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CONTINUOUS ANALOGUES OF FOCK SPACE. III: SINGULAR STATES

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INTRODUCTION

The study of semigroups of endomorphisms of von Neumann algebras was initiated by Powers in [8], [9], and continued by Powers and Robinson [10], Price [11], [12], and the author [1], [2], [3], [4]. In [1], we reduced the problem of classifying E_0-semigroups up to cocycle conjugacy to the problem of classifying certain simpler structures associated to them, called product systems. With every product system E, there is an associated C^*-algebra $C^*(E)$ [4]. These C^*-algebras are in many respects "continuous" analogues of the Cuntz algebra O_∞ [5], [7]. In this paper we analyze the state space of $C^*(E)$, and we obtain a rather explicit description of the space of "singular" states. This allows us to show that the regular representation of $C^*(E)$ is faithful, thereby settling one of the main questions left open in [4].

The following remarks may be helpful in providing a context for the problems taken up in this paper. Let $U = \{U_t : t \geqslant 0\}$ be a strongly continuous semigroup of isometries acting on a Hilbert space H. The Wold decomposition asserts that U has a unique decomposition

$$U_t = V_t \oplus W_t, \quad t \geqslant 0 \tag{0.1}$$

into a direct sum of a semigroup $W = \{W_t : t \geqslant 0\}$ of unitary operators and a semigroup $V = \{V_t : t \geqslant 0\}$ of isometries which is *pure* in the sense that

$$\bigcap_{t>0} \operatorname{ran} V_t = \{0\}.$$

This decomposition is *central* in that the two projections associated with the decomposition (0.1) belong to the center of the von Neumann algebra generated by $\{U_t : t \geqslant 0\}$. Furthermore, the pure summand V is unitarily equivalent to a direct sum of copies of the *shift* semigroup $S = \{S_t : t \geqslant 0\}$, defined on the Hilbert space $L^2(0, \infty)$ by

$$S_t f(x) = \begin{cases} f(x-t), & x > t \\ 0, & 0 < x \leqslant t. \end{cases}$$

The number of S-summands is an invariant of U to within unitary equivalence.

The purpose of this paper is to discuss a corresponding decomposition for the representations of product systems and the C^*-algebras associated with them, and to give certain applications. In more detail, let $E = \{E(t) : t > 0\}$ be a product system in the sense of [1]. This means that E is a measurable family of Hilbert spaces over the open interval $(0, \infty)$ on which there is defined an associative multiplication which *acts like tensoring* in the sense that it is bilinear on fiber spaces, and that for each $s, t > 0$, $E(s + t)$ is spanned by all products $\{uv : u \in E(s),\ v \in E(t)\}$ and we have

$$\langle uv, u'v' \rangle = \langle u, u' \rangle \langle v, v' \rangle,$$

for all $u, u' \in E(s)$, $v, v' \in E(t)$. A *representation* of E is an operator-valued mapping $\varphi : E \to \mathscr{B}(H)$ satisfying

(i) $\varphi(u)\varphi(v) = \varphi(uv)$, $u, v \in E$,
(ii) $\varphi(v)^*\varphi(u) = \langle u, v \rangle 1$, if u and v belong to the same fiber $E(t)$, $t > 0$,

and which is measurable in the sense that $\langle \varphi(v)\xi, \eta \rangle$ is a measurable function of v for fixed $\xi, \eta \in H$. Condition (ii) implies that the restriction of φ to each fiber $E(t)$ is a linear map satisfying $\|\varphi(v)\| = \|v\|$, $v \in E(t)$.

We remark that it is essential that one confine attention to representations $\varphi : E \to \mathscr{B}(H)$ on *separable* Hilbert spaces H, since there are representations of product systems E on inseparable Hilbert spaces with rather pathological properties. In any case, for every (separable) representation $\varphi : E \to \mathscr{B}(H)$, we can define a one-parameter family of subspaces of H by

$$H_t = [\varphi(E(t))H], \quad t > 0.$$

It was shown in [1] that the subspaces H_t are decreasing in t, their union is dense in H, and they are continuous in the sense that the corresponding family of projections $\{P_t : t > 0\}$ is strongly continuous in t. φ is called *singular* if $\bigcap_t H_t = \{0\}$, and is called *nonsingular* in case the opposite extreme occurs, in which $H_t = H$ for every $t > 0$. It was also pointed out in ([1], Proposition 1.14) that every separable representation φ has a unique direct sum decomposition

$$\varphi = \varphi_s \oplus \varphi_n \tag{0.2}$$

where φ_s is singular and φ_n is nonsingular, and moreover that this is a central decomposition in the sense that the two projections arising from this decomposition belong to the center of the von Neumann algebra generated by $\varphi(E)$.

In the case where E is the trivial product system Z, (0.2) is a restatement of the Wold decomposition (0.1). In order to see this, recall that Z is defined as the trivial family of one-dimensional Hilbert spaces $p: Z \to (0, \infty)$, where

$$Z = (0, \infty) \times \mathbf{C}$$

$$p(t, z) = t, \quad t > 0, \ z \in \mathbf{C},$$

having the usual inner product in fiber spaces $Z(t) = p^{-1}(t)$

$$\langle z, w \rangle = z\bar{w}$$

and the multiplication

$$(s, z)(t, w) = (s + t, zw).$$

Every one-parameter semigroup $V = \{V_t : t \geq 0\}$ of isometries in $\mathscr{B}(H)$ gives rise to a representation $\varphi: Z \to \mathscr{B}(H)$ by way of

$$\varphi(t, z) = zV_t, \quad t > 0, \ z \in \mathbf{C}. \tag{0.3}$$

Conversely, every nonzero separable representation φ of Z has the form (0.3) for a unique strongly continuous semigroup of isometries V. φ is a singular (resp. nonsingular) representation of Z iff V is a pure (resp. unitary) semigroup of isometries. Thus, (0.2) is the Wold decomposition for V.

If $\varphi: E \to \mathscr{B}(H)$ is a representation of a nontrivial product system, then the nonsingular summand φ_n in the decomposition (0.2) corresponds to the "unitary" part of the Wold decomposition, and gives rise to an E_0-semigroup as in ([1], Proposition 2.7). The classification of these representations, and the E_0-semigroups associated with them, was taken up in [1]. Here, we want to fix attention on singular representations.

Given a product system E, let $L^2(E)$ denote the Hilbert space of all measurable square-integrable sections $f: t \in (0, \infty) \mapsto f(t) \in E(t)$. The inner product in $L^2(E)$ is given by

$$\langle f, g \rangle = \int_0^\infty \langle f(t), g(t) \rangle \mathrm{d}t,$$

and of course we identify two sections in $L^2(E)$ which agree almost everywhere ($\mathrm{d}t$). $L^2(E)$ admits an obvious direct integral decomposition

$$L^2(E) = \int_{(0,\infty)}^{\oplus} E(t) \mathrm{d}t,$$

and as we have pointed out in [1], it is a "continuous" counterpart of the full Fock space generated by an infinite-dimensional one-particle space. Every $v \in E$ determines a left creation operator $\ell(v)$, defined as follows; for $v \in E(t)$, $t > 0$ and $f \in L^2(E)$,

$$\ell(v)f(x) = \begin{cases} v \cdot f(x-t), & x > t \\ 0, & 0 < x \leqslant t. \end{cases}$$

$\ell : E \to \mathscr{B}(L^2(E))$ is a *singular* representation of E, and it is irreducible in the sense that $\ell(E)$ generates $\mathscr{B}(L^2(E))$ as a von Neumann algebra ([4], Theorem 5.2).

ℓ is called the *regular* representation of E and, relative to other singular representations of E, it occupies a position analogous to that of the semigroup of unilateral shifts $S = \{S_t : t \geqslant 0\}$ introduced above. Since every pure semigrup of isometries is unitarily equivalent to a direct sum of copies of S, one is led to ask if every singular representation $\varphi : E \to \mathscr{B}(H)$ is unitarily equivalent to a direct sum of copies of ℓ. We will show that while the answer is no in general, it is almost yes. More precisely, we show that every nontrivial product system E has singular representations which are not multiples of ℓ (cf. remarks following Proposition 4.3). On the other hand, if $\varphi : E \to \mathscr{B}(H)$ is any singular representation and $t > 0$, then the restriction φ_t of φ to the $\varphi(E)$-invariant subspace $H_t = [\varphi(E(t))H]$ defines a representation of E on H_t which is unitarily equivalent to a direct sum of copies of ℓ (Corollary 4 of Theorem 3.1).

Our proof of these results makes essential use of the spectral C^*-algebra $C^*(E)$ associated to a product system E. In particular, in Section 1 we introduce a contraction semigroup which acts on the *dual* of $C^*(E)$, and which is central to the analysis of singular representations.

1. THE SEMIGROUP β^*

Let E be a product system and let $\varphi : E \to \mathscr{B}(H)$ be a representation of E on a Hilbert space H (cf. Introduction). Let $L^1(E)$ denote the Banach space of all integrable sections

$$f : t \in (0, \infty) \mapsto f(t) \in E(t), \quad t > 0$$

with the natural norm

$$\|f\| = \int_0^\infty \|f(t)\| dt.$$

φ induces a contractive linear map of $L^1(E)$ into $\mathscr{B}(H)$ by integration, and we denote this map of $L^1(E)$ with the same letter φ. Thus,

$$\varphi(f) = \int_0^\infty \varphi(f(t))\mathrm{d}t, \quad f \in L^1(E).$$

There is a C^*-algebra $C^*(E)$ associated with E. While the details of the structure of $C^*(E)$ are not important for our purposes here, we do require the following universal property and some of its elementary consequences. There is a map

$$(f, g) \in L^1(E) \times L^1(E) \mapsto f \otimes \bar{g} \in C^*(E)$$

of $L^1(E) \times L^1(E)$ into $C^*(E)$ which is linear in f, antilinear in g, and satisfies

$$\|f \otimes \bar{g}\| \leqslant \|f\|_1 \|g\|_1$$

(1.1)

$$C^*(E) = \overline{\operatorname{span}}\{f \otimes \bar{g} : f, g \in L^1(E)\}.$$

Moreover, given any representation $\varphi \colon E \to \mathscr{B}(H)$ of E, there is a unique $*$-representation $\pi \colon C^*(E) \to \mathscr{B}(H)$ such that

$$\pi(f \otimes \bar{g}) = \varphi(f)\varphi(g)^*, \quad f, g \in L^1(E). \tag{1.2}$$

π is necessarily nondegenerate. Conversely, given any nondegenerate $*$-representation $\pi \colon C^*(E) \to \mathscr{B}(H)$, there is a unique representation $\varphi \colon E \to \mathscr{B}(H)$ which satisfies (1.2). These properties are established in ([4], §§2—3). $C^*(E)$ is separable, nuclear, and has no unit.

In this section we introduce a contraction semigroup $\beta^* = \{\beta_t^* : t \geqslant 0\}$ which acts on the dual of $C^*(E)$, and which will play a central role in what follows. Let $E = \{E(t) : t > 0\}$ be a product system and let $v \in E(t)$, for some $t > 0$. For every section $f \in L^1(E)$ we can define sections $vf, fv \in L^1(E)$ by

$$vf(x) = \begin{cases} v \cdot f(x - t), & x > t \\ 0, & 0 < x \leqslant t, \end{cases}$$

$$fv(x) = \begin{cases} f(x - t) \cdot v, & x > t \\ 0, & 0 < x \leqslant t. \end{cases}$$

PROPOSITION 1.3. *Fix $t > 0$, and let $\{e_1(t), e_2(t), \ldots\}$ be an orthonormal basis for $E(t)$. For every $\rho \in C^*(E)^*$ there is a unique bounded linear functional $\beta_t^*\rho$ on $C^*(E)$ satisfying*

$$\beta_t^*\rho(f \otimes \bar{g}) = \sum_{n=1}^\infty \rho(fe_n(t) \otimes \overline{ge_n(t)}),$$

for all $f, g \in L^1(E)$, *the series on the right converging absolutely for every* f, g. $\beta_t^* \rho$ *does not depend on the particular choice of basis. For* $t = 0$, *put* $\beta_0^* \rho = \rho$.

$\beta^* = \{\beta_t^* : t \geq 0\}$ *is a semigroup of contractions on the Banach space* $C^*(E)^*$ *satisfying* $\beta_t^* \rho \geq 0$ *for* $\rho \geq 0$, *and*

$$\lim_{t \to 0} \beta_t^* \rho(x) = \rho(x),$$

for every $\rho \in C^*(E)^*$, $x \in C^*(E)$.

Proof. Fix $\rho \in C^*(E)^*$. Let $|\rho|$ be the positive part of the polar decomposition of ρ. The GNS construction provides us with a representation $\pi: C^*(E) \to \mathscr{B}(H)$ and a cyclic vector ξ for π such that

$$|\rho|(x) = \langle \pi(x)\xi, \xi \rangle,$$

and we have $\|\rho\| = \|\xi\|^2$. Moreover, there is a partial isomtery $U \in \pi(C^*(E))''$ satisfying $U^*U\xi = \xi$ with the property that for $\eta = U\xi$ we have

$$\rho(x) = \langle \pi(x)\xi, \eta \rangle.$$

Note that η is also a cyclic vector for π and $\|\eta\|^2 = \|\xi\|^2 = \|\rho\|$. Because $C^*(E)$ is a separable C^*-algebra H is a separable Hilbert space, and π is clearly a nondegenerate representation. By the preceding remarks there is a unique representation $\varphi: E \to \mathscr{B}(H)$ such that

$$\pi(f \otimes g) = \varphi(f)\varphi(g)^*, \quad f, g \in L^1(E).$$

For each $n = 1, 2, \ldots$, $V_n = \varphi(e_n(t))$ is an isometry, and $\sum_n V_n V_n^*$ is the projection P_t of H onto $[\varphi(E(t))H]$. Because φ is multiplicative we have $\varphi(fe_n(t)) = \varphi(f)V_n$ for every $f \in L^1(E)$, $n = 1, 2, \ldots$, and hence

$$\rho(fe_n(t) \otimes \overline{ge_n(t)}) = \langle \varphi(fe_n(t))\varphi(ge_n(t))^*\xi, \eta \rangle =$$

$$= \langle \varphi(f)V_n V_n^* \varphi(g)^*\xi, \eta \rangle = \langle V_n^* \varphi(g)^*\xi, V_n^*\varphi(f)^*\eta \rangle.$$

It follows that for fixed $f, g \in L^1(E)$,

$$\sum_n |\rho(fe_n(t) \otimes \overline{ge_n(t)})| \leq \sum_n \|V_n^*\varphi(g)^*\xi\| \cdot \|V_n^*(\varphi(f)^*\eta\|.$$

The right side is finite because for every $\zeta \in H$ we have

$$\sum_n \|V_n^*\zeta\|^2 = \sum_n \langle V_n V_n^* \zeta, \zeta \rangle = \langle P_t \zeta, \zeta \rangle \leq \|\zeta\|^2,$$

and hence

$$\sum_n \|V_n^*\zeta_1\| \cdot \|V_n^*\zeta_2\| \leqslant \|\zeta_1\| \cdot \|\zeta_2\|.$$

Moreover, we can write

$$(1.4) \qquad \sum_n \rho(fe_n(t) \otimes \overline{ge_n(t)}) = \sum_n \langle \varphi(f)V_nV_n^*\varphi(g)^*\xi, \eta\rangle = \langle \varphi(f)P_t\varphi(g)^*\xi, \eta\rangle,$$

for $f, g \in L^1(E)$.

For $t > 0$, put $H_t = [\varphi(E(t))H]$. Since $E(s+t)$ is spanned by both $E(s)E(t)$ and $E(t)E(s)$, we have for each $s > 0$,

$$\varphi(E(s))H_t \subseteq [\varphi(E(s+t))H] = [\varphi(E(t))\varphi(E(s))H] \subseteq \subseteq [\varphi(E(t))H] = H_t.$$

It follows that H_t is invariant under the set of operators $\varphi(E)$ for each $t > 0$. Hence

$$\varphi_t(v) = \varphi(v)|H_t, \qquad v \in E$$

defines a representation of E on H_t and, since $\varphi(f)P_t\varphi(g)^* = P_t\varphi(f)P_t\varphi(g)^*P_t$ for $f, g \in L^1(E)$, formula (1.4) implies that

$$(1.5) \qquad \sum_n \rho(fe_n(t) \otimes \overline{ge_n(t)}) = \langle \varphi_t(f)\varphi_t(g)^*P_t\xi,\ P_t\eta\rangle.$$

By the universal property of $C^*(E)$, there is a unique representation π_t: $C^*(E) \to \mathscr{B}(H_t)$ such that $\pi_t(f \otimes \bar{g}) = \varphi_t(f)\varphi_t(g)^*$, and thus we can define a bounded linear functional $\beta_t^*\rho$ on $C^*(E)$ by

$$\beta_t^*\rho(x) = \langle \pi_t(x)P_t\xi,\ P_t\eta\rangle, \qquad x \in C^*(E).$$

Clearly $\|\beta_t^*\rho\| \leqslant \|P_t\xi\| \cdot \|P_t\eta\| \leqslant \|\rho\|$, and by (1.5) we have

$$\beta_t^*\rho(f \otimes g) = \sum_{n=1}^{\infty} \rho(fe_n(t) \otimes \overline{ge_n(t)}).$$

It is apparent from its definition that $\beta_t^*\rho$ does not depend on the choice of orthonormal basis $\{e_n(t);\ n = 1, 2, \ldots\}$ for $E(t)$, and the preceding formula determines $\beta_t^*\rho$ uniquely because $\{f \otimes \bar{g} : f, g \in L^1(E)\}$ spans $C^*(E)$.

We claim that $\beta^*_{s+t} = \beta^*_s\beta^*_t$ for $s, t > 0$. Indeed, for every $\rho \in C^*(E)^*$ we can write

$$\beta^*_t\rho(f \otimes \bar{g}) = \sum_n \beta^*_t\rho(fe_n(s) \otimes \overline{ge_n(s)}) =$$

$$= \sum_{m,n} \rho(fe_n(s)e_m(t) \otimes \overline{ge_n(s)e_m(t)}).$$

By virtue of the isomorphism $E(s+t) \cong E(s) \otimes E(t)$, we see that $\{e_n(s)e_m(t) : m, n = 1, 2, \dots\}$ is an orthonormal basis for $E(s+t)$, hence the right side of (1.6) is simply $\beta^*_{s+t}\rho(f \otimes \bar{g})$. The conclusion $\beta^*_s\beta^*_t\rho = \beta^*_{s+t}\rho$ follows from the fact that $C^*(E)$ is spanned by $\{f \otimes \bar{g} : f, g \in L^1(E)\}$.

Thus, $\beta^* = \{\beta^*_t : t \geqslant 0\}$ is a semigroup of contractions. If $\rho \geqslant 0$ then we may take $\eta = \xi$ in the representation of ρ, $\rho(x) = \langle \pi(x)\xi, \xi \rangle$, and hence $\beta^*_t\rho$ has the from

$$\beta^*_t\rho(x) = \langle \pi_t(x)P_t\xi, P_t\xi \rangle,$$

which is clearly a positive linear functional for all $t > 0$.

It remains to show that

$$\lim_{t \to 0+} \beta^*_t\rho(x) = \rho(x)$$

for every $\rho \in C^*(E)^*$, $x \in C^*(E)$. Fixing ρ, we may restrict attention to x's in the spanning set $\{f \otimes \bar{g} : f, g \in L^1(E)\}$. By (1.4) we have

$$\beta^*_t\rho(f \otimes \bar{g}) = \langle \varphi(f)P_t\varphi(g)^*\xi, \eta \rangle = \langle P_t\varphi(g)^*\xi, \varphi(f)^*\eta \rangle.$$

Since $\bigcup_{t>0} H_t$ is dense in H ([1], Corollary of Proposition 2.7), P_t tends strongly to 1 as $t \to 0+$, hence

$$\lim_{t \to 0+} \beta^*_t\rho(f \otimes \bar{g}) = \langle \varphi(g)^*\xi, \varphi(f)^*\eta \rangle = \rho(f \otimes \bar{g}),$$

as required.

REMARK. The semigroup β^* is not strongly continuous. Indeed, we will see later on that for "singular states" ρ of $C^*(E)$ one has

$$\lim_{t \to 0} \|\beta^*_t\rho - \rho\| = 0$$

if and only if the representation π_ρ of $C^*(E)$ associated with ρ is quasi-equivalent to the regular representation (Corollary 2 of Theorem 3.1).

Let ρ be a positive linear functional on $C^*(E)$. If $\|\rho\| = 1$, then ρ is called a *state*. In general, the GNS construction gives rise to a representation $\pi : C^*(E) \to$

$\to \mathcal{B}(H)$ and a cyclic vector ξ_ρ for π_ρ satisfying

$$\rho(x) = \langle \pi_\rho(x)\xi_\rho, \xi_\rho \rangle,$$

$x \in C^*(E)$. By the universal property of $C^*(E)$, there is a unique representation $\varphi_\rho : E \to \mathcal{B}(H)$ such that

$$\pi_\rho(f \otimes \bar{g}) = \varphi_\rho(f)\varphi_\rho(g)^*, \quad f, g \in L^1(E).$$

DEFINITION 1.7. Let ρ be a bounded linear functional on $C^*(E)$ and let $|\rho|$ be the positive linear functional obtained from the polar decomposition of ρ. ρ is called *singular* (resp. *nonsingular*) if $\varphi_{|\rho|}$ is a singular (resp. nonsingular) representation of E. ρ is called *regular singular* if $\varphi_{|\rho|}$ is unitarily equivalent to a direct sum of copies of the regular representation $\ell : E \to \mathcal{B}(L^2(E))$.

REMARKS. Let $\lambda : C^*(E) \to \mathcal{B}(L^2(E))$ be the regular representation of $C^*(E)$, i.e.,

$$\lambda(f \otimes \bar{g}) = \ell(f)\ell(g)^*, \quad f, g \in L^1(E).$$

Since λ is irreducible ([4], Corollary of Theorem 5.2), a representation π of $C^*(E)$ is unitarily equivalent to a direct sum of copies of λ iff π is quasi-equivalent to λ. We conclude that a bounded linear functional $\rho \in C^*(E)^*$ is regular singular iff $\pi_{|\rho|}$ is quasi-equivalent to λ.

The following result gives a convenient characterization of these properties in terms of the semigroup β^*.

PROPOSITION 1.8. *Let ρ be a bounded linear functional on $C^*(E)$.*

(i) *ρ is singular iff* $\lim_{t\to\infty} \|\beta_t^* \rho\| = 0$.

(ii) *ρ is nonsingular iff $\beta_t^* \rho = \rho$ for all $t \geqslant 0$.*

Proof. By the GNS construction we have a representation $\pi : C^*(E) \to \mathcal{B}(H)$ and a cyclic vector $\xi \in H$ such that

$$|\rho|(x) = \langle \pi(x)\xi, \xi \rangle, \quad x \in C^*(E).$$

As in the proof of Proposition 1.3, we can find a second cyclic vector η for π satisfying

$$\rho(x) = \langle \pi(x)\xi \mid \eta \rangle, \quad x \in C^*(E),$$

together with $\|\xi\|^2 = \|\eta\|^2 = \|\rho\|$.

To prove (i), let $\varphi : E \to \mathcal{B}(H)$ be the representation of E associated with π and let P_t be the projection of H onto $H_t = [\varphi(E(t))H]$, $t > 0$. Assume first that φ is singular, so that $P_t \downarrow 0$ as $t \to \infty$. Letting φ_t be the representation of E on H_t

defined by $\varphi_t(v) = \varphi(v)|H_t$, then as in the proof of Proposition 1.3 we have

$$\beta_t^*\rho(f \otimes \bar{g}) = \langle \varphi(f)P_t\varphi(g)^*\xi, \eta \rangle = \langle \varphi_t(f)\varphi_t(g)^*P_t\xi, P_t\eta \rangle.$$

Thus if $\pi_t: C^*(E) \to \mathscr{B}(H_t)$ is the representation of $C^*(E)$ corresponding to φ_t, we have

$$\beta_t^*\rho(x) = \langle \pi_t(x)P_t\xi, P_t\eta \rangle.$$

It follows that $\|\beta_t^*\rho\| \leqslant \|P_t\xi\| \cdot \|P_t\eta\|$ must tend to zero as $t \to \infty$.

Conversely, assume $\|\beta_t^*\rho\| \to 0$ as $t \to \infty$. The projections P_t are decreasing with t, and we have to show that the strong limit $P_\infty = \lim_{t\to\infty} P_t$ is zero. To that end, fix $f, g \in L^1(E)$. Then we have

$$\beta_t^*\rho(f \otimes \bar{g}) = \langle P_t\varphi(g)^*\xi, \varphi(f)^*\eta \rangle = \langle P_tA^*\xi, B^*\eta \rangle, \tag{1.9}$$

where $A = \varphi(g)$, $B = \varphi(f)$. The left side of (1.9) tends to zero as $t \to \infty$, hence

$$\langle P_\infty A^*\xi, B^*\eta \rangle = 0.$$

Since P_∞ commutes with the von Neumann algebra generated by $\varphi(E)$ ([1], Proposition 1.14) the preceding implies that

$$\langle P_\infty\xi, AB^*\eta \rangle = \langle P_\infty A^*\xi, B^*\eta \rangle = 0.$$

Hence $\langle P_\infty\xi, \pi(x)\eta \rangle = 0$ for all $x \in C^*(E)$. Since η is a cyclic vector for π, we conclude that $P_\infty\xi = 0$. Now since $\varphi(E)$ and $\pi(C^*(E))$ generate the same von Neumann algebra ([4], Theorem 3.4), P_∞ must commute with $\pi(C^*(E))$, and hence

$$P_\infty\pi(x)\xi = \pi(x)P_\infty\xi = 0,$$

for every $x \in C^*(E)$. This implies that $P_\infty = 0$ because λ is a cyclic vector for π.

Proof of 1.8 (ii). From the formula

$$\beta_t^*\rho(f \otimes \bar{g}) = \langle \varphi(f)P_t\varphi(g)^*\xi, \eta \rangle,$$

we see that

$$\rho(f \otimes \bar{g}) - \beta_t^*\rho(f \otimes \bar{g}) = \langle \varphi(f)(1 - P_t)\varphi(g)^*\xi, \eta \rangle \tag{1.10}$$

for all $f, g \in L^1(E)$, $t > 0$. Assuming that φ is nonsingular, then we have $1 - P_t = 0$ for every $t > 0$, and hence (1.10) implies that $\beta_t^*\rho(x) = \rho(x)$ for all $x \in C^*(E)$ of the form $f \otimes \bar{g}$, $f, g \in L^1(E)$. Hence $\beta_t^*\rho = \rho$.

Conversely, if $\beta_t^*\rho = \rho$ then (1.10) implies $\langle \varphi(f)(1 - P_t)\varphi(g)^*\xi, \eta\rangle = 0$ for all $t > 0$. Taking the limit as $t \to \infty$ we obtain

$$\langle (1 - P_\infty)\varphi(f)\varphi(g)^*\xi, \eta\rangle = \langle \varphi(f)(1 - P_\infty)\varphi(g)^*\xi, \eta\rangle =$$

$$= \lim_{t\to\infty} [\rho(f \otimes \bar{g}) - \beta_t^*\rho(f \otimes \bar{g})] = 0.$$

Since $C^*(E)$ is spanned by $\{f\otimes\bar{g} : f, g \in L^1(E)\}$ we conclude that $\langle (1 - P_\infty)\pi(x)\xi, \eta\rangle = 0$ for every $x \in C^*(E)$. As in the proof of 1.8(i), P_∞ commutes with $\pi(C^*(E))$, and hence

$$\langle (1 - P_\infty)\pi(x)\xi, \pi(y)\eta\rangle = \langle (1 - P_\infty)\pi(y)^*\pi(x)\xi, \eta\rangle =$$

$$= \langle (1 - P_\infty)\pi(y^*x)\xi, \eta\rangle = 0,$$

for all $x, y \in C^*(E)$. The latter implies that $P_\infty = 1$ because both ξ and η are cyclic vectors for π. Since $P_t \geqslant P_\infty = 1$ for every $t > 0$, we conclude that φ is nonsingular. ▨

2. CONSTRUCTION OF SINGULAR STATES

Let E be a product system and let $\mathscr{S}$ (resp. $\mathscr{N}$) denote the set of singular (resp. nonsingular) elements of $C^*(E)^*$. Proposition 1.8 implies that $\mathscr{S}$ and $\mathscr{N}$ are norm--closed linear subspaces of $C^*(E)^*$.

In fact, we have a direct sum decomposition

$$C^*(E)^* = \mathscr{S} \oplus \mathscr{N}$$

in the sense that every element $\rho \in C^*(E)^*$ decomposes uniquely into a sum $\rho = \sigma + \nu$, $\sigma \in \mathscr{S}$, $\nu \in \mathscr{N}$ where $\|\rho\| = \|\sigma\| + \|\nu\|$. Indeed, it was shown in ([1], Proposition 1.14) that every representation $\varphi\colon E \to \mathscr{B}(H)$ decomposes uniquely into an orthogonal direct sum

$$\varphi = \varphi_{\mathrm{s}} \oplus \varphi_{\mathrm{n}}$$

of a singular representation φ_{s} and a nonsingular representation φ_{n}. Moreover, the projections $1 \oplus 0$ and $0 \oplus 1$ associated with this decomposition belong to the center of the von Neumann algebra generated by $\varphi(E)$. In view of the correspondence between representations of E and nondegenerate representations of $C^*(E)$, it follows that every (separable) nondegenerate representation π of $C^*(E)$ decomposes uniquely into a central direct sum

$$\pi = \pi_{\mathrm{s}} \oplus \pi_{\mathrm{n}}$$

of a singular representation π_s and a nonsingular representation π_n. The indicated decomposition of the dual of $C^*(E)$ follows from these remarks together with the correspondence between positive linear functionals on $C^*(E)$ and cyclic representations of $C^*(E)$. In fact, the direct sum decomposition

$$C^*(E)^* = \mathscr{S} \oplus \mathscr{N}$$

is induced by a central projection in the von Neumann algebra $C^*(E)^{**}$. We conclude that both $\mathscr{S}$ and $\mathscr{N}$ are order ideals in $C^*(E)^*$.

The purpose of this section and the next is to give a more concrete description of the ordered Banach space $\mathscr{S}$ (Theorem 2.4 and Corollary 1 of Theorem 3.1). This will allow us to characterize the states of $C^*(E)$ whose representations are quasi--equivalent to the regular representation. As a consequence, we show that the regular representation of $C^*(E)$ is faithful (Corollary 3 of Theorem 3.1).

Our methods here provide very little information about the summand $\mathscr{N}$. In particular, we still do not know if $\mathscr{N} \neq \{0\}$ for every product system E. This question is equivalent to asking if every product system E is associated to some E_0--semigroup of endomorphisms of $\mathscr{B}(H)$ as in [1], and will be taken up elsewhere.

For $v \in E$, let $\ell(v)$ and $\imath(v)$ be the associated left and right creation operators acting on $L^2(E)$:

$$\ell(v)f = v \cdot f$$

$$\imath(v)f = f \cdot v, \quad f \in L^2(E).$$

There are two semigroups of $*$-endomorphisms α, β of $\mathscr{B}(L^2(E))$ associated with ℓ and $\imath$ as follows. For $t > 0$ and $A \in \mathscr{B}(L^2(E))$, we define

$$\alpha_t(A) = \sum_n \ell(e_n(t))A\ell(e_n(t))^*,$$

$$\beta_t(A) = \sum_n \imath(e_n(t))A\imath(e_n(t))^*,$$

$\{e_1(t), e_2(t), \ldots\}$ being an arbitrary orthonormal basis for $E(t)$. For $t = 0$ we put $\alpha_0(A) = \beta_0(A) = A$. α and β are continuous semigroups of normal $*$-endomorphisms of $\mathscr{B}(L^2(E))$, and $\alpha_t(1) = \beta_t(1)$ is the projection of $L^2(E)$ onto the subspace

$$\{f \in L^2(E) : f(x) = 0 \text{ a.e. on } 0 < x \leqslant t\}$$

([1], Proposition 2.7). In this paper we will be concerned primarily with the semigroup β.

Consider the action of β on the predual of $\mathscr{B}(L^2(E))$. More explicitly, since each β_t is a normal $*$-endomorphism, there is a semigroup of contractions $\beta_* = \{\beta_{t*} : t \geq 0\}$ acting on the Banach space $\mathscr{L}^1(L^2(E))$ of all trace-class operators on $L^2(E)$. The action of β_{t*} is defined by

$$\operatorname{tr}(\beta_{t*}(A)B) = \operatorname{tr}(A\beta_t(B)), \quad A \in \mathscr{L}^1(L^2(E)),\ B \in \mathscr{B}(L^2(E)), \tag{2.1}$$

tr denoting the canonical trace on $\mathscr{L}^1(L^2(E))$. β_* is strongly continuous in the sense that

$$\lim_{t\to 0} \operatorname{tr}|\beta_{t*}(A) - A| = 0,$$

for every trace-class operator A ([1], Proposition 2.5(i)). Moreover, letting $P_t = \beta_t(1)$, $t \geq 0$, we have $\beta_{t*}(A) = \beta_{t*}(P_t A P_t)$ for every $t \geq 0$. Since $P_t \downarrow 0$ as $t \to \infty$, we see that $\operatorname{tr}|P_t A P_t| \to 0$, and hence

$$\lim_{t\to\infty} \operatorname{tr}|\beta_{t*}(A)| = 0$$

for every $A \in \mathscr{L}^1(L^2(E))$.

We introduce a Banach space $\mathscr{M}(\beta_*)$ associated with the semigroup β_* which is basic for what follows. $\mathscr{M}(\beta_*)$ is defined as the space of all bounded functions

$$A : t \in (0, \infty) \mapsto A(t) \in \mathscr{L}^1(L^2(E))$$

satisfying

$$A(s + t) = \beta_{t*}(A(s)), \quad s > 0,\ t \geq 0. \tag{2.2}$$

The norm in $\mathscr{M}(\beta_*)$ is the sup norm

$$\|A\| = \sup_{t>0} \operatorname{tr}|A(t)|.$$

Because β_* is strongly continuous, (2.2) implies that $\mathscr{M}(\beta_*)$ consists of bounded continuous functions from the open interval $(0, \infty)$ to the separable Banach space $\mathscr{L}^1(L^2(E))$; moreover, the preceding paragraph implies that each element $A \in \mathscr{M}(\beta_*)$ vanishes at infinity in the sense that

$$\lim_{t\to\infty} \operatorname{tr}|A(t)| = 0.$$

The $*$-operation on trace-class operators induces an isometric involution in $\mathscr{M}(\beta_*)$, and $\mathscr{M}(\beta_*)$ is partially ordered by $A \geq 0$ iff $A(t) \geq 0$ for every $t > 0$.

$\mathscr{M}(\beta_*)$ contains $\mathscr{L}^1(L^2(E))$. Indeed, every trace-class operator A determines an element $\tilde{A}$ in $\mathscr{M}(\beta_*)$ by

$$\tilde{A}(t) = \beta_{t*}(A), \quad t > 0.$$

$A \mapsto \tilde{A}$ is an isometric order-preserving isomorphism of $\mathcal{L}^1(L^2(E))$ onto the subspace of $\mathcal{M}(\beta_*)$ consisting of all functions $F \in \mathcal{M}(\beta_*)$ for which the limit

$$F(0+) = \lim_{t \to 0} F(t)$$

exists relative to the trace-norm on $\mathcal{L}^1(L^2(E))$. Note finally that, because of the relation (2.2), a function in $\mathcal{M}(\beta_*)$ is completely determined by its restriction to arbitrarily small intervals $0 < t < \delta$, $\delta > 0$: thus $\mathcal{M}(\beta_*)$ is a Banach space which embodies the "limiting behaviour" of the semigroup β_* at time zero.

We begin by giving an explicit formula for the states of $C^*(E)$ which are of the form $\omega \circ \lambda$, where λ is the regular representation of $C^*(E)$ on $L^2(E)$ and ω is a normal state of $\mathcal{B}(L^2(E))$.

PROPOSITION 2.3. *Let T be a trace-class operator on $L^2(E)$ and let $\lambda: C^*(E) \to \mathcal{B}(L^2(E))$ be the regular representation. Then for every pair of functions $f, g \in L^1(E) \cap L^2(E)$ we have*

$$\int_0^\infty |\langle \beta_{t*}(T)f, g\rangle| \mathrm{d}t \leq \operatorname{tr}|T| \cdot \|f\|_1 \|g\|_1,$$

and

$$\int_0^\infty \langle \beta_{t*}(T)f, g\rangle \mathrm{d}t = \operatorname{tr}(T\lambda(f \otimes \bar{g})).$$

Proof. We may find sequences of vectors $\xi_n, \eta_n \in L^2(E)$ such that

$$\sum_n \|\xi_n\| \cdot \|\eta_n\| = \operatorname{tr}|T|,$$

and

$$T = \sum_n \xi_n \otimes \bar{\eta}_n.$$

For each $n \geq 1$ and $t > 0$ we have

$$\langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle = \operatorname{tr}(\beta_{t*}(\xi_n \otimes \bar{\eta}_n)(f \otimes \bar{g})) =$$

$$= \operatorname{tr}((\xi_n \otimes \bar{\eta}_n)\beta_t(f \otimes \bar{g})) = \langle \beta_t(f \otimes \bar{g})\xi_n, \eta_n\rangle.$$

Now since g is integrable, the convolution operator $\ell(g)\xi = g * \xi$ is bounded on $L^2(E)$ and has norm at most $\|g\|_1$. Moreover, since g also belongs to $L^2(E)$ we can assign a definite value to $\ell(g)^*\xi(t)$ for every positive t and every $\xi \in L^2(E)$, namely

$$\ell(g)^*\xi(t) = \int_0^\infty g(s)^*\xi(s + t)\mathrm{d}s$$

(for more detail, see the proof of Proposition 6.4 of [4]). Similarly,

$$\ell(f)^*\eta(t) = \int_0^\infty f^*(s)\eta(s + t)\mathrm{d}s,$$

for every $t > 0$ and every $\eta \in L^2(E)$. Moreover, formula (6.7) of [4] asserts that for $t > 0$, $\xi, \eta \in L^2(E)$, we have the relation

$$\langle \beta_t(f \otimes \bar{g})\xi, \eta\rangle = \langle \ell(g)^*\xi(t), \ell(f)^*\eta(t)\rangle.$$

Thus for every $n = 1, 2, \ldots$ and $t > 0$ we have

$$\langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle = \langle \beta_t(f \otimes \bar{g})\xi_n, \eta_n\rangle = \langle \ell(g)^*\xi_n(t), \ell(f)^*\eta_n(t)\rangle,$$

and hence

$$\int_0^\infty |\langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle|\,\mathrm{d}t \leqslant \int_0^\infty \|\ell(g)^*\xi_n(t)\| \cdot \|\ell(f^*)\eta_n(t)\|\mathrm{d}t \leqslant$$

$$\leqslant \|\ell(g)^*\xi_n\| \cdot \|\ell(f)^*\eta_n\| \leqslant \|\xi_n\| \cdot \|\eta_n\| \cdot \|f\|_1\|g\|_1.$$

Summing this inequality on n we obtain the required inequality

$$\int_0^\infty |\langle \beta_{t*}(T)f, g\rangle|\mathrm{d}t \leqslant \int_0^\infty \sum_n |\langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle|\,\mathrm{d}t =$$

$$= \sum_n \int_0^\infty |\langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle|\,\mathrm{d}t \leqslant$$

$$\leqslant \sum_n \|\xi_n\| \cdot \|\eta_n\| \cdot \|f\|_1\|g\|_1 = \mathrm{tr}|T|\,\|f\|_1\|g\|_1.$$

Let $u_n(t) = \langle \beta_{t*}(\xi_n \otimes \bar{\eta}_n)f, g\rangle$, $t > 0$. The preceding estimate shows that

$$\sum_n \int_0^\infty |u_n(t)|\,\mathrm{d}t < \infty,$$

so by the dominated convergence theorem we have

$$\int_0^\infty \langle \beta_{t*}(T)f, g\rangle \, dt = \sum_n \int_0^\infty \langle \beta_{t*}(\xi_n \otimes \overline{\eta_n})f, g\rangle dt.$$

Now as we have seen,

$$\langle \beta_{t*}(\xi_n \otimes \overline{\eta_n})f, g\rangle = \langle \beta_t(f \otimes \bar{g})\xi_n, \eta_n\rangle.$$

By Proposition 6.4 of [4], we have

$$\int_0^\infty \langle \beta_t(f \otimes \bar{g})\xi_n, \eta_n\rangle = \langle \lambda(f \otimes \bar{g})\xi_n, \eta_n\rangle.$$

Thus we may conclude that

$$\int_0^\infty \langle \beta_{t*}(T)f, g\rangle \, dt = \sum_n \langle \lambda(f \otimes \bar{g})\xi_n, \eta_n\rangle = \mathrm{tr}(T\lambda(f \otimes \bar{g})).$$

The following result shows how elements of $\mathcal{M}(\beta_*)$ determine bounded linear functionals on $C^*(E)$. We will see in Section 3 that this construction gives precisely the singular part of $C^*(E)^*$.

THEOREM 2.4. *Let $A \in \mathcal{M}(\beta_*)$. For every $f, g \in L^1(E) \cap L^2(E)$ we have*

$$\int_0^\infty |\langle A(t)f, g\rangle| dt \leqslant \|A\| \cdot \|f\|_1 \|g\|_1,$$

and there is a unique bounded linear functional ρ_A on $C^*(E)$ satisfying

$$\rho_A(f \otimes \bar{g}) = \int_0^\infty \langle A(t)f, g\rangle dt,$$

for all $f, g \in L^1(E) \cap L^2(E)$. ρ_A is singular, and the map $A \mapsto \rho_A$ is a linear isometry such that $\rho_A \geqslant 0$ iff $A \geqslant 0$.

For every $t > 0$ we have

$$\beta_t^* \rho_A(a) = \mathrm{tr}(A(t)\lambda(a)), \quad a \in C^*(E),$$

λ being the regular representation of $C^*(E)$; if the trace-norm limit $A(0+) = \lim_{t\to 0} A(t)$ exists, then

$$\rho_A(x) = \operatorname{tr}(A(0+)\lambda(x)), \quad x \in C^*(E).$$

Proof. Fix $f, g \in L^1(E) \cap L^2(E)$ and choose $A \in \mathcal{M}(\beta_*)$. For every $\delta > 0$ and $x > \delta$ we have $A(x) = \beta_{x-\delta *}(A(\delta))$, hence by Proposition 2.3

$$\int_\delta^\infty |\langle A(x)f, g\rangle| \mathrm{d}x = \int_\delta^\infty |\langle \beta_{x-\delta *}(A(\delta))f, g\rangle| \mathrm{d}x =$$

$$= \int_0^\infty |\langle \beta_{y*}(A(\delta))f, g\rangle| \mathrm{d}y \leqslant \operatorname{tr}|A(\delta)| \, \|f\|_1 \|g\|_1 \leqslant \|A\| \, \|f\|_1 \|g\|_1 .$$

Letting δ tend to zero in the term on the left, we obtain the asserted estimate

$$\int_0^\infty |\langle A(x)f, g\rangle| \mathrm{d}x \leqslant \|A\| \, \|f\|_1 \|g\|_1 .$$

We claim next that there is a unique bounded linear functional ρ_A on $C^*(E)$ satisfying

$$\rho_A(f \otimes g) = \int_0^\infty \langle A(x)f, g\rangle \mathrm{d}x,$$

$f, g \in L^1(E) \cap L^2(E)$. To see this, fix $\delta > 0$ and consider the linear functional $\rho_\delta \in C^*(E)^*$ defined by

$$\rho_\delta(a) = \operatorname{tr}(A(\delta)\lambda(a)), \quad a \in C^*(E).$$

Clearly $\|\rho_\delta\| \leqslant \operatorname{tr}|A(\delta)| \leqslant \|A\|$, and by Proposition 2.3 we have

$$\rho_\delta(f \otimes g) = \int_0^\infty \langle \beta_{t*}(A(\delta))f, g\rangle \mathrm{d}t =$$

$$= \int_0^\infty \langle A(t + \delta)f, g\rangle \mathrm{d}t = \int_\delta^\infty \langle A(x)f, g\rangle \mathrm{d}x,$$

for every $f, g \in L^1(E) \cap L^2(E)$. Thus

$$\lim_{\delta \to 0} \rho_\delta(f \otimes \bar{g}) = \int_0^\infty \langle A(x)f, g\rangle \mathrm{d}x,$$

for all such f, g. Since $\{\rho_\delta : \delta > 0\}$ is uniformly bounded by $\|A\|$ and since $\lim_{\delta \to 0} \rho_\delta(a)$ exists for all $a \in C^*(E)$ belonging to the spanning set $\{f \otimes \bar{g} : f, g \in L^1(E) \cap \cap L^2(E)\}$, it follows that ρ_δ converges weak* to an element ρ_A in $C^*(E)^*$ satisfying $\|\rho_A\| \leqslant \|A\|$. Clearly $\rho_A(f \otimes \bar{g})$ has the asserted form. The uniqueness of ρ_A is apparent from the fact that $\{f \otimes \bar{g} : f, g \in L^1(E) \cap L^2(E)\}$ spans $C^*(E)$.

If A is a positive element of $\mathscr{M}(\beta_*)$ then $A(\delta)$ is a positive trace-class operator for every $\delta > 0$, hence

$$\rho_\delta(a) = \operatorname{tr}(A(\delta)\lambda(a))$$

is a positive linear functional for every $\delta > 0$, hence $\rho_A \geqslant 0$.

We claim next that for every $t > 0$ we have

$$\beta_t^* \rho_A(a) = \operatorname{tr}(A(t)\lambda(a)), \quad a \in C^*(E). \tag{2.5}$$

It suffices to prove this formula for a of the form $f \otimes \bar{g}$ with $f, g \in L^1(E) \cap L^2(E)$. Fixing $t > 0$ and letting $\{e_1, e_2, \ldots\}$ be an orthonormal basis for $E(t)$, the left side of (2.5) can be written

$$\beta_t^* \rho_A(f \otimes \bar{g}) = \sum_n \rho_A(fe_n \otimes \overline{ge_n}) = \sum_n \int_0^\infty \langle A(x)fe_n, ge_n\rangle \mathrm{d}x,$$

because fe_n and ge_n belong to $L^1(E) \cap L^2(E)$ for every $n \geqslant 1$. We want to interchange the order of summation and integration in the latter term. By the dominated convergence theorem, this will be justified if we show that

$$\sum_n \int_0^\infty |\langle A(x)fe_n, ge_n\rangle| \mathrm{d}x \leqslant \|A\| \|f\|_1 \|g\|_2 . \tag{2.6}$$

Fix $\delta > 0$. Then we have

$$\sum_n \int_\delta^\infty |\langle A(x)fe_n, ge_n\rangle| \mathrm{d}x = \sum_n \int_0^\infty |\langle A(y + \delta)fe_n, ge_n\rangle| \mathrm{d}y =$$

$$= \sum_n \int_0^\infty |\langle \beta_{y*}(A(\delta))fe_n, ge_n\rangle| \mathrm{d}y.$$

We claim first that if T is any trace-class operator on $L^2(E)$, then

$$\sum_n \int_0^\infty |\langle \beta_{y*}(T)fe_n, ge_n\rangle| \mathrm{d}y \leq \operatorname{tr}|T|\, \|f\|_1 \|g\|_1 . \tag{2.7}$$

For that, choose sequences ξ_m, η_m, $m \geq 1$, in $L^2(E)$ such that

$$\sum_m \|\xi_m\| \cdot \|\eta_m\| = \operatorname{tr}|T|, \quad \text{and} \quad T = \sum_m \xi_m \otimes \bar{\eta}_m .$$

Noting that fe_n and ge_n belong to $L^1(E) \cap L^2(E)$ for every $n \geq 1$, we see as in the proof of Proposition 2.3 that for each $m \geq 1$ and $y > 0$,

$$\langle \beta_{y*}(\xi_m \otimes \bar{\eta}_m)fe_n, ge_n\rangle = \langle \ell(ge_n)^*\xi_m(y), \ell(fe_n)^*\eta_m(y)\rangle .$$

Thus

$$\int_0^\infty |\langle \beta_{y*}(T)fe_n, ge_n\rangle| \mathrm{d}y \leq \int_0^\infty \sum_m |\langle \beta_{y*}(\xi_m \otimes \bar{\eta}_m)fe_n, ge_n\rangle| \mathrm{d}y \leq$$

$$\leq \sum_m \int_0^\infty \|\ell(ge_n)^*\xi_m(y)\| \cdot \|\ell(fe_n)^*\eta_m(y)\| \mathrm{d}y \leq$$

$$\leq \sum_m \|\ell(ge_n)^*\xi_m\| \cdot \|\ell(fe_n)^*\eta_m\| .$$

Summing on n, we obtain

$$\sum_n \int_0^\infty |\langle \beta_{y*}(T)fe_n, ge_n\rangle| \mathrm{d}y \leq \sum_{m,n} \|\ell(ge_n)^*\xi_m\| \cdot \|\ell(fe_n)^*\eta_m\| . \tag{2.8}$$

Let $V_n = \ell(e_n)$, $n = 1, 2, \ldots$. This defines a sequence of isometries having mutually orthogonal ranges. Moreover,

$$\ell(ge_n)^*\xi_m = V_n^*\ell(g)^*\xi_m ,$$

and

$$\ell(fe_n)^*\eta_m = V_n^*\ell(f)^*\eta_m .$$

So for m fixed, the Schwarz inequality implies

$$\sum_n \|\ell(ge_n)^*\xi_m\| \cdot \|\ell(fe_n)^*\eta_m\| \leq (\sum_n \|V_n^*\ell(g)^*\xi_m\|)^{21/2}(\sum_n \|V_n^*\ell(f)^*\eta_m\|^2)^{1/2}.$$

But since $\sum_n V_n V_n^* \leqslant 1$, we have

$$\sum_n \|V_n^* \ell(g)^* \xi_m\|^2 = \sum_n \langle V_n V_n^* \ell(g)^* \xi_m, \ell(g)^* \xi_m \rangle \leqslant$$

$$\leqslant \|\ell(g)^* \xi_m\|^2 \leqslant \|g\|_1^2 \|\xi_m\|^2,$$

and similarly

$$\sum_n \|V_n^* \ell(f)^* \eta_m\|^2 \leqslant \|f\|_1^2 \|\eta_m\|^2.$$

Thus the last term in (2.8) is dominated by

$$\sum_n \|\xi_m\| \|\eta_m\| \|f\|_1 \|g\|_1 = \operatorname{tr} |T| \|f\|_1 \|g\|_1,$$

proving the claim.

Returning to the proof of (2.6), we have for each $\delta > 0$

$$\sum_n \int_\delta^\infty |\langle A(x) f e_n, g e_n \rangle| \mathrm{d}x = \sum_n \int_0^\infty |\langle \beta_{y*}(A(\delta)) f e_n, g e_n \rangle| \mathrm{d}y \leqslant$$

$$\leqslant \operatorname{tr}|A(\delta)| \|f\|_1 \|g\|_1 \leqslant \|A\| \cdot \|f\|_1 \|g\|_1.$$

Allowing δ to decrease to zero, we obtain (2.6).

It follows that

$$\beta_t^* \rho_A(f \otimes \bar{g}) = \int_0^\infty \sum_n \langle A(x) f e_n, g e_n \rangle \, \mathrm{d}x.$$

Now for each $x > 0$, we have

$$\sum_n \langle A(x) f e_n, g e_n \rangle = \sum_n \operatorname{tr}(A(x)(f e_n \otimes \overline{g e_n})) =$$

$$= \operatorname{tr}(A(x) \beta_t (f \otimes \bar{g})) = \operatorname{tr}(\beta_{t*}(A(x))(f \otimes \bar{g})) =$$

$$= \langle \beta_{t*}(A(x)) f, g \rangle = \langle A(x + t) f, g \rangle = \langle \beta_{x*}(A(t)) f, g \rangle.$$

Hence

$$\beta_t^* \rho_A(f \otimes \bar{g}) = \int_0^\infty \langle \beta_{x*}(A(t)) f, g \rangle \mathrm{d}x.$$

By Proposition 2.3, the right side is $\operatorname{tr}(A(t) \lambda(f \otimes \bar{g}))$, establishing (2.5).

To see that ρ_A is singular, it suffices to check that $\|\beta_t^*\rho_A\|$ tends to zero as $t \to \infty$ (Proposition 1.8(i)). But by (2.5) we have

$$\beta_t^*\rho_A(x) = \operatorname{tr}(A(t)\lambda(x)), \quad x \in C^*(E)$$

for every $t > 0$, hence $\|\beta_t^*\rho_A\| \leqslant \operatorname{tr}|A(t)|$, and we have already seen that $\lim_{t\to\infty} \operatorname{tr}|A(t)| = 0$ for every $A \in \mathscr{M}(\beta_*)$.

Assuming that the trace-norm limit $A(0+) = \lim_{t\to 0} A(t)$ exists, then for every $a \in C^*(E)$

$$\operatorname{tr}(A(0+)\lambda(a)) = \lim_{t\to 0} \operatorname{tr}(A(t)\lambda(a)) = \lim_{t\to 0} \beta_t^*\rho_A(a) = \rho_A(a)$$

because $\beta_t^*\rho_A$ tends weak* to ρ_A as $t \to 0+$.

It remains to prove that $\|A\| \leqslant \|\rho_A\|$, and that $\rho_A \geqslant 0$ implies $A \geqslant 0$. Now $\omega_t(B) = \operatorname{tr}(A(t)B)$ defines a normal linear functional on $\mathscr{B}(L^2(E))$ for every $t > 0$. Since $\lambda(C^*(E))'' = \mathscr{B}(L^2(E))$ ([4], Corollary of Theorem 5.2), Kaplansky's density theorem implies that the norm of the restriction of ω_t to $\lambda(C^*(E))$ agrees with $\|\omega_t\| = \operatorname{tr}|A(t)|$. Since λ maps the unit ball of $C^*(E)$ onto that of $\lambda(C^*(E))$, we conclude from formula (2.5) that

$$\|\beta_t^*\rho_A\| = \operatorname{tr}|A(t)|, \quad t > 0,$$

and therefore $\|\rho_A\| \geqslant \|\beta_t^*\rho_A\| \geqslant \operatorname{tr}|A(t)|$ for every $t > 0$. Hence

$$\|\rho_A\| \geqslant \sup_{t>0} \operatorname{tr}|A(t)| = \|A\|.$$

Finally, assume $\rho_A \geqslant 0$. Then $\beta_t^*\rho_A \geqslant 0$ for every $t > 0$. Since

$$\beta_t^*\rho_A(a) = \operatorname{tr}(A(t)\lambda(a)), \quad a \in C^*(E),$$

an argument similar to that of the preceding paragraph shows that $A(t) \geqslant 0$ for every $t > 0$, hence $A \geqslant 0$. ▨

3. CHARACTERIZATION OF SINGULAR STATES

In this section we prove that the map

$$A \in \mathscr{M}(\beta_*) \mapsto \rho_A \in C^*(E)^*$$

defined in Theorem 2.4 is an isomorphism of $\mathscr{M}(\beta_*)$ onto the singular part of the dual of $C^*(E)$, and we deduce some consequences. The key step is the assertion that $A \mapsto \rho_A$ is surjective, and is basically the following result.

THEOREM 3.1. *For every bounded linear functional ρ on $C^*(E)$ there is a unique function $A \in \mathcal{M}(\beta_*)$ satisfying*

$$\rho(f \otimes g) - \beta_t^* \rho(f \otimes g) = \int_0^t \langle A(x)f, g\rangle \mathrm{d}x$$

for all $f, g \in L^1(E) \cap L^2(E)$, $t > 0$. One has $\|A\| \leqslant \|\rho\|$, and $A \geqslant 0$ if ρ is positive.

Our proof of Theorem 3.1 is based on the following lemma, which provides a representation for additive cocycles associated with contraction semigroups on certain Banach spaces.

LEMMA 3.2. *Let E be a separable Banach space which is the dual of a Banach space E_*.*

Let $\gamma = \{\gamma_t : t \geqslant 0\}$ be a strongly continuous contraction semigroup which acts on E and let $\{b(t) : t \geqslant 0\}$ be a family of elements of E satisfying

(i) $b(s + t) = b(s) + \gamma_s(b(t))$, $s, t \geqslant 0$

(ii) $\|b(t)\| \leqslant Mt$, $t \geqslant 0$,

M being a positive constant. Then there is a norm-continuous function $a: (0, \infty) \to E$ such that

(i)′ $a(s + t) = \gamma_s(a(t))$,

(ii)′ $\|a(t)\| \leqslant M$, $s \geqslant 0$, $t > 0$,

and for which

$$b(t) = \int_0^t a(s)\mathrm{d}s, \quad t > 0.$$

REMARKS. Notice first that for a separable Hilbert space H, the Banach space $E = \mathscr{T}^1(H)$ of trace-class operators satisfies the hypotheses of Lemma 3.2.

We also want to point out that any Banach space E satisfying the hypothesis of Lemma 3.2 has the following property: for every bounded linear functional F on E, there is a sequence $\{x_1, x_2, \ldots\} \subseteq E_*$ such that

$$F(a) = \lim_n \langle a, x_n\rangle, \quad a \in E,$$

$\langle \cdot, \cdot \rangle$ denoting the canonical pairing of E and its predual E_*. To see this, fix F and choose a sequence $a_1, a_2, \ldots$ in E which is norm-dense in the unit ball of E. Now for any Banach space X, the natural map of X into X^{**} carries the ball of radius r in X onto a subset of X^{**} which is weak*-dense in the ball of radius r in X^{**}. It follows that for every $n = 1, 2, \ldots$ we can find an element x_n of E_* satisfying $\|x_n\| \leqslant \|F\|$ and

$$|F(a_j) - \langle a_j, x_n\rangle| \leqslant 1/n, \quad 1 \leqslant j \leqslant n.$$

The sequence $x_1, x_2, \ldots$ has the asserted property.

Proof of Lemma 3.2. We claim first that there is a unique bounded linear map L: : $L^1(0,\infty) \to E$ satisfying

$$L(\chi_{(s,t)}) = b(t) - b(s), \quad 0 \leqslant s < t,$$

χ_A denoting the characteristic function of the set A. To see this, suppose first that f is a step function in $L^1(0, \infty)$, say

$$f = \sum_{j=1}^{n} \lambda_j \delta_{(t_{j-1}, t_j)},$$

$\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, $0 \leqslant t_0 < t_1 < \ldots < t_n$. By property (i), we have $b(t_j) - b(t_{j-1}) = \gamma_{t_{j-1}}(b(t_j - t_{j-1}))$, and hence

$$\|\sum_j \lambda_j(b(t_j) - b(t_{j-1}))\| \leqslant \sum |\lambda_j| \cdot \|b(t_j) - b(t_{j-1})\| =$$

$$= \sum |\lambda_j| \, \|\gamma_{t_{j-1}}(b(t_j - t_{j-1}))\| \leqslant$$

$$\leqslant \sum |\lambda_j| \cdot \|b(t_j - t_{j-1})\| \leqslant M \sum |\lambda_j| \, |t_j - t_{j-1}|.$$

It follows that

$$L(f) = \sum_j \lambda_j(b(t_j) - b(t_{j-1}))$$

defines a linear operator on step functions, having norm at most M. L extends to $L^1(0, \infty)$ because the step functions are dense. The uniqueness of L is apparent.

Now E is the dual of E_* and E_* must be separable because its dual is separable. Therefore we may apply a known Radon-Nikodym theorem ([6], Theorem 2, p. 499) to infer that there is a bounded function a: $(0, \infty) \to E$ such that $\langle a(t), x\rangle$ is measurable in t for each $x \in E_*$ and such that

$$\langle L(f), x\rangle = \int_0^\infty f(t)\langle a(t), x\rangle \mathrm{d}t$$

for every $f \in L^1(0, \infty)$, $x \in E_*$. Moreover, $\|a(t)\| \leqslant M$ for every $t > 0$.

Let F be a bounded linear functional on E. We claim that $F(a(s))$ is a measurable function of $s > 0$ and that

$$F(L(f)) = \int_0^\infty f(s) F(a(s)) \mathrm{d}s \tag{3.3}$$

for every $f \in L^1(0, \infty)$. Indeed, by the preceding remarks there is a sequence $x_1, x_2, \ldots \in E_*$ with

(3.4)
$$F(a) = \lim_n \langle a, x_n \rangle, \quad a \in E.$$

The uniform boundedness principle implies that $\{\|x_n\| : n \geq 1\}$ is bounded, and (3.4) clearly implies that F is measurable relative to the Borel structure on E generated by its weak*-topology. Hence $s \mapsto F(a(s))$ is a bounded measurable function $(0, \infty)$. We have

$$F(a(s)) = \lim_n \langle a(s), x_n \rangle$$

for every $s > 0$ and

$$|\langle a(s), x_n \rangle| \leq \sup_s \|a(s)\| \cdot \sup_n \|x_n\| < \infty.$$

So the dominated convergence theorem implies that

$$F(L(f)) = \lim_n \langle L(f), x_n \rangle =$$

$$= \lim \int_0^\infty f(s) \langle a(s), x_n \rangle \mathrm{d}s = \int_0^\infty f(s) F(a(s)) \mathrm{d}s,$$

as asserted.

Notice next that for $f \in L^1(0, \infty)$,

(3.5)
$$\gamma_t(L(f)) = L(S_t f)$$

S_t denoting the shift on $L^1(0, \infty)$ defined by

$$S_t f(x) = \begin{cases} f(x - t), & x > t \\ 0, & 0 < x \leq t. \end{cases}$$

Both sides of (3.5) are bounded linear operators on $L^1(0, \infty)$, and so it suffices to check (3.5) when f is a characteristic function $\chi_{(r,s)}$, $0 \leq r < s$. Here, we have

$$\gamma_t L(\chi_{(r,s)}) = \gamma_t(b(s) - b(r)),$$

while

$$L(S_t \chi_{(r,s)}) = L(\chi_{(r+t, s+t)}) = b(s + t) - b(r + t) =$$

$$= (b(s + t) - b(t)) - (b(r + t) - b(t)) =$$

$$= \gamma_t(b(s)) - \gamma_t(b(r)) = \gamma_t(b(s) - b(r)).$$

(3.5) follows.

Now we claim that for every fixed $t > 0$, we have

$$\gamma_t(a(s)) = a(s + t) \text{ almost everywhere } (ds). \tag{3.6}$$

Since E_* and $L^1(0, \infty)$ are separable, it suffices to show that

$$\int_0^\infty f(s)\langle \gamma_t(a(s)), x\rangle ds = \int_0^\infty f(s)\langle a(s + t), x\rangle ds$$

for every $f \in L^1(0, \infty)$ and every $x \in E_*$. Fix f and x. The linear functional $F(a) = \langle \gamma_t(a), x \rangle$ is bounded on E, and so by (3.3) we have

$$\int_0^\infty f(s)\langle \gamma_t(a(s)), x\rangle ds = \int_0^\infty f(s)F(a(s))ds =$$

$$= F(L(f)) = \langle \gamma_t(L(f)), x\rangle.$$

By (3.5), the right side is

$$\langle L(S_t f), x\rangle = \int_t^\infty f(\lambda - t)\langle a(\lambda), x\rangle d\lambda = \int_0^\infty f(s)\langle a(s + t), x\rangle ds,$$

as required.

By the Fubini theorem, there is a Borel set $N \subseteq (0, \infty)$ of measure zero such that for all $s \notin N$, we have

$$\gamma_t(a(s)) = a(s + t) \quad \text{a.e. } (dt).$$

Because N has measure zero, we can find a sequence $s_n \in (0,\infty)\setminus N$ which decreases to zero. The preceding formula implies that for every n, the restriction of $a(\cdot)$ to the interval (s_n, ∞) agrees almost everywhere with the norm-continuous function

$$a_n(t) = \gamma_{t-s_n}(a(s_n)), \quad t > s_n.$$

Thus the continuous functions $a_1, a_2, \ldots$ are compatible, and there is a unique continuous function $a_\infty\colon (0, \infty) \to E$ such that $a_\infty|_{(s_n,\infty)} = a_n$, $n = 1, 2, \ldots$. a_∞ has the properties

$$a_\infty(t) = a(t) \text{ almost everywhere } (dt),$$

and

$$\gamma_s(a_\infty(t)) = a_\infty(s + t) \quad \text{for all } s \geq 0,\ t > 0.$$

Finally, because a_∞ is bounded and continuous it must be Bochner-integrable, and the formula

$$b(t) = \int_0^t a_\infty(s)\,ds, \quad t > 0$$

follows from (3.3) by taking $f = \chi_{(0,t)}$. ∎

Proof of Theorem 3.1. Choose a bounded linear functional ρ on $C^*(E)$. In view of Lemma 3.2, it suffices to show that there is a family $\{B(t): t \geq 0\}$ of trace-class operators on $L^2(E)$ satisfying

$$B(s+t) = B(s) + \beta_{s**}(B(t)), \quad s, t \geq 0,$$

$$\operatorname{tr}|B(t)| \leq \|\rho\| t, \quad t \geq 0,$$

and

$$\rho(f \otimes g) - \beta_t^* \rho(f \otimes g) = \langle B(t) f, g \rangle$$

for every $f, g \in L^1(E) \cap L^2(E)$, $t > 0$.

Now by the polar decomposition for linear functionals on C^*-algebras and the GNS construction applied to $|\rho|$, we can find a representation $\pi: C^*(E) \to \mathscr{B}(H)$ and a pair of cyclic vectors $\xi_1, \xi_2 \in H$ for π such that $\|\xi_1\| = \|\xi_2\| = \|\rho\|^{1/2}$ and

$$\rho(x) = \langle \pi(x)\xi_1, \xi_2 \rangle, \quad x \in C^*(E).$$

By ([4], Corollary 2 of Theorem 3.4), there is a representation $\varphi: E \to \mathscr{B}(H)$ such that

$$\pi(f \otimes g) = \varphi(f)\varphi(g)^*, \quad f, g \in L^1(E).$$

Let P_t be the projection onto $[\varphi(E(t))H]$, for every $t > 0$. For each $t > 0$, we define a pair of antilinear transformations $B_1(t)$, $B_2(t)$ of $L^1(E) \cap L^2(E)$ into H as follows:

(3.7) $$B_j(t) f = (1 - P_t)\varphi(f)^* \xi_j, \quad j = 1, 2.$$

We claim first that $B_1(t)$ and $B_2(t)$ extend uniquely to antilinear Hilbert-Schmidt operators from $L^2(E)$ into H, such that

(3.8) $$\operatorname{tr}(B_j(t)^* B_j(t)) = \|\rho\| t, \quad j = 1, 2.$$

Granting that for a moment, we can then define a *linear* trace-class operator $B(t)$ on $L^2(E)$ by

(3.9) $$B(t) = \begin{cases} B_1(t)^* B_2(t), & t > 0, \\ 0, & t = 0, \end{cases}$$

and we will have $\operatorname{tr}|B(t)| \leq \|\rho\| t$, $t \geq 0$.

We now prove (3.8). Let $e_1, e_2, \ldots$ be a sequence of measurable sections of the map $p: E \to (0, \infty)$ such that $\{e_1(t), e_2(t), \ldots\}$ is an orthonormal basis for $E(t)$ for every $t > 0$ ([1], Proposition 1.15). Let $f_1, f_2, \ldots$ be a orthonormal basis for $L^2(0, \infty)$ consisting of functions in $L^1(0, \infty) \cap L^2(0, \infty)$. Define $g_{mn}: (0, \infty) \to E$ by

$$g_{mn}(s) = f_m(s)e_n(s).$$

Then $\{g_{mn} : m, n \geqslant 1\}$ is an orthonormal basis for $L^2(E)$ consisting of functions in $L^1(E) \cap L^2(E)$. (3.8) will follow if we prove

$$\sum_{m,n=1}^{\infty} \|B_j(t)g_{mn}\|^2 \leqslant \|\rho\|t \tag{3.10}$$

for $j = 1, 2$.

In order to establish (3.10), we make use of the semigroup of endomorphisms $\gamma = \{\gamma_s : s \geqslant 0\}$ of $\mathscr{B}(H)$ defined by

$$\gamma_s(A) = \sum_{n=1}^{\infty} \varphi(e_n(s))A\varphi(e_n(s))^*, \quad A \in \mathscr{B}(H),$$

$\gamma_0(A) = A$. We have $\gamma_s(1) = P_s, s > 0$, and hence $\gamma_s(1 - P_t) = P_s - P_{s+t}$. Moreover,

$$\varphi(e_n(s))A = \gamma_s(A)\varphi(e_n(s)), \tag{3.11}$$

for every $s > 0$, $A \in \mathscr{B}(H)$, $n = 1, 2, \ldots$.

We prove (3.10) for $j = 1$. For each $m, n \geqslant 1$, write

$$\|B_1(t)g_{mn}\|^2 = \langle(1 - P_t)\varphi(g_{mn})^*\xi_1, (1 - P_t)\varphi(g_{mn})^*\xi_1\rangle =$$

$$= \int_0^\infty\int_0^\infty \overline{f_m(x)}f_m(y)\langle(1 - P_t)\varphi(e_n(x))^*\xi_1, (1 - P_t)\varphi(e_n(y))^*\xi_1\rangle \mathrm{d}x\mathrm{d}y =$$

$$= \int_0^\infty\int_0^\infty \overline{f_m(x)}f_m(y)k_n(x, y)\mathrm{d}x\mathrm{d}y,$$

where

$$k_n(x, y) = \langle(1 - P_t)\varphi(e_n(x))^*\xi_1, (1 - P_t)\varphi(e_n(y))^*\xi_1\rangle .$$

Therefore, if we can show that each k_n is the kernel of a positive trace-class operator $K_n \in \mathscr{L}^1(L^2(E))$ for which

$$\sum_{n=1}^{\infty} \operatorname{tr} K_n \leqslant \|\rho\|t, \tag{3.12}$$

then

$$\sum_{m,n} \|B_1(t)g_{mn}\|^2 = \sum_n \operatorname{tr} K_n,$$

and (3.10) will follow.

In order to prove (3.12) we will exhibit, for each $n \geqslant 1$, a sequence of functions $u_{n1}, u_{n2}, \ldots$ in $L^2(0, \infty)$ such that

(3.13)

(i) $$k_n(x, y) = \sum_{p=1}^{\infty} u_{np}(x)\overline{u_{np}(y)},$$

and

(ii) $$\sum_{n,p} \|u_{np}\|^2 \leqslant \|\rho\|t.$$

Let $\{\zeta_1, \zeta_2, \ldots\}$ be an orthonormal basis for H, and put

$$u_{np}(x) = \langle (1 - P_t)\varphi(e_n(x))^* \xi_1, \zeta_p \rangle.$$

The formula (3.13)(i) follows from the fact that

$$\langle \eta_1, \eta_2 \rangle = \sum_p \langle \eta_1, \zeta_p \rangle \overline{\langle \eta_2, \zeta_p \rangle}$$

for any pair of vectors η_1, η_2 in H. Moreover, for each $x > 0$ we have

$$\sum_{n,p} |u_{np}(x)|^2 = \sum_n \|(1 - P_t)\varphi(e_n(x))^* \xi_1\|^2 =$$

$$= \sum_n \langle \varphi(e_n(x))(1 - P_t)\varphi(e_n(x))^* \xi_1, \xi_1 \rangle =$$

$$= \langle \gamma_x(1 - P_t)\xi_1, \xi_1 \rangle = \langle (P_x - P_{x+t})\xi_1, \xi_1 \rangle.$$

Integrating the latter formula over $0 < x < \infty$, we obtain

$$\sum_{n,p} \|u_{np}\|^2 = \int_0^\infty \langle (P_x - P_{x+t})\xi_1, \xi_1 \rangle \mathrm{d}x.$$

Now the function

$$w(x) = \langle P_x \xi_1, \xi_1 \rangle$$

is non-negative and decreasing on $(0, \infty)$. Hence

$$\int_0^\infty (w(x) - w(x + t))\mathrm{d}x = \sum_{k=0}^{\infty} \int_{kt}^{(k+1)t} (w(x) - w(x + t))\mathrm{d}x =$$

$$= \sum_{k=0}^{\infty} \int_0^t (w(kt + s) - w((k + 1)t + s))\mathrm{d}s =$$

$$= \int_0^t \sum_{k=0}^{\infty} (w(kt + s) - w((k + 1)t + s))\mathrm{d}s =$$

$$= \int_0^t (w(s) - w(\infty))\mathrm{d}s \leqslant \int_0^t w(s)\mathrm{d}s \leqslant \|\xi_1\|^2 t = \|\rho\| t,$$

and hence we obtain the required estimate

$$\sum_{n,p} \|u_{np}\|^2 \leqslant \|\rho\| t.$$

We claim now that the family of operators $\{B(t) : t \geqslant 0\}$ defined by (3.9) obeys

$$B(s + t) = B(s) + \beta_{s*}(B(t)). \tag{3.14}$$

For this, we will show first that for $s > 0$ and $v \in E(s)$, we have

$$B_j(t)\mathrm{r}(v) = \varphi(v)^* B_j(s + t), \tag{3.15}$$

r denoting the right regular antirepresentation of E on $L^2(E)$. Indeed, for $f \in L^1(E) \cap \cap L^2(E)$ and $v \in E(s)$, we have

$$B_j(t)\mathrm{r}(v)f = B_j(t)(fv) = (1 - P_t)\varphi(fv)^*\xi_j =$$
$$= (1 - P_t)\varphi(v)^*\varphi(f)^*\xi_j .$$

Taking the semigroup $\gamma = \{\gamma_t : t \geqslant 0\}$ of endomorphisms of $\mathscr{B}(H)$ defined above, we have

$$\varphi(v)(1 - P_t) = \gamma_s(1 - P_t)\varphi(v) =$$

$$= (P_s - P_{s+t})\varphi(v) = (1 - P_{s+t})P_s\varphi(v) = (1 - P_{s+t})\varphi(v).$$

Hence $(1 - P_t)\varphi(v)^* = \varphi(v)^*(1 - P_{s+t})$, and the last term of the preceding equation becomes

$$\varphi(v)^*(1 - P_{s+t})\varphi(f)^*\xi_j = \varphi(v)^* B_j(s + t)f,$$

as required.

In terms of the orthonormal basis $\{e_1(s), e_2(s), \ldots\}$ for $E(s)$, the action of β_s on $\mathscr{B}(L^2(E))$ is given by

$$\beta_s(T) = \sum_n \mathrm{r}(e_n(s))T\mathrm{r}(e_n(s))^*,$$

and hence the action of β_{s*} on $\mathscr{L}^1(L^2(E))$ is given by

$$\beta_{s*}(B) = \sum_n \mathrm{r}(e_n(s))^* B \mathrm{r}(e_n(s)).$$

So by (3.15) we have

$$\beta_{s*}(B(t)) = \sum_n \mathrm{r}(e_n(s))^* B_1(t)^* B_2(t) \mathrm{r}(e_n(s)) =$$

$$= \sum_n B_1(s+t)^* \varphi(e_n(s))\varphi(e_n(s))^* B_2(s+t) = B_1(s+t)^* P_s B_2(s+t) =$$

$$= B_1(s+t)^* B_2(s+t) - B_1(s+t)^*(1-P_s)B_2(s+t) =$$

$$= B(s+t) - B_1(s+t)^*(1-P_s)B_2(s+t).$$

Now since the projections $1 - P_s$ are increasing with s, a glance at Definition 3.7 of $B_j(t)$ shows that $(1-P_s)B_j(s+t) = B_j(s)$ for all $t \geqslant 0$, $s > 0$. Thus the above formula implies

$$\beta_{s*}(B(t)) = B(s+t) - B(s),$$

as required.

It follows that the function $t \in [0, \infty) \mapsto B(t) \in \mathscr{L}^1(L^2(E))$ is norm continuous. It remains to show that

$$\rho(f \otimes \bar{g}) + \beta_t^* \rho(f \otimes \bar{g}) = \langle B(t)f, g\rangle,$$

for f, $g \in L^1(E) \cap L^2(E)$. But for $t > 0$ we have

$$\beta_t^* \rho(f \otimes \bar{g}) = \sum_n \rho(f e_n(t) \otimes \overline{g e_n(t)}) = \sum_n \langle \varphi(f e_n(t))\varphi(g e_n(t))^* \xi_1, \xi_2\rangle =$$

$$= \sum_n \langle \varphi(f)\varphi(e_n(t))\varphi(e_n(t))^* \varphi(g)^* \xi_1, \xi_2\rangle = \langle \varphi(f) P_t \varphi(g)^* \xi_1, \xi_2\rangle.$$

Hence,

$$\rho(f \otimes \bar{g}) - \beta_t^* \rho(f \otimes \bar{g}) = \langle \varphi(f)(1-P_t)\varphi(g)^* \xi_1, \xi_2\rangle =$$

$$= \langle (1-P_t)\varphi(g)^* \xi_1, (1-P_t)\varphi(f)^* \xi_2\rangle =$$

$$= \langle B_1(t)g, B_2(t)f\rangle = \overline{\langle g, B_1(t)^* B_2(t) f\rangle} = \langle B(t)f, g\rangle.$$

Finally, assuming ρ is a positive linear functional, we have to show that $A(s) \geqslant 0$ for every $s > 0$. Because $A(\cdot)$ is continuous, it suffices to show that

$$\int_s^t \langle A(x)f, f\rangle \mathrm{d}x \geqslant 0$$

for all $0 < s < t$ and all $f \in L^1(E) \cap L^2(E)$. Because $\rho \geqslant 0$ we have $\xi_1 = \xi_2 = \xi$ in the GNS representation of ρ, and hence

$$\int_s^t \langle A(x)f, f\rangle \mathrm{d}x = \int_0^t \langle A(x)f, f\rangle \mathrm{d}x - \int_0^s \langle A(x)f, f\rangle \mathrm{d}x =$$

$$= \beta_s^*\rho(f \otimes \bar{f}) - \beta_t^*\rho(f \otimes \bar{f}) = \langle \varphi(f)(P_s - P_t)\varphi(f)^*\xi, \xi\rangle = \|(P_s - P_t)\varphi(f)^*\xi\|^2 \geqslant 0.$$

▨

COROLLARY 1. *The map $A \mapsto \rho_A$ defined in Theorem 2.4 is an isometric order isomorphism of $\mathcal{M}(\beta_*)$ onto the singular part $C^*(E)^*$.*

Proof. In view of Theorem 2.4, it remains to show that every singular element ρ of $C^*(E)^*$ has the form $\rho = \rho_A$ for some $A \in \mathcal{M}(\beta_*)$.

Fixing such a ρ, Theorem 3.1 implies that there is an element $A \in \mathcal{M}(\beta_*)$ such that

$$\rho(f \otimes \bar{g}) - \beta_t^*\rho(f \otimes \bar{g}) = \int_0^t \langle A(x)f, g\rangle \mathrm{d}x,$$

for every $f, g \in L^1(E) \cap L^2(E)$, $t > 0$. By (1.8)(i) we have

$$\lim_{t\to\infty} \beta_t^*\rho(f \otimes \bar{g}) = 0,$$

and by Theorem 3.1,

$$\lim_{t\to\infty} \int_0^t \langle A(x)f, g\rangle \mathrm{d}x = \int_0^\infty \langle A(x)f, g\rangle \mathrm{d}x.$$

Hence $\rho(f \otimes \bar{g}) = \rho_A(f \otimes \bar{g})$, and so $\rho = \rho_A$. ▨

COROLLARY 2. *For every singular state ρ of $C^*(E)$ and every $t > 0$, $\beta_t^*\rho$ is a regular singular state.*

ρ is a regular singular state iff

$$\lim_{t\to 0} \|\beta_t^*\rho - \rho\| = 0.$$

Proof. Let ρ be a singular state and choose $t > 0$. We have to show that $\beta_t^*\rho$ has the form

$$\beta_t^*\rho(x) = \mathrm{tr}(A\lambda(x)), \quad x \in C^*(E)$$

where A is a positive trace-class operator on $L^2(E)$. By Corollary 1 and Theorem 2.4, there is a positive element A in $\mathcal{M}(\beta_*)$ such that $\rho = \rho_A$. The last assertion of Theorem 2.4 gives the required representation

$$\beta_t^*\rho(x) = \operatorname{tr}(A(t)\lambda(x)).$$

If $\|\rho - \beta_t^*\rho\| \to 0$ as $t \to 0+$, then ρ is the norm limit of a sequence of positive linear functionals of the form

$$\rho_n(x) = \operatorname{tr}(A_n\lambda(x)), \quad x \in C^*(E)$$

where A_n is a positive trace-class operator on $L^2(E)$. Since the space of normal functionals on any von Neumann algebra is a Banach space, it follows that ρ must have the same form.

Conversely, if ρ has the form

$$\rho(x) = \operatorname{tr}(A\lambda(x))$$

where A is a positive trace-class operator on $L^2(E)$, then

$$\beta_t^*\rho(x) = \operatorname{tr}(\beta_{t*}(A)\lambda(x))$$

for every $t > 0$ and hence

$$\|\rho - \beta_t^*\rho\| \leqslant \operatorname{tr}|A - \beta_{t*}(A)| \to 0$$

as $t \to 0$. ∎

Corollary 3. *For every non-trivial product system E, the regular representation*

$$\lambda: C^*(E) \to \mathcal{B}(L^2(E))$$

is faithful.

Proof. It suffices to show that every state ρ of $C^*(E)$ satisfies

$$|\rho(x)| \leqslant \|\lambda(x)\|, \quad x \in C^*(E).$$

Fixing ρ, we can write $\rho = \rho_s + \rho_n$ where ρ_s and ρ_n are positive linear functions which are respectively singular and non-singular, and which satisfy $\|\rho_s\| + \|\rho_n\| = \|\rho\| = 1$. We will show that $|\rho_s(x)| \leqslant \|\rho_s\|\|\lambda(x)\|$ and $|\rho_n(x)| \leqslant \|\rho_n\|\|\lambda(x)\|$, for all $x \in C^*(E)$.

For each $t > 0$, Corollary 2 implies that for each $x \in C^*(E)$,

$$|\beta_t^*\rho_s(x)| \leqslant \|\beta_t^*\rho_s\| \cdot \|\lambda(x)\| \leqslant \|\rho_s\| \cdot \|\lambda(x)\|.$$

Since $\beta_t^*\rho_s$ converges weak* to ρ_s as $t \to 0+$, we conclude that

$$|\rho_s(x)| \leqslant \overline{\lim_{t \to 0+}} |\beta_t^*\rho_s(x)| \leqslant \|\rho_s\| \cdot \|\lambda(x)\|.$$

Now consider ρ_n. Applying the GNS construction, we obtain a separable representation $\pi_n: C^*(E) \to \mathscr{B}(H)$ and a cyclic vector ξ_n for π_n such that $\|\xi_n\|^2 = \|\rho_n\|$ and

$$\rho_n(x) = \langle \pi_n(x)\xi_n, \xi_n \rangle, \quad x \in C^*(E).$$

The representation π_n is non-singular, so by ([4], Theorem 7.1 and succeeding remarks) we have $\|\pi_n(x)\| \leqslant \|\lambda(x)\|$. The inequality $|\rho_n(x)| \leqslant \|\xi_n\|^2\|\lambda(x)\| = \|\rho_n\| \, \|\lambda(x)\|$ follows. ▨

COROLLARY 4. *Let $\varphi: E \to \mathscr{B}(H)$ be a singular representation of a non-trivial product system E. For every $t > 0$, let φ_t be the representation of E on $H_t = [\varphi(E(t))H]$ defined by*

$$\varphi_t(v) = \varphi(v)|H_t, \quad v \in E.$$

Then φ_t is unitarily equivalent to a direct sum of copies of the regular representation $\ell: E \to \mathscr{B}(L^2(E))$.

Proof. Fix $t > 0$, and let $\pi_t: C^*(E) \to \mathscr{B}(H_t)$ be the corresponding representation of $C^*(E)$. Since λ is an irreducible representation of $C^*(E)$ ([4], Corollary of Theorem 5.2), it suffices to show that for every vector $\xi \in H_t$, there is a positive trace-class operator $T = T_\xi$ on $L^2(E)$ such that

$$\langle \pi_t(x)\xi, \xi \rangle = \operatorname{tr}(T\lambda(x)), \quad x \in C^*(E). \tag{3.16}$$

Let $\pi: C^*(E) \to \mathscr{B}(H)$ be the representation defined by $\pi(f \otimes \bar{g}) = \varphi(f)\varphi(g)^*$ for $f, g \in L^1(E)$, and consider the positive linear functional ρ on $C^*(E)$ defined by $\rho(x) = \langle \pi(x)\xi, \xi \rangle$. ρ is obviously singular, and we have

$$\beta_t^*\rho(f \otimes \bar{g}) = \langle \varphi(f)P_t\varphi(g)^*\xi, \xi \rangle = \langle \pi_t(f \otimes \bar{g})\xi, \xi \rangle$$

for all $f, g \in L^1(E)$, P_t denoting the projection onto H_t. Hence

$$\beta_t^*\rho(x) = \langle \pi_t(x)\xi, \xi \rangle, \quad x \in C^*(E).$$

Formula (3.16) follows after an application of Corollary 2.

4. IRREGULAR SINGULAR STATES

Let E be a nontrivial product system. It is natural to ask at this point if every singular state of $C^*(E)$ is a regular singular state. In view of the isomorphism

$$\mathscr{S} \cong \mathscr{M}(\beta_*)$$

and the properties of $\mathcal{M}(\beta_*)$ discussed in Section 2, this is equivalent to asking if every positive function $A \in \mathcal{M}(\beta_*)$ has a trace-norm limit at $t = 0$:

$$A(0+) = \lim_{t \to 0} A(t).$$

In this section we exhibit a class of examples which show that this is not the case. The notation of Section 3 remains in force.

LEMMA 4.1. *Let $t_1, t_2, \ldots$ be a sequence of positive real numbers. There is a sequence of rank-one projections $e_1, e_2, \ldots$ in $\mathcal{B}(L^2(E))$ which converges to zero in the weak operator topology such that*

$$\beta_{t_n *}(e_{n+1}) = e_n, \quad n = 1, 2, \ldots .$$

Proof. Choose any faithful normal state ω of $\mathcal{B}(L^2(E))$. Let e_1 be an arbitrary rank-one projection. Inductively, we will construct a sequence $e_2, e_3, \ldots$ of rank-one projections such that

$$\omega(e_k) \leqslant 1/k,$$

$$\beta_{t_k *}(e_{k+1}) = e_k, \quad k \geqslant 1.$$

Assume that $e_1, e_2, \ldots, e_n$ have been defined and satisfy the above conditions insofar as they make sense. Since E is not the trivial product system, each fiber space $E(t)$ is infinite-dimensional ([3], Corollary of Lemma 7.3). Therefore since each t_n is positive, the von Neumann algebras $\beta_{t_n}(\mathcal{B}(L^2(E)))$ are (degenerate) type I_∞ factors of infinite multiplicity. Hence $\beta_{t_n}(e_n)$ is an infinite-dimensional projection. Let $f_1, f_2, \ldots$ be mutually orthogonal one-dimensional projections such that

$$\sum_k f_k = \beta_{t_n}(e_n).$$

By normality of the state ω we have

$$\sum_k \omega(f_k) = \omega(\beta_{t_n}(e_n)) < \infty,$$

and hence $\omega(f_k) \to 0$ as $k \to \infty$. Choosing k_0 so that

$$\omega(f_{k_0}) \leqslant 1/(n+1),$$

we put $e_{n+1} = f_{k_0}$.

We claim that $\beta_{t_n *}(e_{n+1}) = e_n$ or, what is the same,

$$\mathrm{tr}(e_{n+1}\beta_{t_n}(B)) = \mathrm{tr}(e_n B) \tag{4.2}$$

for every $B \in \mathscr{B}(L^2(E))$. Indeed, we have $e_{n+1} = f_{k_0} \leqslant \beta_{t_n}(e_n)$, hence $e_{n+1} = \beta_{t_n}(e_n)e_{n+1}\beta_{t_n}(e_n)$. So for B fixed, the left side of (4.2) can be written

$$\operatorname{tr}(e_{n+1}\beta_{t_n}(e_n)\beta_{t_n}(B)\beta_{t_n}(e_n)) = \operatorname{tr}(e_{n+1}\beta_{t_n}(e_nBe_n)).$$

Since e_n is one-dimensional we have $e_nBe_n = \operatorname{tr}(e_nB)e_n$, and the right side of the preceding formula becomes

$$\operatorname{tr}(e_nB)\operatorname{tr}(e_{n+1}\beta_{t_n}(e_n)) = \operatorname{tr}(e_nB)\operatorname{tr}(e_{n+1}) = \operatorname{tr}(e_nB),$$

as required. ▨

Proposition 4.3. *Let $t_1 > t_2 > \ldots$ be a sequence of positive reals which decreases to zero. There is a positive element $A \in \mathscr{M}(\beta_*)$ such that $\operatorname{tr}(A(t)) = 1$ for all $0 < t \leqslant 1$, and*

$$\lim_{n\to\infty} \langle A(t_n)\xi, \eta\rangle = 0, \quad \xi, \eta \in L^2(E).$$

ρ_A is a singular state of $C^*(E)$ which is not a regular singular state.

Proof. By adding an initial term to the sequence $\{t_n\}$ if necessary, we can assume that $t_1 \geqslant 1$. By Lemma 4.1, there are rank-one projections $e_1, e_2, \ldots$ in $\mathscr{B}(L^2(E))$ such that $e_n \to 0$ weakly and

$$\beta_{t_n - t_{n+1}*}(e_{n+1}) = e_n, \quad n \geqslant 1.$$

For each $n \geqslant 1$, define $A_n\colon [t_n, \infty) \to \mathscr{L}^1(L^2(E))$ by

$$A_n(t) = \beta_{t-t_n*}(e_n).$$

$A_n(t)$ is positive for every $t \geqslant t_n$ and $\operatorname{tr}(A_n(t)) \leqslant \operatorname{tr}(e_n) \leqslant 1$. Note that the restriction of A_{n+1} to $[t_n, \infty)$ agrees with A_n. Indeed, if $t \geqslant t_n$ then using the semigroup property of β_* we have

$$A_{n+1}(t) = \beta_{t-t_{n+1}*}(e_{n+1}) =$$

$$= \beta_{t-t_n*}(\beta_{t_n-t_{n+1}*}(e_{n+1})) = \beta_{t-t_n*}(e_n) = A_n(t).$$

We conclude that there is a unique positive element $A \in \mathscr{M}(\beta_*)$ which agrees with each A_n on its domain.

We claim that $\operatorname{tr}(A(t)) = 1$ for every $t \in (0,1]$. Fixing such a t, we have $t \leqslant t_1$ so that

$$e_1 = A(t_1) = \beta_{t_1-t*}(A(t)).$$

Hence

$$\operatorname{tr}(A(t)) \geqslant \operatorname{tr}(\beta_{t_1-t*}(A(t))) = \operatorname{tr}(e_1) = 1.$$

The opposite inequality $\operatorname{tr}(A(t)) \leqslant 1$ was pointed out already.

Finally, $A(t_n) = e_n$ converges weakly to zero by the choice of $\{e_n\}$.

Let ρ_A be the singular element of $C^*(E)^*$ determined by A. ρ_A is positive because $A(t) \geqslant 0$ for all $t > 0$, and $\|\rho_A\| = 1$ because $\operatorname{tr} A(t) = 1$ near $t = 0$. Hence ρ_A is a singular state.

We claim that ρ_A cannot have the form

$$\rho_A(x) = \operatorname{tr}(T\lambda(x)), \quad x \in C^*(E) \tag{4.4}$$

for any positive trace-class operator T on $L^2(E)$. Indeed, if (4.4) were to hold then for $t > 0$ we would have

$$\operatorname{tr}(\beta_{t*}(T)\lambda(x)) = \beta_t^*\rho_A(x), \quad x \in C^*(E),$$

while by Theorem 2.4,

$$\beta_t^*\rho_A(x) = \operatorname{tr}(A(t)\lambda(x)).$$

It follows that $\operatorname{tr}((A(t) - \beta_{t*}(T))B) = 0$ for all B in the irreducible C^*-algebra $\lambda(C^*(E))$, and hence $A(t) = \beta_{t*}(T)$, $t > 0$. By strong continuity of the semigroup β_* we conclude that $\operatorname{tr}|A(t) - T| \to 0$ as $t \to 0$; while on the other hand, $A(t_n) \to 0$ in the weak operator topology. It follows that $T = 0$ which is obviously absurd. ▨

REMARK. Let E be a non-trivial product system. In view of the correspondence between representations of $C^*(E)$ and representations of E, we conclude that there is a singular representation $\varphi: E \to \mathscr{B}(H)$ which is not unitarily equivalent to a direct sum of copies of the regular representation of E on $L^2(E)$.

5. LOCALLY NORMAL STATES

We conclude by giving a description of $\mathscr{M}(\beta_*)$ as the space of all locally normal linear functionals on a certain C^*-algebraic inductive limit of type I_∞ von Neumann algebras. Taken together with the results of §§ 2—3, this provides a realization of the singular part of $C^*(E)^*$ which is tied rather closely to the regular representation (cf. Corollary 5.2).

Let E be a product system, which will be fixed throughout this section. Let M denote the von Neumann algebra $\mathscr{B}(L^2(E))$ of all bounded operators on $L^2(E)$, and let $\beta = \{\beta_t : t \geqslant 0\}$ be the semigroup of $*$-endomorphisms of M determined by the right regular anti-representation $\mathrm{r}: E \to \mathscr{B}(H)$ as in Section 2.

For every $t > 0$, $\beta_t(M)$ is a (degenerate) type I_∞ subfactor of M whose unit is $\beta_t(1) = P_t \neq 1$, and we have $\beta_t(M) \subseteq \beta_s(M)$ for $s \leqslant t$. Define

$$\mathscr{A} = \overline{\bigcup_{t>0} \beta_t(M)}^{\|\cdot\|}.$$

$\mathscr{A}$ is an irreducible unitless C^*-algebra which is the C^*-algebraic inductive limit of the increasing sequence of von Neumann algebras $M_n = \beta_{1/n}(M)$, $n = 1, 2, \ldots$. The inclusion of M_n into M_{n+1} is isometric and normal, but not unit-preserving.

A bounded linear functional ρ on $\mathscr{A}$ is called *locally normal* if the restriction of ρ to $\beta_t(M)$ is normal for every $t > 0$. $\mathscr{A}_*$ will denote the Banach space of all locally normal elements of $\mathscr{A}^*$. $\mathscr{A}_*$ is an *order ideal* in the sense that if ρ_1 and ρ_2 are positive linear functionals on $\mathscr{A}$ satisfying $\rho_1 \leqslant \rho_2$ and ρ_2 is locally normal, then ρ_1 is locally normal. Finally, we say that an element $\rho \in \mathscr{A}^*$ is *normal* if there is a trace-class operator A on $\mathscr{B}(H)$ such that $\rho(B) = \operatorname{tr}(AB)$, $B \in \mathscr{A}$. A is necessarily unique, whenever it exists.

The following result shows that the Banach space $\mathscr{M}(\beta_*)$ introduced in Section 2 can be identified with $\mathscr{A}_*$.

PROPOSITION 5.1. *For every element $A \in \mathscr{M}(\beta_*)$ there is a unique bounded linear functional ω_A on $\mathscr{A}$ satisfying*

$$\omega_A(\beta_t(B)) = \operatorname{tr}(A(t)B), \qquad B \in M,\ t > 0.$$

ω_A is locally normal and $A \mapsto \omega_A$ is an isometric order isomorphism of $\mathscr{M}(\beta_*)$ onto $\mathscr{A}_*$. ω_A is normal iff the limit

$$A(0+) = \lim_{t\to 0+} A(t)$$

exists relative to the trace-norm.

Proof. Choose $A \in \mathscr{M}(\beta_*)$ and fix $t > 0$. Since $B \mapsto \beta_t(B)$ is an isometric $*$-isomorphism of M onto $\beta_t(M)$, it follows that there is a unique bounded linear functional ω_t on $\beta_t(M)$ satisfying

$$\omega_t(\beta_t(B)) = \operatorname{tr}(A(t)B), \qquad B \in M.$$

ω_t is normal because it is the composition of the normal map $\beta_t^{-1}\colon \beta_t(M) \to M$ and the normal linear functional $B \in M \mapsto \operatorname{tr}(A(t)B)$.

Clearly $\|\omega_t\| = \operatorname{tr}|A(t)| \leqslant \|A\|$, and we claim that if $0 < s < t$ then $\omega_s|\beta_t(M) = \omega_t$. Indeed, every element of $\beta_t(M)$ has the form $\beta_t(B)$ for some $B \in M$, hence

$$\omega_s(\beta_t(B)) = \omega_s(\beta_s(\beta_{t-s}(B))) = \operatorname{tr}(A(s)\beta_{t-s}(B)) =$$

$$= \operatorname{tr}(\beta_{t-s*}(A(s))B) = \operatorname{tr}(A(t)B) = \omega_t(\beta_t(B)).$$

Thus, there is a unique element $\omega_A \in \mathscr{A}^*$ satisfying $\omega|\beta_t(M) = \omega_t$ for $t > 0$, and we have $\|\omega_A\| \leqslant \|A\|$.

ω_A is obviously locally normal, and since $\bigcup_{t>0} \beta_t(M)$ is norm-dense in $\mathscr{A}$ we have

$$\|\omega_A\| = \sup_{t>0}\|\omega_t\| = \sup_{t>0} \operatorname{tr}|A(t)| = \|A\|.$$

Finally, $\omega_A \geqslant 0$ iff $\omega_t \geqslant 0$ for every $t > 0$ iff $A(t) \geqslant 0$ for every $t > 0$ iff $A \geqslant 0$.

Conversely, let $\omega \in \mathscr{A}_*$ and fix $t > 0$. Since $\omega|\beta_t(M)$ is normal,

$$B \mapsto \omega(\beta_t(B))$$

defines a normal linear functional on M. Hence there is a unique trace-class operator $A(t) \in M$ satisfying

$$\omega(\beta_t(B)) = \operatorname{tr}(A(t)B), \quad B \in M.$$

We claim that $\beta_{*s}(A(t)) = A(s + t)$ for all $s \geqslant 0$, $t > 0$. To see that, choose $B \in M$ and write

$$\operatorname{tr}(\beta_{*s}(A(t))B) = \operatorname{tr}(A(t)\beta_s(B)) =$$

$$= \omega(\beta_t(\beta_s(B))) = \omega(\beta_{s+t}(B)) = \operatorname{tr}(A(s + t)B),$$

and the assertion follows. A is a bounded function because

$$\operatorname{tr}|A(t)| = \sup_{\|B\|\leqslant 1} |\operatorname{tr}(A(t)B)| = \sup_{\|B\|\leqslant 1} |\omega(\beta_t(B))| \leqslant \|\omega\|.$$

Hence $A \in \mathscr{M}(\beta_*)$ and $\omega = \omega_A$.

Finally, we show that ω_A is normal iff the trace-norm limit $A(0+) = \lim_{t\to 0} A(t)$ exists. Suppose first that $A(0+)$ exists. Then $A(t) = \beta_{t*}(A(0+))$ for every $t > 0$, and hence for $B \in M$ we have

$$\omega_A(\beta_t(B)) = \operatorname{tr}(A(t)B) =$$

$$= \operatorname{tr}(\beta_{t*}(A(0+)B)) = \operatorname{tr}(A(0+)\beta_t(B)).$$

This implies that for the normal functional on M defined by $\omega_0(T) = \operatorname{tr}(A(0+)T)$, we have

$$\omega_A|\beta_t(M) = \omega_0|\beta_t(M),$$

for every $t > 0$. Hence $\omega_A = \omega_0$ on $\mathscr{A}$, proving that ω_A is normal.

Conversely, if ω_A is normal then there is a trace-class operator $A_0 \in M$ such that $\omega_A(B) = \text{tr}(A_0 B)$, $B \in \mathscr{A}$. For each $t > 0$ and $B \in M$ we have

$$\text{tr}(A(t)B) = \omega_A(\beta_t(B)) = \text{tr}(A_0\beta_t(B)) = \text{tr}(\beta_{t*}(A_0)B),$$

which implies that $A(t) = \beta_{t*}(A_0)$. Hence

$$\lim_{t\to 0} \text{tr}|A(t) - A_0| = \lim_{t\to 0} \text{tr}|\beta_{t*}(A_0) - A_0| = 0,$$

by strong continuity of the semigroup β_*.

COROLLARY 5.2. *$\lambda(C^*(E))$ is contained in $\mathscr{A}$. Moreover, letting $A \in \mathscr{M}(\beta_*) \mapsto \rho_A \in \mathscr{S}$ and $A \in \mathscr{M}(\beta_*) \mapsto \omega_A \in \mathscr{A}_*$ be the isomorphisms defined by Theorem* 2.4 *and Proposition* 5.1, *then we have*

$$\rho_A = \omega_A \circ \lambda, \quad A \in \mathscr{M}(\beta_*).$$

In particular, a bounded linear functional ρ on $C^*(E)$ is singular iff it has the form $\rho = \omega \circ \lambda$ for some $\omega \in \mathscr{A}_*$.

Proof. We show first that $\mathscr{A}$ contains $\lambda(C^*(E))$. Since $C^*(E)$ is spanned by elements of the form $f \otimes \bar{g}$ with f, g compactly supported functions in $L^2(E)$ and since $\mathscr{A}$ is norm-closed, it suffices to show that $\lambda(f \otimes \bar{g}) \in \mathscr{A}$ for all $f, g \in L^2(E)$ of compact support. Fix f and g. Then by ([4], Proposition 6.4), for every $\xi, \eta \in L^2(E)$ the integral

$$\int_0^\infty \langle \beta_t(f \otimes \bar{g})\xi, \eta \rangle \mathrm{d}t$$

is absolutely convergent and agrees with $\langle \lambda(f \otimes \bar{g})\xi, \eta \rangle$. Moreover, for every $t > 0$ we have

$$\langle (\lambda(f \otimes \bar{g}) - \beta_t(\lambda(f \otimes \bar{g})))\xi, \eta \rangle = \int_0^t \langle \beta_s(f \otimes \bar{g})\xi, \eta \rangle \mathrm{d}s.$$

Fix $\varepsilon > 0$ and choose t small enough so that

$$\int_0^t \|\beta_s(f \otimes \bar{g})\| \mathrm{d}s = \int_0^t \|f \otimes \bar{g}\| \mathrm{d}s \leqslant \varepsilon.$$

The preceding expression implies that

$$\|\lambda(f \otimes \bar{g}) - \beta_t(\lambda(f \otimes \bar{g}))\| \leqslant \varepsilon.$$

and since $\beta_t(\lambda(f\otimes\bar{g})) \in \beta_t(M) \subseteq \mathscr{A}$, we see that the distance from $\lambda(f\otimes\bar{g})$ to $\mathscr{A}$ is at most ε. Since ε is arbitrary and $\mathscr{A}$ is norm-closed, we conclude that $\lambda(f\otimes\bar{g})\in\mathscr{A}$.

Notice that the preceding argument implies that

$$\lim_{t\to 0}\|\lambda(x) - \beta_t(\lambda(x))\| = 0$$

for every $x \in C^*(E)$. We also point out that, for $t > 0$, the operators $\beta_t(\lambda(x))$ belong to A but they do not belong to $\lambda(C^*(E))$.

Now fix $A \in \mathscr{M}(\beta_*)$. It remains to show that $\rho_A = \omega_A \circ \lambda$. Again, it is enough to prove that

$$\rho_A(f\otimes\bar{g}) = \omega_A(\lambda(f\otimes\bar{g})),$$

for $f, g \in L^1(E)\cap L^2(E)$. Fixing f and g, we see from the preceding paragraph that

$$\omega_A(\lambda(f\otimes\bar{g})) = \lim_{t\to 0}\omega_A(\beta_t(\lambda(f\otimes\bar{g}))).$$

Now for $t > 0$ and $B \in M$ we have

$$\omega_A(\beta_t(B)) = \mathrm{tr}(A(t)B).$$

Hence

$$\omega_A(\beta_t(\lambda(f\otimes\bar{g}))) = \mathrm{tr}(A(t)\lambda(f\otimes\bar{g})) =$$

$$= \int_0^\infty \mathrm{tr}(A(t)\beta_s(f\otimes\bar{g}))\mathrm{d}s.$$

But for $s > 0$ we have

$$\mathrm{tr}(A(t)\beta_s(f\otimes\bar{g})) = \mathrm{tr}(\beta_{s*}(A(t))(f\otimes\bar{g})) = \langle A(s+t)f, g\rangle.$$

Hence the integral on the right is

$$\int_0^\infty \langle A(s+t)f, g\rangle\mathrm{d}s = \int_t^\infty \langle A(x)f, g\rangle\mathrm{d}x.$$

We conclude that

$$\omega_A(\lambda(f\otimes\bar{g})) = \lim_{t\to 0}\int_t^\infty \langle A(x)f, g\rangle\mathrm{d}x = \int_0^\infty \langle A(x)f, g\rangle\mathrm{d}x,$$

and the latter is $\rho_A(f\otimes\bar{g})$.

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