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CHARACTERISTIC FUNCTIONS FOR INFINITE SEQUENCES OF NONCOMMUTING OPERATORS

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This note deals with the "characteristic function" of an infinite sequence $\mathscr{T} = \{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space $\mathscr{H}$, when the matrix $[T_1, T_2, \ldots]$ is a contraction. In connexion with this, we extend to our setting the results from [2] for two operators and many of the results from [9, Chapter VI] for one operator.

As the main result of this note, we obtain a model for a completely non-co-isometric (c.n.c.) sequence $\mathscr{T}$ (in our notation $\mathscr{T} \in C^{(1)}$) in which the "characteristic function" θ occurs explicitly.

Further, it is shown when an operator $\theta\colon \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ ($\mathscr{E}$, $\mathscr{E}_*$ are Hilbert spaces and $\mathscr{F}$ is defined in Section 1) generates a c.n.c. sequence $\mathscr{T}$ as above. Using these theorems, we prove that two c.n.c. sequences $\mathscr{T}$ and $\mathscr{T}'$ are unitarily equivalent if and only if their characteristic functions coincide.

Finally, by using the above-mentioned model and the Sz.-Nagy—Foiaş lifting theorem [8], [9], [1], [2], [6], we give explicit forms for the commutants of an infinite sequence $\mathscr{T}$ of noncommuting operators.

We point out that an important role in this paper is played by a sequence $\mathscr{S} = \{S_1, S_2, \ldots\}$ of unilateral shifts on a Hilbert space $\ell^2(\mathscr{F}, \mathscr{H})$, with orthogonal final spaces and such that the operator matrix $[S_1, S_2, \ldots]$ is nonunitary (see Section 1).

Let us mention that A. E. Frazho uses (in [3]) a countable number of shifts in a Fock space, in an algebraic setting, to solve a realization problem. Reference [4] also uses two shifts on an ℓ^2 space to solve certain problem in stochastic processes.

Although the Fock space setting is natural for transfer functions of certain system, as explained in [7], or to certain problems in control, it is not the best space to use in dilation theory. The framework of this paper is that of an $\ell^2(\mathscr{F}, \mathscr{H})$ space.

1. NOTATION AND PRELIMINARIES

To put our work in perspective, let us recall from [9], [2], [5], [6], some facts from dilation theory for an infinite sequence $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$ of noncommuting operators on a Hilbert space $\mathcal{H}$, when the matrix $[T_1, T_2, \ldots]$ is a contraction.

Let Λ be the set $\{1, 2, \ldots, k\}$ $(k \in \mathbf{N})$ or $\mathbf{N} = \{1, 2, 3, \ldots\}$ and, for every $n \in \mathbf{N}$, let $F(n, \Lambda)$ be the set of all functions from the set $\{1, 2, \ldots, n\}$ to the set Λ. Denote the set $\bigcup_{n=0}^{\infty} F(n, \Lambda)$ by $\mathcal{F}$, where $F(0, \Lambda) = \{0\}$.

A subspace $\mathcal{L}$ of $\mathcal{H}$ will be called an wandering subspace for the sequence $\mathcal{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ of isometries on $\mathcal{H}$ if for any distinct functions $f, g \in \mathcal{F}$ we have $V_f\mathcal{L} \perp V_g\mathcal{L}$, where for each $f \in \mathcal{F}$, V_f stands for the product $V_{f(1)}V_{f(2)} \ldots V_{f(n)}$ and $V_0 = I_{\mathcal{H}}$ (the identity on $\mathcal{H}$).

We say that $\mathcal{V}$ is a Λ-orthogonal shift on $\mathcal{H}$ if there exists a subspace $\mathcal{L} \subset \mathcal{H}$ which is wandering for $\mathcal{V}$ and

$$\mathcal{H} = M_{\mathcal{F}}(\mathcal{L}) \overset{\text{def}}{=} \bigoplus_{f\in\mathcal{F}} V_f\mathcal{L}. \tag{1.1}$$

Now let $\mathcal{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ be the Λ-orthogonal shift with the wandering subspace $\mathcal{H}$, defined on the Hilbert space

$$\ell^2(\mathcal{F}, \mathcal{H}) = \{(h_f)_{f\in\mathcal{F}}\ ;\ \sum_{f\in\mathcal{F}} \|h_f\|^2 < \infty,\ h_f \in \mathcal{H}\} \tag{1.2}$$

as follows.

For each $\lambda \in \Lambda$ we put $S_\lambda((h_f)_{f\in\mathcal{F}}) = (h'_g)_{g\in\mathcal{F}}$, where $h'_0 = 0$ and for $g \in F(n, \Lambda)$ $(n \geqslant 1)$

$$h'_g = \begin{cases} h_0; & \text{if } g \in F(1, \Lambda) \text{ and } g(1) = \lambda, \\ h_f; & \text{if } g \in F(n, \Lambda),\ (n \geqslant 2),\ f \in F(n-1, \Lambda) \text{ and } g(1) = \lambda, \\ & \quad g(2) = f(1),\ g(3) = f(2), \ldots, g(n) = f(n-1), \\ 0\ ; & \text{otherwise.} \end{cases}$$

This model will play an important role in our investigation.

We can easily see how acts the Λ-orthogonal shift with the wandering subspace $\mathcal{H}$, if we consider another model. For this, let us form the Hilbert space of all formal power series with noncommuting indeterminates X_λ $(\lambda \in \Lambda)$

$$\mathcal{S}^2(\mathcal{F}, \mathcal{H}) = \{\sum_{f\in\mathcal{F}} a_f X_f\ ;\ a_f \in \mathcal{H},\ \sum_{f\in\mathcal{F}} \|a_f\|^2 < \infty\},$$

with the inner product

$$\langle \sum_{f \in \mathcal{F}} a_f X_f , \sum_{f \in \mathcal{F}} b_f X_f \rangle = \sum_{f \in \mathcal{F}} (a_f , b_f),$$

where for any $f \in F(n, \Lambda)$, X_f stands for $X_{f(1)} X_{f(2)} \dots X_{f(n)}$.

Define the Λ-orthogonal shift $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ on $\mathcal{S}^2(\mathcal{F}, \mathcal{H})$ by setting

$$S_\lambda(\sum_{f \in \mathcal{F}} a_f X_f) = \sum_{f \in \mathcal{F}} a_f X_\lambda X_f , \quad (\lambda \in \Lambda).$$

When $\Lambda = \{1\}$ we find again the unilateral shift S defined by

$$S\left(\sum_{n=0}^{\infty} a_n X^n \right) = \sum_{n=0}^{\infty} a_n X^{n+1}$$

which is unitarily equivalent with the usual unilateral shift on the Hardy space $H^2(\mathbf{D}, \mathcal{H})$, where $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$.

In the case when $\Lambda = \{1, 2\}$ the Λ-orthogonal shift $\mathcal{S} = \{S_1, S_2\}$ will be unitarily equivalent with the shifts $\{S, E\}$ defined in [2] on a Fock space.

We recall from [6] that for any sequence $\mathcal{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ of noncommuting operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda \in \Lambda} T_\lambda T_\lambda^* \leqslant I_{\mathcal{H}}$, there exists a minimal isometric dilation (m.i.d.) $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, which is uniquely determined up to an isomorphism, i.e., the following conditions hold

(1.3)
$$\begin{cases} \text{a) Each operator } V_\lambda \ (\lambda \in \Lambda) \text{ is an isometry,} \\ \text{b) } \sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* \leqslant I_{\mathcal{K}}, \\ \text{c) For each } \lambda \in \Lambda,\ V_\lambda^*(\mathcal{H}) \subset \mathcal{H} \text{ and } V_\lambda^*|\mathcal{H} = T_\lambda^*, \\ \text{d) } \mathcal{K} = \bigvee_{f \in \mathcal{F}} V_f \mathcal{H}, \end{cases}$$

(see [6, Theorem 2.1]).

If we consider the following subspaces of $\mathcal{K}$

$$\mathcal{L} = \bigvee_{\lambda \in \Lambda} (V_\lambda - T_\lambda)\mathcal{H}; \quad \mathcal{L}_* = \overline{(I_{\mathcal{K}} - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*)\mathcal{H}}, \tag{1.4}$$

we have the orthogonal decompositions

$$\mathcal{K} = \mathcal{R} \oplus M_{\mathcal{F}}(\mathcal{L}_*) = \mathcal{H} \oplus M_{\mathcal{F}}(\mathcal{L}) \tag{1.5}$$

and $\mathcal{R}$ reduces each operator V_λ $(\lambda \in \Lambda)$.

Moreover, $\mathscr{R} = \{0\}$ if and only if $\mathscr{H}_0 = \{0\}$, where

$$\mathscr{H}_0 = \{h \in \mathscr{H} \; ; \lim_{n\to\infty} \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = 0\}. \tag{1.6}$$

Further, we have

$$\mathscr{L} \cap \mathscr{L}_* = \{0\} \tag{1.7}$$

and

$$M_{\mathscr{F}}(\mathscr{L}) \vee M_{\mathscr{F}}(\mathscr{L}_*) = \mathscr{H} \ominus \mathscr{H}_1, \tag{1.8}$$

where

$$\mathscr{H}_1 = \{h \in \mathscr{H} \; ; \sum_{f\in F(n,\Lambda)} \|T_f^* h\|^2 = \|h\|^2 \text{ for every } n \in \mathbf{N}\}. \tag{1.9}$$

For any sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of operators on $\mathscr{H}$ with $\sum_{\lambda\in\Lambda} T_\lambda T_\lambda^* \leqslant I_{\mathscr{H}}$ we have the following orthogonal decomposition ([6])

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2,$$

where $\mathscr{H}_0$, $\mathscr{H}_1$ are given by (1.6), (1.9) and $\mathscr{H}_2 = \mathscr{H} \ominus (\mathscr{H}_0 \oplus \mathscr{H}_1)$.

We shall say that $\mathscr{T} \in C^{(k)}$ ($\mathscr{T} \in C_{(k)}$) if $\mathscr{H}_k = \{0\}$ ($\mathscr{H} = \mathscr{H}_k$), where $k \in \{0, 1, 2\}$. A sequence $\mathscr{T} \in C^{(1)}$ will be called a completely non-coisometric (c.n.c.) sequence.

2. Λ-ORTHOGONAL SHIFTS

Let $\mathscr{E}$, $\mathscr{E}_*$ be Hilbert spaces and $\mathscr{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{E})$ or $\ell^2(\mathscr{F}, \mathscr{E}_*)$.

An operator $A\colon \ell^2(\mathscr{F}, \mathscr{E}) \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ which commutes with each S_λ ($\lambda \in \Lambda$) is uniquely determined by the operator $\theta\colon \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$, $\theta = A|\mathscr{E}$. This follows because for every $f \in \mathscr{F}$, $h \in \mathscr{E}$ we have $AS_f h = S_f \theta h$ and $\bigvee_{f\in\mathscr{F}} S_f \mathscr{E} = \ell^2(\mathscr{F}, \mathscr{E})$.

Now, let us consider an operator $\theta\colon \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$. We define $M_\theta\colon \ell^2(\mathscr{F}, \mathscr{E}) \to$ $\to \ell^2(\mathscr{F}, \mathscr{E}_*)$ by the relation

$$M_\theta S_f h = S_f \theta h = S_f M_\theta h \quad (h \in \mathscr{H},\ f \in \mathscr{F}).$$

In this paper we only work with θ such that M_θ is a contraction. One can show that

$$M_\theta((h_f)_{f\in\mathscr{F}}) = \sum_{f\in\mathscr{F}} S_f \theta h_f \quad \text{for} \quad (h_f)_{f\in\mathscr{F}} \in \ell^2(\mathscr{F}, \mathscr{E}).$$

Throughout this paper an operator $\theta: \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ will be called

(i) *inner* if M_θ is an isometry,

(ii) *outer* if $\overline{M_\theta \ell^2(\mathscr{F}, \mathscr{E})} = \ell^2(\mathscr{F}, \mathscr{E}_*)$,

(iii) *purely contractive* if $\|P_{\mathscr{E}_*}\theta h\| < \|h\|$ for every $h \in \mathscr{H}$, $h \neq 0$.

PROPOSITION 2.1. *Let $\theta: \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ be an operator such that M_θ is a contraction.*

(i) *θ is inner if and only if θ is an isometry and $\theta\mathscr{E}$ is a wandering subspace for $\mathscr{S}$.*

(ii) *θ is outer if and only if $\theta\mathscr{E}$ is cyclic for $\mathscr{S}$,* i.e.,

$$\bigvee_{f \in \mathscr{F}} S_f(\theta\mathscr{E}) = \ell^2(\mathscr{F}, \mathscr{E}_*).$$

(iii) *θ is inner and outer if and only if θ is a unitary operator from $\mathscr{E}$ to $\mathscr{E}_*$.*

Proof. Straightforward.

The version of the Beurling-Lax theorem [9], [2] in to our setting is

THEOREM 2.2. *A subspace $\mathscr{M} \subset \ell^2(\mathscr{F}, \mathscr{E})$ is invariant for each S_λ $(\lambda \in \Lambda)$ if and only if there exists a Hilbert space $\mathscr{G}$ and an inner operator $\theta: \mathscr{G} \to \ell^2(\mathscr{F}, \mathscr{E})$ such that*

$$\mathscr{M} = M_\theta \ell^2(\mathscr{F}, \mathscr{G}).$$

Proof. Using the Wold decomposition for an infinite sequence $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of isometries with orthogonal final spaces ([6]) this proof is a simple extension of that of Theorem 3.3 in [9, Chapter V] or Theorem 2 in [2].

Let $\mathscr{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be a Λ-orthogonal shift acting on a Hilbert space $\mathscr{K}$ such that $\mathscr{L} \subset \mathscr{K}$ is wandering subspace for $\mathscr{V}$, that is, $\mathscr{K} = M_{\mathscr{F}}(\mathscr{L}) = \bigoplus_{f \in \mathscr{F}} V_f \mathscr{L}$.

Denote by $\Phi^{\mathscr{L}}$ the unitary operator from $M_{\mathscr{F}}(\mathscr{L})$ to $\ell^2(\mathscr{F}, \mathscr{L})$ defined by

$$\Phi^{\mathscr{L}}\Big(\sum_{f \in \mathscr{F}} V_f l_f\Big) = \sum_{f \in \mathscr{F}} S_f l_f \quad \Big(l_f \in \mathscr{L};\ \sum_{f \in \mathscr{F}} \|l_f\|^2 < \infty\Big),$$

where $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{L})$.

Then for any $\lambda \in \Lambda$ we have

$$\Phi^{\mathscr{L}} V_\lambda = S_\lambda \Phi^{\mathscr{L}}.$$

The following extension of Lemma 3.2 in [9, Chapter V] will be used in the sequel. We omit the proof which is simple to deduce.

LEMMA 2.3. *Let $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ and $\mathscr{V}' = \{V'_\lambda\}_{\lambda\in\Lambda}$ be Λ-orthogonal shifts on the Hilbert spaces $\mathscr{K}$ and $\mathscr{K}'$, with the wandering subspaces $\mathscr{L}$ and $\mathscr{L}'$, respectively. Let Q be a contraction of $\mathscr{K}$ into $\mathscr{K}'$ such that for any $\lambda \in \Lambda$*

$$QV_\lambda = V'_\lambda Q.$$

Then there exists θ a contraction of $\mathscr{L}$ into $\ell^2(\mathscr{F}, \mathscr{L}')$ such that

$$\Phi^{\mathscr{L}'}Q = M_\theta \Phi^{\mathscr{L}}.$$

In order that θ be

(a) *purely contractive,*

(b) *inner,*

(c) *outer,*

(d) *a unitary from $\mathscr{L}$ to $\mathscr{L}'$,*

it is necessary and sufficient that the following conditions hold, respectively:

(a) $\|P_{\mathscr{L}'}Ql\| < \|l\|$ *for every $l \in \mathscr{L}$, $l \neq 0$,*

(b) *Q is an isometry,*

(c) $\overline{Q\mathscr{K}} = \mathscr{K}'$,

(d) *Q is a unitary.*

3. THE CHARACTERISTIC FUNCTION FOR $\mathscr{T}$

Let $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ be a sequence of noncommuting operators on a Hilbert space $\mathscr{H}$ such that the matrix $[T_1, T_2, \ldots]$ is a contraction. Let us recall from [6] that the defect operators of $\mathscr{T}$ are

$$D_* = (I_{\mathscr{H}} - \sum_{f\in\Lambda} T_\lambda T_\lambda^*)^{1/2}; \quad D = D_T,$$

where T stands for the matrix $[T_1, T_2, \ldots]$ and $D_T = (I - T^*T)^{1/2}$. The defect spaces of $\mathscr{T}$ are

$$\mathscr{D}_* = \overline{D_*\mathscr{H}}; \quad \mathscr{D} = \overline{D(\bigoplus_{\lambda\in\Lambda}\mathscr{H}_\lambda)},$$

where each $\mathscr{H}_\lambda$ ($\lambda \in \Lambda$) is a copy of $\mathscr{H}$.

We define the characteristic function of $\mathscr{T}$ as the operator $\theta_{\mathscr{T}}: \mathscr{D} \to \ell^2(\mathscr{F}, \mathscr{D}_*)$ by

$$\theta_{\mathscr{T}}(h) = -\sum_{\lambda\in\Lambda} T_\lambda P_\lambda h + \sum_{\lambda\in\Lambda} S_\lambda((D_* T_f^* P_\lambda Dh)_{f\in\mathscr{F}}) \quad (h \in \mathscr{D}),$$

where P_λ stands for the orthogonal projection of $\mathscr{D} \subset \bigoplus_{\lambda\in\Lambda} \mathscr{H}_\lambda$ onto $\mathscr{H}_\lambda$ and $\mathscr{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{D}_*)$.

It is easy to see that $\theta_{\mathscr{T}}$ is a contraction and moreover $\theta_{\mathscr{T}}$ is purely contractive.

Let us remark that if $\mathscr{T} = \{T\}$ ($\|T\| \leqslant 1$), the "characteristic function" of $\mathscr{T}$ is the operator $\theta_{\mathscr{T}}: \mathscr{D}_T \to \ell^2(\mathrm{N}, \mathscr{D}_{T^*})$ given by the following matrix

$$\begin{bmatrix} -T \\ D_{T^*}D_T \\ D_{T^*}T^*D_T \\ D_{T^*}T^{*2}D_T \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

We remark that $M_{\theta_{\mathscr{T}}}$ is unitarily equivalent to $(\theta_T)_+ : L^2_+(\mathscr{D}_T) \to L^2_+(\mathscr{D}_{T^*})$, where θ_T is the classical characteristic function of the contraction T and $(\theta_T)_+$ is defined in [9, Chapter V, Section 2].

Let us consider another sequence $\mathscr{T}' = \{T'_\lambda\}_{\lambda\in\Lambda}$ on a Hilbert space $\mathscr{H}'$ such that the matrix $[T'_1, T'_2, \ldots]$ is a contraction.

We say that the characteristic functions $\theta_{\mathscr{T}}$ and $\theta_{\mathscr{T}'}$ coincide if there exists two unitary operators

$$W: \mathscr{D} \to \mathscr{D}', \quad W_*: \mathscr{D}_* \to \mathscr{D}'_*$$

such that

$$M_{W_*}\theta_{\mathscr{T}} = \theta_{\mathscr{T}'}W,$$

where M_{W_*} is defined as in Section 2.

One can easily show that if $\mathscr{T}$ and $\mathscr{T}'$ are unitarily equivalent, i.e., $T'_\lambda = UT_\lambda U^*$ for any $\lambda \in \Lambda$, where U is a unitary operator from $\mathscr{H}$ to $\mathscr{H}'$, then their characteristic functions coincide. The converse is not true, at least not in this generality. Notice also that if $\mathscr{T} \in C_{(1)}$ then $\theta_{\mathscr{T}} = 0$.

We are now going to show that the definition of the characteristic function for $\mathscr{T}$ arises in a natural way in the context of the theory of isometric dilation of a sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ of noncommuting operators on $\mathscr{H}$ such that the matrix $[T_1, T_2, \ldots]$ is a contraction (see [6]).

Let $\mathscr{V} = \{V_\lambda\}_{\lambda\in\Lambda}$ be the m.i.d. of $\mathscr{T}$ on the Hilbert space $\mathscr{K} \supset \mathscr{H}$.

By (1.5) we have that $\{V_\lambda | M_{\mathscr{F}}(\mathscr{L}_*)\}_{\lambda\in\Lambda}$ and $\{V_\lambda | M_{\mathscr{F}}(\mathscr{L})\}_{\lambda\in\Lambda}$ are Λ-orthogonal shifts acting on $M_{\mathscr{F}}(\mathscr{L}_*)$ and $M_{\mathscr{F}}(\mathscr{L})$, respectively.

Moreover, for each $\lambda \in \Lambda$

$$(P^{\mathscr{L}_*}|M_{\mathscr{F}}(\mathscr{L}))(V_\lambda|M_{\mathscr{F}}(\mathscr{L})) = (V_\lambda|M_{\mathscr{F}}(\mathscr{L}_*))(P^{\mathscr{L}_*}|M_{\mathscr{F}}(\mathscr{L})),$$

where $P^{\mathscr{L}_*}$ stands for the orthogonal projection of $\mathscr{K}$ onto $M_{\mathscr{F}}(\mathscr{L}_*)$.

Setting $Q = P^{\mathscr{L}_*}|M_{\mathscr{F}}(\mathscr{L})$, we can apply Lemma 2.3 and we obtain that there exists a contraction $\theta_{\mathscr{L}}: \mathscr{L} \to \ell^2(\mathscr{F}, \mathscr{L}_*)$ such that

$$\Phi^{\mathscr{L}_*}Q = M_{\theta_{\mathscr{L}}}\Phi^{\mathscr{L}},$$

where $\Phi^{\mathscr{L}_*}$, $\Phi^{\mathscr{L}}$ have been defined in Section 2.

Hence we deduce that

$$\theta_{\mathscr{L}} = \Phi^{\mathscr{L}_*}(P^{\mathscr{L}_*}|\mathscr{L})(\Phi^{\mathscr{L}})^*|\mathscr{L}. \tag{3.1}$$

We remark first that $\theta_{\mathscr{L}}$ is purely contractive. Indeed, if $P_{\mathscr{L}_*}$ denotes the orthogonal projection onto $\mathscr{L}_*$, we have $\|P_{\mathscr{L}_*}P^{\mathscr{L}_*}l\| < \|l\|$ for every $l \in \mathscr{L}$, $l \neq 0$. Otherwise there would exist an $l \in \mathscr{L}$, $l \neq 0$ such that $l = P_{\mathscr{L}_*}P^{\mathscr{L}_*}l$, i.e., $l \in \mathscr{L}_*$, and this contradicts the relation (1.7).

Let us recall from [6] that the operator Φ_* defined from $\mathscr{L}_*$ to $\mathscr{D}_*$ by

$$\Phi_*(I_{\mathscr{K}} - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*)h = D_*h; \quad (h \in \mathscr{H}) \tag{3.2}$$

is unitary and the operator Φ defined form $\mathscr{L}$ to $\mathscr{D}$ by

$$\Phi(\sum_{\lambda\in\Lambda}(V_\lambda - T_\lambda)h_\lambda) = D((h_\lambda)_{\lambda\in\Lambda}); \quad (h_\lambda)_{\lambda\in\Lambda} \in \bigoplus_{\lambda\in\Lambda}\mathscr{H}_\lambda \tag{3.3}$$

is unitary too.

We are ready for proving the following theorem which is a generalization of Proposition 2.2 in [9, Chapter VI].

THEOREM 3.1. *The characteristic function $\theta_{\mathscr{T}}$ for $\mathscr{T}$ coincides with $\theta_{\mathscr{L}}$.*

Proof. We show that

$$M_{\Phi_*}\theta_{\mathscr{L}} = \theta_{\mathscr{T}}\Phi, \tag{3.4}$$

where Φ_*, Φ are the unitary operators in (3.2), (3.3), respectively. For this, it is necessarily to prove that

$$P_{\mathscr{D}_*}S_f^*M_{\Phi_*}\theta_{\mathscr{L}} = P_{\mathscr{D}_*}S_f^*\theta_{\mathscr{T}}\Phi \quad (f \in \mathscr{F}), \tag{3.5}$$

where $P_{\mathscr{D}_*}$ stands for the orthogonal projection of $\ell^2(\mathscr{F}, \mathscr{D}_*)$ onto $\mathscr{D}_*$.

By (3.1) and by the Wold decomposition (1.5), the relation (3.5) is equivalent to

$$\Phi_* P_{\mathscr{L}_*} V_f^* | \mathscr{L} = P_{\mathscr{D}_*} S_f^* \theta_{\mathscr{T}} \Phi \quad (f \in \mathscr{F}). \tag{3.6}$$

In what follows we shall prove this relation. First let us notice that

$$\begin{aligned} P_{\mathscr{D}_*} \theta_{\mathscr{T}} &= -\sum_{\lambda \in \Lambda} T_\lambda P_\lambda , \\ P_{\mathscr{D}_*} S_f^* S_\lambda \theta_{\mathscr{T}} &= D_* T_f^* P_\lambda D \quad (\lambda \in \Lambda,\ f \in \mathscr{F}). \end{aligned} \tag{3.7}$$

For $f = 0$ the relation (3.6) holds true. Indeed, for

$$l = \sum_{\lambda \in \Lambda} (V_\lambda - T_\lambda) h_\lambda = \Phi^* D(\bigoplus_{\lambda \in \Lambda} h_\lambda) \quad (\sum_{\lambda \in \Lambda} \|h_\lambda\|^2 < \infty) \tag{3.8}$$

we have that $l + (I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*) \sum_{\lambda \in \Lambda} T_\lambda h_\lambda \in \bigoplus_{\lambda \in \Lambda} V_\lambda \mathscr{K}$ and by (1.5) we obtain that

$$P_{\mathscr{L}_*} l = -(I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*) \sum_{\lambda \in \Lambda} T_\lambda h_\lambda .$$

Hence, by (3.7) we have

$$\Phi_* P_{\mathscr{L}_*} l = - D_* [T_1, T_2, \ldots](\bigoplus_{\lambda \in \Lambda} h_\lambda) = [T_1, T_2, \ldots]\, D(\bigoplus_{\lambda \in \Lambda} h_\lambda) =$$

$$= -[T_1, T_2, \ldots] \Phi l = P_{\mathscr{D}_*} \theta_{\mathscr{T}} \Phi l.$$

It remains to show that for any $f \in \mathscr{F}$, $\lambda \in \Lambda$

$$\Phi_* P_{\mathscr{L}_*} V_f^* V_\lambda^* l = P_{\mathscr{D}_*} S_f^* S_\lambda^* \theta_{\mathscr{T}} \Phi l \quad (l \in \mathscr{L}). \tag{3.9}$$

Let l be as in (3.8); then according to (3.7) the relation (3.9) becomes

$$\Phi_* P_{\mathscr{L}_*} V_f^* V_\lambda^* l = D_* T_f^* P_\lambda D^2 (\bigoplus_{\lambda \in \Lambda} h_\lambda).$$

Since

$$D_* T_f^* P_\lambda D^2 (\bigoplus_{\lambda \in \Lambda} h_\lambda) = \Phi_* (I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*) T_f^* P_\lambda D^2 (\bigoplus_{\lambda \in \Lambda} h_\lambda),$$

we have only to show that

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* l = (I - \sum_{\lambda \in \Lambda} V_\lambda T_\lambda^*) T_f^* P_\lambda D^2 (\bigoplus_{\lambda \in \Lambda} h_\lambda). \tag{3.10}$$

Let us notice that for any $\lambda \in \Lambda$

$$P_\lambda D^2(\bigoplus_{\lambda\in\Lambda} h_\lambda) = -\sum_{\substack{\mu\in\Lambda\\ \mu\neq\lambda}} T_\lambda^* T_\mu h_\mu + D_{T_\lambda}^2 h_\lambda .$$

Consequently, the relation (3.10) holds if and only if the following relations hold

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* (V_\lambda h_\lambda - T_\lambda h_\lambda) = (I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*) T_f^* D_{T_\lambda}^2 h_\lambda \quad (\lambda \in \Lambda)$$

and

$$P_{\mathscr{L}_*} V_f^* V_\lambda^* (V_\mu h_\mu - T_\mu h_\mu) = -(I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*) T_f^* T_\lambda^* T_\mu h_\mu \quad (\lambda \neq \mu).$$

These relations hold since the elements

$$V_f^* V_\lambda^* (V_\lambda h_\lambda - T_\lambda h_\lambda) - (I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*) T_f^* D_{T_\lambda} h_\lambda \quad (\lambda \in \Lambda)$$

and

$$V_f^* V_\lambda^* (V_\mu h_\mu - T_\mu h_\mu) + (I - \sum_{\lambda\in\Lambda} V_\lambda T_\lambda^*) T_f^* T_\lambda^* T_\mu h_\mu \quad (\lambda \neq \mu)$$

are orthogonal on $\mathscr{L}_*$. This follows by a simple computation. The proof is complete.

REMARK 3.2. If $\mathscr{T} \in C_{(0)}$ then $\Theta_{\mathscr{T}}$ is inner.

Proof. Taking into account Theorem 2.8 in [6], it follows that the m.i.d. $\mathscr{V}$ of $\mathscr{T}$ is pure, i.e., $\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}_*)$. By relation (3.1) and Theorem 3.1 it follows that $\Theta_{\mathscr{T}}$ is inner.

4. FUNCTIONAL MODEL FOR A GIVEN C.N.C. $\mathscr{T}$

In this section we make the additional assumption that $\mathscr{T}$ is c.n.c. on $\mathscr{H}$. Then the relation (1.8) implies

$$\mathscr{K} = M_{\mathscr{T}}(\mathscr{L}) \vee M_{\mathscr{T}}(\mathscr{L}_*)$$

and consequently,

$$(I - P^{\mathscr{L}_*}) M_{\mathscr{T}}(\mathscr{L}) = \mathscr{R} \qquad \text{(cf. (1.5))}.$$

Consider the operator $\Delta_{\mathscr{L}}$ defined on $\ell^2(\mathscr{F}, \mathscr{L})$ by

$$\Delta_{\mathscr{L}} = (I - M^*_{\theta_{\mathscr{L}}} M_{\theta_{\mathscr{L}}})^{1/2},$$

where $\theta_{\mathscr{L}}$ is given by (3.1).

For $k \in M_{\mathscr{F}}(\mathscr{L})$ we have

$$\|(I - P^{\mathscr{L}*})k\|^2 = \|k\|^2 - \|P^{\mathscr{L}*}k\|^2 = \|\Phi^{\mathscr{L}}k\|^2 - \|\Phi^{\mathscr{L}*}P^{\mathscr{L}*}k\|^2 =$$

$$= \|\Phi^{\mathscr{L}}k\|^2 - \|M_{\theta_{\mathscr{L}}}\Phi^{\mathscr{L}}k\|^2 = \|\Delta_{\mathscr{L}}\Phi^{\mathscr{L}}k\|^2.$$

We can define the unitary operator $\Phi_{\mathscr{R}}$ from $\mathscr{R}$ onto $\overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})}$ by the relation

$$\Phi_{\mathscr{R}}(I - P^{\mathscr{L}*})k = \Delta_{\mathscr{L}}\Phi^{\mathscr{L}}k \quad (k \in M_{\mathscr{F}}(\mathscr{L})).$$

Consequently,

$$\Phi = \Phi^{\mathscr{L}*} \oplus \Phi_{\mathscr{R}}$$

is a unitary operator from the space $\mathscr{K} = M_{\mathscr{F}}(\mathscr{L}_*) \oplus \mathscr{R}$ to the Hilbert space

$$K = \ell^2(\mathscr{F}, \mathscr{L}_*) \oplus \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})}.$$

Let us find the image of the space $\mathscr{H}$ under the operator Φ. Since $\mathscr{H} = \mathscr{K} \ominus \ominus M_{\mathscr{F}}(\mathscr{L})$ and for each $k \in M_{\mathscr{F}}(\mathscr{L})$

$$\Phi k = \Phi^{\mathscr{L}*}P^{\mathscr{L}*}k \oplus \Phi_{\mathscr{R}}(I - P^{\mathscr{L}*})k = M_{\theta_{\mathscr{L}}}\Phi^{\mathscr{L}}k \oplus \Delta_{\mathscr{L}}\Phi^{\mathscr{L}}k$$

we have

$$\Phi\mathscr{H} = H = [\ell^2(\mathscr{F}, \mathscr{L}_*) \oplus \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})}] \ominus \{M_{\theta_{\mathscr{L}}}u \oplus \Delta_{\mathscr{L}}u \ ; \ u \in \ell^2(\mathscr{F}, \mathscr{L})\}.$$

Because $P^{\mathscr{L}*}$ commutes with each V_λ ($\lambda \in \Lambda$) it follows that

$$\Phi_{\mathscr{R}}V_\lambda(I - P^{\mathscr{L}*})k = \Phi_{\mathscr{R}}(I - P^{\mathscr{L}*})V_\lambda k = \Delta_{\mathscr{L}}\Phi^{\mathscr{L}}V_\lambda k = \Delta_{\mathscr{L}}S_\lambda\Phi^{\mathscr{L}}k$$

for every $k \in M_{\mathscr{F}}(\mathscr{L})$, where $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift on $\ell^2(\mathscr{F}, \mathscr{L})$.

Therefore,

$$\Phi_{\mathscr{R}}V_\lambda\Phi^*_{\mathscr{R}}(\Delta_{\mathscr{L}}v) = \Delta_{\mathscr{L}}S_\lambda v \quad (v \in \ell^2(\mathscr{F}, \mathscr{L}))$$

and

$$\Phi V_\lambda \Phi^* = \mathbf{V}_\lambda = S_\lambda \oplus C_\lambda \quad \text{for every } \lambda \in \Lambda,$$

where each operator C_λ is an isometry defined on $\overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})}$ by the relation

$$C_\lambda(\Delta_{\mathscr{L}}v) = \Delta_{\mathscr{L}}S_\lambda v \quad \text{for} \quad v \in \ell^2(\mathscr{F}, \mathscr{L}).$$

Now, since $(\sum_{\lambda\in\Lambda} V_\lambda V_\lambda^* - I)|\mathscr{R} = 0$ we have

$$\sum_{\lambda\in\Lambda} C_\lambda C_\lambda^* = I_{\overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F},\mathscr{L})}}, \quad \text{whence } \overline{\Delta_{\mathscr{L}}\ell^2(\mathscr{F}, \mathscr{L})} = \overline{\Delta_{\mathscr{L}}(\ell^2(\mathscr{F}, \mathscr{L}) \ominus \mathscr{L})}.$$

It is easy to see that for every $v \in \ell^2(\mathscr{F}, \mathscr{L})$ and $\lambda, \mu \in \Lambda$

$$C_\lambda^*(\Delta_{\mathscr{L}}S_\mu v) = \begin{cases} \Delta_{\mathscr{L}}v & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

According to (1.3) we have $\mathbf{T}_\lambda^* = \mathbf{V}_\lambda^*|H$, where $\mathbf{T}_\lambda$ is the transform of T_λ by Φ.

Therefore, for $u \oplus \Delta_{\mathscr{L}}S_\mu v \in H$ we can write that

$$\mathbf{T}_\lambda^*(u \oplus \Delta_{\mathscr{L}}S_\mu v) = \begin{cases} S_\lambda^*u \oplus \Delta_{\mathscr{L}}v & \text{if } \mu = \lambda \\ S_\lambda^*u \oplus 0 & \text{if } \mu \neq \lambda, \end{cases}$$

where $\lambda, \mu \in \Lambda$.

The above results and that of Section 3 permit us to construct a model for a c.n.c. sequence $\mathscr{T}$, in which the characteristic function occurs explicitly. We obtain a generalization of Theorem 2.3 in [9, Chapter VI], namely:

THEOREM 4.1. *Every completely non-coisometric sequence $\mathscr{T} = \{T_\lambda\}_{\lambda\in\Lambda}$ on the Hilbert space $\mathscr{H}$ is unitarily equivalent to a sequence $\mathbf{T} = \{\mathbf{T}_\lambda\}_{\lambda\in\Lambda}$ on the Hilbert space*

$$H = [\ell^2(\mathscr{F}, \mathscr{D}_*) \oplus \overline{\Delta_{\mathscr{T}}\ell^2(\mathscr{F}, \mathscr{D})}] \ominus \{M_{\theta_{\mathscr{T}}}u \oplus \Delta_{\mathscr{T}}u \; ; \; u \in \ell^2(\mathscr{F}, \mathscr{D})\},$$

where $\Delta_{\mathscr{T}} = (I - M_{\theta_{\mathscr{T}}}^* M_{\theta_{\mathscr{T}}})^{1/2}$.

For each $\lambda \in \Lambda$ the operator $\mathbf{T}_\lambda$ is defined by

$$\mathbf{T}_\lambda^*(u \oplus \Delta_{\mathscr{T}}S_\mu v) = \begin{cases} S_\lambda^*u \oplus \Delta_{\mathscr{T}}u & \text{if } \mu = \lambda, \\ S_\lambda^*u \oplus 0 & \text{if } \mu \neq \lambda, \end{cases}$$

where $\mathscr{S} = \{S_\lambda\}_{\lambda\in\Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{D})$ or $\ell^2(\mathscr{F}, \mathscr{D}_*)$.

If $\mathscr{T} \in C_{(0)}$, and only in this case, $\theta_{\mathscr{T}}$ is inner, and this model reduces to

$$H = \ell^2(\mathscr{F}, \mathscr{D}_*) \ominus M_{\theta_{\mathscr{T}}}\ell^2(\mathscr{F}, \mathscr{D}); \quad \mathbf{T}_\lambda^*u = S_\lambda^*u \quad (u \in H).$$

Proof. By virtue of the relation (3.4) it follows that

$$M_{\Phi_*} M_{\theta_{\mathscr{L}}} = M_{\theta_{\mathscr{T}}} M_{\Phi}.$$

Hence we obtain that $\Delta_{\mathscr{T}} = M_{\Phi} \Delta_{\mathscr{L}} M_{\Phi}$.

On the other hand the operators Φ and Φ_* defined by (3.2) and (3.3) generate the unitary operator

$$U = M_{\Phi_*} \oplus M_{\Phi}$$

from the space $\ell^2(\mathscr{F}, \mathscr{L}_*) \oplus \overline{\Delta_{\mathscr{L}} \ell^2(\mathscr{F}, \mathscr{L})}$ to the space $\ell^2(\mathscr{F}, \mathscr{D}_*) \oplus \overline{\Delta_{\mathscr{T}} \ell^2(\mathscr{F}, \mathscr{D})}$ such that

$$U\{M_{\theta_{\mathscr{L}}} u \oplus \Delta_{\mathscr{L}} u \ ; \ u \in \ell^2(\mathscr{F}, \mathscr{L})\} = \{M_{\theta_{\mathscr{T}}} v \oplus \Delta_{\mathscr{T}} v \ ; \ v \in \ell^2(\mathscr{F}, \mathscr{D})\}.$$

By means of this unitary operator we can rewrite the results obtained before this theorem and, in this way, we complete the proof.

Let us remark that for $\mathscr{T} = \{T\}$, we find a model for completely non-coisometric contractions, which coincides with the Sz.-Nagy—Foiaş model. Indeed, if T is a completely non-coisometric contraction, that is, if there is no non-zero invariant subspace for T^* on which T^* is an isometry, then it is easy to see that

$$\Delta_T H^2(\mathscr{D}_T) = \Delta_T L^2(\mathscr{D}_T)$$

(for notation see Theorem 2.3 in [9, Chapter VI]).

Let us note that the Sz.-Nagy—Foiaş model is given for a larger class of contractions, namely for completely non-unitary contractions.

5. MODEL FOR A GIVEN θ

The aim of this section is to show that any contraction $\theta\colon \mathscr{E} \to \ell^2(\mathscr{F}, \mathscr{E}_*)$ ($\mathscr{E}$, $\mathscr{E}_*$ Hilbert spaces) such that M_θ is contraction generates, as in Section 4, a c.n.c. sequence $\boldsymbol{T} = \{\mathrm{T}_\lambda\}_{\lambda \in \Lambda}$.

In the case when θ is purely contractive and

$$\overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})} = \overline{\Delta_\theta[\ell^2(\mathscr{F}, \mathscr{E}) \ominus \mathscr{E}]} \tag{5.1}$$

we shall show that θ coincides with the characteristic function of $\mathscr{T}$.

The main result of this section is the following generalization of Theorem 3.1 in [9, Chapter VI].

THEOREM 5.1, *Let θ be a contraction from $\mathscr{E}$ to $\ell^2(\mathscr{F}, \mathscr{E}_*)$ such that M_θ is a contraction. Setting $\Delta_\theta = (I - M_\theta^* M_\theta)^{1/2}$, the sequence $\boldsymbol{T} = \{\mathbf{T}_\lambda\}_{\lambda \in \Lambda}$ of operators defined on the Hilbert space*

$$H = [\ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})}] \ominus \{M_\theta w \oplus \Delta_\theta w \ ; \ w \in \ell^2(\mathscr{F}, \mathscr{E})\}$$

by

$$\mathbf{T}_\lambda(u \oplus \Delta_\theta v) = S_\lambda^* u \oplus C_\lambda^*(\Delta_\theta v) \quad (\lambda \in \Lambda),$$

where each operator C_λ is defined by $C_\lambda(\Delta_\theta g) = \Delta_\theta S_\lambda g$ $(g \in \ell^2(\mathscr{F}, \mathscr{E}))$ and $\mathscr{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ is the Λ-orthogonal shift acting on $\ell^2(\mathscr{F}, \mathscr{E})$ or $\ell^2(\mathscr{F}, \mathscr{E}_*)$, is completely non-coisometric.

If θ is purely contractive and (5.1) *holds, then θ coincides with the characteristic function of $\mathscr{T}$. In this case, considering H as a subspace of*

$$K = \ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})}$$

we have that the sequence $\boldsymbol{V} = \{\mathbf{V}_\lambda\}_{\lambda \in \Lambda}$ of operators defined on K by

$$\mathbf{V}_\lambda = S_\lambda \oplus C_\lambda \quad (\lambda \in \Lambda)$$

is the minimal isometric dilation of $\boldsymbol{T}$.

Proof. A. Let us consider the following Hilbert spaces

$$K = \ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})},$$

$$G = \{M_\theta w \oplus \Delta_\theta w \ ; \ w \in \ell^2(\mathscr{F}, \mathscr{E})\},$$

and let $\boldsymbol{V} = \{\mathbf{V}_\lambda\}_{\lambda \in \Lambda}$ be a sequence of isometries defined on K by $\mathbf{V}_\lambda = S_\lambda \oplus C_\lambda$ $(\lambda \in \Lambda)$, where each C_λ is given by

$$C_\lambda(\Delta_\theta g) = \Delta_\theta S_\lambda g \quad \text{for } g \in \ell^2(\mathscr{F}, \mathscr{E}).$$

It is easy to see that

$$\sum_{\lambda \in \Lambda} \mathbf{V}_\lambda \mathbf{V}_\lambda^* \leqslant I_K$$

and that G is invariant for each $\mathbf{V}_\lambda$ $(\lambda \in \Lambda)$.

Setting $H = K \ominus G$ and $\mathbf{T}_\lambda^* = \mathbf{V}_\lambda^* | H$ $(\lambda \in \Lambda)$ we see that $\boldsymbol{V}$ is an isometric dilation of $\boldsymbol{T} = \{\mathbf{T}_\lambda\}_{\lambda \in \Lambda}$.

Let us show that $\boldsymbol{T}$ is c.n.c. For this, let $u \oplus \Delta_\theta v \in H$ such that for every $n \in \mathbf{N}$ we have

$$\sum_{f \in F(n,\Lambda)} \|T_f^*(u \oplus \Delta_\theta v)\|^2 = \|u \oplus \Delta_\theta v\|^2. \tag{5.2}$$

Since $\lim\limits_{n\to\infty} \sum\limits_{f \in F(n,\Lambda)} \|S_f^* u\|^2 = 0$ and $\sum\limits_{f \in F(n,\Lambda)} \|C_f^* \Delta_\theta v\|^2 \leqslant \|\Delta_\theta v\|^2$ it follows that $u = 0$. But, $(0 \oplus \Delta_\theta v, M_\theta w \oplus \Delta_\theta w) = 0$ for any $w \in \ell^2(\mathscr{F}, \mathscr{E})$ implies $\Delta_\theta v = 0$. Thus $\boldsymbol{T}$ is c.n.c.

B. We assume from now on that θ is purely contractive and that (5.1) holds. Let us show that $\boldsymbol{V}$ is m.i.d. of $\boldsymbol{T}$, i.e.

$$K = \bigvee_{f \in \mathscr{F}} \mathbf{V}_f H.$$

First we note that (5.1) implies

$$\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^* = I_{\overline{\Delta_\theta \ell^2(\mathscr{F}, \mathscr{E})}}. \tag{5.3}$$

Suppose $u \oplus \Delta_\theta v \in K$ and for every $f \in \mathscr{F}$, $u \oplus \Delta_\theta v \perp \mathbf{V}_f H$, i.e., $\mathbf{V}_f^*(u \oplus \Delta_\theta v) \in G$. This means that for each $f \in \mathscr{F}$ there exists $w_{(f)} \in \ell^2(\mathscr{F}, \mathscr{E})$ such that

$$\mathbf{V}_f^*(u \oplus \Delta_\theta v) = M_\theta w_{(f)} \oplus \Delta_\theta w_{(f)}.$$

Therefore, for each $\lambda \in \Lambda$, $f \in \mathscr{F}$ there exists $w_{(f,\lambda)} \in \ell^2(\mathscr{F}, \mathscr{E})$ such that

$$\mathbf{V}_\lambda^*(M_\theta w_{(f)} \oplus \Delta_\theta w_{(f)}) = M_\theta w_{(f,\lambda)} \oplus \Delta_\theta w_{(f,\lambda)}.$$

By using the definition of $\mathbf{V}_\lambda$ $(\lambda \in \Lambda)$ we obtain

$$(\sum_{\lambda \in \Lambda} S_\lambda S_\lambda^*) M_\theta w_{(f)} \oplus (\sum_{\lambda \in \Lambda} C_\lambda C_\lambda^*) \Delta_\theta w_{(f)} =$$

$$= M_\theta (\sum_{\lambda \in \Lambda} S_\lambda w_{(f,\lambda)}) \oplus \Delta_\theta (\sum_{\lambda \in \Lambda} S_\lambda w_{(f,\lambda)}).$$

Hence, according to (5.3), we have

$$M_\theta \omega_{(f)} = P_{\mathscr{E}_*} M_\theta w_{(f)} \quad \text{and} \quad \Delta_\theta \omega_{(f)} = 0, \tag{5.4}$$

where $\omega_{(f)}$ stands for $w_{(f)} - \sum\limits_{\lambda \in \Lambda} S_\lambda w_{(f,\lambda)}$.

Since M_θ commutes with each S_λ ($\lambda \in \Lambda$) it follows that

$$P_{\mathscr{E}_*} M_\theta w_{(f)} = P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}$$

and (5.4) gives

$$\omega_{(f)} = M_\theta^* P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}, \tag{5.5}$$

hence $P_{\mathscr{E}} w_{(f)} = P_{\mathscr{E}} \omega_{(f)} = P_{\mathscr{E}} M_\theta^* P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}$.

Consequently, $\|P_{\mathscr{E}} w_{(f)}\| = \|P_{\mathscr{E}_*} M_\theta P_{\mathscr{E}} w_{(f)}\|$ and since θ is purely contractive it follows that

$$P_{\mathscr{E}} w_{(f)} = 0. \tag{5.6}$$

Now, the relation (5.5) implies $\omega_{(f)} = 0$, i.e.

$$w_{(f)} = \sum_{\lambda \in \Lambda} s_\lambda w_{(f,\lambda)} \quad \text{for } f \in \mathscr{F}.$$

Hence, we obtain that

$$w_{(0)} = \sum_{\lambda \in \Lambda} S_\lambda w_{(\lambda)} = \sum_{\lambda \in \Lambda} S_\lambda \Big(\sum_{\mu \in \Lambda} S_\mu w_{(\lambda,\mu)} \Big) = \sum_{g \in F(2,\Lambda)} S_g w_{(g)} = \ldots$$

$$\ldots = \sum_{f \in F(n,\Lambda)} S_f w_{(f)} \qquad \text{for any } n \in \mathbf{N}.$$

We deduce that $S_f^* w_{(0)} = w_{(f)}$ for every $f \in \mathscr{F}$. By (5.6) we find $P_{\mathscr{E}} S_f^* w_{(0)} = P_{\mathscr{E}} w_{(f)} = 0$ for every $f \in \mathscr{F}$.

It follows that $w_{(0)} = 0$ and $u \oplus \Delta_\theta v = M_\theta w_{(0)} \oplus \Delta_\theta w_{(0)} = 0$, which implies the minimality of $\boldsymbol{V}$.

C. Our next step is to determine

$$L_* = \overline{(I_H - \sum_{\lambda \in \Lambda} \mathbf{V}_\lambda \mathbf{T}_\lambda^*) H}.$$

Taking into account (5.3), for $u \oplus \Delta_\theta v \in H$ we have

$$(I_H - \sum_{\lambda \in \Lambda} \mathbf{V}_\lambda \mathbf{T}_\lambda^*)(u \oplus \Delta_\theta v) = P_{\mathscr{E}_*} u \oplus 0$$

and hence $L_* \subset \mathscr{E}_* \oplus \{0\}$.

Let $e_* \in \mathscr{E}_*$ and let us choose $u = (I - M_\theta M_\theta^*) e_*$ and $\Delta_\theta v = -\Delta_\theta M_\theta^* e_*$. Since $M_\theta^* u + \Delta_\theta^2 v = 0$ it follows that $u \oplus \Delta_\theta v \in H$.

Thus,

$$(I_H - \sum_{\lambda \in \Lambda} \mathbf{V}_\lambda \mathbf{T}_\lambda^*)(u \oplus \Delta_\theta v) = (I_{\mathscr{E}_*} - P_{\mathscr{E}_*} M_\theta M_\theta^*) e_* \oplus 0 .$$

Now, the elements of the form $(I_{\mathscr{E}_*} - P_{\mathscr{E}_*} M_\theta M_\theta^*) e_*$, $(e_* \in \mathscr{E}_*)$, are dense in $\mathscr{E}_*$. Otherwise there exists an $e'_* \in \mathscr{E}_*$, $e'_* \neq 0$, such that $e'_* = P_{\mathscr{E}_*} M_\theta M_\theta^* e'_*$ and hence $\|e'_*\| = \|M_\theta^* e'_*\| = \|P_{\mathscr{E}_*} M_\theta M_\theta^* e'_*\|$; $e'_* = M_\theta M_\theta^* e'_*$. Since $M_\theta^* e'_* \in \mathscr{E}_*$ and θ is purely contractive it follows that $M_\theta^* e'_* = 0$ and $e'_* = 0$ which is a contradiction.

Thus

$$L_* = \mathscr{E}_* \oplus \{0\} \tag{5.7}$$

and $M_{\mathscr{F}}(L_*) = \ell^2(\mathscr{F}, \mathscr{E}_*) \oplus \{0\}$.

Denoting by P^{L_*} the orthogonal projection of K onto $M_{\mathscr{F}}(L_*)$, we have for $u \oplus \Delta_\theta v \in K$

$$(5.8) \quad \begin{aligned} P^{L_*}(u \oplus \Delta_\theta v) &= u \oplus 0, \\ \Phi^{L_*} P^{L_*}(u \oplus \Delta_\theta v) &= \Phi^{\mathscr{E}_*} u \oplus 0 = u \oplus 0. \end{aligned}$$

D. Next we show that

$$L = \bigvee_{\lambda \in \Lambda} (\mathbf{V}_\lambda - \mathbf{T}_\lambda) H = \{M_\theta e \oplus \Delta_\theta e \; ; e \in \mathscr{E}\}.$$

Notice that an element $u \oplus \Delta_\theta v$ in K belongs to H if and only if

$$M_\theta^* u \oplus \Delta_\theta^2 v = 0. \tag{5.9}$$

For $u \oplus \Delta_\theta v \in H$ and $\lambda \in \Lambda$ we have

$$\mathbf{T}_\lambda(u \oplus \Delta_\theta v) = P_H \mathbf{V}_\lambda (u \oplus \Delta_\theta v) = (S_\lambda u \oplus \Delta_\theta S_\lambda v) - (M_\theta w_\lambda \oplus \Delta_\theta w_\lambda),$$

where each $w_\lambda \in \ell^2(\mathscr{F}, \mathscr{E})$ is defined by

$$\langle (S_\lambda u - M_\theta w_\lambda) \oplus (\Delta_\theta S_\lambda v - \Delta_\theta w_\lambda),\; M_\theta w' \oplus \Delta_\theta w' \rangle = 0$$

for every $w' \in \ell^2(\mathscr{F}, \mathscr{E})$.

Hence, we find that

$$w_\lambda = M_\theta^* S_\lambda u + \Delta_\theta^2 S_\lambda v$$

and

$$(\mathbf{V}_\lambda - \mathbf{T}_\lambda)(u \oplus \Delta_\theta v) = M_\theta w_\lambda \oplus \Delta_\theta w_\lambda .$$

By (5.9) an easy computation shows that $\langle w_\lambda, S_f e_* \rangle = 0$ for every $e_* \in \mathscr{E}_*$, $f \in \mathscr{F}$, $f \neq 0$. Consequently, $w_\lambda \in \mathscr{E}$.

Let us show that if $u \oplus \Delta_\theta v$ varies over H and λ over Λ, then the corresponding elements w_λ vary over a set dense in $\mathscr{E}$.

It is easy to see that for $e \in \mathcal{E}$ and $\lambda \in \Lambda$ the element $w_\lambda = M_\theta^* S_\lambda S_\lambda^* M_\theta e + \Delta_\theta C_\lambda C_\lambda^* \Delta_\theta e$ is the corresponding element of $S_\lambda^* M_\theta e \oplus C_\lambda^* \Delta_\theta e \in H$.

Thus, for $e \in \mathcal{E}$ we have

$$\sum_{\lambda \in \Lambda} w_\lambda = M_\theta^*(I - P_{\mathcal{E}_*}) M_\theta e + \Delta_\theta^2 e = e - M_\theta^* P_{\mathcal{E}_*} M_\theta e \in \mathcal{E}.$$

It remains to prove that the set

$$\{(I_{\mathcal{E}} - M_\theta^* P_{\mathcal{E}_*} \theta)e \ ; \ e \in \mathcal{E}\}$$

is dense in $\mathcal{E}$.

Indeed, otherwise there exists $e' \in \mathcal{E}$, $e' \neq 0$ such that $e' = M_\theta^* P_{\mathcal{E}_*} M_\theta e'$. It follows that $\|e'\| = \|P_{\mathcal{E}_*} M_\theta e'\|$, which contradicts that θ is purely contractive.

E. The last step is to prove that the characteristic function of $\boldsymbol{T}$ coincides with θ.

It is easy to see that the operator ω defined from $\mathcal{E}$ to L by $\omega(e) = M_\theta e \oplus \Delta_\theta e$ $(e \in \mathcal{E})$ is a unitary one.

On the other hand, from (5.7) it follows that the operator ω_* defined from $\mathcal{E}_*$ to L_* by $\omega_*(e_*) = e_* \oplus 0$ $(e_* \in \mathcal{E}_*)$ is unitary too.

According to (5.8), for $l = M_\theta e \oplus \Delta_\theta e$ $(e \in \mathcal{E})$ we have

$$\Phi^{L_*} P^{L_*}(M_\theta e \oplus \Delta_\theta e) = \Phi^{L_*}(M_\theta e \oplus 0) = M_\theta e \oplus 0 = M_{\omega_*} M_\theta e = M_{\omega_*} \theta \omega^{-1} e.$$

Hence, using Theorem 3.1, we deduce that the characteristic function of $\boldsymbol{T}$ coincides with θ.

The proof is completed.

REMARK 5.2. $H \neq 0$ if and only if M_θ is nonunitary.

PROPOSITION 5.3. *Let $\theta : \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ and $\theta' : \mathcal{E}' \to \ell^2(\mathcal{F}, \mathcal{E}'_*)$ be some operators such that M_θ and $M_{\theta'}$ be contractions.*

If θ and θ' coincide, then the sequences $\boldsymbol{T}$ and $\boldsymbol{T}'$ which they generate in the sense of Theorem 5.1 are unitarily equivalent.

Proof. If $\chi : \mathcal{E} \to \mathcal{E}'$ and $\chi_* : \mathcal{E}_* \to \mathcal{E}'_*$ are unitary operators such that

$$M_{\chi_*} \theta = \theta' \chi$$

then $U = M_{\chi_*} \oplus M_\chi$ is a unitary operator from H to H' such that $\mathbf{T}'_\lambda = U \mathbf{T}_\lambda U^*$ for every $\lambda \in \Lambda$.

The proof is just the same as in the particular case considered in the proof of Theorem 4.1.

Applying this result to characteristic functions and by using Theorem 4.1 we obtain a generalization of Theorem 3.4 in [9, Chapter VI] and Corollary 2 in [2], namely:

THEOREM 5.4. *Two completely non-coisometric sequences $\mathcal{T}$ and $\mathcal{T}'$ are unitarily equivalent if and only if their characteristic functions coincide.*

Finally, let us show when the characteristic function is outer.

PROPOSITION 5.5. *For a* c.n.c. *sequence $\mathcal{T}$ we have that $\theta_{\mathcal{T}}$ is outer if and only if $\mathcal{T} \in C_{(2)}$.*

Proof. It suffices to prove our assertion for the functional model of $\mathcal{T}$.

Accordingly, let $\boldsymbol{T} = \{\mathbf{T}_\lambda\}_{\lambda \in \Lambda}$ be the sequence defined in Theorem 5.1. For every $u \oplus \Delta_{\mathcal{T}} v \in H$ we have

$$\lim_{n \to \infty} \sum_{f \in F(n, \Lambda)} \|T_f^*(u \oplus \Delta_{\mathcal{T}} v)\|^2 = \|\Delta_{\mathcal{T}} v\|^2.$$

This shows that $\boldsymbol{T} \in C_{(2)}$ if and only if $u \oplus 0 \in H$ implies $u = 0$. On the other hand, $u \oplus 0 \in H$ means $u \perp M_{\theta_{\mathcal{T}}} \ell^2(\mathcal{F}, \mathcal{D})$.

The last condition implies $u = 0$ if and only if

$$\overline{M_{\theta_{\mathcal{T}}} \ell^2(\mathcal{F}, \mathcal{D})} = \ell^2(\mathcal{F}, \mathcal{D}_*),$$

i.e., $\theta_{\mathcal{T}}$ is outer.

6. AN APPLICATION

Using our functional model for a c.n.c. sequence $\mathcal{T} = \{T_\lambda\}_{\lambda \in \Lambda}$ and the litfting theorem [8], [9], [1], [2] to our setting [6, Theorem 3.2], we provide explicit forms for the commutants of $\mathcal{T}$.

For the sake of simplicity we only consider the case when $\mathcal{T} \in C_{(0)}$. Thus, assume that $\theta : \mathcal{E} \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ is a purely contractive inner operator. Let $\boldsymbol{T} = \{\mathbf{T}_\lambda\}_{\lambda \in \Lambda}$ be a sequence of operators defined on the Hilbert space

$$H = \ell^2(\mathcal{F}, \mathcal{E}_*) \ominus M_\theta \ell^2(\mathcal{F}, \mathcal{E}),$$

by

$$\mathbf{T}_\lambda^* u = S_\lambda^* u \quad (u \in H)$$

for every $\lambda \in \Lambda$.

By Theorem 5.1, the Λ-orthogonal shift $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ acting on $K = \ell^2(\mathcal{F}, \mathcal{E}_*)$ is a minimal isometric dilation of $\boldsymbol{T}$.

Let H', $\boldsymbol{T}'$ etc. corresponding similarly to an operator $\theta' : \mathcal{E}' \to \ell^2(\mathcal{F}, \mathcal{E}'_*)$ the same kind.

According to Section 2, we have that every operator

$$Y:\ell^2(\mathcal{F}, \mathcal{E}'_*) \to \ell^2(\mathcal{F}, \mathcal{E}_*)$$

such that

$$S_\lambda Y = YS_\lambda \quad (\lambda \in \Lambda)$$

can be represented in the form $Y = M_\chi$, where $\chi: \mathcal{E}'_* \to \ell^2(\mathcal{F}, \mathcal{E})$ is an operator such that M_χ is bounded.

Combining this fact with Theorem 3.2 in [6], we obtain a generalization of Theorem 3.6 in [9, Chapter VI].

THEOREM 6.1. *Every operator $X: H' \to H$ satisfying*

$$\mathbf{T}_\lambda X = X\mathbf{T}'_\lambda \quad \textit{for every } \lambda \in \Lambda, \tag{6.1}$$

can be represented in the form

$$Xu = P_H M_\chi u \quad (u \in H'), \tag{6.2}$$

where P_H is the orthogonal projection of $\ell^2(\mathcal{F}, \mathcal{E}_*)$ onto H, and $\chi: \mathcal{E}'_* \to \ell^2(\mathcal{F}, \mathcal{E}_*)$ is an operator such that the following conditions hold

a) *M_χ is a bounded operator,*

b) *$M_\chi M_{\theta'}\ell^2(\mathcal{F}, \mathcal{E}') \subset M_\theta \ell^2(\mathcal{F}, \mathcal{E})$.*

Conversely, every χ satisfying the above-mentioned conditions yields, by (6.2), *a solution X of* (6.1).

REFERENCES

1. DOUGLAS, R. G.; MUHLY, P. S.; PEARCY, C. M., Lifting commuting operators, *Michigan Math. J.*, **15**(1968), 385–395.
2. FRAZHO, A. E., Complements to models for noncommuting operators, *J. Funct. Anal.*, **59**(1984), 445–461.
3. FRAZHO, A. E., A shift operator approach to state-affine system theory, *IEEE Trans. Automat. Control*, AC-27(1982), 117–122.
4. FRAZHO, A. E., On stochastic bilinear systems, in *Modeling and applications of stochastic processes*, Ed. U.B. Desai, Kluwer Academic Publishers, Boston, 1986, pp. 215–241.
5. POPESCU, G., Models for infinite sequences of noncommuting operators, *Acta Sci. Math. (Szeged)*, to appear.
6. POPESCU, G., Isometric dilations for infinite sequences of noncommuting operators, *Trans. Amer. Math. Soc.*, to appear.

7. RUGH, W. J., *Nonlinear system theory, the Volterra-Wiener approach*, The Johns Hopkins University Press, Baltimore, 1981.
8. SZ.-NAGY, B.; FOIAŞ, C., Dilation des commutants, *C.R. Acad. Sci. Paris Ser. A*, **266**(1968), 201–212.
9. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

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