*) \leqslant \beta(T)$ and $\alpha(T) \leqslant \beta(T^*)$ (see [5]),

(3) $\alpha(T(B_E)) = \alpha(T^*(B_{F^*}))$ (see [1]),

(4) $\max\{\alpha(T), \beta(T)\} \leqslant \|T\|_e$, where $\|T\|_e$ denotes the essential norm of T.

In this paper we are interested in $\alpha(T)$ and $\beta(T)$ for a special class of operators on Banach lattices. For general information on Banach lattices we refer to the monographs [4], [7] and [10]. For specific results on measures of non-compactness of operators on Banach lattices we refer to [6], [8] and [9]. From now on E and F will denote Banach lattices. A linear operator T from E into F is called *disjointness preserving* if $x \wedge y = 0$ implies $|Tx| \wedge |Ty| = 0$. It was shown in [6, Theorem 3.10], that if E^* is non-atomic and $T: E \to F$ is a norm bounded disjointness preserving operator, then $\beta(T) \geqslant (1/2)\|T\|$. It was indicated in [6] that no example was known for which $\beta(T) < \|T\|$. Moreover for special classes of spaces (e.g. $F = L_p$, $1 \leqslant p < \infty$) it was indicated in [6] that one always has $\beta(T) = \|T\|$. It will be shown in this paper that in fact under the above hypotheses one always has $\beta(T) = \|T\|$. Our approach follows [6], with one major difference: we employ the Kuratowski measure of non-compactness α, whereas [6] only used the Hausdorff measure of non-compactness β. It is this difference which allows us to obtain the improved result.

2. THE MAIN RESULT

We denote by E^* the dual space of E and by E_n^* the space of order continuous linear functionals on E. For $0 \leqslant \varphi \in E^*$ we denote by p_φ the seminorm $p_\varphi(f) = \varphi(|f|)$. The following lemma is an easy consequence of the result [3, Theorem 4] that a probability measure μ on a complete Boolean algebra has a continuous spectral resolution. For the benefit of the reader we provide a direct short proof.

LEMMA 2.1. *Let E be a Dedekind complete non-atomic Banach lattice and let $0 \leqslant u \in E$ and $0 \leqslant \varphi \in E_n^*$ with $\varphi(u) = 1$. Then for all $t \in [0, 1]$ there exists a band projection P_t such that $\varphi(P_t u) = t$ and such that $t \leqslant s$ implies $P_t \leqslant P_s$.*

Proof. Let $P_0 = 0$ and P_1 be the band projection on $\{u\}^{dd}$. By Zorn's lemma we can find a maximal chain $\{P_\tau\}$ of band projections such that $0 \leqslant P_\tau \leqslant P_1$. Then we note that for each $0 < t < 1$ there exists $\tau_0 \in \{\tau\}$ such that $\varphi(P_{\tau_0} u) = t$, since E is non-atomic and φ is order continuous. Define now $P_t = \sup\{P_\tau: \varphi(P_\tau u) = t\}$. The order continuity of φ implies now $\varphi(P_t u) = t$ and obviously $t \leqslant s$ implies $P_t \leqslant P_s$. ▨

In the following lemma we denote by $\mathbf{S}^n$ the n-sphere in $\mathbf{R}^{n+1}$, i.e. $\mathbf{S}^n = \{(x_1, \ldots, x_{n+1}) : (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \text{ with } x_1^2 + \ldots + x_{n+1}^2 = 1\}$.

LEMMA 2.2. *Let E, u and φ be as in Lemma 2.1. Then for all $n \in \mathbf{N}$ there exists a p_φ-continuous map $F_n: \mathbf{S}^n \to \{v \in E : |v| = u\}$ such that $F_n(-x_1, \ldots, -x_{n+1}) = -F_n(x_1, \ldots, x_{n+1})$ for all $(x_1, \ldots, x_{n+1})$ in $\mathbf{S}^n$.*

Proof. Let P_t be a collection of band projections as in Lemma 2.1. We shall construct F_n inductively. To define F_1 we will parametrize $\mathbf{S}^1$ as $\{e^{2\pi i t}: 0 \leqslant t \leqslant 1\}$.

Define then

$$F_1(e^{2\pi i t}) = \begin{cases} 2P_{2t}u - u & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ -2P_{2t-1}u + u & \text{for } \frac{1}{2} < t < 1. \end{cases}$$

Note that $|2P_{2t}u - u| = |P_{2t}u + P_{2t}u - P_1u| = |P_{2t}u + (P_1 - P_{2t})u| = u$, since $P_{2t} \perp P_1 - P_{2t}$ so that $|F_1(e^{2\pi i t})| = u$ for all t. Also observe that if $0 \leqslant t < 1/2$, then $F_1(-e^{2\pi i t}) = F_1(e^{2\pi i\left(t+\frac{1}{2}\right)}) = -2P_{2t}u + u = -F_1(e^{2\pi i t})$. To show that F_1 is p_φ-continuous, we only have to show that P_tu is a p_φ-continuous function of t which is obvious from the fact that $p_\varphi(P_tu - P_su) = |t - s|$ for all $t, s \in [0, 1]$. Hence F_1 satisfies all the requirements. Assume now that $F_{n-1}: S^{n-1} \to \{v \in E : |v| = u\}$ has been constructed. Then define F_n as follow:

$$F_n(x_1, \ldots, x_{n+1}) = \begin{cases} u & \text{if } x_{n+1} = 1 \\ (P_1 - P_{x_{n+1}})F_{n-1}\left(\frac{x_1}{(1 - x_{n+1}^2)^{\frac{1}{2}}}, \ldots, \frac{x_n}{(1 - x_{n+1}^2)^{\frac{1}{2}}}\right) + P_{x_{n+1}}u & \text{if } 0 \leqslant x_{n+1} < 1 \\ -F_n(-x_1, \ldots, -x_{n+1}) & \text{if } x_{n+1} < 0. \end{cases}$$

It is easy to see that for all $(x_1, \ldots, x_{n+1}) \in S^n$ we have $|F_n(x_1, \ldots, x_{n+1})| = u$ and $F_n(-x_1, \ldots, -x_{n+1}) = -F_n(x_1, \ldots, x_{n+1})$, since $F_n(x_1, \ldots, x_n, 0) = F_{n-1}(x_1, \ldots, x_n)$. To show that F_n is p_φ-continuous at all $(x_1, \ldots, x_{n+1}) \in S^n$ one has to consider three cases: $x_{n+1} = 0$, $0 < x_{n+1} < 1$ and $x_{n+1} = 1$. First we consider the case $x_{n+1} = 0$. Then $F_n(x_1, \ldots, x_{n+1}) = F_{n-1}(x_1, \ldots, x_n)$. The continuity of F_n at $(x_1, \ldots, x_{n+1})$ follows now from the continuity of F_{n-1} and the fact that $P_{x_{n+1,k}}u \downarrow 0$ as $k \to \infty$ for any sequence $x_{n+1,k} \downarrow^k 0$. In case $0 < x_{n+1} < 1$, we denote by

$$X = \left(\frac{x_1}{(1 - x_{n+1}^2)^{\frac{1}{2}}}, \ldots, \frac{x_n}{(1 - x_{n+1}^2)^{\frac{1}{2}}}\right)$$

the corresponding point in S^{n-1}. Let now $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ be points in S^n with $0 < x_{n+1}, y_{n+1} < 1$. Then we have

$$p_\varphi(F_n(x_1, \ldots, x_{n+1}) - F_n(y_1, \ldots, y_{n+1})) \leqslant$$

$$\leqslant \varphi(|(P_1 - P_{x_{n+1}})F_{n-1}(X) - (P_1 - P_{y_{n+1}})F_{n-1}(Y)|) + \varphi(P_{x_{n+1}}u - P_{y_{n+1}}u) \leqslant$$

$$\leqslant \varphi(|F_{n-1}(X) - F_{n-1}(Y)|) + 2\varphi(|P_{x_{n+1}}u - P_{y_{n+1}}u|),$$

which implies that F_n is p_φ-continuous at $(x_1, \ldots, x_{n+1})$. We leave it to the reader to verify that F_n is continuous at $(0, \ldots, 1)$. ▨

We now define a measure of non-compactness associated to p_φ. If $D \subset E$ is norm bounded, then define:

$$\alpha_\varphi(D) = \inf\left\{\lambda : D \subset \bigcup_{j=1}^{m} D_j,\, p_\varphi\text{-diam}(D_j) \leqslant \lambda\right\}.$$

It is easy to see that $\alpha_\varphi(D) \leqslant \|\varphi\|\alpha(D)$.

LEMMA 2.3. *Let E, u and φ be as above. Then* $\alpha_\varphi([-u, u]) = 2$.

Proof. Since the p_φ-diameter of $[-u, u]$ is 2, we have $\alpha_\varphi([-u, u]) \leqslant 2$. Assume now that $[-u, u] \subset \bigcup_{j=1}^{n} D_j$. Decompose E as $N_\varphi \oplus N_\varphi^{\mathrm{d}}$, where N_φ denotes $\{x \in E : \varphi(|x|) = 0\}$. We can then assume that the principal ideal E_u generated by u is contained in N_φ^{d} and then replace D_j by $D_j \cap E_u$. Then we denote by $\tilde{D}_j$ the p_φ-closure of D_j in the completion of (E_u, p_φ). Let F_{n-1} be the map constructed in the previous lemma. Then $\bigcup_{j=1}^{n} F_{n-1}^{-1}(\tilde{D}_j)$ is a covering of S^{n-1} with n closed sets. By the Lusternik-Schnirelman-Borsuk theorem ([2]) there exists an index j_0 and $(x_1, \ldots, x_n) \in S^{n-1}$ so that $\pm (x_1, \ldots, x_n) \in F_{n-1}^{-1}(\tilde{D}_{j_0})$, i.e. $\pm F_{n-1}(x_1, \ldots, x_n) \in \tilde{D}_{j_0}$. Hence

$$p_\varphi\text{-diam}(D_{j_0}) = p_\varphi\text{-diam}(\tilde{D}_{j_0}) \geqslant 2p_\varphi(F_{n-1}(x_1, \ldots, x_n)) = 2,$$

and the proof of the lemma is complete. ▨

REMARK. The above lemma says essentially that $\alpha([-\chi_X, \chi_X]) = 2$ in the space $L_1(X, \mu)$, where μ is a non-atomic probability measure. The next proposition show how to compute $\alpha([-u, u])$ in a large class of Banach lattices, in particular the following proposition holds for $E = L_p(X, \mu)$, where $1 \leqslant p \leqslant \infty$.

PROPOSITION 2.4. *Let E be a Dedekind complete non-atomic Banach lattice and assume $\|u\| = \sup\{\langle\varphi, u\rangle : 0 \leqslant \varphi \in E_n^*,\ \|\varphi\| = 1\}$ for all $0 \leqslant u \in E$. Then* $\alpha([-u, u]) = 2\|u\|$.

Proof. Let $\varepsilon > 0$ and $0 \leqslant u \in E$ with $u \neq 0$. Then by assumption there exists $0 \leqslant \varphi \in E_n^*$, $\|\varphi\| = 1$ with $\varphi(u) > (1 - \varepsilon)\|u\|$. It follows now from the above lemma, using a scaling of φ, that $\alpha_\varphi([-u, u]) = 2\varphi(u)$. Hence $\alpha([-u, u]) \geqslant \alpha_\varphi([-u, u]) > 2(1 - \varepsilon)\|u\|$ for all $\varepsilon > 0$. Hence $\alpha([-u, u]) = 2\|u\|$. ▨

Recall now that a positive linear operator T from a Banach lattice E into a Banach lattice F is called a *Maharam operator* (or *interval preserving*) if $T[0, u] = [0, Tu]$ for all $0 \leqslant u \in E$.

PROPOSITION 2.5. *Let E and F be Banach lattices with F Dedekind complete, non-atomic and such that* $\|f\| = \sup\{\langle |f|, \varphi\rangle : 0 \leqslant \varphi \in F_n^*, \|\varphi\| \leqslant 1\}$ *for all* $f \in F$. *If* $0 \leqslant T : E \to F$ *is a Maharam operator, then* $\alpha(T(B_E)) = 2\|T\|$.

Proof. Let $\varepsilon > 0$. Then there exists $0 \leqslant u \in E$ such that $\|u\| = 1$ and $\|Tu\| \geqslant \|T\| - \varepsilon$. Then $[-Tu, Tu] = T[-u, u] \subseteq T(B_E)$ implies that $\alpha(T(B_E)) \geqslant \alpha([-Tu, Tu]) = 2\|Tu\| \geqslant 2(\|T\| - \varepsilon)$, and hence $\alpha(T(B_E)) = 2\|T\|$. ▨

We now derive, along the same lines as in [6], the main result of this paper.

THEOREM 2.6. *Let E and F be Banach lattices such that E^* is non-atomic. If $T : E \to F$ is a norm bounded disjointness preserving operator, then* $\alpha(T) = \beta(T) = \|T\|_e = \|T\|$.

Proof. As noted in [6], $|T^*|$ is an order continuous Maharam operator and there exists $\pi \in Z(F^*)$, the center of F^*, such that $T^* = |T^*| \circ \pi$ and $|\pi| = I$. Now E^* satisfies the hypotheses of the previous proposition, so $\alpha(|T^*|(B_{F^*})) = 2\| \, |T^*| \, \|$. Since π is an isometry, we conclude that $\alpha(T^*(B_{F^*})) = 2\|T^*\|$. From [1] we know that $\alpha(T^*(B_{F^*})) = \alpha(T(B_E))$, so that we conclude that $\alpha(T(B_E)) = 2\|T\|$. Now the inequalities $\alpha(T(B_E)) \leqslant 2\alpha(T)$ and $\alpha(T(B_E)) \leqslant 2\beta(T(B_E)) = 2\beta(T)$ imply that $\beta(T) = \alpha(T) = \|T\|$. The theorem follows now, since we always have $\beta(T) \leqslant \|T\|_e \leqslant \|T\|$. ▨

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