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THE BAND METHOD FOR POSITIVE AND CONTRACTIVE EXTENSION PROBLEMS

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0. INTRODUCTION

This paper develops further a unifying abstract approach to positive definite and strictly contractive extension problems initiated by [10]. Here we include in the general framework the description of all extensions via a linear fractional formula. The main idea is that the coefficients of the linear fractional map which describes all solutions can be read off from a special extension (the band extension), more precisely, from its left and right spectral factorizations. For concrete cases this connection appears in a preliminary form in [1] (where, by the way, the notion of band extension is not present). Further steps clarifying this connection may be found successively in [11], [12] and [5]. For contractive extensions of matrix functions on the unit circle a direct derivation of the linear fractional form using this idea appears in [13].

The abstract results derived in the present paper are used to review a number of concrete extension problems (which were solved earlier) and to solve some new ones, e.g., we solve here the continuous analog of the four block problem on the line and the contractive extension problem for a class of Fredholm integral operators.

The paper consists of two chapters. The first chapter contains the general scheme and applications to Carathéodory-Toeplitz type of extension problems. In the second chapter the general scheme is specified further for contractive extension problems of Nehari type. In both chapters the applications deal with finite operator matrices, matrix functions on the unit circle or on the line, and Fredholm integral operators.

CHAPTER I: POSITIVE EXTENSIONS

I.1. THE ABSTRACT SETTING

Let $\mathscr{M}$ be a Banach algebra with a unit e and an involution *. We suppose that $\mathscr{M}$ admits a direct sum decomposition of the form

$$\mathscr{M} = \mathscr{M}_1 \dotplus \mathscr{M}_2 \dotplus \mathscr{M}_3^0 \dotplus \mathscr{M}_4, \tag{1.1}$$

where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3^0$ and $\mathcal{M}_4$ are closed linear subspaces of $\mathcal{M}$ and the following conditions are satisfied:

(i) $e \in \mathcal{M}_2$, $\mathcal{M}_1^* = \mathcal{M}_4$, $\mathcal{M}_2^0 := (\mathcal{M}_3^0)^* \subset \mathcal{M}_2$, $\mathcal{M}_2^* \subset \mathcal{M}_2 \dot{+} \mathcal{M}_3^0$;

(ii) the following multiplication table describes some additional rules on the multiplication in $\mathcal{M}$:

$$\begin{array}{c|cccc} & \mathcal{M}_1 & \mathcal{M}_2 & \mathcal{M}_3^0 & \mathcal{M}_4 \\ \hline \mathcal{M}_1 & \mathcal{M}_1 & \mathcal{M}_1 & \mathcal{M}_+^0 & \mathcal{M} \\ \mathcal{M}_2 & \mathcal{M}_1 & \mathcal{M}_+ & \mathcal{M}_c & \mathcal{M}_-^0 \\ \mathcal{M}_3^0 & \mathcal{M}_+^0 & \mathcal{M}_c & \mathcal{M}_-^0 & \mathcal{M}_4 \\ \mathcal{M}_4 & \mathcal{M} & \mathcal{M}_-^0 & \mathcal{M}_4 & \mathcal{M}_4 \end{array}, \tag{1.2}$$

where

$$\mathcal{M}_+^0 := \mathcal{M}_1 \dot{+} \mathcal{M}_2^0, \quad \mathcal{M}_c := \mathcal{M}_2 \dot{+} \mathcal{M}_3^0,$$
$$\mathcal{M}_-^0 := \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \quad \mathcal{M}_+ := \mathcal{M}_1 \dot{+} \mathcal{M}_2; \tag{1.3}$$

(iii) $\mathcal{M}_2^0\mathcal{M}_2 \subset \mathcal{M}_+^0$; $\mathcal{M}_2\mathcal{M}_2^0 \subset \mathcal{M}_+^0$.

Some additional notations are

$$\mathcal{M}_3 := \mathcal{M}_2^*, \quad \mathcal{M}_- := \mathcal{M}_3 \dot{+} \mathcal{M}_4, \quad \mathcal{M}_d := \mathcal{M}_2 \cap \mathcal{M}_3. \tag{1.4}$$

Note that $e \in \mathcal{M}_d$, $\mathcal{M}_3 = \mathcal{M}_d \dot{+} \mathcal{M}_3^0$, $\mathcal{M}_2 = \mathcal{M}_2^0 \dot{+} \mathcal{M}_d$, $\mathcal{M}_d\mathcal{M}_j \subset \mathcal{M}_j$ ($j = 1, 2, 3, 4$), $\mathcal{M}_d\mathcal{M}_j^0 \subset \mathcal{M}_j^0$ ($j = 2, 3$) and $\mathcal{M}_d\mathcal{M}_d \subset \mathcal{M}_d$. Further, note that $\mathcal{M}_1$ (resp. $\mathcal{M}_4$) is a two-sided ideal of the subalgebra $\mathcal{M}_+$ (resp. $\mathcal{M}_-$). Also, if $d \in \mathcal{M}_d$ is invertible, then $d^{-1} \in \mathcal{M}_d$.

We say that an element $b \in \mathcal{M}$ is *positive definite* if there exists an invertible element $c \in \mathcal{M}$ such that $b = c^*c$. If $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{M}$, then $b \in \mathcal{A}$ is said to be *positive definite in* $\mathcal{A}$ if $b = c^*c$ with $c^{\pm 1} \in \mathcal{A}$. With respect to positive elements $\mathcal{M}$ is assumed to satisfy the following three axioms:

(A1) If $e - g^*g$ is positive definite and $g \in \mathcal{M}_\pm$, then $e - g$ is invert ble and $(e - g)^{-1} \in \mathcal{M}_\pm$;

(A2) If $\|g\| < 1$, then $e - g^*g$ is positive definite;

(A3) If a is positive definite, then $e + a$ is also positive definite.

Axiom (A3) implies that the sum of two positive definite elements is again positive definite. If $\mathcal{M}$ is a B^*-algebra, then the conditions (A1), (A2) and (A3) are fulfilled automatically. Indeed, if $\mathcal{M}$ is a B^*-algebra an element $e - g^*g$ is positive definite if and only if $\|g\| < 1$. So then (A2) is satisfied. For (A1) one notes in addition that if $g \in \mathcal{M}_\pm$ and $\|g\| < 1$, then the element $e - g$ is invertible and the von Neumann series yields that $(e - g)^{-1} \in \mathcal{M}_\pm$. Axiom (A3), finally, one proves by using that an element in a B^*-algebra is positive definite if and only if its spectrum is positive. Exampes of Banach algebras with a decomposition (1.1) that satisfies conditions (i), (ii) (being the multiplication table (1.2)) and (iii), and for which the axioms (A1)—(A3) hold true, will appear in Sections I.2—I.5.

Let us introduce the following two types of factorizations for positive elements. Let $b \in \mathcal{M}$ be positive definite. We shall say that b admits a *left spectral factorization* (relative to the decomposition (1.1)) if $b = b_+ b_+^*$ for some $b_+ \in \mathcal{M}_+$ with $b_+^{-1} \in \mathcal{M}_+$. We shall say that b admits a *right spectral factorization* (relative to the decomposition (1.1)) if $b = b_- b_-^*$ for some $b_- \in \mathcal{M}_-$ with $b_-^{-1} \in \mathcal{M}_-$. Note that b admits a left spectral factorization if and only if b^{-1} admits a right spectral factorization. In all our examples the Banach algebra will be so that any positive definite elements admits a left and a right spectral factorization.

We shall use the symbols P_i $(i = 1, \ldots, 4)$, P_i^0 $(i = 2, 3)$, $P_\pm$, $P_\pm^0$, P_c and P_d to denote the natural projections of $\mathcal{M}$ onto the subspaces of the same index along their natural complement in $\mathcal{M}$. Thus, for instance,

$$P_+ = P_1 + P_2, \quad P_- = P_3 + P_4, \quad P_c = P_2 + P_3^0 = P_2^0 + P_d + P_3^0 = P_2^0 + P_3.$$

Let $k = k^* \in \mathcal{M}_c$. An element $b \in \mathcal{M}$ is called a *positive extension* of k if

(i) $P_c b = k$;

(ii) b is positive definite.

A positive extension b of k is called a *band extension* of k if in addition

(iii) $b^{-1} \in \mathcal{M}_c$.

LEMMA I.1.1. *If $b_\pm \in \mathcal{M}_\pm$ is invertible with inverse in $\mathcal{M}_\pm$ and $b = b_\pm b_\pm^*$ belongs to $\mathcal{M}_c$, then $b_\pm \in \mathcal{M}_c \cap \mathcal{M}_\pm$.*

Proof. Since $b_+ = bb_+^{*-1}$, $b \in \mathcal{M}_c$ and $b_+^{*-1} \in \mathcal{M}_-$, we get that $b_+ \in \mathcal{M}_2 \dotplus \mathcal{M}_0^3 \dotplus \mathcal{M}_4$. But then, since $b_+ \in \mathcal{M}_+$, we obtain that $b_+ \in \mathcal{M}_2 = \mathcal{M}_c \cap \mathcal{M}_+$. The proof of the minus version is analogous. ▨

LEMMA I.1.2. *Let $x_\pm \in \mathcal{M}_\pm$ be invertible with $x_\pm^{-1} \in \mathcal{M}_\pm$. Then $P_d x_\pm$ is invertible and $(P_d x_\pm)^{-1} = P_d x_\pm^{-1}$.*

Proof. Write $x_\pm = P_d x_\pm + P_\pm^0 x_\pm$ and $x_\pm^{-1} =: y_\pm = P_d y_\pm + P_\pm^0 y_\pm$. Writing out the products $x_\pm y_\pm$ and $y_\pm x_\pm$, which are equal to e, and by applying the projection P_d one obtains that $P_d x_\pm P_d y_\pm = P_d y_\pm P_d x_\pm = P_d e = e$, and the lemma is proved. ▨

THEOREM I.1.3. *Let $k = k^* \in \mathcal{M}_c$. The element k has a band extension b which admits a left and a right spectral factorization if and only if there exists a pair of solutions $x \in \mathcal{M}_2$ and $y \in \mathcal{M}_3$ of the equations*

$$P_2(kx) = e \tag{1.5}$$

$$P_3(ky) = e, \tag{1.6}$$

such that x and y are invertible, $x^{-1} \in \mathcal{M}_+$, $y^{-1} \in \mathcal{M}_-$, and $P_d x$ and $P_d y$ are positive definite in M_d. Moreover, if such an element b exists, then b is unique and given by

$$(1.7) \qquad b = x^{*-1}(P_d x)x^{-1} = y^{*-1}(P_d y)y^{-1}.$$

Proof. Let b be a band extension of k, and let $b^{-1} = uu^* = vv^*$, where $u^{\pm 1} \in \mathcal{M}_+$ and $v^{\pm 1} \in \mathcal{M}_-$. Since $b^{-1} \in \mathcal{M}_c$, Lemma I.1.1 yields that $u \in \mathcal{M}_2$ and $v \in \mathcal{M}_3$. Put $x = u(P_d u^*)$ and $y = v(P_d v^*)$. Then $x \in \mathcal{M}_2$, $y \in \mathcal{M}_3$, and (by Lemma I.1.2) $x^{-1} \in \mathcal{M}_+$, $y^{-1} \in \mathcal{M}_-$. Further, $P_d x = (P_d u)(P_d u)^*$ and $P_d y = (P_d v)(P_d v)^*$ are positive definite in $\mathcal{M}_d$, and

$$(x^*)^{-1}(P_d x)x^{-1} = (u^*)^{-1}u^{-1} = b = (v^*)^{-1}v^{-1} = (y^*)^{-1}(P_d y)y^{-1}.$$

Since $P_c b = k$ we have that $b = P_1 b + k + P_4 b$. So using multiplication table (1.2)

$$P_2(kx) = P_2(bx - (P_1 b)x - (P_4 b)x) = P_2(bx) = P_2((x^*)^{-1}P_d x) = e,$$

where for the last equality one uses Lemma I.1.2. In much the same way one proves that (1.6) holds.

Conversely, suppose that x and y exist such that all the conditions in the theorem are fulfilled. Let $\hat{b}$ be defined by $\hat{b} = b_1 + k + b_1^*$, where $b_1 = - P_1(kx)x^{-1} \in \mathcal{M}_1$. Then $\hat{b}x = - P_1(kx) + kx + b_1^* x$, and using the multiplication table (1.2) we get that $P_1(\hat{b}x) = 0$ and $P_2(\hat{b}x) = P_2(kx) = e$. So $\hat{b}x \in e + \mathcal{M}_-^0$. Since $P_d x$ is positive definite (in $\mathcal{M}_d$), $P_d x = P_d x^*$, and hence $x^*\hat{b}x \in P_d x + \mathcal{M}_-^0$. From $k = k^*$ it follows that $(x^*\hat{b}x)^* = x^*\hat{b}x$, and hence $x^*\hat{b}x \in P_d x + \mathcal{M}_+^0$. This can only happen when $x^*\hat{b}x = P_d x$. So we get that

$$(1.8) \qquad \hat{b} = x^{*-1}(P_d x)x^{-1}.$$

Now $\hat{b}^{-1} = x(P_d x)^{-1}x^* \in \mathcal{M}_c$. Further, using that $P_d x$ is positive definite in $\mathcal{M}_d$, we see from (1.8) that $\hat{b}$ admits a right spectral factorization.

Analogously, one proves that

$$\tilde{b} := -(y^{-1})^*(P_4(ky))^* + k - P_4(ky)y^{-1}$$

is equal to $(y^*)^{-1}(P_d y)y^{-1}$. But then $\tilde{b}$ is a band extension of k which admits a left spectral factorization.

We finish the proof by proving that if k has a band extension f which admits a right spectral factorization and a band extension g which admits a left spectral factorization, then $f = g$. From this it then follows that $\hat{b} = \tilde{b} =: b$ and also the uniqueness of b. So let f and g be as above and write $f^{-1} = uu^*$, $u \in \mathcal{M}_2$, $u^{-1} \in \mathcal{M}_+$, and $g^{-1} = vv^*$, $v \in \mathcal{M}_3$, $v^{-1} \in \mathcal{M}_-$ (use Lemma I.1.1). Put $h := f^{-1} - g^{-1}$. Then h belongs to $\mathcal{M}_c$. Since $P_c f = P_c g = k$, we have that $g - f = z_1 + z_4$ for some $z_1 \in \mathcal{M}_1$ and $z_4 \in \mathcal{M}_4$. Using that $h = f^{-1}(z_1 + z_4)g^{-1}$, we obtain $u^{-1}hv^{*-1} = u^*(z_1 + z_4)v$. Because of the multiplication table (1.2) the left hand side belongs to $\mathcal{M}_+ \dot{+} \mathcal{M}_3^0$ and hence $0 = P_4(u^*(z_1 + z_4)v) = u^*z_4 v$. Thus $z_4 = 0$. By writing $h = g^{-1}(g - f)f^{-1}$ one obtains analogously that $z_1 = 0$. But then $f = g$ follows. ▨

Note that in the last paragraph of the proof of Theorem I.1.3 we actually proved the following result.

THEOREM I.1.4. *Let $k = k^* \in \mathcal{M}_c$, and suppose that k has a band extension f which admits a right spectral factorization and a band extension g which admits a left spectral factorization. Then $f = g$.*

So far we did not use the axioms (A1)—(A3). They will be needed in the proof of the next theorem. This theorem describes the set of all positive extensions of an element $k \in \mathcal{M}_c$ for the case that k has a positive extension band which admits a left and a right spectral factorization.

THEOREM I.1.5. *Let $k = k^* \in \mathcal{M}_c$, and suppose that k has a positive band extension b whose inverse admits a left and a right spectral factorization:*

$$b^{-1} = uu^* = vv^*, \quad u^{\pm 1} \in \mathcal{M}_+, \; v^{\pm 1} \in \mathcal{M}_-. \tag{1.9}$$

Then each positive extension of k is of the form

$$T(g) = (g^*v^* + u^*)^{-1}(e - g^*g)(vg + u)^{-1}, \tag{1.10}$$

where g is an element of $\mathcal{M}_1$ such that $e - g^*g$ is positive definite. Furthermore formula (1.10) gives a 1-1 correspondence between all such g and all positive extensions of k. Alternatively, each positive extension of k is of the form

$$S(f) = (f^*u^* + v^*)^{-1}(e - f^*f)(uf + v)^{-1}, \tag{1.11}$$

where f is an element of $\mathcal{M}_4$ such that $e - f^*f$ is positive definite. Furthermore, formula (1.11) gives a 1-1 correspondence between all such f and all positive extensions of k.

In the proof of Theorem I.1.5 we need the following lemma.

LEMMA I.1.6. *Let $z \in \mathcal{M}_\pm$ be such that $z + z^*$ is positive definite. Then z is invertible and $z^{-1} \in \mathcal{M}_\pm$.*

Proof. Write $z + z^* = aa^*$ with a invertible. For $\varepsilon > 0$ we have

$$(e - \varepsilon z^*)(e - \varepsilon z) = e - \varepsilon[(z + z^*) - \varepsilon z^* z] = e - \varepsilon a[e - \varepsilon a^{-1}z^*z(a^{-1})^*]a^*.$$

Choose $\varepsilon > 0$ such that $\|\varepsilon^{1/2}z(a^{-1})^*\| < 1$. Then Axiom (A2) yields that $e - \varepsilon a^{-1}z^*z(a^{-1})^* = g_\varepsilon g_\varepsilon^*$ for some invertible g_ε. Now

$$e - (e - \varepsilon z^*)(e - \varepsilon z) = \varepsilon a g_\varepsilon g_\varepsilon^* a^*$$

is positive definite. Using Axiom (A1) we get that $z = \varepsilon^{-1}(e - (e - \varepsilon z))$ is invertible and its inverse belongs to $\mathcal{M}_+$. ▨

Proof of Theorem I.1.5. We define the following linear fractional map. Write $b = c + c^*$ with $c \in \mathcal{M}_+$ and $P_d c = (1/2)P_d k$, and define

$$\tilde{T}(g) = (-c^* vg + cu)(vg + u)^{-1}$$

for all g for which $u + vg$ is invertible. Then

$$\tilde{T}(g) + \tilde{T}(g)^* = (g^*v^* + u^*)^{-1}\{(g^*v^* + u^*)(-c^*vg + cu) +$$

$$+ (-g^*v^*c + u^*c^*)(vg + u)\}(vg + u)^{-1} =$$

$$= (g^*v^* + u^*)^{-1}\{u^*(c^* + c)u - g^*v^*(c^* + c)vg\}(vg + u)^{-1} = T(g).$$

Suppose that $g \in \mathcal{M}_1$ and $e - g^*g$ is positive definite. Since $v \in \mathcal{M}_3$, we have that $vg \in \mathcal{M}_+$, so $u^{-1}vg \in \mathcal{M}_+$. Since $e - (-u^{-1}vg)^*(-u^{-1}vg) = e - g^*g$ is positive definite, Axiom (A1) implies that $e + u^{-1}vg$ is invertible and its inverse belongs to $\mathcal{M}_+$. In particular, $T(g)$ and $\tilde{T}(g)$ are well defined. Further one sees that $T(g)$ is positive definite. Using $a(e + ba)^{-1} = (e + ab)^{-1}a$ and $(e + ba)^{-1} = e - b(e + ab)^{-1}a$, we get that

$$\tilde{T}(g) = -c^*vg(u^{-1}vg + e)^{-1}u^{-1} + cu(u^{-1}vg + e)^{-1}u^{-1} =$$

$$= -c^*v(gu^{-1}v + e)^{-1}gu^{-1} + cu(e - u^{-1}v(e + gu^{-1}v)^{-1}g)u^{-1} =$$

$$= c + (-c^*v - cv)(e + gu^{-1}v)^{-1}gu^{-1} = c - v^{*-1}(e + gu^{-1}v)^{-1}gu^{-1}.$$

Since $gu^{-1} \in \mathcal{M}_1$ and $v \in \mathcal{M}_3$, the element $gu^{-1}v$ belongs to $\mathcal{M}_+$. Note that $e - (-gu^{-1}v)^*(-gu^{-1}v) = (u^{-1}v)^*(e - g^*g)u^{-1}v$ is positive definite, and thus Axiom (A1) yields $(e + gu^{-1}v)^{-1} \in \mathcal{M}_+$. Now the multiplication table (1.2) gives that $\tilde{T}(g) \in c + \mathcal{M}_1$. But then $T(g) = \tilde{T}(g) + \tilde{T}(g)^* \in b + \mathcal{M}_1 + \mathcal{M}_\Delta$. Hence $T(g)$ is a positive extension of k.

Conversely, suppose that a is a positive extension of k. Then write $a = z + z^*$ with $z \in \mathcal{M}_+$ and $P_d z = (1/2)P_d k$, and put $w := z - c \in \mathcal{M}_1$. Since $b + a$ is positive definite (by Axiom (A3)), we get that $v^*(b + a)v = v^*(b + b + w + w^*)v = 2e + v^*wv + v^*w^*v$ is positive definite. From Lemma I.1.1 it follows that $v \in \mathcal{M}_- \cap \mathcal{M}_c$, and thus $v^*wv \in \mathcal{M}_+$. Lemma I.1.6 yields that $e + v^*wv$ is invertible and its inverse belongs to $\mathcal{M}_+$. Put now $g := -(e + v^*wv)^{-1}v^*wu$. By Lemma I.1.1 the elements v^* and u are in $\mathcal{M}_2$. Since $w \in \mathcal{M}_1$ we get that $g \in \mathcal{M}_1$. Further $vg + u = - v(e + v^*wv)^{-1}v^*wu + u = (e + vv^*w)^{-1}u$ is invertible, and

$$\tilde{T}(g) = (c^*v(e + v^*wv)^{-1}v^*wu + cu)u^{-1}(e + vv^*w) =$$

$$= (c^* - c^*(e + vv^*w)^{-1} + c)(e + vv^*w) = b(e + b^{-1}w) - c^* = z.$$

Hence $a = \tilde{T}(g) + \tilde{T}(g)^* = T(g)$. Since a is positive definite, it follows that $e - g^*g$ is positive definite. Since the map $g \to T(g)$ is one-one we have established the desired 1-1 correspondence.

In order to prove the alternative representation (1.11) one proceeds in an analogous way. Let

$$\tilde{S}(f) = (- cuf + c^*v)(v + uf)^{-1},$$

and use Axiom (A1) to prove that for $f \in \mathcal{M}_4$ with $e - f^*f$ positive definite the element $\tilde{S}(f)$ is well defined. Then calculations show that $S(f) = \tilde{S}(f) + \tilde{S}(f)^* \in b + \mathcal{M}_1 + \mathcal{M}_4$ is a positive extension of k. Conversely, let a be a positive extension of k and write $a = z + z^*$ with $z \in \mathcal{M}_-$ and $P_d z = (1/2)P_d k$. One uses Lemma I.1.6 to show that $e + u^*(z - c^*)u$ is invertible, and one introduces $f := -(e + u^*(z - c^*)u)^{-1}u^*(z - c^*)v$. This f will appear to be in $\mathcal{M}_4$, and to be such that $e - f^*f$ is positive definite and $\tilde{S}(f) = z$. But then the desired 1-1 correspondence (1.11) is established. ▨

Theorem I.1.3 and Theorem I.1.4 are similar to some results in [10] but now concern positive extensions in the setting of an algebra with an involution. Theorem I.1.5 is a new result inspired by earlier concrete versions (see [5]).

I.2. THE OPERATOR MATRIX CASE

In this section we specify the results of Section I.1 for the algebra Ω_N of $N \times N$ operator matrices. An element of this algebra has the following form:

$$T = \begin{pmatrix} A_{11} & \dots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \dots & A_{NN} \end{pmatrix}.$$

Here A_{ij}, $1 \leqslant i, j \leqslant N$, is a bounded linear operator from a Hilbert space H_j into a Hilbert space H_i, shortly $A_{ij} \in \mathcal{L}(H_j, H_i)$. Note that T is an operator on the Hilbert space $H_1 \oplus \dots \oplus H_N$. The symbol $T > 0$ means that T is positive definite. We write I_j for the identity operator on H_j.

THEOREM I.2.1. *For $1 \leqslant i, j \leqslant N$, $|j-i| \leqslant p$, let $A_{ij} = A_{ji}^*$ be a given operator acting from a Hilbert space H_j into a Hilbert space H_i, and suppose that*

$$\begin{pmatrix} A_{ii} & \dots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \dots & A_{i+p,i+p} \end{pmatrix} > 0, \quad i = 1, \dots, N-p. \tag{2.1}$$

For $q = 1, \dots, N$, let

$$\begin{pmatrix} Y_{qq} \\ \vdots \\ Y_{\beta(q),q} \end{pmatrix} = \begin{pmatrix} A_{qq} & \dots & A_{q,\beta(q)} \\ \vdots & & \vdots \\ A_{\beta(q),q} & \dots & A_{\beta(q),\beta(q)} \end{pmatrix}^{-1} \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{2.2}$$

and

$$\begin{pmatrix} X_{\gamma(q),q} \\ \vdots \\ X_{q,q} \end{pmatrix} = \begin{pmatrix} A_{\gamma(q),\gamma(q)} & \dots & A_{\gamma(q),q} \\ \vdots & & \vdots \\ A_{q,\gamma(q)} & \dots & A_{qq} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_q \end{pmatrix}, \tag{2.3}$$

where $\beta(q) = \min\{N, p+q\}$ and $\gamma(q) = \max\{1, q-p\}$. Let the $N \times N$ triangular operator matrices U and V be defined by

$$V_{ij} = \begin{cases} Y_{ij} Y_{jj}^{-1/2}, & j \leqslant i \leqslant \beta(j) \\ 0 \quad, & \textit{elsewhere}; \end{cases} \tag{2.4}$$

$$U_{ij} = \begin{cases} X_{ij} X_{jj}^{-1/2}, & \gamma(j) \leqslant i \leqslant j \\ 0 \quad, & \textit{elsewhere}. \end{cases} \tag{2.5}$$

Then the $N \times N$ operator matrix F given by the following factorizations of its inverse

$$F := U^{*-1} U^{-1} = V^{*-1} V^{-1} \tag{2.6}$$

is the unique positive definite operator matrix with $F_{ij} = A_{ij}$, $|j-i| \leqslant p$, and $(F^{-1})_{ij} = 0$, $|j-i| > p$.

Proof. We will obtain this theorem as a special case of Theorem I.1.3. Let $\mathcal{M}$ be the algebra Ω_N with involution $*$ being the usual adjoint of an operator between Hilbert spaces. The norm on $\mathcal{M}$ is the usual operator norm. Then $\mathcal{M}$ is a B^*-algebra. Put

$$\mathcal{M}_1 = \{(F_{ij})_{i,j=1}^n \mid F_{ij} = 0, j-i \leqslant p\},$$

$$\mathcal{M}_2 = \{(F_{ij})_{i,j=1}^n \mid F_{ij} = 0,\ j-i > p \text{ and } j-i < 0\},$$

$$\mathcal{M}_3^0 = \{(F_{ij})_{i,j=1}^n \mid F_{ij} = 0,\ j-i \geqslant 0 \text{ and } j-i < -p\},$$

$$\mathcal{M}_4 = \{(F_{ij})_{i,j=1}^n \mid F_{ij} = 0,\ j-i \geqslant -p\}.$$

It is easy to see that

$$\Omega_N = \mathcal{M}_1 \dotplus \mathcal{M}_2 \dotplus \mathcal{M}_3^0 \dotplus \mathcal{M}_4$$

and that the above subspaces satisfy the conditions (i), (ii) and (iii) in Section I.1. In particular, the multiplication table (1.2) holds true. Since Ω_N is a B^*-algebra Axioms (A1)—(A3) are fulfilled automatically.

Let $K = (K_{ij})_{i,j=1}^N$, where $K_{ij} = A_{ij}$ for $|j-i| \leqslant p$ and $K_{ij} = 0$ otherwise. A direct computation shows that

$$P_2(KX) = I, \quad P_3(KY) = I,$$

where $X = (X_{ij})_{i,j=1}^N$ and $Y = (Y_{ij})_{i,j=1}^N$ are the upper and lower block band matrices of which the entries in the band $|j-i| \leqslant p$ are given by (2.2) and (2.3), respectively, and which have zero entries outside this band. Since Y_{qq} is the (1,1)-block element in the left upper corner of the inverse of a positive definite operator matrix, the element Y_{qq} is positive definite. Similarly X_{qq} is positive definite, and hence the main diagonals of X and Y are positive definite. But then X and Y are invertible and $X^{-1} \in \mathcal{M}_+$ and $Y^{-1} \in \mathcal{M}_-$. In this way it follows from Theorem I.1.3 that the operator matrix F defined in Theorem I.2.1 is precisely the unique band extension of K. ▨

We say that an $N \times N$ operator matrix F is a *positive extension* of the band $\{A_{ij},\ |j-i| \leqslant p\}$ if F is positive definite and $F_{ij} = A_{ij}$ for $|j-i| \leqslant p$. Note that condition (2.1) is clearly a necessary condition for the existence of a positive extension of the band $\{A_{ij},\ |j-i| \leqslant p\}$. By applying Theorem I.1.5 in the setting described in the proof of Theorem I.2.1 we obtain the following description for the set of all positive extensions of a given band.

THEOREM I.2.2. *Let $A_{ij} = A_{ji}^*$, $1 \leqslant i, j \leqslant N$, $|j-i| \leqslant p$ be given operators acting from a Hilbert space H_j into a Hilbert space H_i. In order that there exists a positive extension of the band $\{A_{ij},\ |j-i| \leqslant p\}$ it is necessary and sufficient that*

$$\begin{pmatrix} A_{ii} & \cdots & A_{i,i+p} \\ \vdots & & \vdots \\ A_{i+p,i} & \cdots & A_{i+p,i+p} \end{pmatrix} > 0, \quad i = 1, \ldots, N-p. \tag{2.7}$$

Assume that the latter conditions hold. Let U and V be the $N\times N$ operator matrices defined by (2.2)—(2.5). *Then each positive extension F of the given band is of the form*

$$F = (G^*V^* + U^*)^{-1}(I - G^*G)(VG + U)^{-1}, \tag{2.8}$$

where G is a strictly contractive $N \times N$ operator matrix with $G_{ij} = 0$, $j - i \leqslant p$. Furthermore, formula (2.8) *gives a* 1-1 *correspondence between all such G and all positive extensions F.*

Let us remark that there is an alternative description for the set of all positive extensions, which one obtains from (1.11).

For the case of block matrices Theorem I.2.1 has been proved in [9]. Also for the matrix case a linear description of all positive extensions of a given band was obtained in [2] using a finite dimensional version of the Ball-Helton method (see [3]). Using different methods a linear fractional description was also obtained by [5, Theorem 10.4], by [17, Theorem 6.1], and by [4, Theorem 3]. In these papers the coefficients of the linear fractional map are presented in a form which is less explicit than the one appearing here.

I.3. THE WIENER ALGEBRA ON THE CIRCLE

In this section we apply Theorems I.1.3 and I.1.5 to the Wiener algebra on the circle. By $W_{j\times k}(\mathbf{T})$ we denote the Wiener space of all continuous $j \times k$ matrix valued function

$$f(z) = \sum_{j=-\infty}^{\infty} f_j z^j,$$

defined on the unit circle $\mathbf{T} = \{z \in \mathbf{C}, |z| = 1\}$ with the constraint

$$\|f\| := \sum_{j=-\infty}^{\infty} \|f_j\| < \infty.$$

Here $\|f_j\|$ is the spectral norm on the matrix f_j, i.e., $\|f_j\|$ is the largest singular value of f_j. If A is a matrix we write A^* for the usual adjoint of A, and we write $A > 0$ if A is positive definite.

THEOREM I.3.1. *Let $\gamma_j = \gamma_{-j}^*$, $|j| \leqslant m$, be a given set of $N \times N$ matrices, and suppose that the block Toeplitz matrix*

$$\Gamma := \begin{pmatrix} \gamma_0 & \gamma_{-1} & \cdots & \gamma_{-m} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_m & \gamma_{m-1} & \cdots & \gamma_0 \end{pmatrix} \tag{3.1}$$

is positive definite. Let

(3.2) $$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_{-m} \\ \vdots \\ y_{-1} \\ y_0 \end{pmatrix} = \Gamma^{-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \end{pmatrix},$$

and put

(3.3) $$u(z) = \sum_{j=0}^{m} x_j x_0^{-1/2} z^j, \quad v(z) = \sum_{j=-m}^{0} y_j y_0^{-1/2} z^j.$$

Then the function $f \in W_{N\times N}(\mathbf{T})$ given by the following factorizations

(3.4) $$f(z) := u(z)^{*-1}u(z)^{-1} = v(z)^{*-1}v(z)^{-1}$$

is the unique $f \in W_{N\times N}(\mathbf{T})$ such that $f(z) > 0$, $|z| = 1$, for $|j| \leqslant m$ its j^{th} coefficient f_j is equal to γ_j, and for $|j| > m$ the j^{th} coefficient of its inverse $(f^{-1})_j$ is equal to zero.

Proof. We will obtain this theorem as a special case of Theorem I.1.3. Let $\mathcal{M}$ be the Wiener algebra $W_{N\times N}(\mathbf{T})$ with unit $e(z) = I$ for $|z| = 1$ and involution $*$ given by $f^*(z) = f(z)^*$, $|z| = 1$. Put

$$\mathcal{M}_1 = \{f \in W_{N\times N}(\mathbf{T}) \mid f_j = 0,\ j \leqslant m\},$$

$$\mathcal{M}_2 = \{f \in W_{N\times N}(\mathbf{T}) \mid f_j = 0,\ j < 0 \text{ and } j > m\},$$

$$\mathcal{M}_3^0 = \{f \in W_{N\times N}(\mathbf{T}) \mid f_j = 0,\ j < -m \text{ and } j \geqslant 0\},$$

$$\mathcal{M}_4 = \{f \in W_{N\times N}(\mathbf{T}) \mid f_j = 0,\ j \geqslant -m\}.$$

Then clearly (1.1) holds and this decomposition satisfies the conditions (i), (ii) and (iii). Note that an element $f \in W_{N\times N}(\mathbf{T})$ is positive definite in this algebra if and only if $f(z) > 0$ for $|z| = 1$. Let us check that the axioms (A1)—(A3) are fulfilled. Assume that $g \in \mathcal{M}_+$ and $e - g^*g$ is positive definite in $W_{N\times N}(\mathbf{T})$. The latter property implies that $\|g(z)\| < 1$ for $|z| = 1$. Since $g \in \mathcal{M}_+$, the function g is analytic on $|z| < 1$ and continuous on $|z| \leqslant 1$. Thus we can apply the maximum modulus principle to show that $\|g(z)\| < 1$ for $|z| \leqslant 1$. Hence $e - g$ has no zeroes in the closed disk, and by Wiener's theorem $e - g$ is invertible in $\mathcal{M}_+$. For $g \in \mathcal{M}_-$ one reasons in a similar way. Hence Axiom (A1) is fulfilled. For (A2) one notes that $\|g(z)\| < \|g\|$ for $z \in \mathbf{T}$. So, if $\|g\| < 1$, the matrix $I - g(z)^*g(z)$ is positive definite for $z \in \mathbf{T}$. But this means that $e - g^*g$ is positive definite in $W_{N\times N}(\mathbf{T})$. Checking (A3) is a triviality.

In order to apply Theorem I.1.3 we let $k(z) = \sum_{j=-m}^{m} \gamma_j z^j \in \mathcal{M}_c$. Define $x(z) := \sum_{j=0}^{m} x_j z^j \in \mathcal{M}_2$ and $y(z) = \sum_{j=-m}^{0} y_j z^j \in \mathcal{M}_3$, where the coefficients x_j and y_j are given by (3.2). It is straightforward to check that with these choices of k, x and y the equations (1.5) and (1.6) hold. Furthermore, $P_d x = x_0$ is positive definite in $\mathcal{M}_d = \mathbf{C}^{N\times N}$, because x_0 is the (1,1)th block element of the positive definite matrix Γ^{-1}. Similarly, $P_d y$ is positive definite in $\mathcal{M}_d$. The fact that x^{-1} belongs to $\mathcal{M}_+$ and y^{-1} belongs to $\mathcal{M}_-$ is proved in [7, p. 522/3]. Now one can apply Theorem I.1.3 and the proof is completed. ▨

Let $\gamma_j = \gamma^*_{-j}$, $|j| \leqslant m$, be a given set of $N \times N$ matrices. A matrix valued function $f \in W_{N\times N}(\mathbf{T})$ is called a *positive extension* of the given band $\{\gamma_j, |j| \leqslant m\}$ if f is positive definite and $f_j = \gamma_j$, $|j| \leqslant m$. Note that if f is positive definite, then the Toeplitz matrix $(f_{j-i})_{i,j=0}^{\infty}$ is a positive definite matrix. So if f is a positive extension of the given band $\{\gamma_j, |j| \leqslant m\}$ the matrix $(f_{j-i})_{i,j=0}^{m} = (\gamma_{j-i})_{i,j=0}^{m}$ should be positive definite. This remark and Theorems I.3.1 and I.1.5 add up to the following theorem, which gives a description of all positive extensions of a given band.

THEOMEM I.3.2. *Let $\gamma_j = \gamma^*_{-j}$, $|j| \leqslant m$, be a given set of $N \times N$ matrices. In order that there exists a positive extension of the given band it is necessary and sufficient that the matrix*

$$\Gamma := \begin{pmatrix} \gamma_0 & \gamma_{-1} & \cdots & \gamma_{-m} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_m & \gamma_{m-1} & \cdots & \gamma_0 \end{pmatrix}$$

is positive definite. Let this condition hold and let $u(z)$ and $v(z)$ be defined by (3.2)—(3.3). *Then each positive extension of the given band is of the form*

$$(3.5) \qquad f(z) = (v(z)g(z) + u(z))^{*-1}(I - g(z)^* g(z))(v(z)g(z) + u(z))^{-1},$$

where g is an element in $W_{N\times N}(\mathbf{T})$ such that $\|g(z)\| < 1$, $|z| = 1$, and $g_j = 0$, $j \leqslant m$. Furthermore, formula (3.5) *gives a* 1-1 *correspondence between all such g and all positive extensions f of the given band.*

Let us remark that there is an alternative description for the set of all positive extensions, which one obtains from (1.11).

Theorem I.3.1 was obtained before in [7] (see also [15], [16]). A linear fractional description of all positive extensions of a given band appears in [5, Theorem 11.3] without proof. (For the proofs see [6].)

I.4. THE WIENER ALGEBRA ON THE LINE

Let $W_{n\times m}(\mathbf{R})$ denote the Wiener space of continuous $n \times m$ matrix functions f of the form

$$f(\lambda) = d + \hat{k}(\lambda), \quad \lambda \in \mathbf{R}. \tag{4.1}$$

Here $d = (d_{ij})_{i=1,j=1}^{n\ \ m}$ is a constant $n \times m$ matrix and $\hat{k}$ is the Fourier transform of an $n \times m$ matrix valued function $k = (k_{ij})_{i=1,j=1}^{n\ \ m} \in L^1_{n\times m}(\mathbf{R})$, i.e., the entries k_{ij} are integrable on $\mathbf{R}$. The norm in $W_{n\times m}(\mathbf{R})$ is defined by

$$\|f\|_{W_{n\times m}(\mathbf{R})} := \|d\| + \int_{-\infty}^{\infty} \|k(s)\|\mathrm{d}s,$$

where $\|A\|$ denotes the spectral norm of the matrix A. Given an $f \in W_{n\times m}(\mathbf{R})$, the d and k appearing in (4.1) are uniquely determined by f. In fact, by the Riemann-Lebesgue lemma $d_f := d = \lim_{\lambda\to\infty} f(\lambda)$ and hence $k_f(s) := k(s) = (f - d_f)^{\vee}(s)$. Here $g^{\vee}$ stands for the inverse Fourier transform of g. Given $\tau > 0$ we write $W^c_{n\times m}(\tau)$ for the set of all $f \in W_{n\times m}(\mathbf{R})$ such that $k_f(s) = 0$ a.e. on $|s| > \tau$. It is well known that $W_{n\times m}(\mathbf{R})$ is a Banach algebra under pointwise matrix multiplication.

In this section we deal with the following extension problem. Let $k \in L^1_{n\times n}[-\tau, \tau]$. A matrix valued function $\hat{h}$, with $h \in L^1_{n\times n}(\mathbf{R})$, is called a *positive extension* of $\hat{k}$ (where k is understood to be zero outside the interval $[-\tau, \tau]$) if $h(s) = k(s)$, $|s| < \tau$, and the matrix $I_n - \hat{h}(\lambda)$ is positive definite for all $\lambda \in \mathbf{R}$. The symbol I_n stands for the $n \times n$ identity matrix.

THEOREM I.4.1. *Let $k \in L^1_{n\times n}[-\tau, \tau]$ be given, and put*

$$(K_\tau\varphi)(t) = \int_0^\tau k(t - s)\varphi(s)\mathrm{d}s, \quad t \in [0, \tau]. \tag{4.2}$$

Assume that $I - K_\tau$ is a positive operator on $L^2_n[0, \tau]$. Let x and y be the solutions of the equations

$$x(t) - \int_0^\tau k(t - s)x(s)\mathrm{d}s = k(t), \quad 0 \leqslant t \leqslant \tau, \tag{4.3}$$

$$y(t) - \int_{-\tau}^0 k(t - s)y(s)\mathrm{d}s = k(t), \quad -\tau \leqslant t \leqslant 0, \tag{4.4}$$

and let $x(t) = 0$ *and* $y(t) = 0$ *elsewhere. Then the matrix valued function* $\hat{h}$ *given by the following equation:*

$$I_n - \hat{h}(\lambda) = (I_n + \hat{x}(\lambda))^{*-1}(I_n + \hat{x}(\lambda))^{-1} = (I_n + \hat{y}(\lambda))^{*-1}(I_n + \hat{y}(\lambda))^{-1}, \quad \lambda \in \mathbf{R},$$

is the unique positive extension $\hat{h}$ *of* $\hat{k}$ *with* $(I_n - \hat{h})^{-1} \in W^c_{n\times n}(\tau)$.

Proof. We will obtain this theorem as a special case of Theorem I.1.3. Let $\mathcal{M}$ be the Wiener algebra $W_{n\times n}(\mathbf{R})$ with unit $e(\lambda) = I_n$ for $\lambda \in \mathbf{R}$ and involution $*$ given by $f^*(\lambda) = f(\lambda)^*$ (= the usual adjoint of the matrix $f(\lambda)$), $\lambda \in \mathbf{R}$. Put

$$\mathcal{M}_1 = \{f \in W_{n\times n}(\mathbf{R}) \mid c_f = 0,\ k_f(s) = 0,\ s < \tau\},$$

$$\mathcal{M}_2 = \{f \in W_{n\times n}(\mathbf{R}) \mid k_f(s) = 0,\ s > \tau \text{ and } s < 0\},$$

$$\mathcal{M}_3^0 = \{f \in W_{n\times n}(\mathbf{R}) \mid c_f = 0,\ k_f(s) = 0,\ s < -\tau \text{ and } s = 0\},$$

$$\mathcal{M}_4 = \{f \in W_{n\times n}(\mathbf{R}) \mid c_f = 0,\ k_f(s) = 0,\ s > -\tau\}.$$

Then clearly (1.1) holds and this decomposition satisfies the conditions (i), (ii) and (iii). Note that an element $f \in W_{n\times n}(\mathbf{R})$ is positive definite if $f(\lambda) > 0$ for $\lambda \in \mathbf{R} \cup \{\infty\}$. Let us check the axioms (A1)—(A3). First remark that $e - g^*g$ is positive definite in $W_{n\times n}(\mathbf{R})$ if and only if for $\lambda \in \mathbf{R} \cup \{\infty\}$ it holds that $\|g(\lambda)\| < 1$. If $g \in \mathcal{M}_+$, the function $g(\lambda)$ is analytic in the closed upper half plane including ∞. But then the maximum modulus principle gives that $\|g(\lambda)\| < 1$ for λ in the closed upper half plane including ∞. Hence $e - g$ has no zeroes in the closed upper half plane including ∞, but this gives that $e - g$ is invertible in $\mathcal{M}_+$. For an element $g \in \mathcal{M}_-$ one can reason in a similar way. Hence Axiom (A1) is fulfilled. Further, since $\|g(\lambda)\| \leqslant \|g\|$, one obtains from the assumption that $\|g\| < 1$ immediately that $e - g^*g$ is positive definite. This proves (A2). Checking (A3) is a triviality.

In order to apply Theorem I.1.3 one checks that from (4.3) and (4.4) it follows that

$$P_2((e - \hat{k})(e + \hat{x})) = e, \quad P_3((e - \hat{k})(e + \hat{y})) = e,$$

where P_2 and P_3 are as in Section I.1. Further $P_d(e + \hat{x}) = e = P_d(e + \hat{y})$ is clearly positive definite in $\mathcal{M}_d$. The fact that $(e + \hat{x})^{-1}$ belongs to $\mathcal{M}_+$ and $(e + \hat{y})^{-1}$ belongs to $\mathcal{M}_-$ is proved in [8, Theorem 8.1]. Now one can apply Theorem I.1.3 to obtain the theorem. ▨

The description for the set of all positive extensions is now a special case of Theorem I.1.5.

THEOREM I.4.2. *Let* $k \in L^1_{n\times n}[-\tau, \tau]$ *be given. In order that there exists a positive extension* $\hat{k}$ *it is necessary and sufficient that the operator* $I - K_\tau$ *is positive* *definite on* $L_n^2[0,\tau]$. *Here* K_τ *is defined in* (4.2). *Let this condition hold and let* x *and* y *be the solutions of* (4.3) *and* (4.4), *respectively. Then each positive extension* $\hat{h}$ *of the given band is of the form*

$$(4.5)\qquad \begin{aligned}(I_n-\hat{h}(\lambda)) &= ((I_n+\hat{y}(\lambda))\hat{g}(\lambda)+(I_n+\hat{x}(\lambda)))^{*-1}\\ &(I_n-\hat{g}(\lambda)^*\hat{g}(\lambda))((I_n+\hat{y}(\lambda))\hat{g}(\lambda)+(I_n+\hat{x}(\lambda)))^{-1},\end{aligned}$$

where g *is an element in* $L^1_{n\times n}[\tau,\infty)$ *such that* $\|\hat{g}(\lambda)\|<1$, $\lambda\in\mathbf{R}$. *Furthermore, formula* (4.5) *gives a* 1-1 *correspondence between all such* g *and all positive extensions* $\hat{h}$ *of* $\hat{k}$.

Proof. Note that if $\hat{h}$ is a positive extension of $\hat{k}$ then the operator $I-H$ where

$$(H\varphi)(t)=\int_{-\infty}^{\infty} h(t-s)\varphi(s)\mathrm{d}s,\qquad t\in\mathbf{R},$$

is a positive definite operator on $L_n^2(\mathbf{R})$. But then $I-H_\tau$, where H_τ is defined in the spirit of (4.2) with k replaced by h, is a positive definite operator on $L_n^2[0,\tau]$. Since $I-K_\tau=I-H_\tau$ we obtain the necessity of the condition "$I-K_\tau$ is positive definite" in Theorem I.4.2. The rest of the theorem is a direct consequence of Theorems I.4.1 and I.1.5. ▨

Theorem I.4.1 was obtained in [8]. A fractional description as in Theorem I.4.2 appears in [5, Theorem 11.4].

I.5. FREDHOLM INTEGRAL OPERATORS

In this section we apply the abstract results of Section I.1 to functions f which may be viewed as kernels of integral operators. Let $\mathscr{F}=\mathscr{F}_{T,\tau}$, $0<\tau<T<\infty$, denote the class of $n\times n$ matrix valued functions $f(t,s)$ which are defined in the square

$$\Delta=\{(t,s):0<t,s<T\},$$

are continuous on each of the open regions

$$\Delta_1=\{(t,s)\in\Delta:t+\tau<s\},$$

$$\Delta_2=\{(t,s)\in\Delta:t<s<t+\tau\},$$

$$\Delta_3=\{(t,s)\in\Delta:t-\tau<s<t\},$$

$$\Delta_4=\{(t,s)\in\Delta:s<t-\tau\},$$

and the restriction f_i of $f \in \Delta$ to Δ_i extends continuously to the closure $\bar{\Delta}_i$. The set $\mathcal{F}$ is an algebra with multiplication defined by

$$(f \bigstar g)(t, s) = \int_0^T f(t, u)g(u, s)\mathrm{d}u.$$

Also, $\mathcal{F}$ has a natural involution *, namely

$$f^*(t, s) := f(s, t)^*. \tag{5.1}$$

The * in the right hand side of (5.1) is the usual adjoint of a matrix. We shall say that $f \in \mathcal{F}$ is *regular* in $\mathcal{F}$ if there exists a $g \in \mathcal{F}$ such that

$$f + g + f \bigstar g = 0, \quad g + f + g \bigstar f = 0.$$

In that case g is uniquely determined by f and denoted by $f^\dagger$.

Given $f \in \mathcal{F}$ we shall write F for the integral operator on $L_n^2[0, T]$ with kernel f. Thus

$$(F\varphi)(t) = \int_0^T f(t, s)\varphi(s)\mathrm{d}s, \quad 0 \leqslant t \leqslant T.$$

Similarly, G stands for the integral operator with kernel g. If f is regular in $\mathcal{F}$, then $f^\dagger$ is precisely the kernel of the integral operator $(I - F)^{-1} - I$; in other words, $f^\dagger$ is the resolvent kernel. Furthermore, F^* is the integral operator with kernel f^*.

We shall deal with the following extension problem. Let

$$k \in \mathcal{F}_c := \{f \in \mathcal{F} \mid f(t, s) = 0, \ (t, s) \in \Delta_1 \cup \Delta_4\}.$$

A matrix valued function $g \in \mathcal{F}$ is called a *positive extension* of k if $k(t, s) = g(t, s)$, $(t, s) \in \Delta_2 \cup \Delta_3$ and $I - G$ is a positive operator on $L_n^2[0, T]$. To find all such g we need some additional notation. For $-\tau \leqslant \xi \leqslant T$ let J_ξ denote the interval

$$J_\xi = \{t : \max\{\xi, 0\} < t < \min\{\xi + \tau, T\}\}.$$

For $k \in \mathcal{F}_c$ and $-\tau \leqslant \xi \leqslant T$ let $A_{k,\xi}$ denote the integral operator on $L_n^2(J_\xi)$ which is defined by

$$(A_{k,\xi}\varphi)(t) = \varphi(t) - \int_{J_\xi} k(t, s)\varphi(s)\mathrm{d}s, \quad t \in J_\xi. \tag{5.2}$$

THEOREM I.5.1. *Let $k \in \mathscr{F}_c$ be given, and suppose that for every ξ in the interval $0 \leqslant \xi \leqslant T-\tau$ the operator $A_{k,\xi}$ in (5.2) is positive definite. Let x and y be given by*

$$x(t,s) - \int_{J_{s-\tau}} k(t,u)x(u,s)\mathrm{d}u = k(t,s), \tag{5.3}$$

for $t \in J_{s-\tau}$, $0 \leqslant s \leqslant T$, and $x(t,s) = 0$ elsewhere,

$$y(t,s) - \int_{J_s} k(t,u)y(u,s)\mathrm{d}u = k(t,s), \tag{5.4}$$

for $t \in J_s$, $0 \leqslant s \leqslant T$ and $y(t,s) = 0$, elsewhere. Then x and y are regular in $\mathscr{F}$ and the function $f \in \mathscr{F}$ given by

$$-f = x^\dagger + (x^\dagger)^* + (x^\dagger)^* \bigstar x^\dagger = y^\dagger + (y^\dagger)^* + (y^\dagger)^* \bigstar y^\dagger \tag{5.5}$$

is the unique positive extension $f \in \mathscr{F}$ of k with $f^\dagger \in \mathscr{F}_c$.

Proof. We will obtain this theorem as a special case of Theorem I.1.3. We let $\mathscr{M}$ be the linear span of $\{\mathscr{F}, I_n\}$, where I_n denotes the $n \times n$ identity matrix. The multiplication on $\mathscr{M}$ is defined by

$$(\lambda I_n + f)(\mu I_n + g) := \lambda\mu I_n + \lambda g + \mu f + f \bigstar g.$$

The unit in $\mathscr{M}$ is I_n and the involution * is defined by $(\lambda I_n + f)^* := \bar{\lambda} I_n + f^*$. The norm on $\mathscr{M}$ is defined by

$$\|\lambda I_n + f\| := |\lambda| + \sup_{(t,s)\in\Delta} \|f(t,s)\|. \tag{5.6}$$

Note that the norm of the operator $\lambda I + F$ is majorized by the number $\|\lambda I_n + f\|$. Let

$$\mathscr{M}_1 = \{f \in \mathscr{M};\ f \mid \Delta_j = 0,\ j = 2,3,4\},$$

$$\mathscr{M}_2 = \{\lambda I_n + f \in \mathscr{M};\ f \mid \Delta_j = 0,\ j = 1,3,4\},$$

$$\mathscr{M}_3^0 = \{f \in \mathscr{M};\ f \mid \Delta_j = 0,\ j = 1,2,4\},$$

$$\mathscr{M}_4 = \{f \in \mathscr{M};\ f \mid \Delta_j = 0,\ j = 1,2,3\}.$$

Then clearly (1.1) holds and this decomposition satisfies the conditions (i), (ii) and (iii). Note that $\lambda I_n + f \in \mathscr{M}$ is positive definite if $\lambda I + F$ is a positive definite integral operator on $L_n^2[0, T]$. Let us check the axioms (A1)—(A3). Let $g = \lambda I + f \in \mathscr{M}_+$ such that $e - g^* g$ is positive definite. Then, in particular, $|\lambda| < 1$ and the integral operator F is of Volterra type. But then $(1 - \lambda)I - F$ is invertible and the inverse is also of Volterra type. Use now Theorem 3.2 in [10] to conclude, that $(e - g)^{-1}$ belongs to $\mathscr{M}_+$. If $g \in \mathscr{M}_-$ one reasons in the same way. Thus Axiom (A1) holds. For Axiom (A2) note that if $g = \lambda I_n + f$ has norm less than one, then the integral operator $\lambda I + F$ has norm less than one. But then the positive definiteness of $e - g^* g$ follows immediately. Axiom (A3) is a triviality.

In order to apply Theorem I.1.3 one notes that Equations (5.3) and (5.4) imply that

$$P_2((I_n - k)(I_n + x)) = I_n, \quad P_3((I_n - k)(I_n + y)) = I_n,$$

where P_2 and P_3 are as in Section I.1. Further, $P_d(I_n + x) = I_n = P_d(I_n + y)$ is clearly positive definite in $\mathscr{M}_d$. The fact that x and y are regular, $x^\dagger$ belongs to $\mathscr{M}_+$, and $y^\dagger$ belongs to $\mathscr{M}_-$, is proved in [10, Theorems 5.2 and 5.4]. Apply now Theorem I.3.1 and one obtains the desired result. ∎

The description for the set of all positive extensions is now a special case of Theorem I.1.5.

THEOREM I.5.2. *Let $k \in \mathscr{F}_c$ be given. In order that there exists a positive extension of k it is necessary and sufficient that for every ξ in the interval $0 \leqslant \xi \leqslant T - \tau$ the operator $A_{k,\xi}$, which is defined in* (5.2), *is positive definite. Let this condition hold and let x and y be given via* (5.3) *and* (5.4), *respectively. Then each positive extension f of k is of the form*

$$\begin{aligned} f = &-(g^* + g^* \star y^* + x^*)^\dagger \star (g + y \star g + x)^\dagger + \\ &+ (g^* + g^* \star y^* + x^*)^\dagger \star g^* \star g \star (g + y \star g + x)^\dagger + (g + y \star g + x)^\dagger - \\ &- g^* \star g \star (g + y \star g + x)^\dagger + (g^* + g^* \star y^* + x^*)^\dagger - \\ &- (g^* + g^* \star y^* + x^*)^\dagger \star g^* \star g + g^* g, \end{aligned} \tag{5.7}$$

where g is an element of $\mathscr{F}$ such that $g(t, s) = 0$, $(t, s) \in \Delta_2 \cup \Delta_3 \cup \Delta_4$, and $\|g\| < 1$. Furthermore, Equation (5.7) *gives a* 1-1 *correspondence between all such g and all positive extensions f of k.*

Proof. Note that if f is a positive extension of k, the integral operator $I - F$ is positive definite. But then restrictions of $I - F$ from $L_n^2(J_\xi)$ into $L_n^2(J_\xi)$ are also positive definite. From this one sees immediately that the positive definiteness of the operators $A_{k,\xi}$ in Theorem I.5.2 is indeed a necessary condition for the existence of a positive extension. The rest of the theorem is a direct consequence of Theorems I.1.5 and I.5.1. ▨

Note that (5.7) is equivalent with

$$I - F = (G^*(I + Y) + (I + X))^{-1}(I - G^*G)((I + Y)G + (I + X))^{-1},$$

where X and Y denote the integral operators on $L_n^2[0, T]$ with kernel x and y, respectively.

Theorem I.5.1 is already proved in [10]. Theorem I.5.2 seems to be a new result.

CHAPTER II: CONTRACTIVE EXTENSIONS

II.1. THE ABSTRACT SETTING

Let $\mathscr{B}$ be a Banach space, and suppose that $\mathscr{B}$ admits a direct sum decomposition

$$\mathscr{B} = \mathscr{B}_- \dotplus \mathscr{B}_+,$$

where $\mathscr{B}_-$ and $\mathscr{B}_+$ are closed subspaces of $\mathscr{B}$. We are interested in the following problem: given $\varphi \in \mathscr{B}_-$, when does there exist an element $\psi \in \mathscr{B}$ such that $\|\psi\| < 1$ (for some specified norm) and $\psi - \varphi \in \mathscr{B}_+$? Such an element ψ is called a strictly contractive extension of φ. Furthermore, if a strictly contractive extension of φ exists, we want to describe all strictly contractive extensions of φ. In order to give a solution of this problem we need some more structure on $\mathscr{B}$. In what follows we shall assume that $\mathscr{B}$ can be embedded in a Banach algebra of 2×2 matrices with a unit and an involution.

We shall assume that the Banach space $\mathscr{B}$ appears as the space of (1, 2)-elements of the following Banach algebra of 2×2 block matrices:

$$\mathscr{M} = \left\{ f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in \mathscr{A},\ b \in \mathscr{B},\ c \in \mathscr{C},\ d \in \mathscr{D} \right\}.$$

Here $\mathscr{A}$ and $\mathscr{D}$ are Banach algebras with identities $e_{\mathscr{A}}$ and $e_{\mathscr{D}}$, respectively, and involutions *, and $\mathscr{C}$ is a Banach space which is isomorphic to $\mathscr{B}$ via an operator * whose inverse is also denoted by *, such that for every choice of $a \in \mathscr{A}$, $b \in \mathscr{B}$, $c \in \mathscr{C}$ and $d \in \mathscr{D}$:

$$\begin{aligned} & bc \in \mathscr{A}, \quad (bc)^* = c^*b^*; \quad ab \in \mathscr{B}, \quad (ab)^* = b^*a^*; \\ (1.1)\qquad & bd \in \mathscr{B}, \quad (bd)^* = d^*b^*; \quad ca \in \mathscr{C}, \quad (ca)^* = a^*c^*; \\ & dc \in \mathscr{C}, \quad (dc)^* = c^*d^*; \quad cb \in \mathscr{D}, \quad (cb)^* = b^*c^*. \end{aligned}$$

It is easy to see that $\mathcal{M}$ is an algebra (with respect to the natural rules for matrix multiplication and addition) with unit

$$e := \begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & e_{\mathcal{D}} \end{pmatrix}.$$

We define an involution * on $\mathcal{M}$ by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* := \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

On $\mathcal{M}$ there will be a norm given, which is assumed to be submultiplicative and such that the natural embeddings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and into $\mathcal{M}$ are norm preserving. With such a norm $\mathcal{M}$ is a Banach algebra. We will assume some additional structure within each of the four Banach spaces $\mathcal{A}$—$\mathcal{D}$. The algebras $\mathcal{A}$ and $\mathcal{D}$ are assumed to admit direct sum decompositions

$$\mathcal{A} = \mathcal{A}^0_- \dotplus \mathcal{A}_d \dotplus \mathcal{A}^0_+, \quad \mathcal{D} = \mathcal{D}^0_- \dotplus \mathcal{D}_d \dotplus \mathcal{D}^0_+ \tag{1.2}$$

in which all six of the newly indicated spaces are closed subalgebras and are such that

$$\begin{aligned} &e_{\mathcal{A}} \in \mathcal{A}_d, \quad (\mathcal{A}^0_-)^* = \mathcal{A}^0_+, \quad (\mathcal{A}_d)^* = \mathcal{A}_d, \\ &e_{\mathcal{D}} \in \mathcal{D}_d, \quad (\mathcal{D}^0_-)^* = \mathcal{D}^0_+, \quad (\mathcal{D}_d)^* = \mathcal{D}_d, \end{aligned} \tag{1.3}$$

and the inclusions

$$\mathcal{A}_d\mathcal{A}^0_\pm \subset \mathcal{A}^0_\pm, \quad \mathcal{A}^0_\pm\mathcal{A}_d \subset \mathcal{A}^0_\pm, \quad \mathcal{D}_d\mathcal{D}^0_\pm \subset \mathcal{D}^0_\pm, \quad \mathcal{D}^0_\pm\mathcal{D}_d \subset \mathcal{D}^0_\pm \tag{1.4}$$

are in force. It is then readily checked that

$$\mathcal{A}_\pm := \mathcal{A}^0_\pm \dotplus \mathcal{A}_d, \quad \mathcal{D}_\pm := \mathcal{D}^0_\pm \dotplus \mathcal{D}_d$$

are algebras. Moreover, if $a \in \mathcal{A}_d$ (resp. $d \in \mathcal{D}_d$) and is invertible, then $a^{-1} \in \mathcal{A}_d$ (resp. $d^{-1} \in \mathcal{D}_d$). Finally, we suppose that $\mathcal{B}$ and $\mathcal{C}$ admit decompositions

$$\mathcal{B} = \mathcal{B}_- \dotplus \mathcal{B}_+, \quad \mathcal{C} = \mathcal{C}_- \dotplus \mathcal{C}_+, \tag{1.5}$$

where $\mathcal{B}_\pm \subset \mathcal{B}$ and $\mathcal{C}_\pm \subset \mathcal{C}$ are closed subspaces satisfying

$$\begin{aligned} &\mathcal{C}_- = \mathcal{B}^*_+, \quad \mathcal{C}_+ = \mathcal{B}^*_-, \\ &\mathcal{B}_\pm\mathcal{D}_\pm \subset \mathcal{B}_\pm, \quad \mathcal{A}_\pm\mathcal{B}_\pm \subset \mathcal{B}_\pm, \\ &\mathcal{C}_\pm\mathcal{B}_\pm \subset \mathcal{D}^0_\pm, \quad \mathcal{B}_\pm\mathcal{C}_\pm \subset \mathcal{A}^0_\pm, \\ &\mathcal{C}_\pm\mathcal{A}_\pm \subset \mathcal{C}_\pm, \quad \mathcal{D}_\pm\mathcal{C}_\pm \subset \mathcal{C}_\pm. \end{aligned} \tag{1.6}$$

Now let us introduce the following subspaces of $\mathcal{M}$:

$$\mathcal{M}_1 = \begin{pmatrix} 0 & \mathcal{B}_+ \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathcal{B}_+ \right\},$$

$$\mathcal{M}_2 = \begin{pmatrix} \mathcal{A}_+ & \mathcal{B}_- \\ 0 & \mathcal{D}_+ \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \, a \in \mathcal{A}_+, \, b \in \mathcal{B}_-, \, d \in \mathcal{D}_+ \right\},$$

$$\mathcal{M}_3^0 = \begin{pmatrix} \mathcal{A}_-^0 & 0 \\ \mathcal{C}_+ & \mathcal{D}_-^0 \end{pmatrix} = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \middle| \, a \in \mathcal{A}_-^0, \, c \in \mathcal{C}_+, \, d \in \mathcal{D}_-^0 \right\},$$

$$\mathcal{M}_4 = \begin{pmatrix} 0 & 0 \\ \mathcal{C}_- & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \middle| \, c \in \mathcal{C}_- \right\},$$

and introduce $\mathcal{M}_c$, $\mathcal{M}_d$ etc. as in Section I.1. Note that $\mathcal{M}_d$ consists of all matrices

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with $a \in \mathcal{A}_d$ and $d \in \mathcal{D}_d$. It is easily checked that $\mathcal{M}$ satisfies the conditions (i), (ii) and (iii) in Section I.1. In particular, $\mathcal{M}$ satisfies the rules laid down in the multiplication table (I.1.2). In fact, the following more stricter multiplication table holds:

(1.7)

	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	0	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_c$
$\mathcal{M}_2$	$\mathcal{M}_1$	$\mathcal{M}_+$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_-$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}_c$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	0

With respect to positive elements we assume that the Banach algebra $\mathcal{M}$ satisfies the following axiom

(A0) If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is positive definite in $\mathcal{M}$, then a is positive definite in $\mathcal{A}$ and d is positive definite in $\mathcal{D}$.

Furthermore, we also assume that the axioms (A1)—(A3) in Section I.1 are satisfied. Note that Axiom (A0) implies that for $b \in \mathcal{B}$ the element $e_{\mathcal{D}} + b^*b$ is positive definite in $\mathcal{D}$ and the element $e_{\mathcal{A}} + bb^*$ is positive definite in $\mathcal{A}$. This follows immediately from the observation that

$$\begin{pmatrix} e_{\mathcal{A}} & b \\ 0 & e_{\mathcal{D}} \end{pmatrix} \begin{pmatrix} e_{\mathcal{A}} & 0 \\ b^* & e_{\mathcal{D}} \end{pmatrix}, \quad \begin{pmatrix} e_{\mathcal{A}} & 0 \\ b^* & e_{\mathcal{D}} \end{pmatrix} \begin{pmatrix} e_{\mathcal{A}} & b \\ 0 & e_{\mathcal{D}} \end{pmatrix}$$

are positive definite elements in $\mathscr{M}$. Further, from the equations

$$\begin{pmatrix} e_{\mathscr{A}} & h \\ h^* & e_{\mathscr{D}} \end{pmatrix} = \begin{pmatrix} e_{\mathscr{A}} & 0 \\ h^* & e_{\mathscr{D}} \end{pmatrix}\begin{pmatrix} e_{\mathscr{A}} & 0 \\ 0 & e_{\mathscr{D}} - h^*h \end{pmatrix}\begin{pmatrix} e_{\mathscr{A}} & h \\ 0 & e_{\mathscr{D}} \end{pmatrix} =$$

$$= \begin{pmatrix} e_{\mathscr{A}} & h \\ 0 & e_{\mathscr{D}} \end{pmatrix}\begin{pmatrix} e_{\mathscr{A}} - hh^* & 0 \\ 0 & e_{\mathscr{D}} \end{pmatrix}\begin{pmatrix} e_{\mathscr{A}} & 0 \\ h^* & e_{\mathscr{D}} \end{pmatrix}$$

one obtains that for $h \in \mathscr{B}$ the element $\begin{pmatrix} e_{\mathscr{A}} & h \\ h^* & e_{\mathscr{D}} \end{pmatrix}$ is positive definite in $\mathscr{M}$ if and only if $e_{\mathscr{D}} - h^*h$ is positive definite in $\mathscr{D}$, or equivalently, $e_{\mathscr{A}} - hh^*$ is positive definite in $\mathscr{A}$. Finally, note that when a and d are invertible elements of $\mathscr{A}$ and $\mathscr{D}$, respectively, we have that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & 0 \\ -d^{-1}ca^{-1} & d^{-1} \end{pmatrix}.$$

But then it is easy to see that in order to prove that $\mathscr{M}$ satisfies Axiom (A1) it suffices to prove that $\mathscr{A}$ and $\mathscr{D}$ satisfy Axiom (A1).

Let $\varphi \in \mathscr{B}_-$ be given. An element $\psi \in \mathscr{B}$ will be called a *strictly contractive extension* of φ if $e_{\mathscr{D}} - \psi^*\psi$ is positive definite in $\mathscr{D}$ and $\psi - \varphi \in \mathscr{B}_+$. Using Axiom (A0) one easily checks that ψ is a strictly contractive extension of φ if and only if $\begin{pmatrix} e_{\mathscr{A}} & \psi \\ \psi^* & e_{\mathscr{D}} \end{pmatrix}$ is a positive extension of $\begin{pmatrix} e_{\mathscr{A}} & \varphi \\ \varphi^* & e_{\mathscr{D}} \end{pmatrix} \in \mathscr{M}_c$. The following theorem gives sufficient conditions for the existence of a special strictly contractive extension of an element $\varphi \in \mathscr{B}_-$. With this special strictly contractive extension we shall be able to give a description of the set of all strictly contractive extensions of φ. In the examples treated in Sections II.2—II.7 below this sufficient condition will also prove to be necessary for the existence of a stricly contractive extension. To state the conditions we need some additional notation. If $\mathscr{E}_0$ is a subspace of the Banach space $\mathscr{E}$, we let $P_{\mathscr{E}_0}$ denote the projection in $\mathscr{E}$ on $\mathscr{E}_0$ along a natural complement. So, for instance, $P_{\mathscr{A}_+}$ is the projection on $\mathscr{A}_+$ along $\mathscr{A}_-^0$. If $\varphi \in \mathscr{B}$, we let $H := P_{\mathscr{A}_-}\varphi P_{\mathscr{C}_+}$ denote the operator from $\mathscr{C}_+$ into $\mathscr{A}_-$ defined by the following action:

$$H(c) = (P_{\mathscr{A}_-}\varphi P_{\mathscr{C}_+})(c) := P_{\mathscr{A}_-}(\varphi c), \quad c \in \mathscr{C}_+.$$

We shall employ this notation also for other subspaces.

THEOREM II.1.1. *Let $\varphi \in \mathscr{B}_-$ be given. Introduce the following operators*

$$(1.8)\qquad \begin{aligned} H &:= P_{\mathscr{A}_-}\varphi P_{\mathscr{C}_+} : \mathscr{C}_+ \to \mathscr{A}_-; \quad H_* := P_{\mathscr{C}_+}\varphi^* P_{\mathscr{A}_-} : \mathscr{A}_- \to \mathscr{C}_+;\\ \tilde{H} &:= P_{\mathscr{B}_-}\varphi P_{\mathscr{D}_+} : \mathscr{D}_+ \to \mathscr{B}_-; \quad \tilde{H}_* := P_{\mathscr{D}_+}\varphi^* P_{\mathscr{B}_-} : \mathscr{B}_- \to \mathscr{D}_+ .\end{aligned}$$

Suppose that for each $0 \leqslant \varepsilon \leqslant 1$ the operators $I - \varepsilon^2 HH_*$ and *$I - \varepsilon^2 \tilde{H}_*\tilde{H}$ are invertible, and that the elements*

$$(1.9)\qquad P_{\mathscr{A}_d}[(I-\varepsilon^2 HH_*)^{-1}e], \quad P_{\mathscr{D}_d}[(I-\varepsilon^2\tilde{H}_*\tilde{H})^{-1}e]$$

are positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. Let $r \in \mathscr{A}_d$ and $s \in \mathscr{D}_d$ be such that

$$P_{\mathscr{A}_d}[(I-HH_*)^{-1}e] = r^*r, \quad P_{\mathscr{D}_d}[(I-\tilde{H}_*\tilde{H})^{-1}e] = s^*s,$$

and put

$$(1.10)\qquad \begin{aligned} \alpha &:= ((I-HH_*)^{-1}e)r^{-1}, \quad \gamma := P_{\mathscr{C}_+}(\varphi^*\alpha),\\ \delta &:= ((I-\tilde{H}_*\tilde{H})^{-1}e)s^{-1}, \quad \beta := P_{\mathscr{B}_-}(\varphi\delta).\end{aligned}$$

Then

$$(1.11)\qquad \begin{aligned} \begin{pmatrix} e_{\mathscr{A}} & g\\ g^* & e_{\mathscr{D}}\end{pmatrix} &:= \begin{pmatrix} \alpha^* & -\gamma^*\\ 0 & e_{\mathscr{D}}\end{pmatrix}^{-1}\begin{pmatrix} \alpha & 0\\ -\gamma & e_{\mathscr{D}}\end{pmatrix}^{-1} =\\ &= \begin{pmatrix} e_{\mathscr{A}} & 0\\ -\beta^* & \delta^*\end{pmatrix}^{-1}\begin{pmatrix} e_{\mathscr{A}} & -\beta\\ 0 & \delta\end{pmatrix}^{-1}\end{aligned}$$

is the unique band extension of $\begin{pmatrix} e_{\mathscr{A}} & \varphi\\ \varphi^* & e_{\mathscr{D}}\end{pmatrix} \in \mathscr{M}_c$. In particular, $g = \beta\delta^{-1} = \alpha^{*-1}\gamma^*$ is the unique strictly contractive extension of φ with the property that $g(e_{\mathscr{D}} - g^*g)^{-1} \in \in \mathscr{B}_-$.

Proof. We will use the notations of Section I.1. For $0 \leqslant \varepsilon \leqslant 1$ put

$$a_\varepsilon := (I-\varepsilon^2 HH_*)^{-1}e, \quad c_\varepsilon := P_{\mathscr{C}_+}(\varepsilon\varphi^* a_\varepsilon),$$

$$d_\varepsilon := (I-\varepsilon^2\tilde{H}_*\tilde{H})^{-1}e, \quad b_\varepsilon := P_{\mathscr{B}_-}(\varepsilon\varphi d_\varepsilon),$$

$$x_\varepsilon := \begin{pmatrix} e_{\mathscr{A}} & -b_\varepsilon\\ 0 & d_\varepsilon\end{pmatrix}, \quad y_\varepsilon := \begin{pmatrix} a_\varepsilon & 0\\ -c_\varepsilon & e_{\mathscr{D}}\end{pmatrix}.$$

Note that $x_\varepsilon \in \mathcal{M}_+$ and $y_\varepsilon \in \mathcal{M}_-$ for $0 \leqslant \varepsilon \leqslant 1$. Clearly, the elements introduced above are analytic in the real variable ε. For $k_\varepsilon = \begin{pmatrix} e_{\mathscr{A}} & \varepsilon\varphi \\ \varepsilon\varphi^* & e_{\mathscr{D}} \end{pmatrix}$ we have that

$$P_3(k_\varepsilon y_\varepsilon) = P_3 \begin{pmatrix} a_\varepsilon - \varepsilon\varphi c_\varepsilon & \varepsilon\varphi \\ \varepsilon\varphi^* a_\varepsilon - c_\varepsilon & e_{\mathscr{D}} \end{pmatrix} = e,$$

since $P_{\mathscr{C}_+}(\varepsilon\varphi^* a_\varepsilon - c_\varepsilon) = 0$ and

$$P_{\mathscr{A}_-}(a - \varepsilon\varphi c_\varepsilon) = P_{\mathscr{A}_-}(a_\varepsilon - \varepsilon\varphi P_{\mathscr{C}_+}(\varepsilon\varphi^* a_\varepsilon)) = (I - \varepsilon^2 HH_*)a_\varepsilon = e.$$

Analogously, one calculates that $P_2(k_\varepsilon x_\varepsilon) = e$. If ε is small enough, $0 \leqslant \varepsilon < \sigma$ $(\leqslant 1)$, say, then the elements a_ε and d_ε are invertible in $\mathscr{A}_-$ and $\mathscr{D}_+$, respectively. Further, by assumption, for $0 \leqslant \varepsilon \leqslant 1$ the elements $P_{\mathscr{A}_d}(a_\varepsilon)$ and $P_{\mathscr{D}_d}(d_\varepsilon)$ are positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. But then for $0 \leqslant \varepsilon \leqslant 1$ the elements $P_d(x_\varepsilon)$ and $P_d(y_\varepsilon)$ are positive definite in $\mathcal{M}_d$. Now Theorem I.1.1 yields that for $0 \leqslant \varepsilon < \sigma$

$$x_\varepsilon^{*-1}(P_d x_\varepsilon)x_\varepsilon^{-1} = y_\varepsilon^{*-1}(P_d y_\varepsilon)y_\varepsilon^{-1}$$

is the unique band extension of k_ε. It follows that

$$x_\varepsilon(P_d x_\varepsilon)^{-1}x_\varepsilon^* = y_\varepsilon(P_d y_\varepsilon)^{-1}y_\varepsilon^* \tag{1.12}$$

holds for $0 \leqslant \varepsilon < \sigma$, and by analyticity (1.12) also holds for $0 \leqslant \varepsilon \leqslant 1$. By calculating the (1, 1) element of (1.12) we get that

$$a_\varepsilon(P_{\mathscr{A}_d} a_\varepsilon)^{-1}a_\varepsilon^* = e + b_\varepsilon(P_{\mathscr{D}_d} d_\varepsilon)^{-1}b_\varepsilon^*.$$

Axiom (A0) implies that the right hand side is positive definite, which gives that a_ε is invertible for $0 \leqslant \varepsilon \leqslant 1$. Indeed,

$$a_\varepsilon^+ := (P_{\mathscr{A}_d} a_\varepsilon)^{-1}a_\varepsilon^*(e + b_\varepsilon(P_{\mathscr{D}_d} d_\varepsilon)^{-1}b_\varepsilon^*)^{-1}$$

is a right inverse of a_ε. Further, $\varepsilon \to a_\varepsilon^+$ is analytic. Consider

$$a_\varepsilon^+ a_\varepsilon - e.$$

For $0 \leqslant \varepsilon < \sigma$ this equals zero. But then $a_\varepsilon^+ a_\varepsilon - e$ is zero for $\varepsilon \in [0, 1]$, proving the invertibility of a_ε. Further, since $P_{\mathscr{A}_+^0} a_\varepsilon^{-1} = 0$ for $0 \leqslant \varepsilon < \sigma$, we get by analyticity that this also holds for $\varepsilon = 1$. So a_1 is, in fact, invertible in $\mathscr{A}_-$. Analogously, d_1 is invertible in $\mathscr{D}_+$. But then we have shown that $P_2(k_1 x_1) = e = P_3(k_1 y_1)$, $x_1^{-1} \in \mathcal{M}_+$, $y_1^{-1} \in \mathcal{M}_-$ and $P_d x_1$ and $P_d y_1$ are positive definite in $\mathcal{M}_d$. Let $r \in \mathscr{A}_d$

and $s \in \mathcal{D}_d$ be as in the theorem, i.e., such that $r^*r = P_{\mathcal{A}_d}a_1$ and $s^*s = P_{\mathcal{D}_d}d_1$. Theorem I.1.3 gives that

$$f^{-1} := \begin{pmatrix} e_{\mathcal{A}} & -b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & s^*s \end{pmatrix}^{-1} \begin{pmatrix} e_{\mathcal{A}} & 0 \\ -b_1^* & d_1^* \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 & 0 \\ -c_1 & e_{\mathcal{D}} \end{pmatrix} \begin{pmatrix} r^*r & 0 \\ 0 & e_{\mathcal{D}} \end{pmatrix}^{-1} \begin{pmatrix} a_1^* & -c_1^* \\ 0 & e_{\mathcal{D}} \end{pmatrix}$$

is the unique band extension of k_1. Clearly f has to be of the form $\begin{pmatrix} e_{\mathcal{A}} & g \\ g^* & e_{\mathcal{D}} \end{pmatrix}$, with $g \in \mathcal{B}$ such that $g - \varphi \in \mathcal{B}_+$. Now put $\alpha := a_1 r^{-1}$, $\gamma := c_1 r^{-1}$, $\beta := b_1 s^{-1}$, $\delta := d_1 s^{-1}$, and $g = \beta\delta^{-1} = \alpha^{*-1}\gamma^*$. Then the first statement in the theorem is proved.

To prove the second statement it suffices to note that

$$\begin{pmatrix} e_{\mathcal{A}} & g \\ g^* & e_{\mathcal{D}} \end{pmatrix}^{-1} = \begin{pmatrix} * & g(e_{\mathcal{D}} - g^*g)^{-1} \\ * & * \end{pmatrix} \in \mathcal{M}_c$$

if and only if $g(e_{\mathcal{D}} - g^*g)^{-1} \in \mathcal{B}_-$. ▨

The matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is called $\begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & -e_{\mathcal{D}} \end{pmatrix}$-*unitary* if

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & -e_{\mathcal{D}} \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & -e_{\mathcal{D}} \end{pmatrix}.$$

THEOREM II.1.2. *Let $\varphi \in \mathcal{B}_-$, and suppose that*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.13}$$

is a $\begin{pmatrix} e_{\mathcal{A}} & 0 \\ 0 & -e_{\mathcal{D}} \end{pmatrix}$-unitary matrix with $\alpha^{\pm 1} \in \mathcal{A}_-$, $\beta \in \mathcal{B}_-$, $\gamma \in \mathcal{C}_+$, $\delta^{\pm 1} \in \mathcal{D}_+$ and $g := \beta\delta^{-1} = \alpha^{*-1}\gamma^* \in \varphi + \mathcal{B}_+$. Then each strictly contractive extension $\psi \in \mathcal{B}$ of φ is of the form

$$\psi = (\alpha h + \beta)(\gamma h + \delta)^{-1}, \tag{1.14}$$

where h is an element in $\mathcal{B}_+$ such that $e_{\mathcal{D}} - h^*h$ is positive definite in $\mathcal{D}$. Furthermore, Equation (1.14) *gives a* 1-1 *correspondence between all such h and all strictly contractive extensions ψ of φ. Alternatively, each strictly contractive extension*

$\psi \in \mathscr{B}$ of φ is of the form

$$\psi = (\alpha^* + f\beta^*)^{-1}(\gamma^* + f\delta^*), \tag{1.15}$$

where f is an element in $\mathscr{B}_+$ such that $e_{\mathscr{A}} - ff^*$ is positive definite in $\mathscr{A}$. Furthermore, Equation (1.15) gives a 1-1 correspondence between all such f and all strictly contractive extensions ψ of φ.

It is easy to see that the elements $\alpha - \delta$ in Theorem II.1.1 satisfy the requirements in Theorem II.1.2. Thus Theorem II.1.1 gives sufficient conditions on φ for the existence of a matrix (1.13) satisfying these requirements.

Proof. Put

$$k = \begin{pmatrix} e_{\mathscr{A}} & \varphi \\ \varphi^* & e_{\mathscr{D}} \end{pmatrix}, \quad u = \begin{pmatrix} -e_{\mathscr{A}} & \beta \\ 0 & -\delta \end{pmatrix}, \quad v = \begin{pmatrix} \alpha & 0 \\ -\gamma & e_{\mathscr{D}} \end{pmatrix}.$$

Using the assumptions on the matrix (1.13) it is straightforward to check that with these choices of k, u and v the conditions of Theorem I.1.5 are met. Choose $h \in \mathscr{B}_+$ such that

$$\begin{pmatrix} e_{\mathscr{A}} & 0 \\ 0 & e_{\mathscr{D}} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ h^* & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e_{\mathscr{A}} & 0 \\ 0 & e_{\mathscr{D}} - h^*h \end{pmatrix}$$

is positive definite in $\mathscr{M}$. Note that by Axiom (A0) this is equivalent to choosing $h \in \mathscr{B}_+$ such that $e_{\mathscr{D}} - h^*h$ is positive definite in $\mathscr{D}$. Apply now Theorem I.1.5 to obtain that

$$T\left(\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} e_{\mathscr{A}} & \psi^* \\ \psi & \sigma \end{pmatrix},$$

where

$$\sigma = (\delta + \gamma h)^{*-1}\{(\alpha + \gamma h)^*(\alpha + \gamma h) + e - h^*h\}(\delta + \gamma h)^{-1},$$

is a positive extension of k. Then clearly σ must be equal to $e_{\mathscr{D}}$. (The identity $\sigma = e_{\mathscr{D}}$ can also be computed directly by using the $\begin{pmatrix} e_{\mathscr{A}} & 0 \\ 0 & -e_{\mathscr{D}} \end{pmatrix}$-unitary of the matrix (1.13).) Furthermore, Theorem I.1.5 yields that by considering all such h we get all positive extensions of k, and, moreover, the correspondence is 1-1. Since $\begin{pmatrix} e_{\mathscr{A}} & \psi \\ \psi^* & e_{\mathscr{D}} \end{pmatrix}$ is a positive extension of k if and only if ψ is a strictly contractive extension of φ, the theorem follows.

The alternative description (1.15) one proves in the same way, but now with the usage of the description (1.11) in Theorem I.1.5. ▨

The approach used in this section is the same as the abstract approach in [13], except that here the algebra is enriched with an involution. Both Theorems II.1.1 and II.1.2 are new, but they are inspired by earlier concrete versions in [12], [13] (see also [5]).

II.2. THE OPERATOR MATRIX CASE

In this section we specify the results of Section II.1 for the space $\Omega_{N\times M}$ of $N \times M$ operator matrices. An element of this space has the following form

$$T = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1M} \\ \vdots & & \vdots \\ \varphi_{N1} & \cdots & \varphi_{NM} \end{pmatrix}.$$

Here φ_{ij}, $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$, is a bounded linear operator from a Hilbert space H_j into a Hilbert space $\tilde{H}_i$, shortly $\varphi_{ij} \in \mathscr{L}(H_j, \tilde{H}_i)$. We let $\|T\|$ denote the usual operator norm of $T : H_1 \oplus \ldots \oplus H_M \to \tilde{H}_1 \ldots \oplus \tilde{H}_N$.

Fix $-N < p < M$. For $1 \leqslant i \leqslant N, 1 \leqslant j \leqslant M, j-i \leqslant p$, let $\varphi_{ij} \in \mathscr{L}(H_j, \tilde{H}_i)$. An operator matrix $T \in \Omega_{N\times M}$ is called a *strictly contractive extension* of the given lower triangular part $\{\varphi_{ij}, j-i \leqslant p\}$ if $\|T\| < 1$ and for $j-i \leqslant p$ the (i,j)th element of T is equal to φ_{ij}. Let $r(j) = \max\{1, j-p\}$ and $s(j) = \min\{M, j+p\}$. Clearly a necessary condition for the existence of a strictly contractive extension is the following:

$$\|(\varphi_{ij})_{i=k,\, j=1}^{N,\; s(k)}\| < 1, \quad k = r(1), \ldots, N. \tag{2.1}$$

For $j = 1, \ldots, N$ let

$$S_j := \begin{pmatrix} \varphi_{j1} & \cdots & \varphi_{j,s(j)} \\ \vdots & & \vdots \\ \varphi_{N1} & \cdots & \varphi_{N,s(j)} \end{pmatrix} : \bigoplus_{k=1}^{s(j)} H_k \to \bigoplus_{k=j}^{N} \tilde{H}_k,$$

if $j + p \leqslant M$, and

$$S_j = 0 : (0) \to \bigoplus_{k=j}^{N} \tilde{H}_k,$$

if $j + p > M$. For $j = 1, \ldots, M$ let

$$R_j := \begin{pmatrix} \varphi_{r(j),1} & \cdots & \varphi_{r(j),j} \\ \vdots & & \vdots \\ \varphi_{N1} & \cdots & \varphi_{Nj} \end{pmatrix} : \bigoplus_{k=1}^{j} H_k \to \bigoplus_{k=r(j)}^{N} \tilde{H}_k,$$

if $j-p \leqslant N$, and

$$R_j = 0 : \bigoplus_{k=1}^{j} H_k \to (0),$$

if $j-p > N$. Obviously, (2.1) implies that $\|S_j\| < 1$, $1 \leqslant j \leqslant N$, and also $\|R_j\| < 1$, $1 \leqslant j \leqslant M$. The converse statement holds trivially.

THEOREM II.2.1. *For $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$, $j-i \leqslant p$, let φ_{ij} be a given operator acting from a Hilbert space H_j into a Hilbert space $\tilde{H}_i$, and suppose that* (2.1) *holds. Put*

$$\begin{pmatrix} \hat{\alpha}_{ii} \\ \hat{\alpha}_{i+1,i} \\ \vdots \\ \hat{\alpha}_{Ni} \end{pmatrix} = (I - S_i S_i^*)^{-1} \begin{pmatrix} I_{\tilde{H}_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 1, \ldots, N, \tag{2.2}$$

$$\begin{pmatrix} \hat{\beta}_{r(j),j} \\ \vdots \\ \hat{\beta}_{N-1,j} \\ \hat{\beta}_{Nj} \end{pmatrix} = R_j(I - R_j^* R_j)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{H_j} \end{pmatrix}, \quad j = 1, \ldots, M, \tag{2.3}$$

$$\begin{pmatrix} \hat{\gamma}_{1i} \\ \hat{\gamma}_{2i} \\ \vdots \\ \hat{\gamma}_{s(i),i} \end{pmatrix} = S_i^*(I - S_i S_i^*)^{-1} \begin{pmatrix} I_{\tilde{H}_i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad i = 1, \ldots, N, \tag{2.4}$$

$$\begin{pmatrix} \hat{\delta}_{1j} \\ \vdots \\ \hat{\delta}_{j-1,j} \\ \hat{\delta}_{jj} \end{pmatrix} = (I - R_j^* R_j)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_{H_j} \end{pmatrix}, \quad j = 1, \ldots, M, \tag{2.5}$$

and let

$$\alpha := (\alpha_{ij})_{i,j=1}^{N}, \quad \alpha_{ij} = \begin{cases} \hat{\alpha}_{ij}\hat{\alpha}_{jj}^{-1/2}, & i \geqslant j; \\ 0, & i < j; \end{cases} \tag{2.6}$$

$$\beta := (\beta_{ij})_{i=1,j=1}^{N\ \ M}, \quad \beta_{ij} = \begin{cases} \hat{\beta}_{ij}\hat{\delta}_{jj}^{-1/2}, & i \geqslant r(j); \\ 0, & i < r(j); \end{cases} \tag{2.7}$$

$$\gamma := (\gamma_{ij})_{i=1,j=1}^{M\ \ N}, \quad \gamma_{ij} = \begin{cases} \hat{\gamma}_{ij}\hat{\alpha}_{jj}^{-1/2}, & i \leqslant s(j); \\ 0, & i > s(j); \end{cases} \tag{2.8}$$

$$\delta := (\delta_{ij})_{i,j=1}^{M}, \quad \delta_{ij} = \begin{cases} \hat{\delta}_{ij}\hat{\delta}_{jj}^{-1/2}, & i \leqslant j; \\ 0, & i > j. \end{cases} \tag{2.9}$$

Then the operator matrix G defined by

$$G := \beta\delta^{-1} = \alpha^{*-1}\gamma^{*}$$

is the unique strictly contractive extension of the given lower triangular part $\{\varphi_{ij}, j-i \leqslant p\}$ *with* $(G(I-G^{*}G)^{-1})_{ij} = 0$ *for* $j-i > p$.

Proof. We will obtain this theorem as a special case of Theorem II 1.1 Let $\Omega_{\tau,+}(Z_1, \ldots, Z_p; Y_1, \ldots, Y_q)$ denote the set of operator matrices

$$\{(A_{ij})_{i=1,j=1}^{p\ \ q} \mid A_{ij}: Z_j \to Y_i,\ A_{ij} = 0,\ j-i \leqslant \tau\},$$

$\Omega_{\tau,-}(Z_1, \ldots, Z_p; Y_1, \ldots, Y_q)$ denote the set of operator matrices

$$\{(A_{ij})_{i=1,j=1}^{p\ \ q} \mid A_{ij}: Z_j \to Y_i,\ A_{ij} = 0,\ j-i \geqslant \tau\},$$

and let $\Omega_{\tau,0}(Z_1, \ldots, Z_p; Y_1, \ldots, Y_q)$ denote the set of operator matrices

$$\{(A_{ij})_{i=1,j=1}^{p\ \ q} \mid A_{ij} : Z_j \to Y_i,\ A_{ij} = 0,\ j-i = \tau\}.$$

We make the following choices of spaces:

$$\mathscr{B}_{+} = \Omega_{p,+}(H_1, \ldots, H_M; \tilde{H}_1, \ldots, \tilde{H}_N),$$

$$\mathscr{B}_{-} = \Omega_{p+1,-}(H_1, \ldots, H_M; \tilde{H}_1, \ldots, \tilde{H}_N),$$

$$\mathscr{A}^0_{\pm} = \Omega_{0,\pm}(\tilde{H}_1, \ldots, \tilde{H}_N; \tilde{H}_1, \ldots, \tilde{H}_N),$$

$$\mathscr{A}_{\mathrm{d}} = \Omega_{0,0}(\tilde{H}_1, \ldots, \tilde{H}_N; \tilde{H}_1, \ldots, \tilde{H}_N),$$

$$\mathscr{D}^0_{\pm} = \Omega_{0,\pm}(H_1, \ldots, H_M; H_1, \ldots, H_M),$$

$$\mathscr{D}_{\mathrm{d}} = \Omega_{0,0}(H_1, \ldots, H_M; H_1, \ldots, H_M),$$

and let $\mathscr{B}$, $\mathscr{A}$ and $\mathscr{D}$ be given via (1.2) and (1.5). On these spaces we define the operations * as the usual adjoint of an operator between Hilbert spaces. We endow the spaces $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C} := \mathscr{B}^{*}$, $\mathscr{D}$ and $\mathscr{M}$ with the usual operator norm. It is easy to see that the conditions (1.1), (1.3), (1.4) and (1.6) are satisfied. Further, the norm on $\mathscr{M}$ is submultiplicative and the natural embeddings of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ into $\mathscr{M}$ are clearly norm preserving. Let us check the axioms (A0)—(A3). Axiom (A0) is trivially fulfilled. Further, since $\mathscr{M}$ is a B^{*}-algebra, Axioms (A1)—(A3) are fulfilled automatically.

Let $\varphi = (\varphi_{ij})_{i=1,j=1}^{N\ \ M}$, where $\varphi_{ij} = 0$ for $j - i > p$. Let us consider the operator $I - \varepsilon^2 HH_*$, where H and H_* are defined in (1.8). Applying this operator on an element $A \in \mathscr{A}_-$ gives an element in $\mathscr{A}_-$ whose columns are described by the following equations:

$$(((I - \varepsilon^2 HH_*)(A))_{ij})_{i=j}^{N} = (I - \varepsilon^2 S_j S_j^*)(A_{ij})_{i=j}^{N}, \quad j = 1, \ldots, N.$$

It is not hard to see that $I - \varepsilon^2 HH_*$ is invertible for all $0 \leqslant \varepsilon \leqslant 1$ if and only if $\|S_j\| < 1, j = 1, \ldots, N$. Analogously, one shows that the operators $I - \varepsilon^2 \tilde{H}_* \tilde{H}$, $0 \leqslant \varepsilon \leqslant 1$, are all invertible if and only if condition (2.1) holds. Assume that (2.1) holds. Then for $0 \leqslant \varepsilon \leqslant 1$ the first element appearing in (1.9) is the diagonal operator matrix with as (i,i)th element the (1,1) element of the positive operator matrix $(I - \varepsilon^2 S_i S_i^*)^{-1}$. So, clearly this diagonal operator matrix is positive definite in $\mathscr{A}_d$. Analogously, one proves that the second element in (1.9) is positive definite in $\mathscr{D}_d$. Applying now Theorem II.1.1 one obtains (using the above calculations) the operator matrices α, β, γ, and δ given in Theorem II.2.1. ▨

For the description for the set of all strictly contractive extensions of a given lower triangular part one now simply applies Theorem II.1.2. Since condition (2.1) is necessary for the existence of a strictly contractive extension, we obtain the following result.

THEOREM II.2.2. *For $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$, $j - i \leqslant p$, let φ_{ij} be a given operator acting from a Hilbert space H_j into a Hilbert space $\tilde{H}_i$. Then the lower triangular part $\{\varphi_{ij}, j - i \leqslant p\}$ has a strictly contractive extension if and only if* (2.1) *holds. Suppose that* (2.1) *holds, and let α, β, γ, and δ be defined by* (2.2)—(2.9). *Then each strictly contractive extension F of the given lower triangular part is of the form*

$$F = (\alpha E + \beta)(\gamma E + \delta)^{-1}, \tag{2.10}$$

where $E = (E_{ij})_{i=1,j=1}^{N\ \ M}$ is a strictly contractive operator matrix with $E_{ij} = 0$, $j - i \leqslant p$. Furthermore, (2.10) *gives a 1-1 correspondence between all such E and all strictly contractive extensions F.*

Let us remark that there is an alternative description for the set of all positive extensions, which one obtains from (1.15).

For the block matrix case a linear fractional description of all strictly contractive extensions was obtained earlier in [2]. Formulas for the coefficients appearing in the linear fractional map are derived in [17] by using a generalized version of the Schur algorithm. In the latter paper the final formulas are less explicit than the ones given here.

II.3. THE WIENER SPACE ON THE CIRCLE

We use the notations of Section I.3.

THEOREM II.3.1. *Let φ_j, $j \leqslant 0$, be given $N \times M$ matrices, and suppose that the Hankel operator*

$$(3.1)\qquad A := \begin{pmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \cdots \\ \varphi_{-1} & \varphi_{-2} & \varphi_{-3} & \cdots \\ \varphi_{-2} & \varphi_{-3} & \varphi_{-4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \ell_2^M \to \ell_2^N$$

has norm less than one. Put

$$(3.2)\qquad \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}_{-1} \\ \vdots \end{pmatrix} = (I - AA^*)^{-1} \begin{pmatrix} I \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \vdots \end{pmatrix} = A^* \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_{-1} \\ \vdots \end{pmatrix},$$

$$(3.3)\qquad \begin{pmatrix} \hat{\delta}_0 \\ \hat{\delta}_1 \\ \vdots \end{pmatrix} = (I - A^*A)^{-1} \begin{pmatrix} I \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_{-1} \\ \vdots \end{pmatrix} = A \begin{pmatrix} \hat{\delta}_0 \\ \hat{\delta}_1 \\ \vdots \end{pmatrix},$$

and let

$$(3.4)\qquad \begin{aligned} \alpha(z) &= \sum_{j=-\infty}^{0} \hat{\alpha}_j \hat{\alpha}_0^{-1/2} z^j, & \gamma(z) &= \sum_{j=0}^{\infty} \hat{\gamma}_j \hat{\alpha}_0^{-1/2} z^j, \\ \beta(z) &= \sum_{j=-\infty}^{0} \hat{\beta}_j \hat{\delta}_0^{-1/2} z^j, & \delta(z) &= \sum_{j=0}^{\infty} \hat{\delta}_j \hat{\delta}_0^{-1/2} z^j. \end{aligned}$$

Then the function g given by

$$g(z) := \beta(z)\delta(z)^{-1} = \alpha(z)^{*-1}\gamma(z)^*, \quad z \in \mathbf{T},$$

is the unique function $g \in W_{N\times M}(\mathbf{T})$ such that $g_j = \varphi_j$, $j \leqslant 0$, $\|g(z)\| < 1$, $|z| = 1$, and $(g(I - g^*g)^{-1})_j = 0$, $j > 0$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Make the following choices of spaces:

$$\mathscr{B}_+ = (\mathscr{C}_-)^* := \{f \in W_{N\times M}(\mathbf{T}) \mid f_j = 0,\ j \leqslant 0\};$$

$$\mathscr{B}_- = (\mathscr{C}_+)^* := \{f \in W_{N\times M}(\mathbf{T}) \mid f_j = 0,\ j > 0\};$$

$$\mathscr{A}_+^0 = (\mathscr{A}_-^0)^* := \{f \in W_{N\times N}(\mathbf{T}) \mid f_j = 0,\ j \leqslant 0\}; \quad \mathscr{A}_d = \mathbf{C}^{N\times N};$$

$$\mathscr{D}_+^0 = (\mathscr{D}_-^0)^* := \{f \in W_{M\times M}(\mathbf{T}) \mid f_j = 0,\ j \leqslant 0\}; \quad \mathscr{D}_d = \mathbf{C}^{M\times M},$$

and let $\mathscr{B} = \mathscr{C}^*$, $\mathscr{A}$ and $\mathscr{D}$ be given via (1.2) and (1.5). The conditions (1.1), (1.3), (1.4) and (1.6) are clearly satisfied. The above introduced spaces as well as the space $\mathscr{M} = W_{(N+M)\times(N+M)}(\mathbf{T})$ we endow with the usual norm on Wiener spaces. Checking the axioms (A0)—(A3) is here simple: Axiom (A0) evidently holds true. In order to prove (A1) it suffices to prove the axiom for $\mathscr{A}$ and $\mathscr{D}$. For this proof as well as for the proofs of Axioms (A2) and (A3) we refer to Section I.3.

Let us apply Theorem II.1.1 for a given $\varphi(z) = \sum_{j=-\infty}^{0} \varphi_j z^j \in \mathscr{B}_-$. If $a(z) = \sum_{j=-\infty}^{0} a_j z^j \in \mathscr{A}_-$ the element $\hat{a} = (I - \varepsilon^2 HH_*)a \in \mathscr{A}_-$, where H and H_* are defined in (1.8), is given by

$$\begin{pmatrix} \hat{a}_0 \\ \hat{a}_{-1} \\ \vdots \end{pmatrix} = (I - \varepsilon^2 AA^*) \begin{pmatrix} a_0 \\ a_{-1} \\ \vdots \end{pmatrix}.$$

Note that since $(\|\varphi_{-j}\|)_{j=0}^{\infty}$ is an element of $\ell_1 \subset \ell_2$, the operator A is compact as an operator between ℓ_1 spaces as well as between ℓ_2 spaces. Suppose now that $I - \varepsilon^2 AA^*$ is invertible as an operator on ℓ_2^N. Then its kernel is empty, and thus also the kernel of the operator $I - \varepsilon^2 AA^*$ viewed as an operator on ℓ_1^N. But then, using that it concerns a Fredholm operator with index zero, it is invertible. Using this it is not hard to see that the invertibility of all operators $(I - \varepsilon^2 HH_*)$, $0 \leqslant \varepsilon \leqslant 1$, is equivalent to the condition $\|A\| < 1$. Analogously, the operators $I - \varepsilon^2 \tilde{H}_* \tilde{H}$ are treated. The two elements appearing in (1.9) are the (1, 1) element of $(I - \varepsilon^2 AA^*)^{-1}$ and the (1, 1) element of $(I - \varepsilon^2 A^* A)^{-1}$, respectively. So when $\|A\| < 1$ and $0 \leqslant \varepsilon \leqslant 1$ these elements are positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. Now one finds, as expected, that the α, β, γ, and δ, appearing in Theorem II.3.1 coincide with the ones appearing in Theorem II.1.1. ▨

Let φ_j, $j \leqslant 0$, be a given set of $N \times M$ matrices. A matrix valued function $\psi \in W_{N\times M}(\mathbf{T})$ is called a *strictly contractive extension* of φ, where $\varphi(z) = \sum_{j=-\infty}^{0} \varphi_j z^j$, if $\psi_j = \varphi_j$ for $j \leqslant 0$ and $\|\psi(z)\| < 1$ for $z \in \mathbf{T}$. Note that the definition of a strictly contractive extension coincides with the one given in Section II.1. Further, note that if $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j$ is a strictly contractive extension of φ, the doubly infinite matrix $(\psi_{j-i})_{i,j=-\infty}^{\infty}$ has norm less than one. Since A appears as a submatrix (up to reversing the order of the columns) of this doubly infinite matrix, this implies that $\|A\| < 1$. Thus the condition $\|A\| < 1$ is clearly a necessary condition for the existence of a strictly contractive extension of φ. The description of the set of all strictly contractive extensions of a given function φ is a direct corollary of Theorem II.1.2.

THEOREM II.3.2. *Let φ_j, $j \leqslant 0$, be given $N \times M$ matrices. In order that there exists a strictly contractive extension of the given function $\varphi(z) = \sum_{j=-\infty}^{0} \varphi_j z^j$ it is necessary and sufficient that the Hankel operator A, defined in* (3.1), *has norm less than one. Assume that this condition holds, and let $\alpha(z), \beta(z), \gamma(z)$ and $\delta(z)$ be defined by* (3.2)—(3.4). *Then each strictly contractive extension ψ of φ is of the form*

$$\psi(z) = (\alpha(z)h(z) + \beta(z))(\gamma(z)h(z) + \delta(z))^{-1}, \tag{3.5}$$

where h is an element in $W_{N\times M}(\mathbf{T})$ such that $\|h(z)\| < 1$, $|z| = 1$, and $h_j = 0$, $j \leqslant 0$. Furthermore, formula (3.5) *gives a* 1-1 *correspondence between all such h and all strictly contractive extensions ψ of f.*

Let us remark that there is an alternative description for the set of all positive extensions, which one obtains from (1.15).

Formula (3.5) appeared earlier in [1], [11] and [12] (see also [5, Theorem 9.2]).

II.4. THE WIENER SPACE ON THE CIRCLE: THE FOUR BOCK CASE

We use the notations of Section I.3. In this section we view an element f of the Wiener space on the circle $W_{N\times M}(\mathbf{T})$ divided into four subblocks, i.e.,

$$f(z) = \begin{pmatrix} f_{11}(z) & f_{12}(z) \\ f_{21}(z) & f_{22}(z) \end{pmatrix},$$

with $f_{rs} \in W_{\nu_r \times \mu_s}(\mathbf{T})$, where μ_1, μ_2 are positive integers such that $\mu_1 + \mu_2 = M$ and ν_1, ν_2 are positive integers such that $\nu_1 + \nu_2 = N$. The space $W_{N\times M}(\mathbf{T})$ admits the following decomposition

$$W_{N\times M}(\mathbf{T}) = W^-_{N\times M}(\mathbf{T}) \dotplus W^+_{N\times M}(\mathbf{T}),$$

where

$$W^-_{N\times M}(\mathbf{T}) = \{f = (f_{rs})^2_{r,s=1} \in W_{N\times M}(\mathbf{T}) \mid (f_{12})_j = 0,\ j < 0\},$$

and

$$W^+_{N\times M}(\mathbf{T}) = \{f = (f_{rs})^2_{r,s=1} \in W_{N\times M}(\mathbf{T}) \mid f_{rs} = 0,\ (r,s) \neq (1,2);\ (f_{12})_j = 0,\ j \geqslant 0\}.$$

We are interested in the following completion problem. Let $\varphi \in W^-_{N\times M}(\mathbf{T})$ be given. An element $g \in W_{N\times M}(\mathbf{T})$ is called a *strictly contractive extension* of φ if $g - \varphi \in W^-_{N\times M}(\mathbf{T})$ and $\|g(z)\| < 1$, $z \in \mathbf{T}$.

Before we state the results of this section we first introduce some notations. Let p and q denote the projectors which are defined by the rules

$$p\left(\sum_{i=-\infty}^{\infty} f_i z^i\right) = \sum_{i=0}^{\infty} f_i z^i, \quad q\left(\sum_{i=-\infty}^{\infty} f_i z^i\right) = \sum_{i=-\infty}^{0} f_i z^i.$$

These projectors will be applied on the Hilbert space $L_j^2(\mathbf{T})$ of $j \times 1$ vector valued functions on the circle with square summable entries. Let $\varphi \in W_{N\times M}^-(\mathbf{T})$ be given, and introduce the following Hankel like operators which act from $L_{\nu_1}^2(\mathbf{T}) \oplus L_{\nu_2}^2(\mathbf{T})$ to $L_{\mu_1}^2(\mathbf{T}) \oplus L_{\mu_2}^2(\mathbf{T})$:

$$(4.1) \qquad \Gamma_0 := \Phi(q \oplus 0), \quad \Gamma_1 = (p \oplus I_{\mu_2})\Phi(I_{\nu_2} \oplus q), \quad \Gamma_2 := (0 \oplus p)\Phi,$$

where Φ denotes the operator of multiplication with φ.

THEOREM II.4.1. *Let $\varphi \in W_{N\times M}^-(\mathbf{T})$ de given, and suppose that the Hankel like operators Γ_i, $i = 0, 1, 2$, have norm less than one. Put*

$$(4.2) \qquad \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = (I - \Gamma_1\Gamma_1^*)^{-1}\begin{pmatrix} I_{\mu_1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} = (I - \Gamma_2\Gamma_2^*)^{-1}\begin{pmatrix} 0 \\ I_{\mu_2} \end{pmatrix},$$

$$(4.3) \qquad \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = -\Gamma_1^*\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \quad \begin{pmatrix} c_{12} \\ c_{22} \end{pmatrix} = -\Gamma_2^*\begin{pmatrix} 0 \\ a_{22} \end{pmatrix},$$

$$(4.4) \qquad \begin{pmatrix} d_{11} \\ 0 \end{pmatrix} = (I - \Gamma_0^*\Gamma_0)^{-1}\begin{pmatrix} I_{\nu_1} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} d_{12} \\ d_{22} \end{pmatrix} = (I - \Gamma_1^*\Gamma_1)^{-1}\begin{pmatrix} 0 \\ I_{\nu_2} \end{pmatrix},$$

$$(4.5) \qquad \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = -\Gamma_0\begin{pmatrix} d_{11} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = -\Gamma_1\begin{pmatrix} d_{12} \\ d_{22} \end{pmatrix},$$

and put

$$(4.6) \quad \alpha = \begin{pmatrix} a_{11}a_{11}(0)^{-1/2} & 0 \\ a_{21}a_{11}(0)^{-1/2} & a_{22}a_{22}(0)^{-1/2} \end{pmatrix}, \quad \gamma = \begin{pmatrix} c_{11}a_{11}(0)^{-1/2} & c_{12}a_{22}(0)^{-1/2} \\ c_{21}a_{11}(0)^{-1/2} & c_{22}a_{22}(0)^{-1/2} \end{pmatrix},$$

$$(4.7) \quad \delta = \begin{pmatrix} d_{11}d_{11}(0)^{-1/2} & d_{12}d_{22}(0)^{-1/2} \\ 0 & d_{22}d_{22}(0)^{-1/2} \end{pmatrix}, \quad \beta = \begin{pmatrix} b_{11}d_{11}(0)^{-1/2} & b_{12}d_{22}(0)^{-1/2} \\ b_{21}d_{11}(0)^{-1/2} & b_{22}d_{22}(0)^{-1/2} \end{pmatrix}.$$

Then the $k\times k$ block matrix valued function g defined by

$$g(z) = \beta(z)\delta(z)^{-1} := \alpha(z)^{*-1}\gamma(z)^*, \qquad z \in \mathbf{T},$$

is the unique strictly contractive extension of the given lower triangular part φ with $(g(I - g^*g)^{-1}) \in W_{N\times M}^-(\mathbf{T})$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Make the following choices of spaces: Let $\mathscr{B} = \mathscr{C}^* = W_{N\times M}(\mathbf{T})$, $\mathscr{A} = W_{N\times N}(\mathbf{T})$ and $\mathscr{D} = W_{M\times M}(\mathbf{T})$, and we view the elements of these spaces divided in subblocks of sizes $\nu_r\times\mu_s$, $\nu_r\times\nu_s$ and $\mu_r\times\mu_s$, respectively. Further, let

$$\mathscr{B}_- = (\mathscr{C}_+)^* := W^-_{N\times M}(\mathbf{T}); \quad \mathscr{B}_+ = (\mathscr{C}_-)^* := W^+_{N\times M}(\mathbf{T})$$

$$\mathscr{A}^0_- = \{f = (f_{rs})^2_{r,s=1} \in W_{N\times N}(\mathbf{T}) \,|\, f_{12} = 0;\ (f_{rr})_j = 0,\ j < 0\ (r = 1, 2)\},$$

$$\mathscr{D}^0_- = \{f = (f_{rs})^2_{r,s=1} \in W_{M\times M}(\mathbf{T}) \,|\, f_{12} = 0;\ (f_{rr})_j = 0,\ j < 0\ (r = 1, 2)\},$$

$$\mathscr{A}^0_+ := (\mathscr{A}^0_-)^*,\ \mathscr{D}^0_+ := (\mathscr{D}^0_-)^*,\ \mathscr{A}_d = \mathbf{C}^{\nu_1\times\nu_1} \oplus \mathbf{C}^{\nu_2\times\nu_2},\ \mathscr{D}_d = \mathbf{C}^{\mu_1\times\mu_1} \oplus \mathbf{C}^{\mu_2\times\mu_2}.$$

Then clearly (1.1)—(1.6) are satisfied. The above introduced spaces as well as the space $\mathscr{M} = W_{(N+M)\times(N+M)}(\mathbf{T})$ we endow with the usual norm on Wiener spaces. Let us check the Axioms (A0)—(A3). For Axioms (A0), (A2) and (A3) we can refer to Section II.4 since the specific decomposition does not play a role in these axioms. So we are left with Axiom (A1). Let $f = (f_{rs})^4_{r,s=1} \in \mathscr{M}_+$ be such that $\|f(z)\| < 1$ for all $z \in \mathbf{T}$. In order to see that $e - f$ is invertible in $\mathscr{M}_+$ it suffices to show that its diagonal elements are invertible, and that their inverses have positive Fourier coefficients equal to zero. Since we have that $\|f_{rr}(z)\| < 1$, $r = 1, \dots, 4$, for all $z \in \mathbf{T}$ one can simply reason in the same way as was done in Section I.3 when checking (A1). A similar reasoning holds if $f \in \mathscr{M}_-$. So the spaces $\mathscr{A} - \mathscr{D}$ and $\mathscr{M}$ satisfy all the conditions given in Section II.1. We will not write down the exact workout of the application of Theorem II.1.1 for this setting, since this was already done in [13] in the more general $n\times n$ setting. (By the way, the + spaces in [13] correspond to the − spaces here, and vice versa.) ▨

We obtain the following description of the set of all strictly contractive extensions of a given function φ.

THEOREM II.4.2. *Let $\varphi \in W^-_{N\times M}(\mathbf{T})$ be given. In order that there exists a strictly contractive extension of the given function φ it is necessary and sufficient that the operators Γ_i, $i = 0, 1, 2$, defined in* (4.1), *have norm less than one. Let this condition holds and let $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$ be defined by* (4.2)—(4.7). *Then each strictly contractive extension ψ of φ is of the form*

$$\psi(z) = (\alpha(z)h(z) + \beta(z))(\gamma(z)h(z) + \delta(z))^{-1}, \tag{4.8}$$

where h is an element in $W^-_{N\times M}(\mathbf{T})$ such that $\|h(z)\| < 1$, $|z| = 1$. Furthermore, formula (4.8) *gives a 1-1 correspondence between all such h and all strictly contractive extensions ψ of φ.*

Theorem II.4.2 is a direct consequence of Theorems II.1.2 and II.4.1. The observation that the condition mentioned in Theorem II.4.2 is indeed a necessary condition for the existence of a strictly contractive extension is straightforward.

The solution of the more general n^2-block problem, which already appears in [13, Theorem 4.2], may also be derived from the abstract setting derived in Section II.1.

II.5. THE WIENER SPACE ON THE LINE

We use the notations of Section I.4. In this section we will apply the results. of Section II.1 on the following extension problem. Let $\varphi \in L^1_{n\times m}(-\infty, 0]$ be given. A matrix valued function $\hat{\psi}$, with $\psi \in L^1_{n\times m}(\mathbf{T})$ is called a *strictly contractive extension* of $\hat{\varphi}$ if $\psi(s) = \varphi(s)$, a.e. on $(-\infty, 0]$, and $\|\hat{\psi}(\lambda)\| < 1$ for all $\lambda \in \mathbf{R}$.

THEOREM II.5.1. *Let $\varphi \in L^1_{n\times m}(-\infty, 0]$ be given, and suppose that the integral operator $\Phi\colon L^2_m[0, \infty) \to L^2_n(-\infty, 0]$, defined by*

$$(\Phi f)(t) = \int_0^\infty \varphi(t-u)f(u)\mathrm{d}u, \quad t < 0, \tag{5.1}$$

has norm less than one. Put

$$\omega(t,s) := \int_{-\infty}^0 \varphi(t-u)\varphi(s-u)^*\mathrm{d}u, \quad t, s \in \mathbf{R}$$

and

$$\omega_*(t,s) = \int_0^\infty \varphi(u-t)^*\varphi(u-s)\mathrm{d}u, \quad t, s \in \mathbf{R}.$$

Let $\alpha \in L^1_{n\times n}(-\infty, 0]$, $\beta \in L^1_{n\times m}(-\infty, 0]$, $\gamma \in L^1_{m\times n}[0, \infty)$ and $\delta \in L^1_{m\times m}[0, \infty)$ be given by the following equations

$$\alpha(t) - \int_0^\infty \omega(t, s)\alpha(s)\mathrm{d}s = \omega(t, 0), \quad t < 0, \tag{5.2}$$

$$\gamma(t) = \varphi(-t)^* + \int_{-\infty}^0 \varphi(s-t)^*\alpha(s)\mathrm{d}s, \quad t > 0, \tag{5.3}$$

$$\delta(t) - \int_{-\infty}^0 \omega_*(t,s)\delta(s)\mathrm{d}s = \omega_*(t, 0), \quad t > 0, \tag{5.4}$$

$$\beta(t) = \varphi(t) + \int_0^\infty \varphi(t-s)\delta(s)\mathrm{d}s, \quad t < 0. \tag{5.5}$$

Then the matrix valued function $\hat{g} \in (L^1_{n\times m}(\mathbf{R}))^\wedge$ *given by*

$$\hat{g}(\lambda) := \hat{\beta}(\lambda)(I_m + \hat{\delta}(\lambda))^{-1} = (I_n + \hat{\alpha}(\lambda))^{*-1}\hat{\gamma}(\lambda)^*, \quad \lambda \in \mathbf{R},$$

is the unique strictly contractive extension of $\hat{\varphi}$ *such that* $(\hat{g}(I - \hat{g}^*\hat{g})^{-1}) \in$ $\in (L^1_{n\times m}(-\infty, 0])^\wedge$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Put

$$\mathscr{A}^0_+ = (\mathscr{A}^0_-)^* := (L^1_{n\times n}[0, \infty))^\wedge, \quad \mathscr{A}_d = \mathbf{C}^{n\times n},$$

$$\mathscr{D}^0_+ = (\mathscr{D}^0_-)^* := (L^1_{m\times m}[0, \infty))^\wedge, \quad \mathscr{D}_d = \mathbf{C}^{m\times m},$$

$$\mathscr{B}_+ = (\mathscr{C}_-)^* := (L^1_{n\times m}[0,\infty))^\wedge, \quad \mathscr{B}_- = (\mathscr{C}_+)^* := (L^1_{n\times m}(-\infty, 0])^\wedge,$$

and let $\mathscr{B} = \mathscr{C}^*$, $\mathscr{A}$, and $\mathscr{D}$ be given via (1.2) and (1.5). The conditions (1.1), (1.3), (1.4) and (1.6) are clearly satisfied. The above introduced spaces as well as the space $\mathscr{M}$ we endow with the norm $\| \ \|_{W^{p\times q}(\mathbf{R})}$ defined in Section I.4, for all the applicant combinations of p and q. Checking the axioms (A0)—(A3) again consists of pointing out the triviality of Axiom (A0) and the reference to Section I.4 for the others, where it is noted that for (A1) it suffices to prove this axiom for $\mathscr{A}$ and $\mathscr{D}$.

Let $\varphi \in L^1_{n\times m}(-\infty, 0]$ and suppose that $\|\Phi\| < 1$. Let $0 \leqslant \varepsilon \leqslant 1$. Then the operator $I - \varepsilon^2\Phi\Phi^*$ is invertible as an operator on $L^2_n(-\infty, 0] \cap L^1_n(-\infty, 0]$, since its kernel is trivial and Φ is compact. Using that Φ is a compact as an operator acting from $L^1_m[0, \infty)$, we have that $I - \varepsilon^2\Phi\Phi^*$ is a Fredholm operator with index zero on $L^1_n(-\infty, 0]$. Since the dense set $L^2_n(-\infty, 0] \cap L^1_n(-\infty, 0] \subset L^1_n(-\infty, 0]$ is in the image of this operator, $I - \varepsilon^2\Phi\Phi^*$ is surjective. But then invertibility follows Using this it is not hard to see that the condition "$I - \varepsilon^2 HH_*$ is invertible for $0 \leqslant \varepsilon \leqslant 1$", where H and H_* are defined in (1.8), is equivalent to the condition $\|\Phi\| < 1$. Analogously, $I - \varepsilon^2\tilde{H}_*\tilde{H}$ is treated. Since the elements in (1.9) are for this case both equal to the identity, they are evidently positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. Hence we can apply Theorem II.1.1 in this setting, and obtain Theorem II.5.1. ▨

Theorem II.1.2 gives now the following description of the set of all strictly contractive extensions of a given function $\hat{\varphi} \in (L^1_{n\times m}(-\infty, 0])^\wedge$.

THEOREM II.5.2. *Let* $\Phi \in L^1_{n\times m}(-\infty, 0]$ *be given. In order that there exists a strictly contractive extension of the function* $\hat{\varphi}$ *it is necesary and sufficient that the integral operator* $\Phi: L^2_m[0, -\infty) \to L^2_n(-\infty, 0]$, *defined by* (5.1), *has norm less than one. Let this condition hold and let* $\alpha \in L^1_{n\times n}(-\infty, 0]$, $\beta \in L^1_{n\times m}(-\infty, 0]$, $\gamma \in L^1_{m\times n}[0, \infty)$ *and* $\delta \in L^1_{m\times m}(-\infty, 0]$ *be given via the equations* (5.2)—(5.5). *Then each strictly contractive* *extension $\hat{\psi}$ of $\hat{\varphi}$ is of the form*

$$(5.6)\qquad \hat{\psi}(\lambda) = ((I_n + \hat{\alpha}(\lambda))\hat{h}(\lambda) + \hat{\beta}(\lambda))(\hat{\gamma}(\lambda)\hat{h}(\lambda) + I_m + \hat{\delta}(\lambda))^{-1},\ \lambda \in \mathbf{R},$$

where $h \in L^1_{n\times m}[0, -\infty)$ is such that $\|\hat{h}(\lambda)\| < 1$, $\lambda \in \mathbf{R}$. Furthermore, formula (5.6) *gives a* 1-1 *correspondence between all such $\hat{h}$ and all strictly contractive extensions $\hat{\psi}$ of $\hat{\varphi}$.*

The above theorem is a direct consequence of Theorem II.1.2. The observation that the condition $\|\Phi\| < 1$ is a necessary condition for the existence of a strictly contractive extension is straightforward. A linear fractional description as in (5.6) was obtained in [5, Theorem 9.4].

II.6. THE WIENER SPACE ON THE LINE: THE FOUR BLOCK CASE

We use the notations of Section I.6. In this section we view an element f of the Wiener space on the line $W_{n\times m}(\mathbf{R})$ divided into four subblocks, i.e.,

$$f(\lambda) = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix},$$

with $f_{rs} \in W_{\nu_r\times\mu_s}(\mathbf{R})$, where μ_1, μ_2 are positive integers such that $\mu_1 + \mu_2 = m$ and ν_1, ν_2 are positive integers such that $\nu_1 + \nu_2 = n$. We consider the subset $W_{n\times m,0}(\mathbf{R})$ of $W_{n\times m}(\mathbf{R})$ of elements f with constant term d_f equal to zero. The space $W_{n\times m,0}(\mathbf{R})$ admits the following decomposition

$$W_{n\times m,0}(\mathbf{R}) = W^-_{n\times m,0}(\mathbf{R}) \dotplus W^+_{n\times m,0}(\mathbf{R})$$

where

$$W^-_{n\times m,0}(\mathbf{R}) = \{f = (f_{rs})^2_{r,s=1} \in W_{n\times m}(\mathbf{R}) \mid f_{12} \in (L^1_{\nu_1\times\mu_2}[0, \infty))\hat{}\},$$

and

$$W^+_{n\times m,0}(\mathbf{R}) = \{f = (f_{rs})^2_{r,s=1} \in W_{n\times m,0}(\mathbf{R}) \mid f_{rs} = 0,\ (r,s) \neq (1,2);\ f_{12} \in (L^1_{\nu_1\times\mu_2}(-\infty, 0])\hat{}\}.$$

We are interested in the following completion problem. Let $\varphi \in W^-_{n\times m,0}(\mathbf{R})$ be given. An element $g \in W_{n\times m,0}(\mathbf{R})$ is called a *strictly contractive extension* of φ if $g - \varphi \in W^+_{n\times m,0}(\mathbf{R})$ and $\|g(\lambda)\| < 1$, $\lambda \in \mathbf{R}$.

Before we state the results of this section we first introduce some notations. Let χ_+ and χ_- denote the characteristic function of $\mathbf{R}$ of the interval $[0, \infty)$ and

$(-\infty, 0]$, respectively. With $D_1 \oplus D_2$ we indicate the diagonal matrix with diagonal entries D_1 and D_2. Let $\varphi \in W^-_{n\times m,0}(\mathbf{R})$ be given, and introduce the following Hankel like operators which act from $L^2_{\nu_1}(\mathbf{R}) \oplus L^2_{\nu_2}(\mathbf{R})$ to $L^2_{\mu_1}(\mathbf{R}) \oplus L^2_{\mu_2}(\mathbf{R})$:

$$\left(\Xi_0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)(t) = \int_{-\infty}^{0} \varphi(t-s) \begin{pmatrix} \chi_-(s) f_1(s) \\ 0 \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R},$$

$$\left(\Xi_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)(t) = (\chi_+(t) \oplus I) \int_{-\infty}^{\infty} \varphi(t-s) \begin{pmatrix} f_1(s) \\ \chi_-(s) f_2(s) \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R}, \tag{6.1}$$

$$\left(\Xi_2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)(t) = (0 \oplus \chi_+(t)) \int_{-\infty}^{\infty} \varphi(t-s) \begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R}.$$

THEOREM II.6.1. *Let $\varphi \in W^-_{n\times m,0}(\mathbf{R})$ be given, and suppose that the Hankel like operators Ξ_i, $i = 0, 1, 2$, have norm less than one. Put*

$$\psi_0(t, u) = \int_{-\infty}^{\infty} (\chi_-(t) \oplus 0)\varphi(s-t)^* \varphi(s-u)(\chi_-(u) \oplus 0)\mathrm{d}s,$$

$$\psi_1(t, u) = \int_{-\infty}^{\infty} (I \oplus \chi_-(t))\varphi(s-t)^*(\chi_+(s) \oplus I)\varphi(s-u)(I \oplus \chi_-(u))\mathrm{d}s,$$

$$\psi_{1,*}(t, u) = \int_{-\infty}^{\infty} (\chi_+(t) \oplus I)\varphi(t-s)(I \oplus \chi_-(s))\varphi(u-s)^*(\chi_+(u) \oplus I)\,\mathrm{d}s,$$

$$\psi_{2,*}(t, u) = \int_{-\infty}^{\infty} (0 \oplus \chi_+(t))\varphi(t-s)\varphi(u-s)^*(0 \oplus \chi_+(u))\mathrm{d}s.$$

Let

$$\alpha = (\alpha_{rs})^2_{r,s=1}, \quad \beta = (\beta_{rs})^2_{r,s=1}, \quad \gamma = (\gamma_{rs})^2_{r,s=1}, \quad \delta = (\delta_{rs})^2_{r,s=1}$$

be given by $\alpha_{12} = 0$, $\delta_{21} = 0$,

$$\begin{pmatrix} \alpha_{11}(t) \\ \alpha_{21}(t) \end{pmatrix} - \int_{-\infty}^{\infty} \psi_{1,*}(t, u) \begin{pmatrix} \alpha_{11}(u) \\ \alpha_{21}(u) \end{pmatrix} \mathrm{d}u = \psi_{1,*}(t, 0) \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t \in \mathbf{R}, \tag{6.2}$$

$$(6.3)\qquad \begin{pmatrix} 0 \\ \alpha_{22}(t) \end{pmatrix} - \int_0^\infty \psi_{2,*}(t,u) \begin{pmatrix} 0 \\ \alpha_{22}(u) \end{pmatrix} \mathrm{d}u = \psi_{2,*}(t,0) \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad t > 0,$$

$$(6.4)\qquad \begin{aligned} \begin{pmatrix} \gamma_{11}(t) \\ \gamma_{21}(t) \end{pmatrix} &= -(I \oplus \chi_-(t))\varphi(-t)^* \begin{pmatrix} I \\ 0 \end{pmatrix} - \\ &- (I \oplus \chi_-(t)) \int_{-\infty}^{\infty} \varphi(s-t)^* \begin{pmatrix} \alpha_{11}(s) \\ \alpha_{21}(s) \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R}, \end{aligned}$$

$$(6.5)\qquad \begin{pmatrix} \gamma_{12}(t) \\ \gamma_{22}(t) \end{pmatrix} = -\varphi(-t)^* \begin{pmatrix} 0 \\ I \end{pmatrix} - \int_{-\infty}^{\infty} \varphi(s-t)^* \begin{pmatrix} 0 \\ \alpha_{22}(s) \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R},$$

$$(6.6)\qquad \begin{pmatrix} \delta_{11}(t) \\ 0 \end{pmatrix} - \int_{-\infty}^{0} \psi_0(t,u) \begin{pmatrix} \delta_{11}(u) \\ 0 \end{pmatrix} \mathrm{d}u = \psi_0(t,0) \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad t < 0,$$

$$(6.7)\qquad \begin{pmatrix} \delta_{12}(t) \\ \delta_{22}(t) \end{pmatrix} - \int_{-\infty}^{\infty} \psi_1(t,u) \begin{pmatrix} \delta_{12}(u) \\ \delta_{22}(u) \end{pmatrix} \mathrm{d}u = \psi_1(t,0) \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad t \in \mathbf{R},$$

$$(6.8)\qquad \begin{pmatrix} \beta_{11}(t) \\ \beta_{21}(t) \end{pmatrix} = -\varphi(t) \begin{pmatrix} I \\ 0 \end{pmatrix} - \int_{-\infty}^{\infty} \varphi(t-s) \begin{pmatrix} \delta_{11}(t) \\ 0 \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R},$$

$$(6.9)\qquad \begin{pmatrix} \beta_{12}(t) \\ \beta_{22}(t) \end{pmatrix} = -(\chi_+(t) \oplus I)\varphi(t) - (\chi_+(t) \oplus I) \int_{-\infty}^{\infty} \varphi(t-s) \begin{pmatrix} \delta_{12}(s) \\ \delta_{22}(s) \end{pmatrix} \mathrm{d}s, \quad t \in \mathbf{R}.$$

Then the $k \times k$ block matrix valued function g defined by

$$\hat{g}(\lambda) = \hat{\beta}(\lambda)(I_m + \hat{\delta}(\lambda))^{-1} = (I_n + \hat{\alpha}(\lambda))^{*-1}\hat{\gamma}(\lambda)^*, \quad \lambda \in \mathbf{R},$$

is the unique strictly contractive extension of the lower triangular part φ with $(\hat{g}(I - \hat{g}^*\hat{g})^{-1}) \in W^-_{n\times m,0}(\mathbf{R})$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Make the following choices of spaces: Let $\mathscr{B} = \mathscr{C}^* = W_{n\times m,0}(\mathbf{R})$, $\mathscr{A} = W_{n\times n}(\mathbf{R})$ and $\mathscr{D} = W_{m\times m}(\mathbf{R})$, and we view the elements of these spaces divided in subblocks of sizes $\nu_r \times \mu_s$, $\nu_r \times \nu_s$ and $\mu_r \times \mu_s$, respectively. Further, let

$$\mathscr{B}_- = (\mathscr{C}_+)^* := W^-_{n\times m,0}(\mathbf{R}); \quad \mathscr{B}_+ = (\mathscr{C}_-)^* := W^+_{n\times m,0}(\mathbf{R}),$$

$$\mathscr{A}^0_- = (\mathscr{A}^0_+)^* := \left\{ \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \,\middle|\, a_{ii} \in (L_{\nu_i \times \nu_i}[0, \infty))\hat{}\,, \; i = 1, 2 \right\},$$

$$\mathscr{D}^0_- = (\mathscr{D}^0_+)^* := \left\{ \begin{pmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{pmatrix} \,\middle|\, d_{ii} \in (L_{\mu_i \times \mu_i}[0, \infty))\hat{}\,, \; i = 1, 2 \right\},$$

$$\mathscr{A}_d = \mathbf{C}^{n\times n}, \quad \mathscr{D}_d = \mathbf{C}^{m\times m}.$$

Then clearly (1.1)—(1.6) are satisfied. The above introduced spaces as well as the space $\mathscr{M}$ we endow with the norm $\|\cdot\|_{W^{p\times q}(\mathbf{R})}$ defined in Section 1.4, for all the applicant combinations of p and q. Let us check the axioms (A0)—(A3). For (A0), (A2), and (A3) we refer to Section II.5, since the specific decomposition of the space is irrelevant when checking these axioms. Let us check Axiom (A1) for the space $\mathscr{A}$. Let $a = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \in \mathscr{A}_-$ and suppose that $e - a^* a$ is positive definite. This means, in particular, that $\|a_{ii}(\lambda)\| < 1$, $i = 1, 2$, for all $\lambda \in \mathbf{R}$. As in Section 1.4 one proves that the function $I - a_{ii}(\lambda)$ is invertible and its inverse belongs to $\mathbf{C}^{\nu_i \times \nu_i} \dotplus (L^1_{\nu_i \times \nu_i}[0, \infty))\hat{}\,$, $i = 1, 2$. But then $(e_{\mathscr{A}} - a)^{-1} \in \mathscr{A}_-$. For $a \in \mathscr{A}_+$ one can reason analogously. Doing the same for $\mathscr{D}$ gives that Axiom (A1) holds in $\mathscr{M}$.

Let Ξ_i, $i = 0, 1, 2$, have norm less than one. Let us first show that the operators $I - \varepsilon^2 HH_*$ and $I - \varepsilon^2 \tilde{H}_* \tilde{H}$ $(0 \leqslant \varepsilon \leqslant 1)$ are invertible. Relative to the decomposition $\mathscr{A}_- = \mathscr{A}_d \dotplus \mathscr{A}^0_-$ the operator $I - \varepsilon^2 HH_*$ is of the form

$$\begin{pmatrix} I & 0 \\ * & * \end{pmatrix}.$$

So in order to show that $I - \varepsilon^2 HH_*$ is invertible it suffices to prove that this operator is invertible on $\mathscr{A}^0_-$. An analogous reasoning shows that for the invertibility of $I - \varepsilon^2 \tilde{H}_* \tilde{H}$, it suffices to prove that it is invertible as an operator on $\mathscr{D}^0_+$. If $a = \begin{pmatrix} \hat{a}_{11} & 0 \\ \hat{a}_{21} & \hat{a}_{22} \end{pmatrix} \in \mathscr{A}^0_-$ and $d = \begin{pmatrix} \hat{d}_{11} & \hat{d}_{12} \\ 0 & \hat{d}_{22} \end{pmatrix} \in \mathscr{D}^0_+$, then

$$\{(I - \varepsilon^2 HH_*)a\}^\vee = \left((I - \varepsilon^2 \Xi_1 \Xi_1^*) \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \; (I - \varepsilon^2 \Xi_2 \Xi_2^*) \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} \right)$$

and

$$\{(I - \varepsilon^2 \tilde{H}_* \tilde{H})d\}^\wedge = \left((I - \varepsilon^2 \Xi_0^* \Xi_0) \begin{pmatrix} d_{11} \\ 0 \end{pmatrix} \; (I - \varepsilon^2 \Xi_1^* \Xi_1) \begin{pmatrix} d_{12} \\ d_{22} \end{pmatrix} \right).$$

Let us show that the invertibility of the operators

$$(6.10) \qquad \begin{pmatrix} I & \varepsilon\Xi_i \\ \varepsilon\Xi_i^* & I \end{pmatrix}, \quad i = 0, 1, 2,$$

seen as an operator on $L^2_{(n+m)\times(n+m)}$ space implies invertibility when seen as an operator on $L^1_{(n+m)\times(n+m)}$ space. For this consider these operators acting on the intersection of $L^1_{(n+m)\times(n+m)}$ and $L^2_{(n+m)\times(n+m)}$. On this intersection these operators are invertible. If on $L^1_{(n+m)\times(n+m)}$ the operators in (6.10) are Fredholm with index zero, we are done: indeed, since the operators acting on the intersection have full images, the operators will have a full image when considered as operators working on the L^1 space. In order to show that the operators are Fredholm with index zero one can reason in the same way as was done in the proof of Theorem 3.1 in [13] for the case of the Wiener algebra on the circle. The main idea there is that the operators are compact perturbations of operators that contain only identities and Toeplitz--like blocks. Using the results in [14] these operators are readily seen to be Fredholm operators with index zero, and thus also all their compact perturbations. This establishes the invertibility of the operators $I - \varepsilon^2 HH_*$ and $I - \varepsilon^2 \tilde{H}_* \tilde{H}$. We can now apply Theorem II.1.1. Straightforward computations show that α - δ in Theorem II.1.1 coincide with $I_n + \hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $I_m + \hat{\delta}$ here. ▨

From Theorem II.1.2 we obtain the following description of the set of all strictly contractive extensions of a given function φ.

THEOREM II.6.2. *Let $\varphi \in W^-_{n\times m,0}(\mathbf{R})$ be given. In order that there exists a strictly contractive extension of the given function φ it is necessary and sufficient that the operators Ξ_i, $i = 0, 1, 2$, defined in* (6.1), *have norm less than one. Let this condition hold and let $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ and $\delta(\lambda)$ be defined by* (6.2)—(6.9). *Then each strictly contractive extension ψ of φ is of the form*

$$(6.11) \qquad \psi(\lambda) = (\alpha(\lambda)h(\lambda) + \beta(\lambda))(\gamma(z)h(\lambda) + \delta(\lambda))^{-1}, \quad \lambda \in \mathbf{R},$$

where h is an element in $W^-_{n\times m,0}(\mathbf{R})$ such that $\|h(\lambda)\| < 1$, $|z| = 1$. Furthermore, formula (6.11) *gives a* 1-1 *correspondence between all such h and all strictly contractive extensions ψ of φ.*

Theorem II.6.2. is a direct consequence of Theorems II.1.2 and II.6.1. The observation that the condition mentioned in Theorem II.6.2 is indeed a necessary condition for the existence of a strictly contractive extension is of a simple nature. Indeed, if ψ is a contractive extension of φ, then if we replace φ by ψ the operators in (6.1) will stay the same, and since $\psi(\lambda)$ has norm less than one for all λ the operators are easily recognized as strict contractions.

The solution of the more general $p \times q$-block version of the present problem may also be derived from the abstract setting derived in Section II.1.

II.7. FREDHOLM INTEGRAL OPERATORS

We use the notations of Section I.5. In this section we apply the abstract results of Section II.1 to functions $f \in \mathscr{F} = \mathscr{F}_{T,0}$ which may be viewed as kernels of integral operators. Thus we consider $n \times n$ matrix valued functions defined in the square Δ which are continuous in the open regions

$$\Delta_+ = \{(t, s) \in \Delta \mid t < s\},$$

$$\Delta_- = \{(t, s) \in \Delta \mid t > s\}$$

and the restrictions $f_\pm$ of f to $\Delta_\pm$ extends continuously to the closure $\bar{\Delta}_\pm$. Put $\mathscr{F}_\pm = \{f \in \mathscr{F} \mid f(t, s) = 0,\ (t, s) \in \Delta_\mp\}$.

We shall deal with the following problem. Let $\varphi \in \mathscr{F}_-$. A matrix valued function $\psi \in \mathscr{F}$ is called a *strictly contractive extension of* φ if $\psi(t, s) = \varphi(t, s)$, $(t, s) \in \Delta_-$, and the integral operator Ψ with kernel ψ has norm less than one. Before we state the theorem we need some additional notation. For $\varphi \in \mathscr{F}_-$ and $0 \leqslant \xi \leqslant T$ let $K_{\varphi,\xi}$ denote the integral operator acting from $L_2^n[0, \xi]$ to $L_2^n[\xi, T]$ which is defined by

$$(7.1) \qquad (K_{\varphi,\xi} f)(t) = \int_0^{\xi} \varphi(t, s) f(s) \mathrm{d}s, \quad \xi < t < T.$$

THEOREM II.7.1. *Let* $\varphi \in \mathscr{F}_-$ *be given, and suppose that for every* ξ *in the interval* $[0, T]$ *the operator* $K_{\varphi,\xi}$ *has norm less than one. Put*

$$\omega(t, s) = \int_0^T \varphi(t, v)\varphi(s, v)^* \mathrm{d}v, \quad \omega_*(t, s) = \int_0^T \varphi(v, t)^* \varphi(v, s) \mathrm{d}v, \quad (s, t) \in \Delta.$$

Let $\alpha \in \mathscr{F}_-$, $\beta \in \mathscr{F}_-$, $\gamma \in \mathscr{F}_+$ *and* $\delta \in \mathscr{F}_+$ *be given via*

$$(7.2) \qquad \alpha(t, s) - \int_0^T \omega(t, u)\overset{\times}{\alpha}(u, s) \mathrm{d}u = \omega(t, s), \quad 0 < s < t < T,$$

$$(7.3) \qquad \gamma(t, s) = \varphi(s, t)^* + \int_s^T \varphi(u, t)^* \alpha(u, s) \mathrm{d}u, \quad 0 < t < s < T,$$

$$(7.4) \qquad \delta(t, s) - \int_0^s \omega_*(t, u)\delta(u, s)du = \omega_*(t, s), \quad 0 < t < s < T,$$

$$(7.5) \qquad \beta(t, s) = \varphi(t, s) + \int_0^s \varphi(t, u)\delta(u, s)ds, \quad 0 < s < t < T.$$

Then the matrix valued function $g \in \mathcal{F}$ defined by

$$g = \beta - \beta \bigstar \delta^\dagger = \gamma^* - (\alpha^*)^\dagger \bigstar \gamma^*$$

is the unique strictly contractive extension of φ such that $g + g \bigstar (g^* \bigstar g)^\dagger \in \mathcal{F}_-$.

Proof. We will obtain this theorem as a special case of Theorem II.1.1. Put

$$\mathcal{A}^0_\pm = \mathcal{D}^0_\pm = \mathcal{C}_\pm = \mathcal{B}_\pm := \mathcal{F}_\pm, \quad \mathcal{A}_d = \mathcal{D}_d := \{\lambda I_n, \ \lambda \in \mathbf{C}\},$$

and let $\mathcal{A} - \mathcal{D}$ be given via (1.2) and (1.5). The conditions (1.1), (1.3), (1.4) and (1.6) are clearly satisfied. The norm on $\mathcal{M}$ is given by

$$\left\| \lambda I_{2n} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| := |\lambda| + \sup_{(t,s)\in\Delta} \left\| \begin{pmatrix} a(t, s) & b(t, s) \\ c(t, s) & d(t, s) \end{pmatrix} \right\|.$$

Checking Axiom (A0) is a triviality. For Axioms (A1) (for $\mathcal{A}$ and $\mathcal{D}$), (A2) and (A3) we refer to Section I.5.

Let $\varphi \in \mathcal{F}_-$, and suppose that $\|K_{\varphi,\xi}\| < 1$ for $\xi \in [0, T]$. We have to show that $I - \varepsilon^2 HH_*$, where H and H_* are defined in (1.8), is invertible for $0 \leqslant \varepsilon \leqslant 1$. Fix $\varepsilon \in [0,1]$, and let $g \in \mathcal{F}_-$ be given. First note that the kernel of the operator $K_{\varphi,\xi}K^*_{\varphi,\xi}: L^n_2[\xi, T] \to L^n_2[\xi, T]$ is the matrix valued function

$$\psi_\xi(t, u) = \int_0^\xi \varphi(t, s)\varphi(u, s)^* ds, \quad t, u \in [\xi, T].$$

Since for each $\xi \in [0, T]$ the operator $I - \varepsilon^2 K_{\varphi,\xi}K^*_{\varphi,\xi}$ on $L^{n\times n}_2[\xi, T]$ (columnwise defined) is invertible, there exists a unique $f_\xi \in L^{n\times n}_2[\xi, T]$ such that

$$((I - \varepsilon^2 K_{\varphi,\xi}K^*_{\varphi,\xi})f_\xi)(t) = g(t, \xi), \quad \xi \leqslant t \leqslant T, \quad \text{a.e.} \ .$$

Since

$$f_\xi(t) = (\varepsilon^2 K_{\varphi,\xi}K^*_{\varphi,\xi}f_\xi)(t) + g(t, \xi), \quad \xi \leqslant t \leqslant T, \quad \text{a.e.},$$

and the right hand side is continuous in t, we have that $f_\xi \in C_{n\times n}[\xi, T]$. Put now

$$f(t, \xi) := f_\xi(t), \quad 0 \leqslant \xi \leqslant t \leqslant T,$$

and let us show that f is continuous.

Let λ be a positive number, and let φ_e be a continuous function on $\breve{\Delta}$ such that $\varphi_e = \varphi$ on Δ_-. Put

$$\psi_{e,\xi}(t, u) = \int_0^\xi \varphi_e(t, s)\varphi_e(u, s)^* \mathrm{d}s, \quad \xi, t, u \in [0, T].$$

Then ψ_e is jointly continuous in ξ, t and u, and moreover $\psi_{e,\xi}(t, u) = \psi_\xi(t, u)$ if $t, u \geqslant \xi$. Fix $\xi \in [0, T]$. Consider the following two equations:

$$f_\eta(t) - \varepsilon^2 \int_0^T \psi_{e,\eta}(t, s) f_\eta(s)\mathrm{d}s = g(t, \eta), \quad t \geqslant \xi. \tag{7.6}$$

$$\tilde{f}_\eta(t) - \varepsilon^2 \int_\eta^T \psi_\xi(t, s)\tilde{f}_\eta(s)\mathrm{d}s = g(t, \xi), \quad t \geqslant \xi. \tag{7.7}$$

If η is close to ξ then the kernels $\psi_{e,\eta}$ and ψ_ξ are close, and, furthermore, the right hand sides are close. Thus for η in a neighbourhood of ξ we have that

$$\sup_{\xi\leqslant t\leqslant T} \|\tilde{f}_\eta(t) - f_\eta(t)\| \leqslant \frac{1}{3}\lambda.$$

Compare now (7.7) with the equation

$$f_\xi(t) - \varepsilon^2 \int_\xi^T \psi_\xi(t, s) f_\xi(s)\mathrm{d}s = g(t, \xi), \quad \xi \leqslant t \leqslant T.$$

Since by Theorem 3.3 in [10] the resolvents of both integral operators are close if η and ξ are, we have that for η in a neighbourhood of ξ it holds that

$$\sup_{\xi\leqslant t\leqslant T} \|\tilde{f}_\eta(t) - f_\xi(t)\| < \frac{1}{3}\lambda.$$

Further, if t' and t are close, then

$$\|f_\eta(t)-f_\eta(t')\| < \frac{1}{3}\cdot\lambda.$$

But then

$$\|f_\xi(t)-f_\eta(t')\| < \lambda,$$

proving the continuity of f on $\bar{\Delta}_-$.

The above reasonings show that if $\|K_{\varphi,\xi}\| < 1$ for all $0 \leqslant \xi \leqslant T$ the operator $I-\varepsilon^2 HH_*$ is invertible ($0 \leqslant \varepsilon \leqslant 1$). The converse is easy: if g is in the null space of $I-\varepsilon^2 HH_*$ then its nonzero columns are eigenvectors of $I-\varepsilon^2 K_{\varphi,\xi}K^*_{\varphi,\xi}$, and thus ε^{-1} is a singular value of $K_{\varphi,\xi}$. But then $\|K_{\varphi,\xi}\| \geqslant \varepsilon^{-1} \geqslant 1$. Analogously, one treates $I-\varepsilon^2\tilde{H}_*\tilde{H}$. Since both elements in (1.9) both equal the identity, they are positive definite in $\mathscr{A}_d$ and $\mathscr{D}_d$, respectively. Going through straightforward calculations one sees that the elements $\alpha-\delta$ in Theorem II.1.1 coincide with the $I_n+\alpha$, β, γ and $I_n+\delta$ in Theorem II.7.1. ▨

The description for the set of all strictly contractive extensions of a given $\varphi \in \mathscr{F}_-$ is now a special case of Theorem II.1.2.

THEOREM II.7.2. *Let $\varphi \in \mathscr{F}_-$ be given. In order that φ has a strictly contractive extension it is necessary and sufficient that*

$$\|K_{\varphi,\xi}\| < 1, \quad 0 \leqslant \xi \leqslant T, \tag{7.8}$$

where $K_{\varphi,\xi}$ is defined in (7.1). *Suppose that* (7.8) *holds. Let $\alpha-\delta$ be given via* (7.2)—(7.5). *Then every contractive extension ψ of φ is of the form*

$$\psi = (g+\alpha\bigstar g+\beta)-(g+\alpha\bigstar g+\beta)(\delta+\gamma\bigstar g)^{\dagger}, \tag{7.9}$$

where g is an element in $\mathscr{F}_+$ such that its corresponding integral operator G has norm less than one. Furthermore, (7.9) *gives a* 1-1 *correspondence between all such g and all strictly contractive extension ψ of φ.*

Theorem II.7.2 is a direct consequence of Theorem II.1.2. Condition (7.8) is obviously a necessary condition for the existence of a strictly contractive extension of φ. The above results are new.

It is easy to deduce similar results for the Banach space of $n\times m$ matrix kernels defined on a rectangle $[0, T_1]\times[0, T_2]$ with $\Delta_- = \{(t, s) \mid t > s-\tau\}$ for some $\tau \in \mathbf{R}$.

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