J. OPERATOR THEORY
23(1990), 21–42

DILATIONS OF HOLOMORPHIC SEMIGROUPS

BRIAN McENNIS

1. INTRODUCTION

In this paper we present a dilation of a semigroup which generalizes that given in the paper [2] by Ch. Davis. As in [2], the dilation space is a Krein space, on which is defined an indefinite inner product, and the dilation is a strongly continuous group which is unitary with respect to this inner product. The construction also follows that in [2], in that it represents the original Hilbert space as a state space and obtains the dilation by incorporating incoming and outgoing shifts into a system.

The paper [2] treats only uniformly continuous semigroups, in which the infinitesimal generator is bounded. Here we consider strongly continuous semigroups for which the infinitesimal generator has numerical range that is contained in a sector in the complex plane of the form

$$\mathrm{S}_{\gamma,\theta} = \{\lambda : |\arg(\gamma - \lambda)| \leqslant \theta\}, \tag{1.1}$$

for some complex number γ (the vertex) and for some real number θ (the semi-angle), with $0 \leqslant \theta < \pi/2$. With this restriction, a dilation is obtained which preserves the symmetry of the Davis construction, in that the incoming and outgoing shifts act on functions taking their values in the same Krein spaces.

The condition that the numerical range of the infinitesimal generator be contained in a sector $\mathrm{S}_{\gamma,\theta}$ is necessary in order to apply the techniques used in this paper. There is, however, a sense in which this restriction is the minimal restriction on the numerical range that will guarantee this symmetry of the dilation. (See Example 5.1 below.) As we will see, in Theorem 2.1, this condtition implies that $T(t)$ is a quasi-bounded holomorphic semigroup, in the sense of [4], Section IX.1.6, with $\|T(t)\| \leqslant \mathrm{e}^{\beta t}$, for some real constant β, and for all $t \geqslant 0$.

2. SEMIGROUPS AND DILATIONS

Suppose $T(t)$ is a strongly continuous semigroup, acting on a separable Hilbert space $\mathscr{H}$. Then, for each $t \geqslant 0$, $T(t)$ is a bounded operator on $\mathscr{H}$, $T(s)T(t) = T(s+t)$ for each $s, t \geqslant 0$, $T(0) = I$, and $T(t)$ converges strongly to zero as $t \to 0^+$. The infinitesimal generator A of $T(t)$ is defined as the strong limit of $t^{-1}(T(t) - I)$, as $t \to 0^+$. A is a closed operator, with domain $\mathscr{D}(A)$ dense in $\mathscr{H}$, and we can write $T(t) = e^{At}$. For such a semigroup, there exists a real number α such that

$$\sigma(A) \subseteq \{\lambda : \operatorname{Re}\lambda \leqslant \alpha\}, \tag{2.1}$$

where $\sigma(A)$ denotes the spectrum of A. (See, for example [3], [4], [5], and [7].)

A stronger condition on A is that its spectrum be contained in a sector $S_{\gamma,\theta}$, defined by (1.1), with $\theta < \pi/2$. Then for a suitable choice of the real number β, the operator $B = A - \beta$ has

$$\sigma(B) \subseteq S_{0,\theta}, \tag{2.2}$$

a sector with vertex at the origin. Suppose that, in addition, we have for each $\varepsilon > 0$,

$$\|(\lambda - B)^{-1}\| \leqslant M_\varepsilon |\lambda|^{-1} \quad \text{for } |\arg(\lambda)| \leqslant \pi - \theta - \varepsilon, \tag{2.3}$$

where M_ε is independent of λ. Then, following [4], Section IX.1.6, we will refer to $T(t)$ as a *holomorphic semigroup, quasi-bounded by* β. It is shown in [4] that such a semigroup has a holomorphic extension to a sector of the form $|\arg(t)| < \pi/2 - \theta$.

Let $W(A)$ denote the numerical range of A, i.e. $W(A) = \{(Ax, x) : x \in \mathscr{D}(A), \|x\| = 1\}$. We will call A *sectorial* if

$$W(A) \subseteq S_{\gamma,\theta}, \tag{2.4}$$

for some vertex γ and semi-angle $\theta < \pi/2$. (This is equivalent to $-A$ being sectorial in the sense of [4], p. 280.)

We have the following:

THEOREM 2.1. *If the infinitesimal generator A of a semigroup $T(t)$ is sectorial, then there is a real constant $\beta \geqslant 0$ such that $T(t)$ is a holomorphic semigroup, quasi-bounded by β, and $\|T(t)\| \leqslant e^{\beta t}$ for all $t \geqslant 0$.*

Proof. We are assuming that (2.4) holds, so for a suitable choice of $\beta \geqslant 0$, we can get $W(B) \subseteq S_{0,\theta}$, where $B = A - \beta$. Let Γ denote the closure of $W(B)$, and let Δ denote the complement of Γ in the complex plane. Since Γ is a convex set contained in a sector with semi-angle less than $\pi/2$, it follows that Δ is connected and, since, by (2.1), $\sigma(B)$ is contained in a half-plane, Δ contains points of the resolvent set of B. Thus, by [4], Theorem V.3.2, we conclude that Δ is contained in the resolvent set of B, or equivalently,

$$\sigma(B) \subseteq \Gamma \subseteq \mathbf{S}_{0,\theta}.$$

Therefore, $-B$ is m-accretive, in the sense of [4], p. 279, and we conclude from [4], Theorem IX.1.24, that B satisfies condition (2.3), with $\|e^{Bt}\| \leqslant 1$. Since $T(t)$ is generated by $A = B + \beta$, the conclusions of the theorem follow. ▨

COROLLARY 2.2. *If A is sectorial, then for every $x \in \mathscr{H}$ and for every $t > 0$, we have $T(t)x \in \mathscr{D}(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = AT(t)x. \tag{2.5}$$

In addition, there is a constant M such that, for all $t > 0$,

$$\|AT(t)\| \leqslant (Mt^{-1} + \beta)e^{\beta t}. \tag{2.6}$$

Proof. Let $B = A - \beta$ be the operator introduced in the proof of Theorem 2.1, and let $S(t) = e^{Bt}$, so that $T(t) = e^{\beta t}S(t)$. Then, since B satisfies (2.2) and (2.3), we can conclude from [4], Section IX.1.6, that for all $x \in \mathscr{H}$ and $t > 0$, $S(t)x \in \mathscr{D}(B) = \mathscr{D}(A)$, $\|S(t)\| \leqslant 1$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t) = BS(t), \quad \left\| \frac{\mathrm{d}}{\mathrm{d}t} S(t) \right\| \leqslant Mt^{-1},$$

for some constant M. Thus $T(t)x \in \mathscr{D}(A)$, and (2.5) and (2.6) follow by differentiating the equation $T(t) = e^{\beta t}S(t)$. ▨

COROLLARY 2.3. *If the infinitesimal generator A of $T(t)$ is sectorial, then it is m-sectorial (see* [4], p. 280*).*

Proof. This follows immediately from the fact that $-B$ is m-accretive. ▨

Under the assumption that $T(t)$ has a sectorial infinitesimal generator A, we are able to construct a dilation $\{U(s) : s \in \mathbf{R}\}$ (where $\mathbf{R}$ denotes the real numbers), which is analogous to the construction in [2] for a semigroup with bounded infinitesimal generator. The dilation acts on a Krein space $\mathscr{K}$, on which is defined an indefinite inner product $[\cdot,\cdot]$ and a Hilbert space inner product $(\cdot,\cdot)$. These inner products are linked by the relations

$$(x, y) = [Jx, y] \text{ and } [x, y] = (Jx, y) \quad \text{for all } x, y \in \mathscr{K},$$

for an operator J which is selfadjoint and unitary, with respect to both inner products. (Such an operator is referred to as a fundamental symmetry. See [1].)

If the numerical range of A lies in a sector contained in the half-plane $\{\lambda : \operatorname{Re}\lambda \leqslant \leqslant 0\}$, then $T(t)$ is contractive, and this paper presents a construction of the minimal unitary dilation of $T(t)$ (see [6]). The fundamental symmetry J is, in this case, equal to the identity operator, so that $\mathcal{K}$ is a Hibert space. Our construction does not cover all contractive semigroups, as can be seen from Example 5.1 below.

We will prove:

THEOREM 2.4. *Let $T(t)$ be a strongly continuous semigroup, acting on a separable Hilbert space $\mathcal{H}$, with a sectorial infinitesimal generator A. Then there exist a Krein space $\mathcal{K} \supseteq \mathcal{H}$, a fundamental symmetry J on $\mathcal{K}$, and a strongly continuous group $U(s)$ on $\mathcal{K}$, such that*

(i) $Jx = x$ *for all* $x \in \mathcal{H}$;

(ii) $[U(s)x, y] = (T(s)x, y)$ *for all* $s \geqslant 0$ *and all* $x, y \in \mathcal{H}$;

(iii) $[U(s)x, U(s)y] = [x, y]$ *for all* $s \in \mathbf{R}$ *and all* $x, y \in \mathcal{K}$;

(iv) $\bigvee \{U(s)\mathcal{H} : s \in \mathbf{R}\} = \mathcal{K}$.

The construction of the dilation is similar to that in [2], except that unbounded operators are involved. In the next section, we use bilinear forms to define these operators and to establish their domains. In Section 4, the dilation is constructed and its properties are verified. Finally, in Section 5, an example is presented to illustrate that the assumption that A be sectorial cannot be relaxed if the symmetry of the dilation is to be preserved.

3. BILINEAR FORMS

When the semigroup $T(t)$ is uniformly continuous, its infinitesimal generator A is bounded, and we can consider the operator

$$G = A + A^*, \tag{3.1}$$

used in the construction of the dilation in [2]. When A is not bounded, the operator G cannot be defined by (3.1), since the intersection of the domains of A and A^* may be too small. However, when A is sectorial, we can use bilinear forms to define an operator which is an analogue of (3.1), and use this to construct the dilation.

A bilinear form f, with domain $\mathcal{D}(f)$ (a linear manifold in $\mathcal{H}$), is a function from $\mathcal{D}(f)\times\mathcal{D}(f)$ into the complex numbers, which is linear in the first variable, and conjugate linear in the second variable (see [4], [5]). We follow the usual convention of denoting $f(x, x)$ by $f(x)$, for any $x \in \mathcal{D}(f)$. We call f *symmetric* if $f(y, x) = \overline{f(x, y)}$ for all $x, y \in \mathcal{D}(f)$.

The numerical range of f, denoted by $W(f)$, is defined by

$$W(f) = \{f(x) : x \in \mathscr{D}(f),\ \|x\| = 1\}.$$

f is symmetric if and only if $W(f)$ is a subset of the real line (see [4], p. 309). f is called *sectorial* if $W(f) \subseteq S_{\gamma,\theta}$ for some vertex γ and semi-angle $\theta < \pi/2$. (This is equivalent to $-f$ being sectorial in the sense of [4], p. 310.) If F is a closed operator on $\mathscr{H}$, and if we define a bilinear form f by $f(x, y) = (Fx, y)$, for $x, y \in \mathscr{D}(F)$, then $W(f) = W(F)$ and f is sectorial if and only if F is.

A sequence x_n in $\mathscr{H}$ is said to be *f-convergent* to $x \in \mathscr{H}$ if $x_n \in \mathscr{D}(f)$, $x_n \to x$, and $f(x_n - x_m) \to 0$ as $n, m \to \infty$. We say that f is *closed* if, whenever x_n is f-convergent to x, it follows that $x \in \mathscr{D}(f)$ and $f(x_n - x) \to 0$. f is called *closable* if it has a closed extension. We will make repeated use of the following results. (See [4], Theorems VI.1.17, VI.1.18, and VI.1.27.)

THEOREM 3.1. *Let f_0 be a sectorial bilinear form. Then f_0 is closable if and only if $f_0(x_n) \to 0$ whenever x_n is f_0-convergent to zero. If f_0 is closable, then it has a smallest closed extension (the closure of f_0), denoted by f, with the following properties:*

(i) *$\mathscr{D}(f)$ is the set of all $x \in \mathscr{H}$ such that there exists a sequence x_n which is f_0-convergent to x.*

(ii) *If x_n is f_0-convergent to x, and y_n is f_0-convergent to y, then $f(x, y) = \lim f_0(x_n, y_n)$.*

(iii) *$W(f_0)$ is dense in $W(f)$.*

THEOREM 3.2. *Let A be a closed operator on $\mathscr{H}$, and define a bilinear form a_0 by $a_0(x, y) = (Ax, y)$, with $\mathscr{D}(a_0) = \mathscr{D}(A)$. If the operator A is sectorial, then the form a_0 is closable.*

If f is a closed sectorial form, a set $\mathscr{D} \subseteq \mathscr{D}(f)$ is called a *core* of f if the restriction of f to $\mathscr{D}$ has closure equal to f. There is an analogous concept for an operator: if A is a closed operator, a set $\mathscr{D} \subseteq \mathscr{D}(A)$ is a *core* of A if the restriction of A to $\mathscr{D}$ has closure equal to A.

We have the following useful criterion for comparing domains of two bilinear forms.

PROPOSITION 3.3. *Suppose f and g are two closed sectorial bilinear forms, and let $\mathscr{D}$ be a core of f that is contained in the domain of g. If there exist real numbers $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that*

$$|g(x)| \leqslant \alpha |f(x)| + \beta \|x\|^2, \tag{3.2}$$

for all $x \in \mathscr{D}$, then $\mathscr{D}(f) \subseteq \mathscr{D}(g)$, (3.2) is valid for all $x \in \mathscr{D}(f)$, and any sequence that is f-convergent is also g-convergent.

Proof. Cf. [4], pp. 319–320. ▨

Let us now return to considering the infinitesimal generator A of a semigroup $T(t)$. Throughout the remainder of this section, we will be assuming that A is sectorial, so that $W(A)$ is contained in some sector with real vertex $\gamma \geqslant 0$. In Theorem 2.1, we can take $\beta = 2\gamma$; we make this choice in order to simplify some subsequent formulas. By Theorem 3.2, the bilinear form

$$a_0(x, y) = (Ax, y), \tag{3.3}$$

defined for $x, y \in \mathcal{D}(A)$, is closable. Denote the closure of a_0 by a, and define a form

$$g(x, y) = a(x, y) + \overline{a(y, x)} \tag{3.4}$$

with $\mathcal{D}(g) = \mathcal{D}(a)$. Then $g = 2\mathrm{Re}\, a$ ([4], pp. 309—310) and, since the form a is closed and sectorial, with vertex $\beta/2$ (by Theorem 3.1 (iii)), it follows that g is closed (see [4], p. 313) and symmetric, with $W(g) \subseteq (-\infty, \beta]$. Also, by [4], p. 313, g-convergence is equivalent to a-convergence, and thus Definitions 3.3 and 3.4 imply:

PROPOSITION 3.4. *$\mathcal{D}(A)$ is a core of g, and for $x, y \in \mathcal{D}(A)$, we have $g(x, y) = (Ax, y) + (x, Ay)$.*

From the form g we can obtain an operator G, which is analogous to that defined by (3.1) when A is bounded.

LEMMA 3.5. *There exists a selfadjoint operator G, such that $\mathcal{D}(G)$ is a core of g, and*

$$g(x, y) = (Gx, y) \tag{3.5}$$

for all $x, y \in \mathcal{D}(G)$. G is bounded above by β, i.e.

$$(Gx, x) \leqslant \beta \|x\|^2 \quad \text{for every } x \in \mathcal{D}(G). \tag{3.6}$$

Proof. The existence of an operator G which has $\mathcal{D}(G)$ as a core of g, and which satisfies (3.5), is guaranteed by [4], Theorem VI.2.1. The fact that G is selfadjoint with the same upper bound as g follows from [4], Theorem VI.2.6. (The results in [4] need to be applied to the form $-g$.) ∎

Thus the operator G given by Lemma 3.5 provides a generalization of the definition (3.1). In fact, we have $G = 2\mathrm{Re}\, A$, as defined in [4], p. 337, where the following important fact is also established:

PROPOSITION 3.6. *If $G = 2\mathrm{Re}\, A$ is the operator given by Lemma 3.5, then we also have $G = 2\mathrm{Re}\, A^*$.*

In other words, the same operator G is obtained regardless of whether the construction is started with A or with its adjoint A^*. This fact is the origin of the symmetry in the dilation referred to in Section 1, and will be used repeatedly in the sequel. An immediate consequence is:

PROPOSITION 3.7. *$\mathscr{D}(A^*)$ is a core of g, and for $x, y \in \mathscr{D}(A^*)$ we have $g(x, y) = (A^*x, y) + (x, A^*y)$.*

We know that $\mathscr{D}(G)$, $\mathscr{D}(A)$, and $\mathscr{D}(A^*)$ are all cores of g, but it is not clear what relationship (if any) exists between these domains. Consequently, we will be working, for the most part, with the form g, whose domain contains $\mathscr{D}(A)$ and $\mathscr{D}(A^*)$, rather than with the operator G.

In [2], where A is assumed to be bounded, G is important because of the fact that the derivatives of $T(t)^*T(t)$ and $T(t)T(t)^*$ are, respectively, $T(t)^*GT(t)$ and $T(t)GT(t)^*$. We have an anlogous result for the bilinear form g:

PROPOSITION 3.8. *For every $x, y \in \mathscr{H}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)x, T(t)y) = g(T(t)x, T(t)y) \tag{3.7}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)^*x, T(t)^*y) = g(T(t)^*x, T(t)^*y). \tag{3.8}$$

Proof. Since the infinitesimal generator of $T(t)^*$ is A^*, we can exploit the symmetry mentioned above and prove only (3.7). This follows immediately from Corollary 2.2 and the calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)x, T(t)y) = \left(\frac{\mathrm{d}}{\mathrm{d}t}T(t)x, T(t)y\right) + \left(T(t)x, \frac{\mathrm{d}}{\mathrm{d}t}T(t)y\right) =$$

$$= (AT(t)x, T(t)y) + (T(t)x, AT(t)y) = g(T(t)x, T(t)y),$$

for all $x, y \in \mathscr{H}$. ▨

The operator G has a unique polar decomposition of the form $G = UH$, where H is selfadjoint and nonnegative, with $\mathscr{D}(H) = \mathscr{D}(G)$, and where U is a selfadjoint partial isometry. Let us define $J = -U$, and let $\mathscr{G}$ be the closure of the range of G, or equivalently $\mathscr{G} = J\mathscr{H}$. Then J is a bounded operator satisfying

$$J = J^*, \tag{3.9}$$

$$J^2x = x, \quad \text{for all } x \in \mathscr{G}, \tag{3.10}$$

and

$$Jx = 0, \quad \text{for all } x \in \mathscr{G}^\perp. \tag{3.11}$$

(See [4], Section VI.2.7.) (We are using J instead of the S used in [2]. We could also define $J = \operatorname{sgn}(-G)$ and $H = |G|$, by using the functional calculus for unbounded selfadjoint operators.)

PROPOSITION 3.9. *For every $x \in \mathscr{D}(G) = \mathscr{D}(H)$ we have $Jx \in \mathscr{D}(G)$, with $Gx = -JHx = -HJx$, and $Hx = -JGx = -GJx$.*

Proof. See [4], p. 335. ▨

PROPOSITION 3.10. *For every $x \in \mathscr{D}(g)$ we have $Jx \in \mathscr{D}(g)$ and*

$$g(Jx) = g(x).$$

Proof. (3.9) and (3.10) imply that, for $x \in \mathscr{D}(G)$,

$$g(Jx) = (GJx, Jx) = (JGx, Jx) = (Gx, x) = g(x). \tag{3.12}$$

Now let $x \in \mathscr{D}(g)$, and let x_n be a sequence in $\mathscr{D}(G)$ which is g-convergent to x. From (3.12) it follows that Jx_n is g-convergent to Jx, and thus, by Theorem 3.1, $Jx \in \mathscr{D}(g)$ and (using (3.12) again) $g(Jx) = \lim g(Jx_n) = \lim g(x_n) = g(x)$. ▨

LEMMA 3.11. *There exist operators G_+ and G_-, with domains containing $\mathscr{D}(G)$, such that for all $x \in \mathscr{D}(G)$, $Gx = G_+x + G_-x$, $Hx = G_+x - G_-x$, and $0 \leqslant (G_+x, x) \leqslant \beta\|x\|^2$.*

Proof. The decomposition of G is that given in [4], p. 335. We can write explicitly $G_+ = GP_+$ and $G_- = GP_-$, where $P_+ = (J^2 - J)/2$ and $P_- = (J^2 + J)/2$ are selfadjoint projections. Since G and J commute, we have $GP_\pm \supseteq P_\pm G$, and thus the domains of G_+ and G_- contain $\mathscr{D}(G)$. An application of Proposition 3.9 then gives $G_+x + G_-x = GJ^2x = -HJx = Gx$ and $G_+x - G_-x = -GJx = Hx$ for all $x \in \mathscr{D}(G)$.

P_+ is a projection with $GP_+ \supseteq P_+G$, so we can conclude that, for all $x \in \mathscr{D}(G)$ $G_+x = P_+Gx = P_+^2Gx = P_+GP_+x$. Thus, by Lemma 3.5,

$$(G_+x, x) = (GP_+x, P_+x) \leqslant \beta\|P_+x\|^2 \leqslant \beta\|x\|^2.$$

On the other hand, the properties $JP_+ = -P_+$ and $JG = -H$ imply that, for all $x \in \mathscr{D}(G)$,

$$(G_+x, x) = -(GP_+x, JP_+x) = (HP_+x, P_+x) \geqslant 0,$$

since H is positive. ▨

Let us define $Q = H^{1/2}$, using the functional calculus for selfadjoint operators. Note that the space $\mathscr{G}$, defined as the closure of the range of G, is also the closure of the range of Q. Also note that, since J and H commute, so do J and Q.

It will be important to know that $\mathscr{D}(A)$ and $\mathscr{D}(A^*)$ are cores of Q, and to develop an estimate for $\|Qx\|$ when $x \in \mathscr{D}(Q)$. Towards these ends, we consider the symmetric bilinear form $h_0(x, y) = (Hx, y)$ defined for $x, y \in \mathscr{D}(H) = \mathscr{D}(G)$.

LEMMA 3.12. *The bilinear form h_0 is closable. If the closure of h_0 is denoted by h, then*

(i) $\mathscr{D}(h) = \mathscr{D}(g)$.
(ii) *h is symmetric, with $W(h) \subseteq [0, \infty)$,*
(iii) *$g(x, y) = -h(Jx, y)$ for all $x, y \in \mathscr{D}(g)$,*
(iv) *$\mathscr{D}(A)$ and $\mathscr{D}(A^*)$ are cores of h, and*
(v) *$h(x) \leqslant -g(x) + 2\beta\|x\|^2 \leqslant |g(x)| + 2\beta\|x\|^2$ for all $x \in \mathscr{D}(g)$.*

Proof. Since $W(H) \subseteq [0, \infty)$, $-H$ is sectorial. From Theorem 3.2 we deduce that h_0 is closable. If h is the closure of h_0, Theorem 3.1 implies that $W(h) \subseteq [0, \infty)$, and hence h is symmetric.

Observe that, by definition, $\mathscr{D}(G)$ is a core of h, and for $x \in \mathscr{D}(G)$, we have

$$|g(x)| = |(Gx, x)| \leqslant (|G|x, x) = (Hx, x) = h(x). \tag{3.13}$$

(See, for example, [4], Problem VI.2.35.) Thus, by Proposition 3.3, $\mathscr{D}(h) \subseteq \mathscr{D}(g)$.

For the reverse inclusion, we apply Lemma 3.11 to obtain, for $x \in \mathscr{D}(G)$, $g(x) + h(x) = (Gx, x) + (Hx, x) = 2(G_+x, x)$. Thus, using Lemma 3.11 again, we have for every $x \in \mathscr{D}(G)$,

$$h(x) = -g(x) + 2(G_+x, x) \leqslant -g(x) + 2\beta\|x\|^2 \leqslant |g(x)| + 2\beta\|x\|^2. \tag{3.14}$$

Since $\mathscr{D}(G)$ is a core of g (Lemma 3.5), it follows from Proposition 3.3 that $\mathscr{D}(g) \subseteq \mathscr{D}(h)$, and thus $\mathscr{D}(g) = \mathscr{D}(h)$. As in Proposition 3.3, we also conclude that (3.13) and (3.14) are valid for all $x \in \mathscr{D}(g)$.

We can also conclude from Proposition 3.3 that g-convergence is equivalent to h-convergence. Since $\mathscr{D}(A)$ and $\mathscr{D}(A^*)$ are cores of g (Propositions 3.4 and 3.7), it follows that they are also cores of h.

It remains to prove (iii). If we restrict x and y to be in $\mathscr{D}(G)$, we can write, using Proposition 3.9,

$$g(x, y) = (Gx, y) = -(HJx, y) = -h(Jx, y). \tag{3.15}$$

Since $\mathscr{D}(G)$ is a core of both g and h, and since g-convergence is equivalent to h-convergence, we can use Theorem 3.1 (ii) and Proposition 3.10 to extend (3.15) to all $x, y \in \mathscr{D}(g)$. ▨

We can now make use of this closed bilinear form h to obtain the information we need about the operator Q.

LEMMA 3.13. *The operator* $Q = H^{1/2}$ *has* $\mathcal{D}(Q) = \mathcal{D}(g)$, *and* $\mathcal{D}(A)$ *and* $\mathcal{D}(A^*)$ *are cores of* Q. *We have*

$$(JQx, Qy) = -g(x, y) \quad \textit{for } x, y \in \mathcal{D}(Q), \tag{3.16}$$

and

$$\|Qx\|^2 \leqslant -g(x) + 2\beta\|x\|^2 \quad \textit{for } x \in \mathcal{D}(Q). \tag{3.17}$$

Proof. Since h is closed, symmetric, and bounded below by zero, we can apply [4], Theorem VI.2.23, and deduce that the operator $Q = H^{1/2}$ has $\mathcal{D}(Q) = \mathcal{D}(h) = \mathcal{D}(g)$, and

$$h(x, y) = (Qx, Qy) \tag{3.18}$$

for all $x, y \in \mathcal{D}(Q)$. (Note that, by the uniqueness assertion in [4], Theorem VI.2.1, H is the selfadjoint operator associated with h.) Using (3.18), and the fact that J and Q commute, we can deduce (3.16) and (3.17) from Lemma 3.12, (iii) and (v).

[4], Theorem VI.2.23 also asserts that any core of h is also a core of Q, and thus Lemma 3.12 (iv) implies that $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are both cores of Q. ▨

4. THE DILATION

In the previous section, under the assumption that the infinitesimal generator A is sectorial, we introduced on $\mathcal{H}$ a closed symmetric bilinear form g, a bounded operator J, and a closed operator Q, with $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ being cores for both the form g and the operator Q. In this section, we again assume that A is a sectorial operator, acting on a separable Hilbert space, and use g, J, and Q to construct the promised dilation $U(s)$ of the semigroup $T(t)$.

We define on the space $\mathcal{G} = J\mathcal{H}$ an indefinite inner product $[x, y] = (Jx, y)$, for $x, y \in \mathcal{G}$, where $(\cdot, \cdot)$ denotes the inner product of $\mathcal{H}$. Then, with the Hilbert space structure inherited from $\mathcal{H}$, $\mathcal{G}$ is a Krein space, with J acting as a fundamental symmetry on $\mathcal{G}$. (See [1].)

Consider the Krein space $\mathcal{L} = L^2(\mathbf{R}, \mathcal{G})$ of (equivalence classes) of functions from the real line $\mathbf{R}$ into $\mathcal{G}$, which are measurable (strongly or weakly, since these are equivalent in a separabe space) and square integrable. In addition to the usual Hilbert space inner product $(\cdot, \cdot)$ on $\mathcal{L}$ we define an indefinite inner product

$$[f_1, f_2] = \int_{-\infty}^{\infty} [f_1(t), f_2(t)]\mathrm{d}t,$$

for $f_1, f_2 \in \mathscr{L}$. The topology on $\mathscr{L}$ is that given by the Hilbert space norm $(f, f)^{1/2}$. If we extend the definition of the operator J to $\mathscr{L}$, by defining $(Jf)(t) = J(f(t))$, then J is a fundamental symmetry on $\mathscr{L}$, with $(f_1, f_2) = [Jf_1, f_2]$ and $[f_1, f_2] = (Jf_1, f_2)$ for all $f_1, f_2 \in \mathscr{L}$.

As in [2], we will be considering two complementary subspaces $\mathscr{L}_+$ and $\mathscr{L}_-$ of $\mathscr{L}$, where $\mathscr{L}_+$ (respectively, $\mathscr{L}_-$) consists of all functions in $\mathscr{L}$ with support in $[0, \infty)$ (respectively, $(-\infty, 0]$).

The dilation $U(s)$ acts on a Krein space $\mathscr{K}$, defined by $\mathscr{K} = \mathscr{H} \oplus \mathscr{L}$. We will use the notation $k = \langle x, f\rangle$, where $x \in \mathscr{H}$ and $f \in \mathscr{L}$, to denote a vector $k \in \mathscr{K}$. We will consider $\mathscr{H}$ and $\mathscr{L}$ as subspaces of $\mathscr{K}$, and freely identify $\langle x, 0\rangle$ with x, and $\langle 0, f\rangle$ with f.

The inner products on $\mathscr{K}$ are given by $[k_1, k_2] = (x_1, x_2) + [f_1, f_2]$ and $(k_1, k_2) = (x_1, x_2) + (f_1, f_2)$, where $k_1 = \langle x_1, f_1\rangle$ and $k_2 = \langle x_2, f_2\rangle$. If we extend, once more, the definition of J to get an operator on $\mathscr{K}$, by defining $J\langle x, f\rangle = \langle x, Jf\rangle$, then J is a fundamental symmetry on $\mathscr{K}$, satisfying property (i) of Theorem 2.4, and with $(k_1, k_2) = [Jk_1, k_2]$ and $[k_1, k_2] = (Jk_1, k_2)$.

The dilation $U(s)$ is defined in a manner which is analogous to that in [2], but we need to be careful about domains, and the verification of the boundedness of some of the operators involved is not as straightforward. Let us begin by writing down, for $s \geqslant 0$, a definition of $U(s)$ which is valid when A is bounded, and which is equivalent to that in [2]. We will then show how this definition generalizes to the case when A is sectorial. As in [2], we will be using a special symbol to represent the "dummy variable" in $\mathscr{L}$; in this paper, the symbol τ will be used for this purpose. Thus, $f(\tau)$ denotes a function f in $\mathscr{L}$, whereas $f(t)$ denotes a vector in $\mathscr{G}$. The function $f(\tau - s)$, in (4.2) below, is the function obtained by shifting f to the right by s units.

When the infinitesimal generator is bounded, we have [2], for $s \geqslant 0$ and for $k = \langle x, f\rangle \in \mathscr{K}$. the definition $U(s)k = \langle x', f'\rangle$, where

$$x' = T(s)x - \int_0^s T(s-t)JQf(-t)\,\mathrm{d}t \tag{4.1}$$

and

$$f' = f(\tau - s) + \chi_{[0,s]}(\tau)\left[QT(s-\tau)x - \int_0^{s-\tau} QT(s-t-\tau)JQf(-t)\,\mathrm{d}t\right]. \tag{4.2}$$

In order to extend this definition to semigroups with sectorial infinitesimal generator, we must take into account that the operator Q may be unbounded. We will be obtaining estimates which will serve to verify the boundedness of the operators $U(s)$, and the strong continuity of the semigroup $\{U(s) : s \geqslant 0\}$.

Let us introduce some notation for one of the functions appearing in (4.2). We define $R(s):\mathscr{H}\to\mathscr{L}$ by

$$R(s)x=\chi_{[0,s)}(\tau)QT(s-\tau)x,$$

for $x\in\mathscr{H}$ and $s>0$, and define $R(0)=0$. $R(s)$ will serve as the output part of $U(s)$, carrying vectors in $\mathscr{H}$ to functions in the output space $\mathscr{L}_+$. It is not readily apparent that the domain of $R(s)$ will be particularly large; it is a key result that $R(s)$ is in fact bounded and that the span of functions of the form $R(s)x$ is dense in $\mathscr{L}_+$.

THEOREM 4.1. *For all $s\geqslant 0$, $R(s)$ is a bounded operator, with*

$$\|R(s)\|\leqslant e^{\beta s}. \tag{4.3}$$

For each $x, y\in\mathscr{H}$ and $s\geqslant 0$, we have

$$\|R(s)x\|^2\leqslant e^{2\beta s}\|x\|^2-\|T(s)x\|^2 \tag{4.4}$$

and

$$[R(s)x, R(s)y]=(x,y)-(T(s)x, T(s)y). \tag{4.5}$$

The closed linear span of $\{R(s)x:x\in\mathscr{H},\ s>0\}$ equals $\mathscr{L}_+$.

Proof. Choose any $x\in\mathscr{H}$ and $s>0$. By Corollary 2.2, we have $T(s-t)x\in\in\mathscr{D}(A)\subseteq\mathscr{D}(Q)$ for every $t\in[0,s)$, and thus the function $R(s)x$ is well-defined. Also, $R(s)x$ is the limit almost everywhere, as $\varepsilon\to 0^+$, of the functions $R_\varepsilon(s)x= =\chi_{[0,s-\varepsilon]}(\tau)QT(\varepsilon)T(s-\tau-\varepsilon)x$, each of which is continuous, because of the strong continuity of $T(\tau)$ and the boundedness (for $\varepsilon>0$) of the operator $QT(\varepsilon)$ (by Corollary 2.2). Thus, $R(s)x$ is strongly measurable. (See, for example, [3], Section 3.5.) We can use Lemma 3.13 to obtain the estimate

$$\int_0^s\|QT(s-t)x\|^2\,dt=\int_0^s\|QT(t)x\|^2\,dt\leqslant$$

$$\leqslant\int_0^s(-g(T(t)x)+2\beta\|T(t)x\|^2)\,dt\leqslant$$

$$\leqslant\int_0^s-\frac{d}{dt}\|T(t)x\|^2\,dt+\int_0^s 2\beta e^{2\beta t}\|x\|^2\,dt,$$

by Proposition 3.8 and Theorem 2.1. Performing the integration gives (4.4), and (4.3) follows immediately. Also, Lemma 3.13 and Proposition 3.8 imply that

$$[QT(t)x, QT(t)y] = -g(T(t)x, T(t)y) = -\frac{d}{dt}(T(t)x, T(t)y),$$

and integrating this from zero to s gives (4.5).

Let us now show that $\{R(s)x : x \in \mathscr{H}, s > 0\}$ has span that is dense in $\mathscr{L}_+$. Suppose $f \in \mathscr{L}_+$ has the property that $[R(s)x, f] = 0$ for every $x \in \mathscr{H}$ and every $s > 0$; we need to show that $f(t)$ must necessarily be zero, for almost all t. Since $(t) \in \mathscr{G}$, and $\mathscr{G}$ is the closure of the range of Q, it suffices to show that $f(t)$ is orthogonal to the range of Q, for almost all t.

Choose $x \in \mathscr{H}$ and $s > 0$. Then, for all u with $0 \leqslant u \leqslant s$ we have

$$\int_0^u [QT(s-t)x, f(t)]\,dt = [R(u)T(s-u)x, f] = 0.$$

Thus

$$[QT(s-t)x, f(t)] = 0, \tag{4.6}$$

for all $t \in [0, s)$, with the exception of a set $E_{x,s}$ (depending on x and s) of measure zero. Since we are assuming $\mathscr{H}$ is separable, we can obtain a set E_s of measure zero by taking the union of the sets $E_{x,s}$ over a countable dense set in $\mathscr{H}$. For $t \in [0, s)$, the operators $QT(s-t)$ are bounded (since, by Corollary 2.2, the range of $T(s-t)$ is in $\mathscr{D}(A) \subseteq \mathscr{D}(Q)$), so we then have (4.6) valid for all $x \in \mathscr{H}$ and for all $t \in [0, s)$, with $t \notin E_s$. If E is the union of the sets E_s with s rational, then E has measure zero, and (4.6) is valid for all $x \in \mathscr{H}$, for all nonnegative $t \notin E$, and for all rational $s > t$. Therefore, for any nonnegative $t \notin E$, and for any $x \in \mathscr{H}$ and $s > 0$, we have

$$[QT(s)x, f(t)] = [QT(r-t)T(s+t-r)x, f(t)] = 0,$$

where r is any rational number between t and $s+t$. Consequently, for almost all $t \geqslant 0$, $f(t)$ is orthogonal to $\mathscr{G}_0$, where

$$\mathscr{G}_0 = \bigvee \{QT(s)x : x \in \mathscr{H}, s > 0\}.$$

For any real number λ and $x \in \mathscr{H}$ for which the integral exists, the vector

$$\int_0^\infty e^{-\lambda t} QT(t)x\,dt \tag{4.7}$$

is contained in $\mathscr{G}_0$. In fact, the integral exists for all $x \in \mathscr{H}$ and for all $\lambda > \beta$: As with the functions $R(s)x$, we can establish the strong measurability of the integrand, and by using Lemma 3.13, Proposition 3.4, Theorem 2.1, and Corollary 2.2, we obtain the estimate, valid for $t > 0$,

$$\|QT(t)x\|^2 \leqslant -g(T(t)x) + 2\beta\|T(t)x\|^2 =$$

$$= -2\,\mathrm{Re}(AT(t)x,\ T(t)x) + 2\beta\|T(t)x\|^2 \leqslant$$

$$\leqslant 2\|AT(t)x\|\,\|T(t)x\| + 2\beta\|T(t)x\|^2 \leqslant$$

$$\leqslant 2(Mt^{-1} + \beta)\mathrm{e}^{2\beta t}\|x\|^2 + 2\beta \mathrm{e}^{2\beta t}\|x\|^2.$$

Thus, if λ is a real number greater than β, we have

$$\|\mathrm{e}^{-\lambda t}QT(t)x\| \leqslant (2Mt^{-1} + 4\beta)^{1/2}\mathrm{e}^{-(\lambda-\beta)t}\|x\|,$$

and so the integral (4.7) exists. On the other hand, we have, for $\lambda > \beta$,

$$(\lambda - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)x\,\mathrm{d}t.$$

(See [7], Section IX.4, Corollary 1, which should be applied to the bounded semigroup $\mathrm{e}^{-\beta t}T(t)$.) Since Q is closed, it follows ([3], Theorem 3.7.12) that the integral (4.7) equals $Q(\lambda - A)^{-1}x$. The range of $(\lambda - A)^{-1}$ is $\mathscr{D}(A)$, and (4.7) is in $\mathscr{G}_0$. We can therefore conclude that, for almost all t, $f(t)$ is orthogonal to the range of Q_0, where $Q_0 = Q|\mathscr{D}(A)$. Consequently we have, for almost all t, $Q_0^*f(t) = 0$, and thus $f(t)$ is orthogonal to the range of Q_0^{**}, the closure of Q_0. We are wanting to show that $f(t)$ is orthogonal to the range of Q, for almost all t; this follows immediately, since $\mathscr{D}(A)$ is a core of Q (Lemma 3.13), and hence $Q_0^{**} = Q$. ▨

We also define, for $x \in \mathscr{H}$ and for $s > 0$, an operator $\Pi(s) : \mathscr{H} \to \mathscr{L}$ by

(4.8) $$\Pi(s)x = -\chi_{(-s,0]}(\tau)QT(s + \tau)^*x,$$

and define $\Pi(0) = 0$. $\Pi(s)$ plays a similar role for $U(s)^*$ as $R(s)$ does for $U(s)$, mapping vectors in $\mathscr{H}$ to functions in $\mathscr{L}_-$. By symmetry (see Proposition 3.6), we have:

COROLLARY 4.2. *For all $s \geqslant 0$, $\Pi(s)$ is a bounded operator, with*

(4.9) $$\|\Pi(s)\| \leqslant \mathrm{e}^{\beta s}.$$

For each $x, y \in \mathscr{H}$ *and* $s \geqslant 0$, *we have*

$$\|\Pi(s)x\|^2 \leqslant e^{2\beta s}\|x\|^2 - \|T(s)^*x\|^2 \tag{4.10}$$

and

$$[\Pi(s)x, \ \Pi(s)y] = (x, \ y) - (T(s)^*x, \ T(s)^*y). \tag{4.11}$$

The closed linear span of $\{\Pi(s)x : x \in \mathscr{H}, s > 0\}$ *equals* $\mathscr{L}_-$.

In defining $U(s)$, we will be using the adjoint of $\Pi(s)$:

$$P(s) = \Pi(s)^* : \mathscr{L} \to \mathscr{H}, \tag{4.12}$$

where the indefinite inner product is used on $\mathscr{L}$. Note that, since functions in the range of $\Pi(s)$ have support in $[-s, 0]$, $P(s)$ annihilates any function whose support is in the complement of $[-s, 0]$. $P(s)$ will serve as the input part of $U(s)$, carrying functions in the input space $\mathscr{L}_-$ to vectors in $\mathscr{H}$.

COROLLARY 4.3. *For all* $s \geqslant 0$, $P(s)$ *is a bounded operator, with*

$$\|P(s)\| \leqslant e^{\beta s}, \tag{4.13}$$

and

$$P(s)\Pi(s) = I - T(s)T(s)^*. \tag{4.14}$$

Proof. This follows immediately from (4.9) and (4.11). ▨

A simple application of the semigroup property yields:

PROPOSITION 4.4. *If* $0 \leqslant s \leqslant t$, *then*

$$\chi_{[0,s]}R(t) = R(s)T(t - s) \tag{4.15}$$

and

$$\chi_{[-s,0]}\Pi(t) = \Pi(s)T(t - s)^*. \tag{4.16}$$

PROPOSITION 4.5. *If* $0 \leqslant s \leqslant t$, *and* $x, y \in \mathscr{H}$, *then*

$$[R(s)x, \ R(t)y] = (x, \ T(t - s)y) - (T(s)x, \ T(t)y) \tag{4.17}$$

and

$$[\Pi(s)x, \ \Pi(t)y] = (x, \ T(t - s)^*y) - (T(s)^*x, \ T(t)^*y). \tag{4.18}$$

Proof. From (4.15) we get $[R(s)x, R(t)y] = [R(s)x, R(s)T(t - s)y]$, and (4.17) follows from (4.5). We prove (4.18) similarly. ▨

Proposition 4.6. *If* $0 \leqslant s \leqslant t$, *then*

$$P(s)\Pi(t) = T(t-s)^* - T(s)T(t)^*. \tag{4.19}$$

If $0 \leqslant t \leqslant s$, *then*

$$P(s)\Pi(t) = T(s-t) - T(s)T(t)^*. \tag{4.20}$$

Proof. (4.19) follows directly from (4.18). (4.20) can be proved from (4.19) by taking adjoints. ▨

We are now in a position to extend the definition of $U(s)$ in (4.1)–(4.2) to semigrous with sectorial infinitesimal generator. Let us consider first the case where $k = \langle x, f\rangle$ for $f \in \mathscr{L}_+$. We define

$$U(s)\langle x, f\rangle = \langle T(s)x, f(\tau - s) + R(s)x\rangle, \tag{4.21}$$

for $s \geqslant 0$, $x \in \mathscr{H}$, and $f \in \mathscr{L}_+$. Theorem 4.1 implies that (4.21) defines a bounded operator into $\mathscr{K}$, with

$$\begin{aligned} \|U(s)\langle x, f\rangle\|^2 &= \|T(s)x\|^2 + \|f\|^2 + \|R(s)x\|^2 \leqslant \\ &\leqslant \|f\|^2 + e^{2\beta s}\|x\|^2, \end{aligned} \tag{4.22}$$

and the dilation property (ii) of Theorem 2.4 is obvious. We also have $\|(U(s) - I)x\|^2 = \|(T(s) - I)x\|^2 + \|R(s)x\|^2 \to 0$ as $s \to 0^+$, by (4.4) and the strong continuity of $T(s)$. Since $U(s)f = f(\tau - s) \to f$ as $s \to 0^+$, we conclude that

$$U(s)\langle x, f\rangle \to \langle x, f\rangle \quad \text{as } s \to 0^+, \tag{4.23}$$

for all $x \in \mathscr{H}$ and $f \in \mathscr{L}_+$.

Verification of the semigroup property $U(s)U(t)\langle x, f\rangle = U(s+t)\langle x, f\rangle$, for $s, t \geqslant 0$, $x \in \mathscr{H}$, and $f \in \mathscr{L}_+$, comes down to a verification of

$$[R(s)T(t)x](\tau) + [R(t)x](\tau - s) = [R(s+t)x](\tau), \tag{4.24}$$

details of which are similar to those in [2]. The unitary property, $[U(s)k_1, U(s)k_2] = [k_1, k_2]$, follows immediately from property (4.5) of $R(s)$ and the fact that $U(s)x \perp f(\tau - s)$.

The definition of $U(s)$ is completed by defining, for $f \in \mathscr{L}$,

$$U(s)f = \langle P(s)f, f(\tau - s) + \chi_{[0,s]}(\tau)QP(s - \tau)f\rangle. \tag{4.25}$$

Three things should be noted. First, since every operator $P(t)$ annihilates $\mathscr{L}_+$, (4.25) agrees with (4.21) when $f \in \mathscr{L}_+$. Second, (4.25) is an extension of (4.1) and (4.2), since (4.8) and (4.12) imply that

$$P(s)f = -\int_0^s T(s-t)JQf(-t)\,\mathrm{d}t,$$

whenever the integral is defined. Third, (4.25) may not be defined for some $f \in \mathscr{L}$, since we have no guarantee that $P(s-t)f$ is in the domain of Q for almost all $t \in [0, s]$, or that the function $\chi_{[0,s]}(\tau)QP(s-\tau)f$ is in $\mathscr{L}$ if it is defined almost everywhere. We do have:

LEMMA 4.7. *For every $s \geqslant 0$, the operator $U(s)$ is densely defined by* (4.21) *and* (4.25). *In particular, we have, for $x \in \mathscr{H}$,*

$$(4.26)\quad U(s)\Pi(u)x = \langle T(s-u)x - T(s)T(u)^*x,\ R(s-u)x - R(s)T(u)^*x\rangle,$$

whenever $0 < u \leqslant s$, and

$$(4.27)\quad U(s)\Pi(u)x = \langle T(u-s)^*x - T(s)T(u)^*x,\ \Pi(u-s)x - R(s)T(u)^*x\rangle,$$

whenever $0 < s \leqslant u$.

For every $x \in \mathscr{H}$ and $u > 0$, $U(s)\Pi(u)x \to \Pi(u)x$ as $s \to 0^+$.

Proof. (4.25) is defined for any function $f \in \mathscr{L}_+$. By Corollary 4.2, the proof that $U(s)$ is densely defined will be completed once (4.26) and (4.27) are established.

For any $s, u > 0$, the part of $U(s)\Pi(u)x$ in $\mathscr{H}$ is $P(s)\Pi(u)x$, and Proposition 4.6 shows that this agrees with (4.26) and (4.27). Note that, if u is constant, then (4.19) and the strong continuity of $T(\tau)$ and $T(\tau)^*$ show that $P(s)\Pi(u)x \to 0$ as $s \to 0^+$.

Suppose $0 < u \leqslant s$. For the part of $U(s)\Pi(u)x$ in $\mathscr{L}$, we have, using Proposition 4.6,

$$\begin{aligned}
&[\Pi(u)x](\tau - s) + \chi_{[0,s]}(\tau)QP(s-\tau)\Pi(u)x = \\
&= -\chi_{[-u,0]}(\tau - s)QT(u+\tau-s)^*x + \\
&+\chi_{[0,s-u]}(\tau)Q(T(s-\tau-u)x - T(s-\tau)T(u)^*x) + \\
&+\chi_{[s-u,s]}(\tau)Q(T(u-s+\tau)^*x - T(s-\tau)T(u)^*x) = \\
&= \chi_{[0,s-u]}(\tau)QT(s-u-\tau)x - \chi_{[0,s]}(\tau)QT(s-\tau)T(u)^*x,
\end{aligned}$$

since $\chi_{[-u,0]}(\tau - s) = \chi_{[s-u,s]}(\tau)$, agreeing with (4.26).

Now suppose that $0 < s \leqslant u$. Proposition 4.6 gives us for the part of $U(s)\Pi(u)x$ in $\mathscr{L}$:

$$(4.28)\quad \begin{aligned}&[\Pi(u)x](\tau - s) + \chi_{[0,s]}(\tau)Q(T(u - s + \tau)^* x - T(s - \tau)T(u)^* x) = \\ &= [\Pi(u)x](\tau - s) - \chi_{[-s,0]}(\tau - s)[\Pi(u)x](\tau - s) - R(s)T(u)^* x.\end{aligned}$$

If we let $s \to 0^+$, then the first term in (4.28) converges to $\Pi(u)x$, whereas the other two terms converge to zero (using (4.4) for the third term). Thus we have $U(s)\Pi(u)x \to \Pi(u)x$ as $s \to 0^+$.

The first two terms of (4.28) can be rewritten as

$$\chi_{[-u,-s]}(\tau - s)[\Pi(u)x]\,(\tau - s) = -\chi_{[-u+s,0]}(\tau)QT(u + \tau - s)^* x = \Pi(u - s)x,$$

and (4.27) is proved. ▨

The definition of $U(s)$ will be completed by showing that, for each $s \geqslant 0$, $U(s)$ is bounded, and therefore can be extended by continuity to all of $\mathscr{K}$. Before doing that, we will verify that, on a dense set of vectors, $U(s)$ has the semigroup property and preserves the indefinite inner product.

We have verified the semigroup property on the subspace of vectors of the form $\langle x, f\rangle$, where $x \in \mathscr{H}$ and $f \in \mathscr{L}_+$. We will now show that $U(s)U(t)f = U(s + t)f$ for any $s, t \geqslant 0$, and for a dense set of f in $\mathscr{L}_-$. The result is trivial if the support of f is contained in $(-\infty, -s-t]$, and so, by Corollary 4.2 and Proposition 4.4, it suffices to verify the result for functions of the form $\Pi(u)x$, where $x \in \mathscr{H}$ and $0 < u \leqslant s + t$.

By Lemma 4.7, we have the following expression for $U(s + t)\Pi(u)x$:

$$(4.29)\quad \langle T(s + t - u)x - T(s + t)T(u)^* x,\ R(s + t - u)x - R(s + t)T(u)^* x\rangle.$$

The expression for $U(t)\Pi(u)x$ depends on whether $u \leqslant t$ or $u \geqslant t$. If $u \leqslant t$, we use (4.26) to calculate $U(s)U(t)\Pi(u)x$. The part in $\mathscr{H}$ is given by $T(s)(T(t - u)x - T(t)T(u)^* x)$, which agrees with (4.29), whereas the part in $\mathscr{L}$ is

$$R(s)(T(t - u)x - T(t)T(u)^* x) + [R(t - u)x - R(t)T(u)^* x](\tau - s).$$

Two applications of the formula (4.24) then give the second part of (4.29).

If $u \geqslant t$, we use (4.27) to calculate $U(t)\Pi(u)x$. One term that is obtained is $\Pi(u - t)x$, and (4.26) is used for this term in the calculation of $U(s)U(t)\Pi(u)x$. After simplification, (4.29) is obtained; details are omitted.

We now wish to verify that $U(s)$ preserves the indefinite inner product, i.e.,

$$(4.30)\quad [U(s)k_1,\ U(s)k_2] = [k_1,\ k_2],$$

for k_1, k_2 belonging to a linear manifold dense in $\mathscr{K}$. Let us consider three subspaces of $\mathscr{K}$, mutually orthogonal in both inner products: $\mathscr{K}_+$, the subspaces of vectors of the form $\langle x, f\rangle$, where $x \in \mathscr{H}$ and $f \in \mathscr{L}_+$; $\mathscr{L}_s$, the subspace of $\mathscr{L}_-$ consisting of functions with support in $[-s, 0]$; and $\mathscr{L}_\infty$, the subspace of $\mathscr{L}$ consisting of functions with support in $(-\infty, -s]$. We have already shown that $U(s)$ preserves the indefinite inner product on $\mathscr{K}_+$, and it is trivial that (4.30) is satisfied if k_1 and k_2 are in $\mathscr{L}_\infty$. It is also trivial that both sides of (4.30) are zero if $k_1 \in \mathscr{L}_\infty$ and $k_2 \in \mathscr{L}_s \cap \cap \mathscr{D}(U(s))$, or if $k_1 \in \mathscr{L}_\infty$ and $k_2 \in \mathscr{K}_+$, or $k_1 \in \mathscr{L}_s \cap \mathscr{D}(U(s))$ and $k_2 \in \mathscr{L}_+$. Functions of the form $\Pi(u)x$, with $x \in \mathscr{H}$ and $0 < u \leqslant s$ span a dense linear manifold of $\mathscr{L}_s$. Thus, to prove (4.30) on a dense linear manifold of $\mathscr{K}$, it suffices to show that

$$[U(s)\Pi(u)x,\ U(s)\Pi(v)y] = [\Pi(u)x,\ \Pi(v)y]$$

and

$$[U(s)\Pi(u)x,\ U(s)y] = 0$$

for all $x, y \in \mathscr{H}$ and for all u, v with $0 < v \leqslant u \leqslant s$. These can easily be proved by using the expression (4.26) for $U(s)\Pi(u)x$ and $U(s)\Pi(v)y$, and (4.21) for $U(s)y$. The inner products are computed with the aid of Proposition 4.5; the details of the routine calculation are omitted.

We now complete the definition of $U(s)$ by showing that it is bounded on its domain of definition. The only part of this demonstration that has not already been verified, or is not trivial, is to show that $\|U(s)f\| \leqslant K\|f\|$, for some constant K, for a dense set of f with support in the interval $[-s, 0]$.

Suppose f is a linear combination of functions of the form $\Pi(u)x$, where $x \in \mathscr{H}$ and $0 < u \leqslant s$. Such f are dense in $\mathscr{L}_s$, and we have shown in Lemma 4.7 that, for such a function f, $U(s)f$ is defined. Using (4.25), we have

$$\begin{aligned}\|U(s)f\|^2 &= \|P(s)f\|^2 + \|f(\tau - s) + m\|^2 \leqslant \\ &\leqslant \|P(s)f\|^2 + 2\|f\|^2 + 2\|m\|^2,\end{aligned} \tag{4.31}$$

where $m(\tau) = \chi_{[0,s]}(\tau)QP(s-\tau)f$. By Lemma 3.13, we have

$$\begin{aligned}\|m\|^2 &= \int_0^s \|QP(s-t)f\|^2\,\mathrm{d}t = \int_0^s \|QP(t)f\|^2\,\mathrm{d}t \leqslant \\ &\leqslant -\int_0^s g(P(t)f)\,\mathrm{d}t + \int_0^s 2\beta\|P(t)f\|^2\,\mathrm{d}t \leqslant \\ &\leqslant -\int_0^s g(P(t)f)\,\mathrm{d}t + (\mathrm{e}^{2\beta s} - 1)\|f\|^2,\end{aligned} \tag{4.32}$$

using the estimate (4.13) in Corollary 4.3 for $\|P(t)f\|$. An estimate for the integral appearing in the last line of (4.32) is found by considering the indefinite inner product of $U(s)f - f(\tau - s)$ with itself:

$$[U(s)f - f(\tau - s),\ U(s)f - f(\tau - s)] = \|P(s)f\|^2 + [m, m] =$$

$$(4.33) \qquad = \|P(s)f\|^2 + \int_0^s [QP(t)f,\ QP(t)f]\,dt =$$

$$= \|P(s)f\|^2 - \int_0^s g(P(t)f)\,dt,$$

by Lemma 3.13. Note that (4.30) has been verified for vectors k_1 and k_2 which are linear combinations of functions of the form $\Pi(u)x$. Thus we can use (4.30) to show that the left side of (4.33) is

$$2[f, f] - 2\mathrm{Re}[U(s)f, f(\tau - s)] \leqslant 2\|f\|^2 + 2\|U(s)f\|\,\|f\|.$$

Combining this with (4.32) and (4.33), we obtain

$$\|m\|^2 \leqslant 2\|U(s)f\|\,\|f\| + (e^{2\beta s} + 1)\|f\|^2 - \|P(s)f\|^2,$$

and therefore, by (4.31),

$$(4.34) \qquad \|U(s)f\|^2 \leqslant 4\|U(s)f\|\,\|f\| + (2e^{2\beta s} + 4)\|f\|^2.$$

This gives us our bound, since we can divide (4.31) by $\|U(s)f\|$ (assuming, without any loss of generality, that this is at least as big as $\|f\|$), and obtain

$$(4.35) \qquad \|U(s)f\| \leqslant (2e^{2\beta s} + 8)\|f\|.$$

The estimates (4.35) and (4.22) show that, for each $s \geqslant 0$, $U(s)$ is a bounded operator; since we have shown $U(s)$ is densely defined, it extends by continuity to be defined on all of $\mathscr{H}$. The semigroup property and property (iii) of Theorem 2.4 have been verified on a dense set of vectors in $\mathscr{H}$, and therefore are valid on the whole space. We have shown that $U(s)k \to k$, as $s \to 0^+$, for a set of vectors dense in $\mathscr{H}$ (Lemma 4.7 and (4.23)). Since the estimates (4.22) and (4.35) for the norm of $U(s)$ show that $U(s)$ is uniformly bounded on the interval $[0, 1]$, it follows that $U(s)$ is strongly continuous.

As in [2], we can show that, for $s \geqslant 0$, $U(s)$ is invertible. Indeed, it is obvious that any function f with support in the complement of $[0, s]$ is in the range of $U(s)$, and for any $x \in \mathscr{H}$, the definition (4.21), and Lemma 4.7, show that we have

$$U(s)\langle T(s)^* x,\ \Pi(s)x\rangle = x,$$

so that $\mathscr{H}$ is in the range of $U(s)$. If $0 < t \leqslant s$, then x is in the range of $U(s - t)$, and the semigroup property then implies that $R(t)x$ is in the range of $U(s)$. Thus, by Theorem 4.1 and Proposition 4.4, $U(s)$ has dense range. Property (iii) of Theorem 2.4, proved earlier, shows that $U(s)^*$ (where the adjoint is computed in the indefinite inner product) is a bounded inverse of $U(s)$ (cf. [2]).

The semigroup $\{U(s): s \geqslant 0\}$ can be extended to a strongly continuous group by defining, for $s > 0$, $U(-s) = U(s)^*$. It is straightforward to verify that we then have, for $x \in \mathscr{H}$ and $s > 0$,

$$U(-s)x = \langle T(s)^* x,\ \Pi(s)x\rangle.$$

With this, and (4.21), we can use Theorem 4.1 and Corollary 4.2 to prove the minimality property (iv), and the proof of Theorem 2.4 is complete.

5. AN EXAMPLE

With the dilation constructed in Section 4, we have an input space $\mathscr{L}_-$ and an output space $\mathscr{L}_+$, characterized by

$$\mathscr{H} \oplus \mathscr{L}_+ = \bigvee \{U(s)\mathscr{H} : s \geqslant 0\}, \quad \mathscr{H} \oplus \mathscr{L}_- = \bigvee \{U(s)\mathscr{H} : s \leqslant 0\}.$$

There is symmetry between these two subspaces: the action of $U(s)$ on $\mathscr{L}_+$ is isomorphic to the action of $U(s)^*$ on $\mathscr{L}_-$. In what sense can we relax the requirement that the infinitesimal generator A be sectorial, and still expect to obtain this symmetry in the dilation? The following example shows that if the numerical range is a half-plane, then the symmetry in the dilation can be lost.

EXAMPLE 5.1. Let $\mathscr{H}$ be the Hilbert space $L^2(0, \infty)$ of square-integrable complex-valued functions, with support in $(0, \infty)$, and let $T(t)$ be the backward shift on $\mathscr{H} : T(t)x = x(\tau + t)$. The infinitesimal generator A of $T(t)$ is the differentiation operator, with domain equal to all absolutely continuous functions ([4], Example IX.1.8). We have $(Ax, x) = -|x(0)|^2$, and so the numerical range of A is the half-plane $\{\lambda : \operatorname{Re}\lambda \leqslant 0\}$. Since $T(t)$ is a contraction semigroup, it has a unique minimal unitary dilation, given by $\mathscr{K} = L^2(-\infty, \infty)$ and $U(s)f = f(\tau + s)$. It is easy to see that, for this dilation, $\mathscr{L}_+ = L^2(-\infty, 0)$, whereas $\mathscr{L}_- = \{0\}$. ▨

In Example 5.1, an attempt to construct the form g of Section 3 gives, for $x, y \in \mathscr{D}(A)$, $g(x, y) = x(0)\overline{y(0)}$, which is not closable (see [4], Example VI.1.26). However, (4.21) can be used to define $U(s)$ on a dense linear manifold of $\mathscr{K}$ if, in the definition of $R(s)$, we let Q be the (non-closed) operator from $\mathscr{H}$ to the complex numbers defined by $Qx = x(0)$. (We do not need (4.25), since $\mathscr{L}_- = \{0\}$ in this case.) This sugests that a non-symmetric extension of the theory should be possible.

Acknowledgement. Research for this paper was done while the author was visiting at the Centre for Mathematical Analysis, at the Australian National University, during 1987. The support of the Centre and the Ohio State University, for this visit, is gratefully acknowledged.

REFERENCES

1. BOGNAR, J., *Indefinite inner product spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.
2. DAVIS, CH., Dilation of uniformly continuous semi-groups, *Rev. Roumaine Math. Pures Appl.*, 15(1970), 975–983.
3. HILLE, E.; PHILLIPS, R. S., *Functional analysis and semi-groups*, American Mathematical Society, Providence, 1957.
4. KATO, T., *Perturbation theory for linear operators*, Second edition, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
5. SCHECHTER, M., *Principles of functional analysis*, Academic Press, New York, London, 1971.
6. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, London, 1970.
7. YOSIDA, K., *Functional analysis*, Sixth edition, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

BRIAN McENNIS
Centre for Mathematical Analysis,
The Australian National University,
Canberra, A.C.T. 2601,
Australia.

current address:

Department of Mathematics,
The Ohio State University,
Marion, Ohio 43302,
U.S.A.

Received December 2, 1987.