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KASPAROV PRODUCTS, KK-EQUIVALENCE, AND PROPER ACTIONS OF CONNECTED REDUCTIVE LIE GROUPS

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INTRODUCTION

The K-theory of C^*-algebras has played an increasingly important role in index theory and its applications. For instance certain important statements, whose original formulations make no mention of C^*-algebras, are known to be implied by statements about the injectivity of maps between the K-theory groups of certain C^*-algebras. KK-theory and Kasparov products have played a significant role in the analysis of the latter statements. Suppose a map $T\colon \mathrm{K}_*(A) \to \mathrm{K}_*(B)$ between the K-groups of C^*-algebras A and B is realized by taking a Kasparov product (on the right) with an element t of $\mathrm{KK}(A, B)$, and suppose one expects that T has a (one-sided) inverse S realized by taking a Kasparov product with an element s of $\mathrm{KK}(B, A)$. Because $S \circ T$ is realized by taking a Kasparov product with the element $t \otimes_B s$ of $\mathrm{KK}(A, A)$, to show that $S \circ T$ is the identity on $\mathrm{K}_*(A)$, it suffices to establish certain properties for $t \otimes_B s$. Often $t \otimes_B s$ is fairly tractable. The method described above has contributed already to significant new theorems, and it plays a central role in a far-reaching program initiated by Baum, Connes, and Kasparov.

The main result of our paper, Theorem 4.1, is the calculation of a Kasparov product that plays a key role in the program described in the first paragraph. Let G be a noncompact connected reductive Lie group, and let K be the subgroup associated to the fixed point set of a Cartan involution. We show that a well-known differential operator on G/K represents the Kasparov product of the KK-element realizing de Rham induction and its "Miscenko dual". These KK-elements are defined at the beginning of Section 4, the rest of which is devoted to a proof of Theorem 4.1. This proof relies heavily on a technique we describe briefly in Section 2 and on facts about analysis on G/K that we collect in Section 3. The framework in which we do our proof is summarized by the observation, made in Section 1, that the connection approach to Kasparov products developed by Connes and Skandalis [10], [38] extends to the equivariant KK_G-theory, where the group G is not necessarily compact. (This observation has now also appeared in [20].) Although this observation and its proof are straightforward generalizations of [38], the observation is a useful one. Taken as a whole the methods in our paper form the foundation for the calculation in (equivariant) KK-theory of Kasparov products involving the most commonly occurring pseudodifferential operators on (noncompact) manifolds.

To enhance the reader's appreciation of Theorem 4.1, we include a discussion placing our work in context (Section 5) and a sketch of an application of our result to a problem in transformation group C^*-algebras (Section 6). This problem is as follows. Let G be a noncompact connected reductive Lie group acting properly on a second countable locally compact Hausdorff space X. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{p}$ be the -1 eigenspace of a Cartan involution of $\mathfrak{g}$ and let K' be the maximal compact subgroup of G associated with the Cartan involution. Assume that the adjoint action of K' on $\mathfrak{p}$ factors through $\mathrm{spin}^c(\mathfrak{p})$. The problem is to show that there is a canonical isomorphism of K-theory groups $K_{i+\dim(G/K')}(C^*(K', C_0(X))) \cong K_i(C^*(G, C_0(X)))$. Such an isomorphism would reduce questions about the K-theory of the C^*-algebra arising from the transformation group G to questions about the better understood compact transformation group K'. The program of Baum, Connes, and Kasparov involves the reduction of questions about transformation group C^*-algebras to questions about transformation group C^*-algebras arising from proper actions. A canonical realization of the isomorphism mentioned above supports the assertion that proper actions can be used to understand more general actions.

In Section 6 we show that de Rham induction realizes a slightly more general canonical isomorphism of K-theory groups, at least when the action of G on X has finitely many orbit types. In the situation of the preceding paragraph, drop the assumption that the action of K' on $\mathfrak{p}$ factors through $\mathrm{spin}^c(\mathfrak{p})$. Let K be the subgroup of G associated with the fixed point set of the Cartan involution. Let C_τ be the algebra of continuous sections vanishing at infinity of the Clifford algebra bundle associated with the complexified cotangent bundle of G/K. G acts on C_τ in a natural manner. We show that de Rham induction (Kasparov product with an element represented by a de Rham operator) gives an isomorphism $K_*(C^*(G, C_0(X) \otimes C_\tau)) \to K_*(C^*(G, C_0(X)))$. The relationship of this result to the isomorphism mentioned in the preceding paragraph is described in Section 5, where we also outline an argument pointed out to us by Chris Phillips that uses results of [1], [2], and [28] to establish the existence of a (noncanonical) isomorphism $K_{i+\dim(G/K')}(C^*(K', C_0(X))) \cong K_i(C^*(G, C_0(X)))$ when the adjoint action is spin^c. Our proof has the advantage of allowing general adjoint actions (note also that Kasparov [20] shows that the semisimple case is fundamental in his program) and of providing a canonical isomorphism. Moreover it is an additional consequence of our techniques and of results of [11] and [36] that de Rham induction realizes a KK-equivalence between $C^*(G, C_0(X) \otimes C_\tau)$ and $C^*(G, C_0(X))$.

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1. IMPLICIT CHARACTERIZATION OF KASPAROV PRODUCTS

In [10] and [38] an implicit characterization of Kasparov products in KK--theory is given using the concept of connection. This approach allows one to avoid explicit stabilization in the calculation of specific examples. We extend this characterization of Kasparov products to equivariant KK-theory, KK_G, where G is a separable locally compact group. When this paper was first submitted for publication, it contained a complete account of the connection approach to products in KK_G. With the appearance of [20], such an account is no longer necessary, and we restrict ourselves to an outline of the ideas involved.

Our C^*-algebras are assumed to have gradings, continuous G actions, and countable approximate units. For C^*-algebras A and B, $KK_G(A, B)$ is defined in [20]. Cycles for $KK_G(A, B)$ are called Kasparov (A, B)-bimodules, the set of which is denoted $\mathscr{E}_G(A, B)$. One usually denotes an element of $\mathscr{E}_G(A, B)$ by (E, T) where E is the Hilbert C^*-module over B and $T \in \mathscr{L}(E)$. Elements of $\mathscr{E}_G(A, B)$ have additional structure, including a grading, which is mentioned once and then omitted from the notation. Tensor products, denoted $\otimes$, and commutators, denoted [,], are graded unless otherwise indicated. It is convenient to use the terminology Hilbert B-module for a countably generated graded Hilbert C^*-module over B and (A, B)-bimodule for a Hilbert B-module E with a homomorphism $A \to \mathscr{L}(E)$. Such a module or bimodule is said to have a continuous G-action if the module and algebra(s) have compatible continuous G-actions. [20], [21] and [22] are good sources of definitions and information.

We now define the fundamental object in the implicit characterization of products.

DEFINITION 1.1. ([10], [38]). Let D and B be C^*-algebras. Let E_1 be a Hilbert D-module and E_2 a (D, B)-bimodule with $F_2 \in \mathscr{L}(E_2)$. Set $E = E_1 \otimes_D E_2$. For $e \in E_1$, define $T_e \in \mathscr{L}(E_2, E)$ by $T_e(f) = e \otimes f$. Let $\tilde{T}_e \in \mathscr{L}(E_2 \oplus E)$ be the operator given by

$$\begin{pmatrix} 0 & T_e^* \\ T_e & 0 \end{pmatrix}.$$

An element $F \in \mathscr{L}(E)$ is said to be an *F_2-connection for E_1* if and only if for all $e \in E_1$, $[\tilde{T}_e, F_2 \oplus F] \in \mathscr{K}(E_2 \oplus E)$.

The stabilization theorem [20] always permits the construction of a G-continuous Grassmann connection, for which Proposition 9 of [38] holds.

DEFINITION 1.2. Let A, B, and D be C^*-algebras with $(E_1, F_1) \in \mathscr{E}_G(A, D)$ and $(E_2, F_2) \in \mathscr{E}_G(D, B)$. Denote by E the (A, B)-bimodule $E_1 \otimes_D E_2$. For $F \in \mathscr{L}(E)$ the pair (E, F) is called a *Kasparov product* of (E_1, F_1) by (E_2, F_2) if and only if:

a) $(E, F) \in \mathscr{E}_G(A, B)$;

b) F is an F_2-connection for E_1;

c) $\forall\, a \in A$, $a[F_1 \otimes 1, F]a^* \geqslant 0$ modulo $\mathscr{K}(E)$.

We generalize Theorem 12 of [38].

THEOREM 1.3. *Let A, B, and D be C^*-algebras with A separable and with B and D having strictly positive elements. Let $(E_1, F_1) \in \mathscr{E}_G(A, D)$ and $(E_2, F_2) \in \mathscr{E}_G(D, B)$.*

a) *There exists a Kasparov product (E, F) of (E_1, F_1) by (E_2, F_2) unique up to an operator homotopy.*

b) *This product determines a well-defined map $\mathrm{KK}_G(A, D) \otimes \mathrm{KK}_G(D, B) \to \mathrm{KK}_G(A, B)$ denoted by $\otimes_D$.*

Proof. [20] provides the "M, N lemma" in this setting. The arguments of [38] are C^*-algebraic arguments. Because we impose only the additional condition of G-continuity, which is a norm-closed condition, the arguments extend to our case.

REMARK 1.4. The general Kasparov product is defined as in Definition 15 of [38].

2. FUNCTIONAL CALCULUS

We discuss, in the generality needed for Section 4, a technique for analyzing commutators involving the —1/2 power of certain differential operators. This technique, which has fairly widespread application, was used for constructing KK-elements in [3]. The observations behind the technique are that the Riemann integral $(1/\pi)\int_0^\infty \lambda^{-1/2}(x + \lambda)^{-1}\mathrm{d}\lambda$ equals $x^{-1/2}$ and that convergence is uniform in $x \geqslant 1$. By uniform convergence we mean that for any $\delta > 0$ there exist ε, N, and m such that for any $x \geqslant 1$ any Riemann sum R of mesh length $\leqslant m$ for $(1/\pi)\int_\varepsilon^N \lambda^{-1/2}(x + \lambda)^{-1}\mathrm{d}\lambda$ satisfies $|R - x^{-1/2}| < \delta$.

Let L^2 be the space of L^2 differential forms on a manifold M. Let D be a self-adjoint operator on L^2 that arises as the closure of an essentially self-adjoint differential operator. Assume that $(1 + D^2)^{-1}$, and thus $(1 + D^2)^{-k}$ for any $k > 0$, is compact. Let S be an element of $\mathscr{L}(L^2)$ that takes smooth compactly supported forms to smooth compactly supported forms and whose restrictions to domain(D) and domain($1 + D^2$) are bounded operators domain(D) $\to$ domain(D) and domain($1 + D^2$) $\to$ domain($1 + D^2$) respectively.

$$(2.1) \qquad (1 + D^2)^{-1/2} = (1/\pi)\int_0^\infty \lambda^{-1/2}(1 + D^2 + \lambda)^{-1}d\lambda$$

in the sense of norm limit of functional calculus expressions arising from Riemann sums for approximating proper integrals for $(1/\pi)\int_0^\infty \lambda^{-1/2}(x + \lambda)^{-1}d\lambda$. Because composing with a bounded operator commutes with taking norm limits,

$$(2.2) \qquad [(1 + D^2)^{-1/2}, S] = (1/\pi)\int_0^\infty \lambda^{-1/2}[(1 + D^2 + \lambda)^{-1}, S]d\lambda.$$

Because $[S, 1 + D^2 + \lambda] = [S, D^2]$,

$$(2.3) \quad [(1 + D^2)^{-1/2}, S] = (1/\pi)\int_0^\infty \lambda^{-1/2}(1 + D^2 + \lambda)^{-1}[S, D^2](1 + D^2 + \lambda)^{-1}d\lambda.$$

Under the further assumptions that $[D, S]$ extends to a bounded operator on L^2 and that $[D^2, S]$ extends to a bounded operator from domain(D) to L^2, one can show that $[D(1 + D^2)^{-1/2}, S]$ is a compact operator on L^2.

$$(2.4) \qquad [D(1 + D^2)^{-1/2}, S] = D[(1 + D^2)^{-1/2}, S] + [D, S](1 + D^2)^{-1/2}.$$

If follows from our assumptions that $[D, S](1 + D^2)^{-1/2}$ is compact. One can check directly that the right-hand side of

$$D[(1 + D^2)^{-1/2}, S] = (1/\pi)\int_0^\infty \lambda^{-1/2}D(1 + D^2 + \lambda)^{-1}[S, D^2](1 + D^2 + \lambda)^{-1}d\lambda$$

(2.5)

is the norm limit of compact operators. Because D is closed, (2.5) holds and establishes that its left-hand side is compact.

REMARK 2.6. When one understands $[D, S]$ better than one understands $[D^2, S]$ it can be can be convenient to replace $[D^2, S]$ by $D[D, S]+(-1)^{\partial D\partial S}[D, S]D$.

The following lemma can help establish the conditions needed to apply this technique.

LEMMA 2.7. *Let D be as above. Let S', defined as a map from smooth compactly supported forms to smooth compactly supported forms, extend to an element, called S, of $\mathscr{L}(L^2)$.*

a) *If $[D, S']$ extends to a bounded operator from* domain(D) *to L^2, the restriction of S is a bounded operator from* domain(D) *to* domain(D).

b) *If $[D^2, S']$ extends to a bounded operator from* domain$(1+D^2)$ *to L^2, the restriction of S is a bounded operator from* domain$(1 + D^2)$ *to* domain$(1 + D^2)$.

Proof. a) Use $DS' = [D, S'] + (-1)^{\partial D\partial S'}S'D$ to show that there exists a constant c such that for every w in domain(D), $\|DSw\|^2_{L^2} \leqslant c(\|w\|^2_{L^2} + \|Dw\|^2_{L^2})$. The proof of b) is analogous using $D^2S' = [D^2, S'] + S'D^2$.

3. NONCOMPACT CONNECTED REDUCTIVE LIE GROUPS

We define a class of groups for which the results of this paper hold, and we discuss some properties of these groups and of certain of their homogeneous spaces.

DEFINITION 3.1. A noncompact, connected Lie group G, with involutive automorphism θ of its Lie algebra $\mathfrak{g}$, is called *reductive* if and only if:

1) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$ is reductive. (For Lie algebra elements $[\cdot, \cdot]$ denotes the usual bracket. $\mathfrak{z}$ is the center of $\mathfrak{g}$.)

2) θ extends a Cartan involution of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces of θ.

3) The connected subgroup K associated to $\mathfrak{k}$ is closed and is a covering group of a compact Lie group.

4) $(X, k) \to (\exp X)\cdot k$ is a diffeomorphism $\mathfrak{p}\times K \to G$.

LEMMA 3.2. *Such a (G, K) is a Riemannian symmetric pair.*

Proof By [17, p. 213], $(\tilde{G}, \tilde{K})$ is a Riemannian symmetric pair. ($\tilde{G}$ is the universal covering group of G.) Because the kernel of the covering map is central [37] it is in $\tilde{K}$ [17, p. 252 ff.], and the involution on $\tilde{G}$ descends to G.

All such groups are unimodular. Examples of such groups include noncompact, connected semisimple Lie groups, whether or not they are linear, and the identity component of $GL(n, \mathbf{R})$. For the groups of Definition 3.1, G/K is diffeomorphic to Euclidean space. Moreover, the following is true.

LEMMA 3.3. *Let H be a subgroup of K. Denote by $N_G(H)$ and $N_K(H)$ the normalizers of H in G and K respectively. Let $c_H(\mathfrak{p})$ denote the vector space $\{X \in \mathfrak{p} : \forall h \in H, \mathrm{Ad}(h)X = X\}$. Then $N_G(H)/N_K(H)$ is diffeomorphic to $c_H(\mathfrak{p})$.*

Proof. That $N_G(H) = \exp(c_H(\mathfrak{p})) \cdot N_K(H)$ follows from the uniqueness inherent in the $G = (\exp \mathfrak{p}) \cdot K$ decomposition.

Henceforth G and K denote groups described in Definition 3.1.

REMARK 3.4. A decomposition like that of [41, p. 44] and results of [17, pp. 129, 252 ff.] show that our G/K can be isometrically identified with the product of a G/K arising from a linear semisimple group and a flat vector space. Whenever we use without comment results on semisimple G, it is implicit that this remark shows how to extend them to the reductive case. Our G/K has sectional curvatures everywhere nonpositive, [17, p. 241]. It is convenient to say G/K has negative curvature.

DEFINITION OF THE OPERATOR D. Let $C_c^\infty(\Lambda^* T^*(G/K))$ denote the smooth compactly supported differential forms with $\mathbf{C}$ coefficients on G/K. Let r denote the function on G/K whose value at a point is the distance of that point from the point representing the identity coset. Let d^* be the formal adjoint of exterior differentiation d. Let $\times$ denote the standard Clifford action, $\mathrm{d}f \times = (\mathrm{d}f \wedge) + (\mathrm{d}f \wedge)^*$. We have an operator

$$\mathrm{d} + \mathrm{d}^* + \mathrm{d}(r^2/2)\times : C_c^\infty(\Lambda^* T^*(G/K)) \to C_c^\infty(\Lambda^* T^*(G/K)).$$

By Stokes' theorem and [27, vol. I, p. 253] this operator has a well-defined closure, which we call D, on $L^2(\Lambda^* T^*(G/K))$.

PROPERTIES OF D. D is self-adjoint, [8].

$(1 + D^2)^{-1/2} : L^2(\Lambda^* T^*(G/K)) \to \mathrm{domain}(D)$, where domain$(D)$ has the graph norm. $D \circ (1 + D^2)^{-1/2}$ is a bounded self-adjoint operator on $L^2(\Lambda^* T^*(G/K))$.

The techniques of [19] and [24] extend to give the following results, analogous to those of [24].

REMARK 3.5. $\mathrm{Domain}(D) = \{w \in L^2(\Lambda^* T^*(G/K)) : \mathrm{d}w, \mathrm{d}^* w, rw \in L^2(\Lambda^* T^*(G/K))\}$.

REMARK 3.6. Domain(D) imbeds compactly in $L^2(\Lambda^* T^*(G/K))$.

REMARK 3.7. The restriction of D to L^2 forms of even degree is a surjective map from forms of even degree to forms of odd degree with a one-dimensional kernel generated by the function $e^{-r^2/2}$.

In order to identify a Kasparov product in Section 4, we must show that various expressions involving commutators and powers of D are compact. (See, e.g., the considerations in Section 2.) To analyze these expressions, we must do explicit calculations on G/K, which we often accomplish by lifting the calculations to G. In what follows we outline the methods we use but leave the calculations themselves to the interested reader. Some of the calculations appear in detail in [24].

ANALYSIS ON G. Viewing $\mathfrak{g}$ as the tangent space to G at the identity, let $X_1, \ldots, X_n$ be an orthonormal basis for $\mathfrak{p}$ and $X_{n+1}, \ldots, X_m$ be an orthonormal basis for $\mathfrak{k}$. Denote by $X_1, \ldots, X_m$ the corresponding left-invariant vector fields on G and by $w_1, \ldots, w_m$ the dual Maurer-Cartan forms. For $f \in C^\infty(G), (Xf)(g) = \{(d/dt)(f(g \cdot \exp(tX)))\}_{t=0}$, [17, p. 104]. The exterior derivative of f, df, equals $\sum_{i=1}^{m} (X_i f) w_i$. The exterior derivative of an arbitrary differential form on G is calculated by using the above equation, the equation $dw_i = -(1/2) \sum_{j,k=1}^{m} c^i_{jk} w_j \wedge w_k$, where c^i_{jk} is defined by $[X_j, X_k] = \sum_{i=1}^{m} c^i_{jk} X_i$, [17, p. 137], and the rules for exterior differentiation.

ANALYSIS ON G/K. The above techniques are used to study analysis on G/K by applying them to functions and forms that are the pullbacks of such from G/K under the natural map $\pi: G \to G/K$. For $g \in G$ the map π intertwines left translation by g on G with left translation by g on G/K. The metric on G/K is defined by pulling back under left translation to the identity coset, pulling back under π to $\mathfrak{p}$, and using the natural metric on $\mathfrak{p}$, [17, pp. 209—210].

General considerations imply that for α and β differential forms on G/K, $\pi^*(d\alpha) = d(\pi^*\alpha)$ and $\pi^*(\alpha \wedge \beta) = (\pi^*\alpha) \wedge (\pi^*\beta)$. It is a consequence of the observations in the preceding paragraph that $\pi^*\alpha$ lies in the span over functions of $\{w_1, \ldots, w_n\}$, that $\pi^*(d^*\alpha) = d^*(\pi^*\alpha)$, and that $\pi^*(\alpha \lrcorner \beta) = (\pi^*\alpha) \lrcorner (\pi^*\beta)$. Here $\lrcorner$ denotes the operation adjoint to $\wedge$. Because θ, the $+1$ eigenspace of which is $\mathfrak{k}$ and the -1 eigenspace of which is $\mathfrak{p}$, is a Lie algebra automorphism, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, and $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$. These observations determine some of the c^i_{jk} and thus allow us to discard many terms that occur in calculations with pullbacks of forms from G/K.

For β a k-form on G/K, $\pi^*\beta$ can be written $\sum_I f_I w_I$ where each $I = i_1, i_2, \ldots, \ldots, i_k$ is a multi-index with $i_j \in \{1, \ldots, n\}$, $w_I = w_{i_1} \wedge \ldots \wedge w_{i_k}$, and f_I is a function on G. The observations of the preceding paragraphs allow us to write

$$\pi^*(d\beta) = \sum_I \sum_{j=1}^{n} X_j(f_I) w_j \wedge w_I \tag{3.8}$$

and

$$\pi^*(d^*\beta) = -\sum_I \sum_{j=1}^{n} X_j(f_I) w_j \lrcorner w_I . \tag{3.9}$$

THE FUNCTIONS x_i. The 1-form $d(r^2/2) = rdr$ has special significance for us. We define functions on G, x_i, $1 \leqslant i \leqslant n$, by

$$\pi^*(rdr) = \sum_{i=1}^{n} x_i w_i. \tag{3.10}$$

$$\pi^*(r^2) = \pi^*(rdr \times rdr) = \sum_{i=1}^{n} x_i^2. \tag{3.11}$$

PROPERTIES OF THE FUNCTIONS x_i. For certain calculations (see, e.g., the proof of Lemma 3.22) we need to understand the growth of derivatives of the x_i. For $X \in \mathfrak{p}$ it is possible to evaluate $X(x_i)$ on exp (any line through the origin in $\mathfrak{p}$) if one defines x_i with respect to a basis of $\mathfrak{p}$ determined by the line. For nonzero $P \in \mathfrak{p}$ choose an orthonormal basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{p}$ by setting $X_1 = P/\|P\|$ and letting $X_2, \ldots, X_n$ be eigenvectors of the self-adjoint transformation $(\operatorname{ad} P)^2 \colon \mathfrak{p} \to \mathfrak{p}$. Define $\lambda_i(tP)$ by $(\operatorname{ad}(tP))^2 X_i = \lambda_i(tP) X_i$ for $t \in \mathbf{R}$. Define x_i with respect to this basis as described previously. $\lambda_i \geqslant 0$ and

$$(X_j x_i)(tP) = \delta_{ji} (\lambda_j(tP))^{1/2} / \tanh((\lambda_j(tP))^{1/2}). \tag{3.12}$$

Interpret this expression to be δ_{ji} when $\lambda_j = 0$. The function $X_j x_i$ is left K-invariant. As $t \to \infty$ the expression in (3.12) grows like t. Justification for the claims in this paragraph appears in [24, p. 105—106].

SAMPLE CALCULATIONS. To establish properties of commutators of D or D^2 with the action of an element of C_τ on $L^2(\Lambda^* T^*(G/K))$, it is usually convenient to establish the properties by direct calculations for functions and sections of $T^*(G/K)$. The properties for general elements of C_τ often follow from a formula such as

$$\begin{aligned}[D, \sum_i f_i v_{i_1} \times \ldots \times v_{i_k} \times] &= \sum_i [D, f_i] \circ v_{i_1} \times \ldots \times v_{i_k} \times + \\ &+ \sum_i \sum_{j=1}^{k} (-1)^{j-1} f_i v_{i_1} \times \ldots \times v_{i_{j-1}} \times \circ [D, v_{i_j} \times] \circ v_{i_{j+1}} \times \ldots \times v_{i_k} \times .\end{aligned} \tag{3.13}$$

We now give examples of the results of calculations done with the methods we have described. In our notation $I_{\hat{k}l}$ refers to the multi-index I with k removed and l put in its place. The Laplace operator $dd^* + d^*d$ is denoted Δ.

$$\begin{aligned}&[D, \sum_i f_i w_i \times](\sum_I f_I w_I) = \\ &= \sum_{i,j,I} X_j(f_i) f_I w_j \wedge (w_i \wedge w_I) - \sum_{i,j,I} X_j(f_i) f_I w_j \lrcorner (w_i \lrcorner w_I) + \\ &+ \sum_{I; i \in I; j \notin I} X_j(f_i) f_I w_{I_{\hat{i}j}} + \sum_{I; j \in I; i \notin I} X_j(f_i) f_I w_{I_{\hat{j}i}} + \\ &+ \sum_{I; i \in I} X_i(f_i) f_I w_I - \sum_{I; i \notin I} X_i(f_i) f_I w_I + 2 \sum_{i,I} x_i f_i f_I w_I .\end{aligned} \tag{3.14}$$

$$
(3.15)\quad \begin{aligned}[D^2.f]\sum_I f_I w_I &= (\Delta f)\sum_I f_I w_I + \\ &+ \sum_{I;\, j\in I;\, k\notin I} [X_j, X_k](f)\cdot f_I w_{I_{\hat{jk}}} - 2\sum_{I,j} X_j(f)X_j(f_I)w_I.\end{aligned}
$$

THE LAPLACE OPERATOR APPLIED TO ROTATIONALLY INVARIANT FUNCTIONS AND FORMS. To use calculations like (3.15) we will need to analyze the growth of Δf and $\Delta(h\mathrm{d}r)$ for certain functions f and forms $h\mathrm{d}r$ that are smooth and rotationally invariant on G/K. There is a set of "polar coordinates" in which we can express the effect of the Hodge $*$-operator everywhere it occurs in the calculation of such $\Delta(f(r))$ and $\Delta(h(r)\mathrm{d}r)$. Let $\mathrm{Exp} = \pi\circ\exp$. For $r > 0$ and $X/\|X\|$ in the unit sphere of $\mathfrak{p}$, map $(r, X/\|X\|)$ to $\mathrm{Exp}(rX/\|X\|)$. If $\{\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_{n-1}\}$ is an orthonormal basis, expressed in local coordinates, for the cotangent space to the sphere at a point on the sphere, $\{\mathrm{d}r, \mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_{n-1}\}$ is identified with a basis $(\mathrm{d}r \perp \mathrm{span}\{\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_{n-1}\})$ of the fiber of $T^*(G/K)$ at a corresponding point on G/K. The form $\mathrm{d}r$ has pointwise norm one. In these coordinates

$$
\begin{aligned}\mathrm{dvol}_{G/K}(\mathrm{Exp}\,X) &= \det(\sinh(\mathrm{ad}\,X)/\mathrm{ad}\,X)_{\mathfrak{p}}\mathrm{d}r \wedge \mathrm{dvol}_{\mathrm{sphere}} = \\ &= (\det(\sinh(\mathrm{ad}\,X)/\mathrm{ad}\,X)_{\mathfrak{p}})r^{n-1}\mathrm{d}r \wedge \mathrm{d}\theta_1 \wedge \ldots \wedge \mathrm{d}\theta_{n-1},\end{aligned}
$$

[16, p. 30].

Let $A = \ln((\det(\sinh(\mathrm{ad}\,X)/\mathrm{ad}\,X)_{\mathfrak{p}})\cdot r^{n-1})$.

$$
(3.16)\quad (\Delta f)(\mathrm{Exp}\,X) = -(f''(\|X\|) + f'(\|X\|)\cdot(\partial A/\partial r)).
$$

$$
(3.17)\quad \begin{aligned}(\Delta(h\mathrm{d}r))(\mathrm{Exp}\,X) &= -h(\|X\|)\sum_{j=1}^{n-1}(\partial/\partial\theta_j)(\partial A/\partial r)\mathrm{d}\theta_j - \\ &- (h''(\|X\|) + h'(\|X\|)\cdot(\partial A/\partial r) + h(\|X\|)\cdot(\partial^2 A/\partial r^2))\mathrm{d}r.\end{aligned}
$$

Because the f and $h\mathrm{d}r$ that concern us in Section 4 are smooth, we are interested in the behavior of Δf and $\Delta(h\mathrm{d}r)$ only as $r\to\infty$. Two lemmas give us the information we need to use (3.16) and (3.17).

LEMMA 3.18. *As $r\to\infty$, $\partial A/\partial r$ and $\partial^2 A/\partial r^2$ are bounded.*

Proof. Because we are working with determinants, it is tempting to diagonalize the operators that appear. We can make enough sense of this to get the estimates we need. For $X\in\mathfrak{p}$ let $\{\lambda_i(X)\}$ denote the set of eigenvalues of the self-adjoint transformation $(\mathrm{ad}\,X)^2|\mathfrak{p}$. Each $\lambda_i \geqslant 0$, [24, p. 106]. Let $e_i(X) = (\lambda_i(X))^{1/2}$. (Because all expressions involving e_i involve only even powers of e_i, there is no problem of choice here.) Because different $(\mathrm{ad}\,X)$'s are not in general simultaneously diagonalizable and because multiple eigenvalues can arise, it is not possible to consider each λ_i or each e_i as a smooth function on $\mathfrak{p}$. However, it is possible to make consistent choices on each line through the origin, and for $X \neq 0$, $e_i(X) = \|X\| e_i(X/\|X\|)$. Thus on each one-dimensional subspace we can write A as a function of r and $e_i(X/\|X\|)$, and we can use this expression to calculate $\partial A/\partial r$ and $\partial^2 A/\partial r^2$. From this calculation we can observe directly that on each one-dimensional subspace $\partial A/\partial r$ and $\partial^2 A/\partial r^2$ are bounded as $r \to \infty$. Because the collection of eigenvalues of $(\operatorname{ad} X)^2|\mathfrak{p}$ depends continuously on X, $\{e_i(X) : \|X\| = 1\}$ is bounded, and inspection of the result of the above-mentioned calculation shows that the bounds of the preceding sentence can be chosen independent of $X/\|X\|$.

LEMMA 3.19. *As* $r \to \infty$, $(\partial/\partial\theta_j)(\partial A/\partial r)$ *is bounded.*

Proof. We use another set of "polar coordinates." Any $X \in \mathfrak{p}$ lies in some maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, and $\operatorname{Exp} \mathfrak{a}$ is a flat submanifold of G/K, [17, p. 215 and arguments of p. 247]. Thus any point of G/K lies in some flat submanifold. Moreover, one can choose the $\partial/\partial\theta_j$ at a point so that some lie in directions tangent to this flat submanifold and others, at least if the point is taken from an open dense subset of the flat submanifold, lie in directions tangent to the orbit of the point under left translation by K, [17, p. 401 ff.], [18, p. 267].

When $\partial/\partial\theta_j$ corresponds to a direction orthogonal to $\operatorname{Exp} \mathfrak{a}$, $(\partial/\partial\theta_j)(\partial A/\partial r)$ equals zero. This follows by continuity once we establish it for directions tangent to left translation by K. Observe that because the left K action fixes the identity coset and acts as isometries on G/K, it commutes with $\partial/\partial r$. To finish the argument of this paragraph, it suffices to show that A is invariant under the left K action. For $k \in K$ and $X \in \mathfrak{p}$, $k \cdot \operatorname{Exp}(X) = \operatorname{Exp}((\operatorname{Ad} k)X)$. Because $(\operatorname{Ad} k)$ is an automorphism of $\mathfrak{g}$, $\operatorname{ad}((\operatorname{Ad} k)X) = (\operatorname{Ad} k) \circ (\operatorname{ad} X) \circ (\operatorname{Ad} k^{-1})$. Properties of the determinant finish the argument.

We now have only to consider $\partial/\partial\theta_j$ representing directions in $\operatorname{Exp} \mathfrak{a}$. Because $\mathfrak{a}$ is abelian, the $(\operatorname{ad} X)$'s we must consider in calculating derivatives in directions in $\operatorname{Exp} \mathfrak{a}$ can be diagonalized simultaneously. Thus the eigenvalue functions $\lambda_i(X)$ are smooth functions on $\mathfrak{a}$. As before we can calculate explicitly, and it suffices to get a bound for one direction at a time. The calculation reveals that a bound as $r \to \infty$ depends on a bound for $r = 1$, which is obtained by observing that the coefficients we are studying arise also in $\Delta(dr)$, which is smooth on $\operatorname{Exp}(\mathfrak{p} \setminus \{\vec{0}\})$. This completes for now our general discussion of differential operators on G/K.

When we apply the methods of Section 2 in Section 4, the bounded operator of interest is often L_g^*, the unitary isomorphism of $L^2(\Lambda^* T^*(G/K))$ arising from the natural left action of $g \in G$ on G/K; $(L_g^*)^{-1} = L_{g^{-1}}^*$. Recall that $D = d + d^* + d(r^2/2)\times$. For x, $y \in G/K$ let $\rho(x, y)$ denote the geodesic distance between x and y. For $g \in G$ let (g) denote the point gK in G/K. Let e be the identity in G.

LEMMA 3.20. *For each* $g \in G$ $[L_g^*, D]$ *extends to a bounded operator on* $L^2(\Lambda^* T^*(G/K))$. *There is a constant* c, *independent of* g, *such that the norm of* $[L_g^*, D]$ *is no greater than* $cp((e),(g))$.

Proof. Each $[L_g^*, \mathrm{d}] = 0$. Thus each $[L_g^*, \mathrm{d}^*] = 0$. An analogous argument with adjoints completes the proof of the lemma once it has been established that each $[L_g^*, \mathrm{d}(r^2/2)\wedge]$ has the desired properties. It suffices to show that $\mathrm{d}(r^2/2) - L_{g^{-1}}^*(\mathrm{d}(r^2/2))$ is bounded in pointwise norm by $cp((e),(g))$. For fixed $g \in G$, we consider $p((g),x)$ to be a function of the variable x alone. $(\mathrm{d}(r^2/2) - L_{g^{-1}}^*(\mathrm{d}(r^2/2)))(x) = (1/2)\mathrm{d}(p^2((e),x) - p^2((g),x))$. Assuming $x \neq (e)$ and $x \neq (g)$ and extending our result to all of G/K by continuity, we estimate $\mathrm{d}(p^2((e),x) - p^2((g),x))$, which equals

$$\begin{aligned}(3.21)\qquad &[\mathrm{d}(p((e),x) + p((g),x))]\cdot[p((e),x) - p((g),x)] + \\ &+ [p((e),x) + p((g),x)]\cdot[\mathrm{d}(p((e),x) - p((g),x))].\end{aligned}$$

Because the pointwise norm of $\mathrm{d}p(y,x)$ is one for fixed y and variable $x \neq y$, and by the triangle inequality, the first term in (3.21) is bounded in norm by $2p((e),(g))$. If x lies on the geodesic through (e) and (g) and is between (e) and (g), the second term in (3.21) has norm $2p((e),(g))$. If x is on this geodesic but is not between (e) and (g), this second term is zero.

When x is not on the geodesic through (g) and (e), we must consider two cases. One case, when the smaller of $p((e),x)$ and $p((g),x)$ is at least one-half the larger is studied by applying [17, p. 50—54, 74, p. 73, Corollary 13.2]. Let θ denote the angle at x in the geodesic triangle with vertices (e), (g), and x. Comparing this geodesic triangle to a flat triangle with the same side lengths, we see that a flat isosceles triangle with angle θ formed by two sides of length minimum$(p((e),x), p((g),x))$ has its side opposite θ of length no greater than $p((g),(e))$. Thus the side opposite θ in the flat triangle formed by $\mathrm{d}p((e),x)$ and $\mathrm{d}p((g),x)$ has length less than or equal to $p((g),(e))/\text{minimum}(p((e),x), p((g),x))$. In this case the second term in (3.21) is bounded in norm by $3p((g),(e))$.

In the remaining case, in which the smaller of $p((e),x)$ and $p((g),x)$ is less than one-half the larger of these distances, $p((g),(e))$ is at least one-half of the larger of these distances. We depart from the framework suggested by (3.21) and analyze $\mathrm{d}(p^2((e),x) - p^2((g),x))$ as $2p((e),x)\mathrm{d}p((e),x) - 2p((g),x)\mathrm{d}p((g),x)$. In norm this is less than or equal to $6p((g),(e))$.

LEMMA 3.22. *For each* $g \in G$ $[L_g^*, D^2]$ *extends to a bounded operator from* domain(D) *to* $L^2(\Lambda^* T^*(G/K))$.

Proof. Let $H(x) = p^2((e),x) - p^2((g),x)$. Our commutator equals

$$(3.23)\qquad [L_g^*, \Delta] + L_g^*\text{-multiplication by } H(x) + (1/2)L_g^*\{[\mathrm{d}^*, \mathrm{d}H\wedge] + [\mathrm{d}, \mathrm{d}H\lrcorner]\}.$$

$[L_g^*, \Delta] = 0$. Because $|H(x)| \leqslant p^2((e), (g)) + 2p((e), (g)) \cdot p((e), x)$, the second term of (3.23) behaves as required by Remark 3.5. To apply Remark 3.5 to the last term of (3.23) we do explicit calculations on G. Observe that $H = r^2 + L_{g^{-1}}^* r^2$ and that the left action of group elements commutes with the (right) action of vector fields. The analysis surrounding (3.12) gives the estimate we need.

4. IDENTIFICATION OF $\beta \otimes_{C_\tau} \alpha$

In the rest of this paper G and K denote the groups of Definition 3.1. C_τ is the algebra of continuous sections vanishing at infinity of the Clifford algebra bundle associated with the complexified cotangent bundle of G/K.

In [20] Kasparov introduces, in greater generality than we will discuss, the elements $\alpha \in \mathrm{KK}_G(C_\tau, \mathbf{C})$ and $\beta \in \mathrm{KK}_G(\mathbf{C}, C_\tau)$. (In [20] α is called $\mathrm{d}_{G/K}$ and β is called $\eta_{G/K}$.) The module defining α is $L^2(\Lambda^* T^*(G/K))$ and the operator is $(\mathrm{d} + \mathrm{d}^*)(1 + \Delta)^{-1/2}$. The module is graded by the parity of the degree of a homogeneous form, and the action of C_τ is pointwise the standard Clifford action. The module defining β is C_τ, also graded by the parity of the degree of a homogeneous section, and the Hilbert C_τ-module structure of C_τ is the usual one. The operator is left Clifford multiplication by $(1 + r^2)^{-1/2}\mathrm{d}(r^2/2)$. The action of $g \in G$ on each of these modules is $L_{g^{-1}}^*$.

In this section we use the characterization of Kasparov products described in Section 1 and the analysis of Sections 2 and 3 to prove the following theorem.

THEOREM 4.1. *$\beta \otimes_{C_\tau} \alpha \in \mathrm{KK}_G(\mathbf{C}, \mathbf{C})$ is represented by $(L^2(\Lambda^* T^*(G/K)), D \circ (1 + D^2)^{-1/2})$, where D is the differential operator discussed in Section 3. The grading and G-action are the same as those of α.*

Proof. The map defined on elementary tensors in $C_\tau \otimes_{C_\tau} L^2(\Lambda^* T^*(G/K))$ by $f \otimes w \to f \times w$ extends to a Hilbert space isomorphism $C_\tau \otimes_{C_\tau} L^2(\Lambda^* T^*(G/K)) \cong L^2(\Lambda^* T^*(G/K))$. Thus by Section 1 to prove Theorem 4.1 it suffices to prove the following three propositions.

PROPOSITION 4.2. $(L^2(\Lambda^* T^*(G/K)), D \circ (1 + D^2)^{-1/2}) \in \mathscr{E}_G(\mathbf{C}, \mathbf{C})$.

PROPOSITION 4.3. *$D \circ (1 + D^2)^{-1/2}$ is a $(\mathrm{d} + \mathrm{d}^*)(1 + \Delta)^{-1/2}$ connection for C_τ.*

PROPOSITION 4.4. $[r(1 + r^2)^{-1/2}\mathrm{d}r \times, D \circ (1 + D^2)^{-1/2}] \geqslant 0$ modulo $\mathscr{K}(L^2(\Lambda^* T^*(G/K)))$.

Proof of Proposition 4.2. By the discussion in Section 3, $D \circ (1 + D^2)^{-1/2}$ is a degree one self-adjoint bounded linear operator on the Hilbert space $L^2(\Lambda^* T^*(G/K))$. Our knowledge of the kernel and cokernel of D and the general spectral theory of the compact self-adjoint operator $(1 + D^2)^{-1}$ [30] show that $(D \circ (1 + D^2)^{-1/2})^2 - 1$ is compact. Thus to prove Proposition 4.2 it suffices to prove the following two lemmas.

LEMMA 4.5. *For each* $g \in G$ $g(D \circ (1 + D^2)^{-1/2}) - D \circ (1 + D^2)^{-1/2}$ *is compact.*

LEMMA 4.6. $D \circ (1 + D^2)^{-1/2}$ *is G-continuous.*

Proof of Lemma 4.5. Lemma 4.5 is equivalent to $[L^*_{g^{-1}}, D \circ (1 + D^2)^{-1/2}] \in \in \mathscr{K}(L^2(\Lambda^* T^*(G/K)))$. The argument of Section 2, with $S = L^*_{g^{-1}}$, applies in this situation because all properties required of $L^*_{g^{-1}}$ are implied by Lemmas 3.20 and 3.22 alone or in combination with Lemma 2.6.

Proof of Lemma 4.6. It suffices to show that the norm of $[D \circ (1 + D^2)^{-1/2}, L^*_g]$ is bounded by a fixed constant multiple of $p((g), (e))$. As above we can use Section 2 to analyze the commutator. By (2.4) the bound on the norm of $[D, L^*_g]$ provided by Lemma 3.20 completes the proof. This claim is obvious for the second term on the right of (2.4). For the first term it follows from (2.5) and Remark 2.6.

Proof of Proposition 4.3. We use: the compact operators form a norm-closed, $*$-closed ideal; $(d + d^*) \circ (1 + \Delta)^{-1/2}$ and $D \circ (1 + D^2)^{-1/2}$ are self-adjoint [8]; the standard Clifford action is pointwise linear; and standard smoothing techniques to reduce this proposition to the statement that for each ξ, a smooth compactly supported element of C_τ that is homogeneous in degree,

$$\xi \times \circ (d + d^*) \circ (1 + \Delta)^{-1/2} - (-1)^{\partial \xi} D \circ (1 + D^2)^{-1/2} \circ \xi \times \quad \text{is in} \quad \mathscr{K}(L^2(\Lambda^* T^*(G/K))). \tag{4.7}$$

$$\begin{aligned} &\xi \times \circ (d + d^*) \circ (1 + \Delta)^{-1/2} - (-1)^{\partial \xi} D \circ (1 + D^2)^{-1/2} \circ \xi \times = \\ &= \xi \times \circ (d + d^*) \{(1 + \Delta)^{-1/2} - (1 + D^2)^{-1/2}\} + [\xi \times, d + d^*] \circ (1 + D^2)^{-1/2} - \\ &\quad -(-1)^{\partial \xi} d(r^2/2) \times \circ \xi \times \circ (1 + D^2)^{-1/2} - (-1)^{\partial \xi} D \circ [(1 + D^2)^{-1/2}, \xi \times]. \end{aligned} \tag{4.8}$$

We proceed by showing that each of the four terms after the equals sign in (4.8) is compact. In the process we show that (4.8) makes sense.

LEMMA 4.9. $D \circ [(1 + D^2)^{-1/2}, \xi \times]$ *is compact.*

Proof. The methods of Section 2 prove the lemma once we show that $\xi \times$ satisfies the appropriate conditions. Apply Lemma 2.7. It suffices to show that $\xi \times$ and $[D, \xi \times]$ are bounded on $L^2(\Lambda^* T^*(G/K))$ and that $[D^2, \xi \times]$ is bounded from domain(D) to $L^2(\Lambda^* T^*(G/K))$. $\xi \times$ is clearly bounded. Direct calculations with the methods of Section 3 show that $[D, \xi \times]$ is bounded.

The methods of Section 3 show that $[D^2, \xi\times]$ is a first order differential operator with bounded coefficients. Remark 3.5 shows that domain(D) imbeds continuously in domain$(\mathrm{d} + \mathrm{d}^*)$.

To show that a first order differential operator with bounded coefficients is continuous from domain$(\mathrm{d}+\mathrm{d}^*)$ to $L^2(\Lambda^* T^*(G/K))$, we use [5] to construct a uniform collection of coordinate neighborhoods and a uniform partition of unity subordinate to the neighborhoods on G/K. The result then follows from the (local) basic elliptic estimate.

Lemma 4.10. *$\mathrm{d}(r^2/2)\times \circ \xi\times \circ (1 + D^2)^{-1/2}$ is compact.*

Proof. $(1 + D^2)^{-1/2}$ is compact. Because ξ is smooth with compact support $\mathrm{d}(r^2/2)\times \circ \xi\times$ is bounded.

Lemma 4.11. *$[\xi\times, \mathrm{d} + \mathrm{d}^*] \circ (1 + D^2)^{-1/2}$ is compact.*

Proof. Use the methods of Section 3 to calculate that $[\xi\times, \mathrm{d} + \mathrm{d}^*]$ is bounded, and argue as above.

Lemma 4.12. *$\xi\times \circ (\mathrm{d} + \mathrm{d}^*) \circ \{(1 + \Delta)^{-1/2} - (1 + D^2)^{-1/2}\}$ is compact.*

Proof. The intuition behind this proof is that because Δ and D^2 have equal principal symbols, $(\mathrm{d} + \mathrm{d}^*)\{(1 + \Delta)^{-1/2} - (1+D^2)^{-1/2}\}$ should behave like a pseudo-differential operator of negative order. Because the support of ξ is compact, Rellich's lemma should make the operator we are considering compact. To make this reasoning rigorous we use the finite propagation speed of solutions of hyperbolic equations to reduce the situation to one involving pseudodifferential operators on a compact manifold without boundary.

Solutions of the equations $\left(\frac{\partial^2}{\partial t^2} + 1 + \Delta\right) b = 0$ and $\left(\frac{\partial^2}{\partial t^2} + 1 + D^2\right) b = 0$ propagate at finite speed. The observations supporting this claim are that the propagation speed depends on the principal symbols of the elliptic operators $1 + \Delta$, resp. $1 + D^2$, that the geometry of G/K is uniform, and that the domains of Δ and D^2 are locally defined. (See [8], [26, Introduction], or [40, p. 70 ff.].)

It follows that the techniques of [7], as used in [15, §4], apply. (Observe that in the present paper Δ is a non-negative operator, while [15] follows the opposite sign convention.) Choose $\varepsilon > 0$ and a large positive integer N. We can write $(1+\Delta)^{-1/2} = X + Y$ and $(1 + D^2)^{-12} = Z + W$, where X and Z increase the support of any form to which they are applied by no more than $\varepsilon/2$ and Y acts like $(1 + \Delta)^{-N}$, W like $(1 + D^2)^{-N}$.

Remarks 3.5 and 3.6 show that $\xi\times \circ (\mathrm{d} + \mathrm{d}^*) \circ W$ is compact. Because ξ has compact support, Rellich's lemma implies that $\xi\times \circ (\mathrm{d} + \mathrm{d}^*) \circ Y$ is compact.

Only $\xi\times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)$ remains to be considered.

Choose a two-element partition of unity $\{\mu_1, \mu_2\}$ on G/K satisfying the conditions: support(μ_1) is compact; and there exists a bounded ball B in G/K, centered at the identity coset and containing support(ξ) such that distance(B, support(μ_2)) $> \varepsilon$. For any $v \in L^2(\Lambda^* T^*(G/K))$ finite propagation speed implies that $\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)(\mu_2 v) = 0$. Thus $\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)(v) = \xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)(\mu_1 v)$, and every step of this last composition is supported in a compact neighborhood of support(ξ).

We now form a compact manifold without boundary on which we apply the calculus of pseudodifferential operators. Take a bounded ball B' centered at the identity coset and containing support(μ_1) as well as all points whose distance from support(μ_1) is no greater than ε. Put a collar on this ball, i.e., make a slightly larger ball called B''. Form the double of B''. This is a compact manifold M without boundary. $\Delta_{G/K}$ on B'' extends to Δ_M on M. $D^2 - \Delta$ is a smooth self-adjoint vector bundle map. By shrinking $r\mathrm{d}r$ to zero as we move from the inside to the outside of the collar, we can deform D^2 to equal Δ on the outside of the collar and beyond. Using Δ on the second copy of B'', we can extend D^2 to an operator $\mathscr{D}$ on M whose principal symbol equals that of Δ_M. $\mathscr{D}$, restricted to smooth forms on M, is the square of a formally self-adjoint operator. By [8] both $1 + \Delta_M$ and $1 + \mathscr{D}$ are (essentially) self-adjoint.

Because $(1 + \Delta_M)^{-1/2}$ and $(1 + \mathscr{D})^{-1/2}$ are pseudodifferential operators of order -1 with equal principal symbols [40, p. 293 ff.], Rellich's lemma implies that $\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ \{(1 + \Delta_M)^{-1/2} - (1 + \mathscr{D})^{-1/2}\}$ is a compact map $L^2(\Lambda^* T^*(M)) \to L^2(\Lambda^* T^*(M))$. We now compare the effect of this map with the effect of $\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)$ on differential forms $\mu_1 v$. Use exactly the same constructions as were used to form the operators X, Y, Z, and W to write $(1 + \Delta_M)^{-1/2} = X_M + Y_M$ and $(1 + \mathscr{D})^{-1/2} = Z_M + W_M$.

$$
\begin{aligned}
\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ \{(1 + \Delta_M)^{-1/2} - (1 + \mathscr{D})^{-1/2}\} &= \xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ \\
\circ (X_M - Z_M) &+ \xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (Y_M - W_M).
\end{aligned} \tag{4.13}
$$

The first sentence of this paragraph shows that the left side of (4.13) is compact. Because Y_M and W_M act like $(1 + \Delta_M)^{-N}$ and $(1 + \mathscr{D})^{-N}$ respectively, the second term on the right of (4.13) is compact. Thus $\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X_M - Z_M)$ is compact.

$(\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ X_M)(\mu_1 v) = (\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ X)(\mu_1 v)$ because they can be written in terms of identical expressions involving the unique solutions to the differential equations $\left(\frac{\partial}{\partial t^2} + 1 + \Delta_M\right) b = 0$ and $\left(\frac{\partial^2}{\partial t^2} + 1 + \Delta_{G/K}\right) b = 0$, which are identical in the region B' with which we are concerned. Similarly $(\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ Z_M)(\mu_1 v) = (\xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ Z)(\mu_1 v)$. Thus the map $L^2(\Lambda^* T^*(G/K)) \to L^2(\Lambda^* T^*(G/K))$ given by $v \to \xi \times \circ (\mathrm{d} + \mathrm{d}^*) \circ (X - Z)(\mu_1 v)$ factors through a compact map and is itself compact.

Proof of Proposition 4.4.

$$[r(1+r^2)^{-1/2}\,\mathrm{d}r\times,\ (1+D^2)^{-1/2}\circ D] =$$

$$(4.14)\qquad = [r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}\circ(\mathrm{d}+\mathrm{d}^*)] +$$

$$+[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}\circ r\mathrm{d}r\times].$$

We show first that the first term on the right of (4.14) is compact.

$$[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}\circ(\mathrm{d}+\mathrm{d}^*)] =$$

$$(4.15)\qquad = [r(1+r^2)^{-1/2}\,\mathrm{d}r\times,\ (1+D^2)^{-1/2}]\circ(\mathrm{d}+\mathrm{d}^*) +$$

$$+\ (1+D^2)^{-1/2}\circ[r(1+r^2)^{-1/2}\,\mathrm{d}r\times,\ \mathrm{d}+\mathrm{d}^*].$$

LEMMA 4.16. $(1+D^2)^{-1/2}\circ[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ \mathrm{d}+\mathrm{d}^*]$ *is compact.*

Proof. A direct calculation using methods of Section 3 and an application of the discussion surrounding (3.12) shows that the commutator is a bounded vector bundle map.

LEMMA 4.17. $[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}]\circ(\mathrm{d}+\mathrm{d}^*)$ *is compact.*

Proof. Use the integral for $(1+D^2)^{-1/2}$ and the arguments introduced in Section 2. Introduce $(1+D^2)^{1/2}\circ(1+D^2)^{-1/2}$ on the end of the integrand and pull the $(1+D^2)^{-1/2}$ outside the integral. Because (2.4) and Remark 2.6 are unnecessary for this argument, we do not need $[D,\ r(1+r^2)^{-1/2}\mathrm{d}r\times]$ bounded. To show that $[D^2,\ r(1+r^2)^{-1/2}\mathrm{d}r\times]$ is bounded from domain(D) to $L^2(\Lambda^*T^*(G/K))$ we must calculate directly as in Section 3 and use the discussion of the effect of Δ.

We return to the second term on the right of (4.14).

$$[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}\circ r\mathrm{d}r\times] =$$

$$(4.18)\qquad =[r(1+r^2)^{-1/2}\mathrm{d}r\times,\ (1+D^2)^{-1/2}]\circ(1+r^2)^{1/2}\circ r(1+r^2)^{-1/2}\mathrm{d}r\times\ -$$

$$-2(1+D^2)^{-1/2}\circ(1+r^2)^{-1/2}+2(1+D^2)^{-1/2}\circ(1+r^2)^{1/2}.$$

Because $r(1+r^2)^{-1/2}\mathrm{d}r\times$ is bounded, the reasoning used to prove Lemma 4.17 shows that the first term on the right of (4.18) is compact. The second term is compact. The third term equals

$$2(1+r^2)^{1/4}\circ(1+D^2)^{-1/2}\circ(1+r^2)^{1/4}+$$

$$(4.19)\qquad +\ 2(1+r^2)^{1/4}\circ[(1+r^2)^{-1/4},\ (1+D^2)^{-1/2}]\circ(1+r^2)^{1/2},$$

which is the sum of a positive operator and an operator we now show to be compact.

Once we establish a few facts, the reasoning of Section 2 shows that the second term of (4.19) is equal to the compact operator

$$\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(1 + D^2 + \lambda)^{-1}\{(1 + r^2)^{1/4} + [D^2, (1 + r^2)^{1/4}](1 + D^2 + \lambda)^{-1}\}\cdot$$

(4.20)

$$\cdot[D^2, (1 + r^2)^{-1/4}](1 + D^2 + \lambda)^{-1}(1 + D^2)^{1/2} d\lambda \circ (1 + D^2)^{-1/2} \circ (1 + r^2)^{1/2}.$$

These facts, that $[D^2, (1 + r^2)^{-1/4}]$, $[D^2, (1+r^2)^{1/4}]$, and $(1 + r^2)^{1/4}[D^2, (1+ r^2)^{-1/4}]$ are bounded from domain(D) to $L^2(A^*T^*(G/K))$, follow from calculation with the methods of Section 3.

5. CONTEXT

This section places our work in the context of a program initiated by Baum, Connes, and Kasparov. (See also the papers by Rosenberg listed in the references for an example of the success of this program.) A full discussion of the constructions and claims used in this section can be found in [20]. (Except for minor changes, e.g. subscript G in place of superscript G, we follow the notation of [20].)

Let G be a separable locally compact group that acts on C^*-algebras A, B, C, and D. Kasparov defines maps

$$\sigma_D\colon \mathrm{KK}_G(A, B) \to \mathrm{KK}_G(A \otimes D, B \otimes D)$$

and

$$j_G\colon \mathrm{KK}_G(A, B) \to \mathrm{KK}(C^*(G, A), C^*(G, B)).$$

$\mathrm{KK}_G(D, D)$ is given a ring structure by the Kasparov product $\otimes_D$. This ring has a unit 1_D.

(5.1) $$\sigma_D(1_{\mathrm{C}}) = 1_D.$$

REMARK 5.2. The map taking $x \in \mathrm{KK}_G(\mathrm{C}, D)$ to $x \otimes_D 1_D$ is the identity map on $\mathrm{KK}_G(\mathrm{C}, D)$.

For $x \in \mathrm{KK}_G(A, C)$ and $y \in \mathrm{KK}_G(C, B)$,

(5.3) $$\sigma_D(x \otimes_C y) = \sigma_D(x) \otimes_{C\otimes D} \sigma_D(y)$$

(5.4) $$j_G(1_A) = 1_{C^*(G,A)}$$

(5.5) $$j_G(x \otimes_D y) = j_G(x) \otimes_{C^*(G,D)} j_G(y).$$

LEMMA 5.6. *For α and β as in Section 4, the map taking $x \in \mathrm{KK}_G(\mathbf{C}, C_\tau)$ to $(x \otimes_{C_\tau} \alpha) \otimes_{\mathbf{C}} \beta$ is the identity map on $\mathrm{KK}_G(\mathbf{C}, C_\tau)$.*

Proof. $\alpha \otimes_{\mathbf{C}} \beta = 1_{C_\tau}$, [20]. Observe that the Kasparov product is associative and use Remark 5.2.

LEMMA 5.7. *For α and β as in Section 4, the map taking $x \in \mathrm{KK}(\mathbf{C}, C^*(G, C_\tau \otimes A))$ to $(x \otimes_{C^*(G, C_\tau \otimes A)} j_G(\sigma_A(\alpha))) \otimes_{C^*(G,A)} j_G(\sigma_A(\beta))$ is the identity map on $\mathrm{KK}(\mathbf{C}, C^*(G, C_\tau \otimes A))$.*

Proof. Add 5.1—5.5 to the line of reasoning used in the proof of Lemma 5.6.

LEMMA 5.8. *Let G be a group for which $\beta \otimes_{C_\tau} \alpha = 1_{\mathbf{C}}$. Then:*

(a) *the map taking $y \in \mathrm{KK}_G(\mathbf{C}, \mathbf{C})$ to $(y \otimes_{\mathbf{C}} \beta) \otimes_{C_\tau} \alpha$ is the identity on $\mathrm{KK}_G(\mathbf{C}, \mathbf{C})$;*

(b) *the map taking $z \in \mathrm{KK}(\mathbf{C}, C^*(G, A))$ to*

$$(z \otimes_{C^*(G,A)} j_G(\sigma_A(\beta))) \otimes_{C^*(G, C_\tau \otimes A)} j_G(\sigma_A(\alpha))$$

is the identity on $\mathrm{KK}_G(\mathbf{C}, C^*(G, A))$.

Proof. The proof is analogous to the proofs of Lemmas 5.6 and 5.7.

REMARK 5.9. Note that for a group G as in Lemma 5.8, we have established that $C^*(G, A)$ and $C^*(G, C_\tau \otimes A)$ are KK-equivalent (definition in [36]). The isomorphism from $\mathrm{KK}(\mathbf{C}, C^*(G, C_\tau \otimes A))$ to $\mathrm{KK}(\mathbf{C}, C^*(G, A))$ is realized by Kasparov product with a KK-element created from $\mathrm{d} + \mathrm{d}^*$, i.e. by "de Rham induction."

LEMMA 5.10. *For many groups, including "most" connected reductive Lie groups, $\beta \otimes_{C_\tau} \alpha \neq 1_{\mathbf{C}}$.*

Proof. There is an operator j_G^r taking $\mathrm{KK}_G(A, B)$ to $\mathrm{KK}(C_r^*(G, A), C_r^*(G, B))$ defined as j_G is defined but using reduced crossed product C^*-algebras. The map j_G^r has properties analogous to those of j_G. Let $f: C^*G \to C_r^*G$ and $g: C^*(G, C_\tau) \to C_r^*(G, C_\tau)$ be the natural maps, and let $f_*: \mathrm{KK}(\mathbf{C}, C^*G) \to \mathrm{KK}(\mathbf{C}, C_r^*G)$ and $g_*: \mathrm{KK}(\mathbf{C}, C^*(G, C_\tau)) \to \mathrm{KK}(\mathbf{C}, C_r^*(G, C_\tau))$ be the associated maps on K-theory. Assume that the subgroup $K \subset G$ used to define C_τ is compact. For $x \in \mathrm{KK}(\mathbf{C}, C^*G)$

$$f_*(x) \otimes_{C_r^*G} j_G^r(\beta) = g_*(x \otimes_{C^*G} j_G(\beta)) \in \mathrm{KK}(\mathbf{C}, C_r^*(G, C_\tau)), \quad [20], \quad [22]$$

If $\beta \otimes_{C_\tau} \alpha = 1_{\mathbf{C}}$, Lemmas 5.7 and 5.8, for $A = \mathbf{C}$, show that $\_\otimes_{C^*G} j_G(\beta)$ and $\_\otimes_{C_r^*G} j_G^r(\beta)$ are isomorphisms. The map g_* is an isomorphism. Because f_* is not an isomorphism in general [35, p. 70], $\beta \otimes_{C_\tau} \alpha$ cannot equal $1_{\mathbf{C}}$ in general.

REMARK 5.11. The above reasoning provides the foundation for showing that if G is a group for which $\beta \otimes_{C_\tau} \alpha = 1_C$ and if G acts on a C^*-algebra A, then $C^*(G, A)$ and $C^*_r(G, A)$ are KK-equivalent.

OUTLINE OF SECTION 6. In Section 6 we use the description of $\beta \otimes_{C_\tau} \alpha$ given in Section 4 to show that for X a locally compact second countable Hausdorff space on which the reductive group G acts properly with finitely many orbit types, the map taking $\mathrm{KK}(\mathbf{C}, C^*(G, C_0(X)))$ to itself defined by

$$z \to z \otimes_{C^*(G, C_0(X))} j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha)) \tag{5.12}$$

is an isomorphism. Because of the general fact that $j_G(\sigma_{C_0(X)}(\alpha \otimes_{\mathbf{C}} \beta)) = 1_{C^*(G, C_\tau \otimes C_0(X))}$ it follows that as maps between $\mathrm{KK}(\mathbf{C}, C^*(G, C_\tau \otimes C_0(X)))$ and $\mathrm{KK}(\mathbf{C}, C^*(G, C_0(X)))$

$$\_\otimes_{C^*(G, C_\tau \otimes C_0(X))} j_G(\sigma_{C_0(X)}(\alpha)) \quad \text{and} \quad \_\otimes_{C^*(G, C_0(X))} j_G(\sigma_{C_0(X)}(\beta))$$

are inverses of each other.

In Section 6 we show directly that when a proper action of G on X has a single orbit type, $j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha)) = 1_{C^*(G, C_0(X))}$. For more general proper actions of G on X, exact sequences arising from the decomposition of X into orbit types show that (5.12) is an isomorphism. Because $j_G(\sigma_{C_0(X)}(\alpha \otimes_{\mathbf{C}} \beta))$ acts as the identity on $\mathrm{KK}(\mathbf{C}, C^*(G, C_\tau \otimes C_0(X)))$, the idempotent [20] $j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha))$ acts as the identity on $\mathrm{KK}(\mathbf{C}, C^*(G, C_0(X)))$. (Use 5.1—5.5.) By [11] the results of [36] apply to show that $j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha)) = 1_{C^*(G, C_0(X))}$, i.e. that de Rham induction realizes a KK-equivalence between $C^*(G, C_\tau \otimes C_0(X))$ and $C^*(G, C_0(X))$.

REMARK 5.13. The algebra C_τ and, in certain cases, all mention of Clifford algebras can be removed from the statement of the theorems in Section 6. Let C_V denote the Clifford algebra of the complexified cotangent space V to G/K at the point representing the identity coset. Note that $V \cong \mathfrak{p}^* \otimes_{\mathbf{R}} \mathbf{C}$. In what follows we switch to the older K-theory notation, $\mathrm{KK}(\mathbf{C}, A) = \mathrm{K}_0(A)$.

LEMMA 5.14. $\mathrm{K}_i(C^*(K, C_V \otimes C_0(X))) \cong \mathrm{K}_i(C^*(G, C_0(X)))$.

Proof. Combine the isomorphism of Section 6 with the strong Morita equivalence between $C^*(G, C_\tau \otimes A)$ and $C^*(K, C_V \otimes A)$, [14]. (See [20].)

LEMMA 5.15. *$K \cong K' \times W$ where K' is a maximal compact subgroup of G and W is a vector group.*

Proof. We defer the proof to the end of this section.

LEMMA 5.16. $K_{i+\dim(K/K')}(C^*(K', C_V \otimes C_0(X))) \cong K_i(C^*(G, C_0(X)))$.

Proof. Use Lemma 5.15 and apply Connes' analogue of the Thom isomorphism [9] to the left side of Lemma 5.14.

LEMMA 5.17. *If the action of K' on $\mathfrak{p}$ factors through* $\mathrm{spin}^c(\mathfrak{p})$, *then*

$$K_{i+\dim(K/K')}(C^*(K', C_V \otimes C_0(X))) \cong$$

$$\cong K_{i+\dim(K/K')}(C^*(K', C_0(X)) \otimes C_V) \cong K_{i+\dim(G/K')}(C^*(K', C_0(X))).$$

Proof. See [20] and [22]. Only the first isomorphism uses the assumption on the K'-action.

LEMMA 5.18. *Under the above assumptions:*

(a) $$K_{i+\dim(G/K')}(C^*(K', C_0(X))) \cong K_i(C^*(G, C_0(X)));$$

and

b) $$K_{i+\dim(G/K)}(C^*(K, C_0(X))) \cong K_i(C^*(G, C_0(X))).$$

Proof. To prove (a) use Lemmas 5.16 and 5.17. Part (b) follows from (a) by another application of [9].

REMARK 5.19. There is another proof of Lemma 5.18 (a). In [1] Abels shows that X has a global K'-slice S, that X is G-homeomorphic to $G \times_{K'} S$, and that X is K'-homeomorphic to $S \times \mathbf{R}^{\dim(G/K')}$. Here the action of K' on $\mathbf{R}^{\dim(G/K')}$ arises from the diffeomorphism of $\mathbf{R}^{\dim(G/K')}$ with G/K'. Using [12] to make the equivariant Thom isomorphism of [2] a statement about the K-theory of transformation group C^*-algebras, we see that the K'-homeomorphism between X and $S \times \mathbf{R}^{\dim(G/K')}$ and the main result of [2] show that $K_{i+\dim(G/K')}(C^*(K', C_0(X))) \cong K_i(C^*(K', C_0(S)))$. Then the strong Morita equivalence between $C^*(K', C_0(S))$ and $C^*(G, C_0(G \times_{K'} S))$ established in [28] and the G-homeomorphism between $G \times_{K'} S$ and X show that $K_i(C^*(K', C_0(S))) \cong K_i(C^*(G, C_0(X)))$. Although this proof uses techniques that are simpler, or at least older, than those of our proof, its failure to use de Rham induction at the level where G acts means that it does not fit naturally into the Connes-Kasparov program. It is within this program that one expects proper actions to achieve their greatest significance, as a tool in understanding actions that are not proper, [4].

Proof of Lemma 5.15. If G is semisimple, then [17, p. 129, 252—253] shows that K satisfies the hypotheses of [6, p. 22—05, Corollary 2] and thus K is the direct product of a compact group and a vector group. For general connected reductive G, K has a universal cover which is the direct product of a K associated with a simply-connected semisimple G and a vector group. It follows that the proof of the same corollary in [6] shows that the K of a reductive G has a universal cover which is the direct product of a compact group with semisimple Lie algebra and a vector group. The proof in [6] shows that such a K is the direct product of a compact group and a vector group. The proof of [17, p. 256, Theorem 2.1] extends to show that the compact factor of K is a maximal compact subgroup of G.

6. PROPER ACTIONS

Throughout this section G is a noncompact connected reductive Lie group acting properly and with finitely many orbit types on a second countable locally compact Hausdorff space X. In this situation the definition of proper given in [25] is equivalent to the following.

DEFINITION 6.1. An action of G on X is said to be *proper* if under the map $G \times X \to X \times X$ given by $(g, x) \to (x, g(x))$ the inverse images of compact sets are compact.

In this section we use the characterization of $\beta \otimes_{C_\tau} \alpha$ given in Section 4 to prove that $j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha)) = 1_{C^*(G, C_0(X))}$. We let (E, T) denote the representative of $\beta \otimes_{C_\tau} \alpha$ described in Theorem 4.1. The properties of T that are essential in Section 6 are that it commutes with the action of K and that its restriction as a map from forms of even degree to forms of odd degree is surjective with a one-dimensional K-invariant kernel.

In working with crossed products, $\sigma_{C_0(X)}$, and j_G, it is often convenient to work with explicit expressions that make sense at least on dense subalgebras and submodules (e.g., $C_c(G \times X) \subset C^*(G, C_0(X))$). With such an approach one can show that $j_G(\sigma_{C_0(X)}(E, T)) = (C^*(G, C_0(X)) \otimes E, 1 \otimes T)$.

Henceforth for $g \in G$ we denote $L^*_{g^{-1}}$ by $\gamma(g)$.

THEOREM 6.2. *Assume $X = G/H$ with H a compact subgroup of G. (We may assume $H \subset K$.)*

Then $j_G(\sigma_{C_0(G/H)}(\beta \otimes_{C_\tau} \alpha)) = 1_{C^*(G, C_0(G/H))}$.

Proof. The proof consists of the proofs of two propositions.

PROPOSITION 6.3. *Define T' on $C_c(G \times (G/H), E)$ by*

$$(T'\varphi)(g_1, [g_2]) = \gamma(g_2) \circ T \circ \gamma(g_2^{-1})(\varphi(g_1, [g_2])).$$

(Because T commutes with K, T' is well-defined.) T' extends, and $(C^*(G, C_0(G/H)) \otimes$ $\otimes E, T')$ represents an element of KK$(C^*(G, C_0(G/H)), C^*(G, C_0(G/H)))$ *equal to the element represented by* $(C^*(G, C_0(G/H)) \otimes E, 1 \otimes T)$.

PROPOSITION 6.4. *$(C^*(G, C_0(G/H)) \otimes E, T')$ represents* $1_{C^*(G,C_0(G/H))}$.

Proof of Proposition 6.3. To understand T' we realize $C^*(G, C_0(G/H))$ as a subalgebra of $\mathscr{K}(L^2(G))$ and $C^*(G, C_0(G/H) \otimes E$ as a submodule of $\mathscr{K}(L^2(G)) \otimes E$. The subalgebra realization arises from

$$C_c(G\times(G/H)) \xrightarrow{p} C_c(G\times G)^H \subset C_c(G\times G) \xrightarrow{K} \mathscr{K}(L^2(G)).$$

For $\xi \in L^2(G)$ and $f\in C_c(G\times G)$ $(K(f)\xi)(t) = \int_G f(t, s)\xi(s)\mathrm{d}s$.

$$C_c(G\times G)^H = \{f\in C_c(G\times G): f(th, vh) = f(t, v) \quad \text{for all } h\in H \text{ and } t, v\in G\}.$$

For $z\in C_c(G\times(G/H))$ $(p(z))(t, v) = z(tv^{-1}, t)$. To see that this realization is faithful on $C^*(G, C_0(G/H))$ compare it to $m\times\lambda_G$, the integrated form of the covariant representation arising from pointwise multiplication and the left regular representation. One shows that $m\times\lambda_G$ is faithful by observing that $m\times\lambda_G$ is equivalent to $\mathrm{ind}_H^G(\varepsilon_H\times\lambda_H)$ (where $\varepsilon_H(f) = f((H))$ and λ_H is again left regular) and by applying [28] and [29]. We realize $C^*(G, C_0(G/H)) \otimes E$ by similar steps involving $C_c(G\times(G/H), E) \to C_c(G\times G, E)^H \to C_c(G\times G, E)$.

Define $V: C_c(G\times G, E) \to C_c(G\times G, E)$ by $(V\xi)(t, v) = \gamma(t^{-1})(\xi(t, v))$. V extends to an endomorphism of $\mathscr{K}(L^2(G)) \otimes E$ with $(V^*\xi)(t, v) = \gamma(t)(\xi(t, v))$. In fact the restriction of $V^*\circ(1\otimes T)\circ V$ maps the closure of $C_c(G\times G, E)^H$ to itself and is identified with T'. The claims of Proposition 6.3 follow from calculations that show that T and T' satisfy the conditions of the following lemma.

LEMMA 6.5. *Suppose $(F, S_1) \in \mathscr{E}(A, B)$, $S_1 = S_1^*$, and the degree one operator $S_0 \in \mathscr{L}(F)$ satisfies $S_0 = S_0^*$ and $a\circ(S_0 - S_1) \in \mathscr{K}(F)$ for every a in A. Then $(F, S_0) \in \mathscr{E}(A, B)$, and (F, S_0) represents the same class in* KK(A, B) *that (F, S_1) does.*

Proof. $tS_1 + (1-t)S_0$ is an operator homotopy.

Proof of Proposition 6.4. Let $E^K = \{e\in E: \gamma(k)e = e \;\; \forall k\in K\}$. Let $P: E\to E^K$ be orthogonal projection. The restriction of $V^*\circ(1\otimes P)\circ V$ to our module is a projection onto the submodule M which is the closure of $M_c = \{\xi\in C_c(G\times G, E)^H: \gamma(t^{-1})(\xi(t, v)) \in E^K \;\; \forall t, v\in G\}$. The range of the complementary projection is the closure N of $N_c = \{\xi\in C_c(G\times G, E)^H : \gamma(t^{-1})(\xi(t, v)) \in (E^K)^\perp \;\; \forall t, v\in G\}$. T' restricts to M and N to give rise to cycles (M, T'_M) and (N, T'_N) for KK$(C^*(G, C_0(G/H)), C^*(G, C_0(G/H)))$, and $(C^*(G, C_0(G/H)) \otimes E, T') = (M, T'_M) \oplus (N, T'_N)$. T'_N is invertible and self-adjoint, and it commutes with the left action of $C^*(G, C_0(G/H))$. An operator homotopy shows that (N, T'_N) represents zero in KK$(C^*(G, C^0(G/H)), C^*(G, C_0(G/H)))$.

LEMMA 6.6. $(M, T'_M) = 1_{C^*(G,C_0(G/H))}$.

Proof. We continue to identify implicitly $C^*(G, C_0(G/H))$ with its image under the map involving p. One can calculate that V is a unitary isomorphism from (M,T'_M) to $\sigma_{C^*(G,C_0(G/H))}(E^K, T)$. Remark 3.7 and [21] complete the proof.

THEOREM 6.7. *Assume the action of G on X has a single orbit type (H). Then* $j_G(\sigma_{C_0(X)}((E, T))) = 1_{C^*(G,C_0(X))}$.

Proof. The slice theorem [25] and the reduction of structure group [39] permitted by Lemma 3.3 show that X is a fiber bundle over $G\backslash X$ with fiber G/H and structure group $N_K(H)/H_0$, where H_0 is the kernel of the left translation map from G to homeomorphisms of G/H. It is then possible to realize $C^*(G, C_0(X))$, resp. $C^*(G, C_0(X)) \otimes E$, as sections of bundles with fiber $C^*(G, C_0(G/H))$, resp. $C^*(G,C_0(G/H)) \otimes E$, and base space $G\backslash X$. The proof for $X = G/H$ can be patched together to give a proof for the X of Theorem 6.7.

THEOREM 6.8. *Assume the action of G on X has finitely many orbit types. Then* $\_\_\otimes_{C^*(G,C_0(X))} j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha))$ *is an isomorphism from* KK$(\mathbf{C}, C^*(G, C_0(X)))$ *to itself.*

Proof. For any subgroup H of G and any proper G-space Y, $Y_{(H)} = \{y \in Y :$ the isotropy group of y is conjugate to a subgroup of $H\}$ is open in Y, [25, p. 313]. Define an order on subgroups of G by saying $H_1 \leqslant H_2$ if H_1 is conjugate to a subgroup of H_2. Let $H_1, \ldots, H_n$ be representatives of the conjugacy classes of the isotropy groups of the action of G on X that are minimal with respect to this order. Then $X_{(H_1)}, \ldots, X_{(H_n)}$ are disjoint open subsets of X, each of a single orbit type, and there is an exact sequence $0 \to \bigoplus_{i=1}^{n} C^*(G, C_0(X_{(H_i)})) \to C^*(G, C_0(X)) \to$ $\to C^*\left(G, C_0\left(X - \bigcup_{i=1}^{n} X_{(H_i)}\right)\right) \to 0$. We can apply the five lemma to the maps given by $\_\_\otimes_{C^*(G,C_0(X))} j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha))$ and its restrictions to G-invariant subspaces of X if we establish that its restriction to KK$\left(\mathbf{C}, C^*\left(G, C_0\left(X - \bigcup_{i=1}^{n} X_{(H_i)}\right)\right)\right)$ is an isomorphism. An inductive argument does this.

THEOREM 6.9. *Assume G and X satisfy the conditions stated at the beginning of Section 6. Then* $j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha)) = 1_{C^*(G,C_0(X))}$. $C^*(G, C_0(X))$ *and* $C^*(G, C_0(X) \otimes C_\tau)$ *are* KK-*equivalent.*

Proof. Because $\alpha \otimes_{\mathbf{C}} \beta = 1_{C_\tau}$, because j_G and σ_D preserve the unit, and by (5.1) and (5.3), $\_\_\otimes_{C^*(G,C_0(X))} j_G(\sigma_{C_0(X)}(\beta \otimes_{C_\tau} \alpha))$ is not only an isomorphism but is the identity on $KK(C, C^*(G, C_0(X)))$. (It is easy to extend the argument to $K_1(C^*(G, C_0(X)))$.) By [11], $C^*(G, C_0(X))$ is an algebra to which [36] applies. Let $D = C^*(G, C_0(X))$. In the ring structure [36] puts on $KK(D, D)$, an idempotent whose $\mathrm{Hom}_Z(K_*(D), K_*(D))$ part is the identity must have its $\mathrm{Ext}^1_Z(K_*(D), K_*(D))$ part equal to zero.

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