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JENSEN'S INEQUALITY IN SEMI-FINITE VON NEUMANN ALGEBRAS

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0. INTRODUCTION

Operator inequalities have been important tools in the study of operator algebras. For example, the following operator inequality due to Hansen, [9], is very useful:

$$a^*f(x)a \leqslant f(a^*xa)$$

where x is a positive operator, a is a contraction, and f is an operator concave function on the interval $[0, \infty)$ with $f(0) = 0$, [3]. Here operator concavity is quite a strong condition. (For example it automatically implies analyticity.)

In many applications, however, operator inequalities such as the Hansen inequality appear only inside a trace. Even under such circumstances, the theory of operator concave (or convex or monotone) functions has been mainly used in the literature on operator algebras. On the other hand, our experience tells (or at least we hope) that inside a trace operators behave "like numbers". Therefore, it may be interesting to prove these inequalities without using the theory of operator concave functions. More importantly, it would allow us to obtain these inequalities assuming only the usual concavity of involved functions.

The main purpose of the article is to prove

$$\tau(a^*f(x)a) \leqslant \tau(f(a^*xa)) \tag{1}$$

for any trace τ on a semi-finite von Neumann algebra and any continuous concave function f on the interval $[0, \infty)$ with $f(0) = 0$. (The assumptions with regard to the existence of the traces in (1) are analyzed in a following section.) From this main result (Theorem 11), many related inequalities involving concave functions are derived (without assuming their operator concavity). Our main technical tools are the notion of spectral dominance (Definition 2) and the classical Jensen inequality. A result analogous to Theorem 11 (but very easy) for monotone rather than concave functions is given in Lemma 4. We also give a simple proof of the equality $\tau(xy) = \tau(yx)$ for operators x, y satisfying $\tau(|xy|) < +\infty$ and $\tau(|yx|) < +\infty$.

As explained in [2], (see also the paragraph after Theorem 14), our main result (1) can be regarded as "Jensen's inequality for traces". When the algebra in question is a factor of type I_∞, (1) as well as some other inequalities in the present article are not so hard to prove. The reason is that for such an algebra the trace can be expressed as $\sum_{i=1}^{\infty} (\cdot\,\xi_i \mid \xi_i)$ with the basis $\{\xi_i\}$ consisting of all the eigenvectors of a suitably chosen (positive compact) operator. Indeed, based on this fact (1) was first obtained by Berezin [2], under the additional assumption that a is a projection and f is increasing.

1. NOTATIONS AND PRELIMINARIES

In this section, we will fix some notations, introduce certain terminologies and prove their basic properties.

Throughout the article, let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a faithful normal semi-finite trace τ. (Our basic reference on operator algebras is [17].) By operators, we shall always mean (not necessarily bounded) τ-measurable operators in the sense of [12] (see also [15]). It is well known that though unbounded, (τ-measurable) operators can be handled like bounded operators. For a self-adjoint operator h and a Borel subset I in $\mathbf{R}$, $e_I(h)$ will denote the spectral projection of h corresponding to I (so that $e_I(h)$ is a projection in $\mathcal{M}$).

Throughout, the letter f will be used to indicate a continuous concave ($\lambda f(s) + (1 - \lambda)f(t) \leqslant f(\lambda s + (1 - \lambda)t)$; $0 < \lambda < 1$, $t, s \geqslant 0$) function on the interval $[0, \infty)$ with $f(0) = 0$ while the letter a will be used to indicate a contraction ($\|a\| \leqslant 1$) in the algebra.

We note that f is one of the following types:

(2) f is monotone increasing ($f(s) \leqslant f(t)$ if $0 \leqslant s < t$) and $f \geqslant 0$.

(3) f is monotone descreasing and $f \leqslant 0$.

(4) f is monotone increasing on $[0, t_1]$ for some $t_1 > 0$ while f is monotone decreasing on $[t_1, \infty)$ and not a constant function there. In this case, there exists a number t_2, $t_2 > t_1$, satisfying $f(t_2) = 0$. (Thus, $f \geqslant 0$ on $[0, t_2]$ and $f \leqslant 0$ on $[t_2, \infty)$.)

We begin with the following easy consequence of the classical Jensen inequality that will be repeatedly used:

LEMMA 1. *For a vector ξ in $\mathcal{H}$ with $\|\xi\| \leqslant 1$, and a positive operator x, we have*

$$(a^* f(x) a\xi \mid \xi) \leqslant f((a^* x a\xi \mid \xi)). \tag{5}$$

Proof. We introduce the following two measures on $[0, \infty)$:

$$\begin{cases} \mu(I) = \|e_I(x)a\xi\|^2, \\ \nu = \mu + (1 - \|a\xi\|^2)\delta_0. \end{cases}$$

We observe that ν is a probability measure and the functions $f(t)$ and t vanish at 0 where the support of δ_0 is. We thus compute

$$(a^*f(x)a\xi \mid \xi) = \int_0^\infty f(t)\,d\mu(t) = \int_0^\infty f(t)\,d\nu(t) \leqslant$$

$$\leqslant f\left(\int_0^\infty t\,d\nu(t)\right) = \qquad \text{(Jensen's inequality)}$$

$$= f\left(\int_0^\infty t\,d\mu(t)\right) = f((a^*xa\xi \mid \xi)).$$

Q.E.D.

Even if x is unbounded, (5) remains valid. Here, $(a^*xa\xi \mid \xi)$ and $(a^*f(x)a\xi \mid \xi)$ should be understood as $\int_0^\infty t\,d\mu(t) \in [0, \infty]$ and $\int_0^\infty f_+(t)d\mu(t) - \int_0^\infty f_-(t)d\mu(t) \in [-\infty, \infty]$ respectively. (If f is of type (4), then $\int_0^\infty f_+(t)d\mu(t) < +\infty$ so that $(a^*f(x)a\xi \mid \xi)$ is well-defined.) Since $f(t)$ is monotone for t large enough, $f((a^*xa\xi \mid \xi))$ can be understood as $\lim_{t\to\infty} f(t)$ if $(a^*xa\xi \mid \xi) = +\infty$. Approximating x by the bounded operators $x_n = \int_0^n t\,de_{[0,t]}(x)$, $n = 1, 2, \ldots$, for which the arguments in the above proof are valid, one gets (5) for any operator x based on the monotone convergence theorem.

DEFINITION 2. (cf. [1], p. 170). For positive operators x, y, we say that y *spectrally dominates* x, denoted by $x \lesssim y$, if for each $s > 0$ we have

$$e_{(s,\infty)}(x) \lesssim e_{(s,\infty)}(y),$$

that is, there exists a projection in $\mathcal{M}$ equivalent to $e_{(s,\infty)}(x)$ and majorized by $e_{(s,\infty)}(y)$ (p. 44 in [13]).

The next result is known; however, for the sake of completeness, we give its proof.

LEMMA 3. *Let x, y be positive operators.*
(i) $x \lesssim y$ if $x \leqslant y$ *(in the usual sense).*
(ii) $a^*xa \lesssim x$.
(iii) $\tau(x) \leqslant \tau(y)$ *if* $x \lesssim y$.

Proof. (i) If $\xi \in e_{(s,\infty)}(x)\mathscr{H} \cap e_{[0,s]}(y)\mathscr{H}$, $\|\xi\| = 1$, then we get

$$+\infty \geqslant (x\xi \mid \xi) > s \geqslant (y\xi \mid \xi) \geqslant (x\xi \mid \xi).$$

Here, $(x\xi \mid \xi)$ and $(y\xi \mid \xi)$ should be understood as forms; that is, $\|x^{1/2}\xi\|^2$ and $\|y^{1/2}\xi\|^2$. The operator inequality $x \leqslant y$ means that the closure of $y - x$ is positive self-adjoint. In particular, one gets $\|x^{1/2}\zeta\|^2 = (x\zeta \mid \zeta) \leqslant (y\zeta \mid \zeta) = \|y^{1/2}\zeta\|^2$ for each $\zeta \in \mathscr{D}(x) \cap \mathscr{D}(y)$. Since $\mathscr{D}(x) \cap \mathscr{D}(y)$ is a core for $y^{1/2}$ ([12], [15]), we get $\|x^{1/2}\zeta\|^2 \leqslant$ $\leqslant \|y^{1/2}\zeta\|^2 \leqslant +\infty$ for any vector ζ. Indeed, if $\|y^{1/2}\zeta\| < +\infty$, choosing a sequence $\{\zeta_n\}$ in $\mathscr{D}(x) \cap \mathscr{D}(y)$ such that $\zeta_n \to \zeta$ and $y^{1/2}\zeta_n \to y^{1/2}\zeta$ we get

$$\|x^{1/2}\zeta\|^2 \leqslant \liminf_{n\to\infty}\|x^{1/2}\zeta_n\|^2 \leqslant \liminf_{n\to\infty}\|y^{1/2}\zeta_n\|^2 = \|y^{1/2}\zeta\|^2$$

due to the lower semi-continuity of the map: $\eta \in \mathscr{H} \to \|x^{1/2}\eta\|^2 \in [0, \infty]$.) We thus have $e_{(s,\infty)}(x) \wedge e_{[0,s]}(y) = 0$ and

$$e_{(s,\infty)}(x) = e_{(s,\infty)}(x) - e_{(s,\infty)}(x) \wedge e_{[0,s]}(y) \sim$$

$$\sim e_{(s,\infty)}(x) \vee e_{[0,s]}(y) - e_{[0,s]}(y) \leqslant$$

$$\leqslant 1 - e_{[0,s]}(y) = e_{(s,\infty)}(y).$$

(ii) It is well-known that

$$e_{(s,\infty)}(a^*xa) \sim e_{(s,\infty)}(x^{1/2}aa^*x^{1/2}),$$

the equivalence being given by the phase part of the polar decomposition of $x^{1/2}a$. Thus, (ii) follows from (i) and the obvious fact $x^{1/2}aa^*x^{1/2} \leqslant x$.

(iii) The assumption implies that

$$\tau(e_{(s,\infty)}(x)) \leqslant \tau(e_{(s,\infty)})(y)), \quad s > 0.$$

It is then elementary to show that

$$\tau(g(x)) \leqslant \tau(g(y)) \tag{6}$$

for any increasing step function g on $[0, \infty)$ with $g(0) = 0$ which is continuous from the left. Approximating the function t from below by such g's the result follows from (6) and the monotone convergence theorem (applied to the measures $\tau(e_I(x))$ and $\tau(e_I(y))$). Q.E.D.

Actually, the argument in the proof of the above (iii) shows the following:

LEMMA 4. *If positive operators x, y satisfy $x \lesssim y$, then we have*

$$\tau(g(x)) \leqslant \tau(g(y))$$

for any continuous increasing function g on the interval $[0, \infty)$ *with* $g(0) = 0$.

REMARK 5. The infimum of numbers $s \geqslant 0$ satisfying $\tau(e_{(s,\infty)}(|x|)) \leqslant t$ is denoted by $\mu_t(x)$, $t > 0$, and called the "*generalized s-number*" of x, [4]. (See also [8], the appendix to [11], or p. 133 in [13].) Although μ_t will not be used in this article, for positive operators x, y one can prove the equivalence of the following three statements:

(i) $\mu_t(x) \leqslant \mu_t(y)$, $t > 0$,
(ii) $\tau(e_{(s,\infty)}(x)) \leqslant \tau(e_{(s,\infty)}(y))$, $s > 0$,
(iii) $\tau(g(x)) \leqslant \tau(g(y))$ for any function g described in Lemma 4.

When $\mathcal{M}$ is a factor, these are obviously equivalent to $x \lesssim y$.

LEMMA 6. (Minimality of the Jordan decomposition). *Let h be a self-adjoint operator with the Jordan decomposition $h = h_+ - h_-$. If positive operators h_1, h_2 satisfy $h = h_1 - h_2$, then h_1 and h_2 spectrally dominate h_+ and h_- respectively.*

Proof. We will prove $h_+ \lesssim h_1$. We notice that $e_{(s,\infty)}(h) = e_{(s,\infty)}(h_+)$ for each $s > 0$. Since $h_1 \geqslant h$, the argument in the proof of Lemma 3, (i) shows that

$$e_{(s,\infty)}(h) \lesssim e_{(s,\infty)}(h_1).$$

Q.E.D.

DEFINITION 7. Let h be a self-adjoint operator with the Jordan decomposition $h = h_+ - h_-$. We say that *$\tau(h)$ is defined* if $\tau(h_+) < +\infty$ or $\tau(h_-) < +\infty$. In this case, we set

$$\tau(h) = \tau(h_+) - \tau(h_-) \in [-\infty, \infty].$$

LEMMA 8. *Let h be a self-adjoint operator.*

(i) *$(a^*ha)_+ \lesssim a^*h_+a$ and $(a^*ha)_- \lesssim a^*h_-a$.*

(ii) *If $\tau(h)$ is defined, then so is $\tau(a^*ha)$ and*

$$\tau(a^*ha) = \tau(a^*h_+a) - \tau(a^*h_-a).$$

(iii) *If $\tau(h)$ is defined and three projections p_i, $i = 1, 2, 3$, in $\mathcal{M}$ satisfy $\sum_{i=1}^{3} p_i = 1$ then we have*

$$\tau(h) = \sum_{i=1}^{3} \tau(p_i h p_i).$$

(In the equalities in (ii), (iii), "$\infty - \infty$" never occurs.)

Proof. (i) follows from Lemma 6 and

$$a^*ha = a^*h_+a - a^*h_-a.$$

(ii) Considering $-h$, if necessary, we may and do assume $\tau(h_+) < +\infty$ (in (iii) below as well). The above (i) and Lemma 3, (ii), (iii), imply that

$$\tau((a^*ha)_+) \leqslant \tau(a^*h_+a) \leqslant \tau(h_+) < \infty$$

so that $\tau(a^*ha)$ is defined. Because

$$(a^*ha)_+ - (a^*ha)_- \ (=a^*ha) = a^*h_+a - a^*h_-a,$$

we get

$$\tau((a^*ha)_+) + \tau(a^*h_-a) = \tau(a^*h_+a) + \tau((a^*ha)_-).$$

Since $\tau(a^*h_+a) < +\infty$ and $((a^*ha)_+) < +\infty$, we get the equality in (ii).

(iii) Setting $a = p_i$ in (ii), one gets

$$\tau(p_i h p_i) = \tau(p_i h_+ p_i) - \tau(p_i h_- p_i) =$$

$$= \tau(h_+^{1/2} p_i h_+^{1/2}) - \tau(h_-^{1/2} p_i h_-^{1/2}).$$

Summing up over $i = 1, 2, 3$, we get (iii). Q.E.D.

LEMMA 9. *Let h, k be self-adjoint operators. If both $\tau(h)$ and $\tau(k)$ are defined and $\tau(h) + \tau(k)$ is well-defined (without having $\infty - \infty$), then $\tau(h + k)$ is defined and*

$$\tau(h + k) = \tau(h) + \tau(k).$$

Its proof is straightforward ($\tau((h + k)_+) \leqslant \tau(h_+) + \tau(k_+)$ is used) so that details are left to the reader.

2. THE MAIN THEOREM

In this section, we will prove the inequality (1) of the introduction.

LEMMA 10. *Let I be an interval (in $[0, \infty)$) on which f is monotone (either increasing or decreasing). We set $p = e_I(a^*xa)$. Here x is a positive operator.*

(i) *If $f \geqslant 0$ on I, then $pf(a^*xa)p$ spectrally dominates $(pa^*f(x)ap)_+$.*

(ii) *If $f \leqslant 0$ on I, then $pa^*f(x)ap \leqslant 0$ and $-pa^*f(x)ap$ spectrally dominates $-pf(a^*xa)p$.*

Proof. The projection p being a spectral projection of a^*xa, we get

$$pf(a^*xa)p = f(pa^*xap),$$

and it is positive or negative depending upon $f \geqslant 0$ or $f \leqslant 0$ on I.

(i) For each $s > 0$, we set

$$\begin{cases} I_1 = \{t \in I\,;\ f(t) > s\}, \\ I_2 = \{t \in I\,;\ f(t) \leqslant s\}. \end{cases}$$

The function f being monotone on I, I is the disjoint union of its two subintervals I_1, I_2. We introduce the following three projections:

$$\begin{cases} q_1 = e_{I_1}((pa^*xap) \mid p\mathscr{H}), \\ q_2 = e_{I_2}((pa^*xap) \mid p\mathscr{H}), \\ q = e_{(s,\infty)}(pa^*f(x)ap) = e_{(s,\infty)}((pa^*f(x)ap)_+). \end{cases}$$

We observe that

$$\begin{cases} q \leqslant p, \ q_1 + q_2 = p, \\ q_1 = e_{I_1}(pa^*xap) = e_{(s,\infty)}(pf(a^*xa)p). \end{cases}$$

Here, the last fact follows from $0 \notin I_1$.

If $\xi \in q_2\mathscr{H}$, $\|\xi\| = 1$, then we have

$$s \geqslant f((pa^*xap\xi \mid \xi)) \geqslant (pa^*f(x)ap\xi \mid \xi)$$

thanks to Lemma 1 applied to the contraction ap. On the other hand, if $\xi \in q\mathscr{H}$, $\|\xi\| = 1$, then we have

$$(pa^*f(x)ap\xi \mid \xi) > s.$$

We thus conclude that $q \wedge q_2 = 0$ and

$$q = q - q \wedge q_2 \sim q \vee q_2 - q_2 \leqslant p - q_2 = q_1.$$

(ii) Lemma 1 implies that

$$(pa^*f(x)ap\xi \mid \xi) \leqslant f((pa^*xap\xi \mid \xi)) \leqslant 0$$

for each $\xi \in p\mathscr{H}$, $\|\xi\| \leqslant 1$, so that we get

$$pa^*f(x)ap \leqslant 0.$$

For each $s > 0$, we set

$$\begin{cases} I_1 = \{t \in I\,;\, f(t) \geqslant -s\}, \\ I_2 = \{t \in I\,;\, f(t) < -s\}, \\ q_1 = e_{I_1}((pa^*xap) \mid p\mathscr{H}), \\ q_2 = e_{I_2}((pa^*xap) \mid p\mathscr{H}), \\ q = p \wedge e_{[-s,0]}(pa^*f(x)ap) = p \wedge e_{[0,s]}(-pa^*f(x)ap). \end{cases}$$

As before, we observe that

$$\begin{cases} q \leqslant p,\ q_1 + q_2 = p, \\ q_2 = e_{I_2}(pa^*xap) = e_{(s,\infty)}(-pf(a^*xa)p). \end{cases}$$

The same arguments as in (i) imply that $q \wedge q_2 = 0$ and

$$q_2 = q_2 - q \wedge q_2 \sim q_2 \vee q - q \leqslant p - q.$$

However, $p - q$ is exactly $e_{(s,\infty)}(-pa^*f(x)ap)$. Q.E.D.

THEOREM 11. *For an arbitrary continuous concave function f on the interval* $0, \infty)$ *with* $f(0) = 0$ *and a faithful normal semi-finite trace τ on a von Neumann algebra, we have*

$$\tau(a^*f(x)a) \leqslant \tau(f(a^*xa))$$

whenever both sides are defined. Here, x is a positive τ-measurable operator and a is a contraction in the algebra. Furthermore, $\tau(f(a^*xa))$ *is defined if* $\tau(a^*f(x)a)$ *is defined and not equal to* $-\infty$.

Proof. We assume that f is of type (4) described in Section 1 and introduce the following three intervals:

$$I_1 = [0, t_1),\quad I_2 = [t_1, t_2),\quad I_3 = [t_2, \infty).$$

Note that if f is of type (2) (resp. (3)) one can consider $I_1 = [0, \infty)$ and $I_2 = I_3 = \emptyset$ (resp. $I_1 = I_2 = \emptyset$ and $I_3 = [0, \infty)$). We then consider the three projections $p_i = e_{I_i}(a^*xa)$, $i = 1, 2, 3$. By the assumption and Lemma 8, (ii), all of $\tau(p_i a^* f(x) a p_i)$ and $\tau(p_i f(a^*xa) p_i)$ are defined.

If $i = 1, 2$, then it follows from Lemma 10, (i), and Lemma 3, (iii), that

$$\tau(p_i f(a^*xa)p_i) \geqslant \tau((p_i a^* f(x) a p_i)_+) \geqslant$$

$$\geqslant \tau(p_i a^* f(x) a p_i).$$

On the other hand, if $i = 3$, then Lemma 10, (ii), and Lemma 3, (iii), imply that

$$\text{(7)} \qquad \begin{cases} p_3 a^* f(x) a p_3 \leqslant 0 \text{ (and } p_3 f(a^*xa) p_3 \leqslant 0), \\ \tau(-p_3 f(a^*xa)p_3) \leqslant \tau(-p_3 a^* f(x) a p_3). \end{cases}$$

Therefore, the desired inequality follows from Lemma 8, (iii).

To show the second statement in the theorem, it suffices to prove that $\tau((a^*f(x)a)_-) < +\infty$ implies $\tau(f(a^*xa)_-) < +\infty$. However, this follows from the following estimates based on (7) and Lemma 8, (i):

$$\tau(f(a^*xa)_-) = \tau(-p_3 f(a^*xa)p_3) \leqslant$$

$$\leqslant \tau(-p_3 a^* f(x) a p_3) = \tau((p_3 a^* f(x) a p_3)_-) \leqslant$$

$$\leqslant \tau(p_3 (a^* f(x) a)_- p_3) = \tau((a^* f(x) a)_-^{1/2} p_3 (a^* f(x) a)_-^{1/2}) \leqslant$$

$$\leqslant \tau((a^* f(x) a)_-). \qquad \text{Q.E.D.}$$

We remark that the proof can easily be adapted to cover the case where the domain of f is any interval containing 0, not necessarily $[0, \infty)$.

3. REMARKS ON THE MAIN THEOREM

In this section, we will further analyze the hypotheses of Theorem 11.

Both of $\tau(a^*f(x)a)$ and $\tau(f(a^*xa))$ are defined if one of the following conditions is fulfilled:

(i) $\tau(1) < +\infty$,

(ii) f is increasing. (In this case, we get $a^*f(x)a \lesssim f(a^*xa)$ by Lemma 10, (i).)

(iii) f is decreasing. (In this case, we get $-f(a^*xa) \lesssim -a^*f(x)a$ by Lemma 10, (ii).)

The next result tells that for a very wide class of operators (see Proposition 13 and the paragraph right after this) the inequality in Theorem 11 holds in a very strong sense.

PROPOSITION 12. *In Theorem* 11, *we further assume* $\tau(e_{(\varepsilon,\infty)}(x)) < +\infty$ *for all* $\varepsilon > 0$. *Then* $\tau(a^*f(x)a)$ *is defined if* $\tau(f(a^*xa))$ *is defined and not equal to* $+\infty$.

Proof. It suffices to show that $\tau(f(a^*xa)_+) < +\infty$ implies $\tau((a^*f(x)a)_+) < +\infty$ (assuming that f is of type (4)). We note that $\tau(e_{(\varepsilon,\infty)}(a^*xa)) < +\infty$ for all $\varepsilon > 0$ because of Lemma 3, (ii). We set

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } 0 \leqslant t \leqslant t_1, \\ f(t_1) & \text{if } t > t_1, \end{cases}$$

and observe that $\tilde{f}$ is of type (2). Theorem 11 applied to $\tilde{f}$, the inequality $\tilde{f} \geqslant f_+$, and Lemma 8, (i), imply that

$$\tau(\tilde{f}(a^*xa)) \geqslant \tau(a^*\tilde{f}(x)a) \geqslant$$

$$\geqslant \tau(a^*f(x)_+a) \geqslant \tau((a^*f(x)a)_+).$$

Thus, the proposition follows from the obvious estimate

$$\tau(f(a^*xa)_+) + f(t_1)\tau(e_{(t_1,\infty)}(a^*xa)) \geqslant \tau(\tilde{f}(a^*xa)).$$

Q.E.D.

From the above proof it is clear that $\tau(e_{(t_1,\infty)}(x)) < +\infty$ (or even, $\tau(e_{(t_1,\infty)}(a^*xa)) < +\infty$) is sufficient to obtain Proposition 12. We now reformulate the extra assumption in Proposition 12 as follows:

PROPOSITION 13. *For an operator* x, *the following three conditions are equivalent:*

(i) $\tau(e_{(\varepsilon,\infty)}(|x|)) < +\infty$ *for all* $\varepsilon > 0$,

(ii) $\lim_{t\to\infty}\mu_t(x) = 0$ *(see Remark* 5*)*,

(iii) *there exists a sequence of (bounded) "finite rank" (relative to* τ*) operators converging to* x *in the measure topology (*[12]*)*.

This result will not be used later, hence its proof is omitted. We just remark that all operators in noncommutative L^p-spaces $L^p(\mathcal{M};\tau)$, $0 < p < \infty$, [12], [15]. (or even in Lorentz type spaces) satisfy the conditions in Proposition 13.

In the rest of the section we will show that the hypothesis of Proposition 12 cannot be omitted, even if x is bounded. Indeed, we will construct f, $\tilde{x} \geqslant 0$, $\tilde{a}$ satisfying

$$\begin{cases} f(\tilde{a}^*\tilde{x}\tilde{a}) \geqslant 0, & \tau(f(\tilde{a}^*\tilde{x}\tilde{a})) < +\infty, \\ \tau((\tilde{a}^*f(\tilde{x})\tilde{a})_+) = +\infty, & \|\tilde{a}\| = 1. \end{cases} \tag{8}$$

For each $0 < \varepsilon \leqslant 1/2$, we consider the following 2×2-matrices:

$$a_\varepsilon = \begin{bmatrix} u & v \\ v & w \end{bmatrix}, \quad x = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix},$$

$$\begin{cases} u = 3^{-1}(1+\sqrt{\varepsilon})^{-1}(1+3\sqrt{\varepsilon}+\varepsilon), \\ v \geqslant 0 \text{ and } v^2 = 9^{-1}(1+\sqrt{\varepsilon})^{-2}(2-5\varepsilon+2\varepsilon^2), \\ w = 3^{-1}(1+\sqrt{\varepsilon})^{-1}(2+3\sqrt{\varepsilon}+2\varepsilon). \end{cases}$$

Elementary computations show that

(9) $a_\varepsilon = a_\varepsilon^*$ has eigenvalues $1, \sqrt{\varepsilon}$,

(10) $a_\varepsilon^* x a_\varepsilon$ has eigenvalues $1, \varepsilon$.

We now set

$$f(t) = \begin{cases} t & \text{if } 0 \leqslant t \leqslant 1/2 \\ 1-t & \text{if } t > 1/2. \end{cases}$$

From (10), we get

(10)′ $f(a_\varepsilon^* x a_\varepsilon)$ has eigenvalues $0, \varepsilon$.

Another computation shows that

(11) $a_\varepsilon^* f(x) a_\varepsilon$ has eigenvalues $\pm \sqrt{\varepsilon/2}$.

We take $0 < \varepsilon_n \leqslant 1/2$, $n = 1, 2, \ldots$, satisfying $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and $\sum_{n=1}^{\infty} \sqrt{\varepsilon_n} = +\infty$. (For example, $\varepsilon_n = (n+1)^{-2}$.) We consider the operators $\tilde{a} = \sum_{n=1}^{\infty}{}^{\oplus} a_{\varepsilon_n}$ and $\tilde{x} = \sum_{n=1}^{\infty}{}^{\oplus} x$ in the von Neumann algebra $\sum_{n=1}^{\infty}{}^{\oplus} M_2(\mathbf{C})$ equipped with the faithful normal semi-finite trace $\tau = \sum_{n=1}^{\infty}{}^{\oplus} \mathrm{Tr}$. Then (8) obviously follows from (9), (10)′, (11). We remark that

$$\tau(e_{(\varepsilon,\infty)}(\tilde{a}^* \tilde{x} \tilde{a})) = +\infty \quad \text{for } 0 \leqslant \varepsilon < 1$$

because of (10). Also note that it was necessary for this example that f not be operator concave.

4. MISCELLANEOUS RESULTS

In this section, we will obtain some consequences of Theorem 11 and prove the equality $\tau(xy) = \tau(yx)$ mentioned in the introduction.

We begin by rephrasing Theorem 11 in the following form that might be more usable:

THEOREM 14. *Let f be a continuous function on the interval $[0, \infty)$ with $f(0) = 0$. The following five conditions are equivalent:*

(i) *f is concave;*

(ii) *$\tau(a^*f(x)a) \leqslant \tau(f(a^*xa))$ for a positive operator x and a contraction a;*

(iii) *$\tau(pf(x)p) \leqslant \tau(f(pxp))$ for a positive operator x and a projection p;*

(iv) *$\lambda\tau(f(x)) + (1 - \lambda)\tau(f(y)) \leqslant \tau(f(\lambda x) + (1 - \lambda)y)$ for positive operators x, y and a real number $0 < \lambda < 1$;*

(v) *$\tau(a^*f(x)a) + \tau(b^*f(y)b) \leqslant \tau(f(a^*xa + b^*yb))$ for positive operators x, y and operators satisfying $a^*a + b^*b \leqslant 1$.*

Here, (ii) ~ (v) should be understood to hold for any faithful normal semi--finite trace τ on any von Neumann algebra and whenever the both sides are defined. (For example, in (v) all the traces must be defined and the sum in the left hand side must be well-defined.)

Proof. The implication (i) $\Rightarrow$ (ii) is precisely Theorem 11 while (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (iv) $\Rightarrow$ (i) are trivial. Therefore, it suffices to prove the implications (ii) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (v) We consider the von Neumann algebra $M_2(\mathcal{M})$ equipped with the trace $\tilde{\tau} = \begin{bmatrix} \tau & 0 \\ 0 & \tau \end{bmatrix}$, and set

$$\tilde{x} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \quad \tilde{a} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}.$$

The assumption $a^*a + b^*b \leqslant 1$ is equivalent to the property that $\tilde{a}$ be a contraction. We note that

$$\tilde{a}^*\tilde{x}\tilde{a} = \begin{bmatrix} a^*xa + b^*yb & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{a}^*f(\tilde{x})\tilde{a} = \begin{bmatrix} a^*f(x)a + b^*f(y)b & 0 \\ 0 & 0 \end{bmatrix}.$$

When both sides in (v) are defined, by Lemma 9

$$\tilde{\tau}(f(\tilde{a}^*\tilde{x}\tilde{a})) = \tau(f(a^*xa) + b^*yb)),$$

$$\tilde{\tau}(\tilde{a}^*f(\tilde{x})\tilde{a}) = \tau(a^*f(x)a + b^*f(y)b) =$$

$$= \tau(a^*f(x)a) + \tau(b^*f(y)b)$$

are defined, and (v) follows from (ii).

(iii) $\Rightarrow$ (iv) In the von Neumann algebra $M_2(\mathcal{M})$ equipped with the above trace $\tilde{\tau}$, we consider

$$\tilde{x} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \quad u = \begin{bmatrix} \lambda^{1/2} & -(1-\lambda)^{1/2} \\ (1-\lambda)^{1/2} & \lambda^{1/2} \end{bmatrix}, \quad p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since u is a unitary and p is a projection, (iii) implies that

$$\tilde{\tau}(pu^*f(\tilde{x})up) = \tilde{\tau}(pf(u^*\tilde{x}u)p) \leqslant \tilde{\tau}(f(pu^*\tilde{x}up)).$$

However, the far left hand side and the far right hand side are $\lambda\tau(f(x)) + (1-\lambda)\tau(f(y))$ and $\tau(f(\lambda x + (1-\lambda)y))$ respectively. Q.E.D.

Generalizing the above implication (ii) $\Rightarrow$ (v) in the obvious way, we can prove: For $x_i \geqslant 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} a_i^*a_i \leqslant 1$, one gets

$$\sum_{i=1}^{n} \tau(a_i^*f(x_i)a_i) \leqslant \tau f\left(\sum_{i=1}^{n} a_i^*x_ia_i\right)$$

when f is concave. This makes it clear that our main result is precisely Jensen's inequality for traces. If f is in addition increasing, the arguments in (ii) $\Rightarrow$ (v) (and (ii) at the beginning of Section 3) yield that

$$\sum_{i=1}^{n} a_i^*f(x)a_i \lesssim f\left(\sum_{i=1}^{n} a_i^*x_ia_i\right).$$

PROPOSITION 15. *Let x, y be positive operators, and f be a continuous concave function on the interval $[0, \infty)$ with $f(0) = 0$. Then we have*

$$\tau(f(x + y)) \leqslant \tau(f(x)) + \tau(f(y))$$

whenever both sides are defined.

Proof. Since $x, y \leqslant x + y$, there exist contractions u, v in $\mathcal{M}$ such that

$$x^{1/2} = u(x + y)^{1/2} \qquad \text{(hence } x = u(x + y)u^*\text{)},$$

$$y^{1/2} = v(x + y)^{1/2} \qquad \text{(hence } y = v(x + y)v^*\text{)},$$

$$u^*u + v^*v = p, \text{ the support projection of } x + y.$$

We compute

$$\tau(f(x)) + \tau(f(y)) = \tau(f(u(x + y)u^*)) + \tau(f(v(x + y)v^*)) \geqslant$$

$$\geqslant \tau(uf(x + y)u^*) + \tau(vf(x + y)v^*)$$

thanks to Theorem 11. Since the supports of $f(x + y)_{\pm}$ are majorized by p, the same computations as the proof of Lemma 8, (iii), show that

$$\tau(uf(x + y)u^*) + \tau(vf(x + y)v^*) = \tau(f(x + y)).$$

Q.E.D.

PROPOSITION 16. *Let x, y be operators. For an increasing concave function f on the interval* $[0, \infty)$ *with* $f(0) = 0$, *we always have*

$$\tau(f(|x + y|)) \leqslant \tau(f(|x|)) + \tau(f(|y|)).$$

Proof. (The arguments in the proofs of) Lemma 4 and Corollary 5 of [11] (the case where x and y are bounded was done in [1]) guarantee the existence of partial isometries u, v in $\mathcal{M}$ satisfying

$$|x + y| \leqslant u|x|u^* + v|y|v^*.$$

We now estimate

$$\tau(f(|x + y|)) \leqslant \tau(f(u|x|u^* + v|y|v^*)) \leqslant \qquad \text{(Lemma 4)}$$

$$\leqslant \tau(f(u|x|u^*)) + \tau(f(v|y|v^*)) \leqslant \qquad \text{(Proposition 15)}$$

$$\leqslant \tau(f(|x|)) + \tau(f(|y|))$$

(Lemma 3, (ii), and Lemma 4). Q.E.D.

For usual compact operators in $B(\mathcal{H})$, this result was first proved in [14]. For an arbitrary semi-finite von Neumann algebra, the result was proved in Remark 6.5, [1], if f is a non-negative operator concave function.

We now describe some typical applications of our results. In what follows we will assume that f (resp. g) is an increasing concave (resp. increasing convex) function on the interval $[0, \infty)$ vanishing at 0. Also, for an operator x, we will denote its p-norm (quasi-norm if $0 < p < 1$) $\tau(|x|^p)^{1/p}$ by $\|x\|_p$, $0 < p < \infty$, [12].

(i) Applying Proposition 15 to $-g(t)^p$, $1 \leqslant p < \infty$, we get

$$\|g(x + y)\|_p^p \geqslant \|g(x)\|_p^p + \|g(y)\|_p^p, \quad 1 \leqslant p < \infty,$$

for positive operators x, y.

(ii) Applying Proposition 16 to $f(t)^p$, $0 < p \leqslant 1$, we get

$$\|f(|x + y|)\|_p^p \leqslant \|f(|x|)\|_p^p + \|f(|y|)\|_p^p, \quad 0 < p \leqslant 1,$$

for any operators x, y.

(iii) Applying Proposition 16 to $\log(1 + f(t))$, we get

$$\tau(\log(1 + f(|x + y|))) \leqslant \tau(\log(1 + f(|x|))) + \tau(\log(1 + f(|y|)))$$

for any operators x, y. If we denote the Fuglede-Kadison determinant, [6], by Δ ($\Delta_x = \exp\tau(\log|x|)$ if x is invertible), then the above inequality reads

$$\Delta_{1+f(|x+y|)} \leqslant \Delta_{1+f(|x|)}\Delta_{1+f(|y|)}.$$

(iv) For a positive functional $\varphi \in \mathcal{M}_*^+$, its entropy $S(\varphi)$ (relative to τ) is defined as $\tau(f(h_\varphi))$, where h_φ is the Radon-Nikodym derivative of φ relative to τ ([15]), and $f(t) = -t\log t$ ($f(0) = 0$). Since this f is concave, many properties of the entropy function can be obtained from our results.

The above (i) ~ (iii) are well-known when $f(t) = g(t) = t$ (see [5], [11], for (i), (ii), and Theorem 6.3 in [1], Theorem 2.1 in [5] for (iii)). Also, a very readable account on the entropy function can be found in [19].

We now recall that the non-commutative L^1-space $L^1(\mathcal{M}; \tau)$ consists of all operators x satisfying $\|x\|_1 = \tau(|x|) < +\infty$ equipped with the norm $\|\cdot\|_1$ ([12], [15]). The predual $\mathcal{M}_*$ can be identified with $L^1(\mathcal{M}; \tau)$ as a Banach space, and the trace extends to a positive linear functional on $L^1(\mathcal{M}; \tau)$.

THEOREM 17. *Let x, y be τ-measurable operators. If both of xy and yx belong to $L^1(\mathcal{M}; \tau)$, then we have*

$$\tau(xy) = \tau(yx).$$

When $\mathcal{M}$ is a factor of type I_∞, this result is usually derived from the (deep) Lidskii theorem [7], [16]. (More information can be found in [10] for operators x, y such that $xy - yx$ is of trace class.) We give here a simple proof of the theorem based on the following two easy consequences of what was mentioned before the theorem:

(12) if $x \in \mathcal{M}$ and $y \in L^1(\mathcal{M}; \tau)$, then $\tau(xy) = \tau(yx)$,

(13) if $y \in L^1(\mathcal{M}; \tau)$ and $\{x_n\}$ is a sequence in $\mathcal{M}$ converging to x in the σ-weak topology, then $\tau(x_n y)$ $(= \tau(yx_n))$ tends to $\tau(xy)$ $(= \tau(yx))$.

Proof. Let p (resp. q) be the left (resp. right) projection of x, and set

$$p_n = e_{[n^{-1},n]}(xx^*), \quad q_n = e_{[n^{-1},n]}(x^*x), \quad n = 1, 2, \ldots .$$

We observe that

$$\begin{cases} px = x = xq, \quad p_n x = p_n x q_n = x q_n, \\ \text{strong-}\lim_{n\to\infty} p_n = p, \quad \text{strong-}\lim_{n\to\infty} q_n = q. \end{cases}$$

We claim that $\tau(p_n xy) = \tau(yxq_n)$, $n = 1, 2, \ldots$. In fact, from $n^{-1}q_n \leqslant x^*x$, we get $y^*q_n y \leqslant ny^*x^*xy$; hence, $q_n y \in L^1(\mathcal{M}; \tau)$ follows from $xy \in L^1(\mathcal{M}; \tau)$. Since $q_n y$, $yxq_n \in L^1(\mathcal{M}; \tau)$ and $p_n x$, $q_n \in \mathcal{M}$, using (12) twice we compute

$$\tau(p_n xy) = \tau(p_n x q_n y) = \tau(q_n y p_n x) =$$

$$= \tau(q_n y x q_n) = \tau(yxq_n).$$

Since xy and yx are in $L^1(\mathcal{M}; \tau)$, using (13) twice together with the claim, we conclude that

$$\tau(xy) = \tau(pxy) = \lim_{n\to\infty} \tau(p_n xy) =$$

$$= \lim_{n\to\infty} \tau(yxq_n) = \tau(yxq) = \tau(yx). \qquad \text{Q.E.D.}$$

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REFERENCES

1. AKEMANN, C.; ANDERSON, J.; PEDERSEN, G., Triangle inequalities in operator algebras, *Linear and Multilinear Algebra*, **11**(1982), 167–178.
2. BEREZIN, F. A., Convex trace functions, *Math. USSR-Sb.*, **17**(1972), 269–277.
3. DONOGHUE JR., W., *Monotone matrix functions and analytic continuation*, Springer, Berlin — Heidelberg — New York, 1974.
4. FACK, T., Sur la notion de valeur caractéristique, *J. Operator Theory*, **7**(1982), 307–333.
5. FACK, T., Proof of the conjecture of A. Grothendieck on the Fuglede-Kadison determinant, *J. Funct. Anal.*, **50**(1983), 215–228.
6. FUGLEDE, B.; KADISON, R., Determinant theory in finite factors, *Ann. of Math.*, **55**(1952), 520–530.

7. GOHBERG, I. C.; KREIN, M. G., *Introduction to linear nonself-adjoint operators*, Translations of Mathematical Monographs, Vol. **18**, Amer. Math. Soc., Rhode Island, 1969.
8. GROTHENDIECK, A., Réarrangements de fonctions et inequalité de convexité dans les algèbres de von Neumann minuies d'une trace, *Seminar Bourbaki*, **113**(1955), 1–13.
9. HANSEN, F., An operator inequality, *Math. Ann.*, **246**(1980), 249–250.
10. HELTON, W.; HOWE, R., Integral operators, traces, index and homology, in *Lecture Notes in Math.*, No. **345**, Springer, Berlin–Heidelberg – New York, 1973.
11. KOSAKI, H., On the continuity of the map $\varphi \to |\varphi|$ from the predual of a W^*-algebra, *J. Funct, Anal.*, **59**(1984), 123–131.
12. NELSON, E., Notes on non-commutative integration, *J. Funct. Anal.*, **15**(1974), 103–116.
13. VON NEUMANN, J., *Collected works, Vol. III*, Pergamon, London, 1961.
14. ROTFEL'D, S. YU., The singular numbers of the sum of completely continuous operators, in *Topics in Math. Phys.*, Vol. **3**, Ed. M. Sh. Birman, Consultants Bureau, New York – – London, 1969.
15. SEGAL, I., A non-commutative extension of abstract integration, *Ann. of Math.*, **57**(1953), 491–457.
16. SIMON, B., *Trace ideals and their applications*, Cambridge Univ. Press, 1979.
17. TAKESAKI, M., *Theory of operator algebras. I*, Springer, Berlin – Heidelberg – New York, 1979.
18. THOMPSON, R. C., Convex and concave functions of singular values of matrix sums, *Pacific J. Math.*, **66**(1976), 285–290.
19. WEHRL, A., General properties of entropy, *Rev. Modern Physics*, **50**(1978), 221–260.

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