J. OPERATOR THEORY
23(1990), 115–123

INVARIANT SUBSPACES FOR AN OPERATOR ON $L^2(\Pi)$ COMPOSED OF A MULTIPLICATION AND A TRANSLATION

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Throughout what follows the set $[0, 1)$ is taken to be identified in the usual way as a topological group with Π, the unit circle in the complex plane. In this paper we give sufficient conditions for an operator on $L^2[0,1)$ composed of a multiplication operator and a translation to possess an invariant subspace. In fact all that follows is valid for $L^p[0, 1)$, $1 \leqslant p \leqslant \infty$.

More specifically, let α be a fixed number in $[0, 1)$ and let φ be a fixed non-zero continuous function on $[0, 1)$. This implies that $\varphi(0) = \varphi(1^-)$. Define an operator T on $L^2[0,1)$ by

$$Tf(x) = \varphi(x)f(x + \alpha) \quad (x \in [0, 1))$$

for each $f \in L^2[0, 1)$. Here addition is modulo 1 of course. Thus if M_φ is the multiplication operator on $L^2[0, 1)$

$$M_\varphi f(x) = \varphi(x)f(x) \quad (x \in [0, 1)),$$

and if S_α is the translation operator

$$S_\alpha f(x) = f(x + \alpha) \quad x \in [0,1)),$$

then we have

$$T = M_\varphi S_\alpha .$$

This operator is related to a class of operators introduced by Bishop as candidates for operators possibly not possessing invariant subspaces. Subsequently almost all of these have been shown by A. M. Davie [2] to have hyperinvariant subspaces.

DEFINITION. An irrational number l is called a *Liouville number* if for each natural number n there exist integers p and q with $q \geqslant 2$ such that

$$|l - p/q| < q^{-n} .$$

One can show [3] that the set of Liouville numbers is dense in $\mathbf{R}$ but has s-dimensional Hausdorff (and consequently also Lebesgue) measure zero for all $s > 0$.

We shall need the following theorem:

THEOREM (Wermer [5], [1]). *Let X be a Banach space and suppose R is an invertible operator on X satisfying the following two conditions:*

(i) *the spectrum of R contains more than one point, and*

(ii) $\sum_{n=-\infty}^{\infty} \log\|R^n\|/(1+n^2) < \infty$.

Then R possesses a non-trivial hyperinvariant subspace.

The result we give below asserts that the operator T defined above possesses an invariant subspace provided that α is not a Liouville number and provided that φ is sufficiently smooth. For a bounded function $g:[0,1)\to\mathbf{C}$ the *modulus of continuity* ω_g of g is defined to be

$$\omega_g(\delta) = \sup\{|g(x) - g(x')| : |x - x'| \leqslant \delta\},$$

for $\delta \geqslant 0$. This is an increasing function of $\delta \geqslant 0$. Note that ω_g is not quite the usual modulus of continuity [6] since the subtraction above on $[0,1)$ is modulo 1; however if $g(1^-) = g(0)$ then ω_g is bounded above and below by a constant multiple of the usual modulus of continuity.

Suppose g is a fixed non-vanishing complex valued function on $[0,1)$, with $g(0) = g(1^-)$, such that g and g^{-1} are bounded. Let ω_g be the modulus of continuity of g. Notice that if $|x - x'| \leqslant t$ then

$$|g(x)| \leqslant |g(x')| + \omega_g(t),$$

and consequently

$$\big|\log|g(x)| - \log|g(x')|\big| = |\log(|g(x)|/|g(x')|)| \leqslant$$

$$\leqslant \log(1 + \|g^{-1}\|_\infty \omega_g(t)) \leqslant \|g^{-1}\|_\infty \omega_g(t).$$

Thus it is clear that the modulus of continuity of $\log|g|$ is dominated by a constant multiple of the modulus of continuity of $|g|$.

THEOREM. *Let $\alpha \in [0,1)$. Let φ be a fixed non-vanishing continuous complex valued function $[0,1)$ (with $\varphi(0) = \varphi(1^-)$). The operator T on $L^2[0,1)$ defined by*

$$Tf(x) = \varphi(x)f(x+\alpha) \qquad (x \in [0,1)),$$

where the addition is modulo 1, possesses an invariant subspace provided that α is not a Liouville number and provided that the modulus of continuity ω of φ, or even of log $|\varphi|$, *satisfies*

$$\int_0^1 (\omega(t)/t)\,\mathrm{d}t < \infty.$$

If, in addition, T is not a scalar multiple of the identity then T possesses a hyperinvariant subspace.

Proof. If α is rational then we do not need the smoothness condition for φ: if $\alpha = p/q$, for some $p, q \in \mathbf{N}$, then T^q is a multiplication operator. If T^q is not a scalar multiple of the identity then T^q has a hyperinvariant subspace, which will also be a hyperinvariant subspace of T. If T^q is a scalar multiple of the identity, but T is not, then the spectrum of T is a finite set with more than one point; then the range of a non-trivial spectral projection gives a hyperinvariant subspace.

Assume henceforth then that α is irrational, and put $\psi = \log|\varphi|$. By the remark immediately before the statement of the theorem we may as well assume the integral condition holds for the modulus of continuity ω of ψ. If n is a non-negative integer and we put

$$\varphi_n = \prod_{k=0}^{n-1} (S_\alpha^k \varphi)$$

hen we have $T^n = M_{\varphi_n} \circ S_\alpha^n$ and so $\|T^n\| = \|\varphi_n\|_\infty$. Similarly $\|T^{-n}\| = \|\varphi_n^{-1}\|_\infty$. Thus as $n \to \infty$

$$\log\|T^n\|/n = \sup\left\{\left(n^{-1}\sum_{k=0}^{n-1} S_\alpha^k(\psi)\right)(x) : x \in [0, 1)\right\} \to \int_0^1 \psi\,\mathrm{d}t$$

by the uniform ergodic theorem (see [4], 1.1), if not by more elementary considerations. We may conclude from this that $r(T)$, the spectral radius of T, satisfies

$$r(T) = \exp\left(\int_0^1 \psi\,\mathrm{d}t\right).$$

If M is the unitary multiplication operator on $L^2[0, 1)$ given by

$$Mf(x) = \mathrm{e}^{2\pi\mathrm{i}x} f(x) \quad (x \in [0, 1)),$$

for $f \in L^2[0, 1)$, then it is easy to see that

$$T - e^{2\pi i\alpha}\lambda I = e^{2\pi i\alpha}M(T - \lambda I)M^{-1}$$

for any $\lambda \in \mathbf{C}$. This shows that the spectrum of T is invariant under rotation by $e^{2\pi i\alpha}$, and so certainly contains more than one point. Indeed it is easy to see that the spectrum $\sigma(T)$ is the circle of radius $r(T)$, centred at 0, but we shall not explicitly need this fact. Normalize the operator T by setting

$$R = r(T)^{-1}T.$$

This is equivalent to scaling φ by a constant

For a bounded function $g : [0, 1) \to \mathbf{C}$ let us write $D(g, n)$ for the discrepancy

$$D(g, n) = \sup\left\{ \left| \left(n^{-1} \sum_{k=0}^{n-1} S_\alpha^k g\right)(x) - \int_0^1 g \, dt \right| : x \in [0, 1)\right\}.$$

We now appeal to Wermer's theorem quoted above to deduce that the operator R has a non-trivial hyperinvariant subspace if

$$\sum_{n=1}^{\infty} n^{-1} \sup\left\{ \left(n^{-1} \sum_{k=0}^{n-1} S_\alpha^k g\right)(x) - \int_0^1 g \, dt : x \in [0, 1)\right\} < \infty$$

for $g = \psi$ and $g = -\psi$. We may rewrite this condition as

$$(*) \qquad \sum_{n=1}^{\infty} n^{-1} D(\psi, n) < \infty.$$

Now since α is not a Liouville number by elementary number theory (see [2]) here exist $K, N \in \mathbf{N}$ such that if n is a positive integer greater than N then there exists $p, q \in \mathbf{N}$, with p and q coprime, such that both

$$n^{1/K} \leqslant q \leqslant n^{1/2}$$

and

$$|\alpha - p/q| \leqslant q^{-2}$$

hold. For such n we may write $n = rq + s$, for some non-negative integers r and s, with $s < q$. We obtain

$$\left| n^{-1} \sum_{k=0}^{n-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt \right| \leqslant$$

$$\leqslant n^{-1} \left| \sum_{k=rq}^{n-1} \psi(x + k\alpha) \right| + \left| (n^{-1} - (rq)^{-1}) \sum_{k=0}^{rq-1} \psi(x + k\alpha) \right| +$$

$$+ \left| (rq)^{-1} \sum_{k=0}^{rq-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt \right| \leqslant$$

$$\leqslant 2(q/n) \|\psi\|_\infty + \left| (rq)^{-1} \sum_{k=0}^{rq-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt \right| \leqslant$$

$$\leqslant r^{-1} \sum_{j=0}^{r-1} \left| q^{-1} \sum_{k=0}^{q-1} \psi(x + jq\alpha + k\alpha) - \int_0^1 \psi \, dt \right| + O(n^{-1/2}).$$

As an integer a runs 1 to q, the number ap/q assumes each of the values 0, $1/q, \ldots, (q-1)/q$ in some order (modulo 1 of course). Since $|a\alpha - ap/q)| \leqslant q^{-1}$ the following assertion is clear: for each $x \in [0, 1)$ there is a partition of $[0, 1)$ into disjoint intervals $I_0, \ldots, I_{q-1}$ each of length q^{-1} such that each of the q numbers $x, x + \alpha, \ldots, x + (q-1)\alpha$ may be associated with a unique interval $I_0, \ldots, I_{q-1}$ respectively which it lies within a distance of q^{-1} from.

By the mean value theorem we may for each $k = 0, \ldots, (q-1)$ choose $\xi_k \in I_k$ such that

$$|I_k|^{-1} \int_{I_k} \psi \, dt = \psi(\xi_k).$$

Then

$$\left| \psi(x + k\alpha) - |I_k|^{-1} \int_{I_k} \psi \, dt \right| \leqslant \omega(2/q),$$

where ω is the modulus of continuity of ψ, and so

$$\left| q^{-1} \sum_{k=0}^{q-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt \right| \leqslant q^{-1} \sum_{k=0}^{q-1} \left| \psi(x + k\alpha) - q \int_{I_k} \psi \, dt \right| \leqslant \omega(2/q).$$

Thus for any $x \in [0, 1)$ we see that

$$r^{-1} \sum_{j=0}^{r-1} \left| q^{-1} \sum_{k=0}^{q-1} \psi(x + jq\alpha + k\alpha) - \int_0^1 \psi \, dt \right| \leqslant \omega(2/q)$$

and so

$$D(\psi, n) \leqslant \omega(2/q) + O(n^{-1/2}) \leqslant$$

$$\leqslant \omega(2n^{-1/K}) + O(n^{-1/2}).$$

Consequently (*) is satisfied if

$$\sum_{n=1}^{\infty} n^{-1}\omega(2n^{-1/K}) < \infty.$$

which proves the theorem after an application of the integral test of elementary undergraduate analysis. ▨

For $s > 0$ let Λ_s be the *Hölder class* [6]: the class of those bounded complex valued functions g on $[0, 1)$ for which there exists a constant $C > 0$ such that the modulus of continuity ω of g satisfies

$$\omega(\delta) \leqslant C\delta^s \quad (\delta \geqslant 0).$$

COROLLARY. *The operator T defined above possesses an invariant subspace if α is not a Liouville number and if the function φ, or even* $\log|\varphi|$, *is in the Hölder class Λ_s for some $s > 0$.*

Proof. If $\varphi \in \Lambda_s$ then $\log|\varphi| \in \Lambda_s$, by the remark above the previous theorem. Thus if either φ or $\log|\varphi|$ is in Λ_s for some $s > 0$ and if ω is the modulus of continuity of $\log|\varphi|$ then

$$\int_0^1 (\omega(t)/t) \, dt < \infty.$$

An application of the theorem above completes the proof. ▨

REMARK It would be of interest if one could enlarge the set of numbers α or the set of functions φ for which the result holds. It is not difficult to see how the result might be adapted to the case when φ is bounded away from zero and permitted to be discontinuous at, at most a finite number of points. It is probably possible to use the method of [2] to extend this result to certain cases when φ is permitted to assume the value 0 on a set of measure zero. It is easy to see that if φ is zero on a set of positive measure then the associated operator T has an invariant subspace.

The hypothesis on α can be weakened at the cost of stronger hypotheses on φ. This is most easily seen using the Fourier series of ψ; for example, one can prove the following:

PROPOSITION. *Suppose α is irrational and $\psi = \log|\varphi|$ has Fourier series*

$$\psi(x) = \sum_{-\infty}^{\infty} a_m e^{2\pi i m x}.$$

Then

(a) *if* $\sum_{m=1}^{\infty} |a_m|(1 + \log^+|1 - e^{2\pi i m\alpha}|^{-1}) < \infty$ *then T has a hyperinvariant subspace;*

(b) *if* $\sum_{m=1}^{\infty} |a_m|\,|1 - e^{2\pi i m\alpha}|^{-1} < \infty$ *then T is similar to a scalar multiple of a unitary operator.*

Proof. Multiplying T by a scalar we may assume that $\int\psi = 0$, so $a_0 = 0$ Since ψ is real, $a_{-m} = \bar{a}_m$.

(a) We have

$$n^{-1}\sum_{k=0}^{n-1}\psi(x + k\alpha) = n^{-1}\sum_{m\neq 0} a_m \sum_{k=0}^{n-1} e^{2\pi i m(x+k\alpha)} =$$

$$= n^{-1}\sum_{m\neq 0} a_m(1 - e^{2\pi i m n\alpha})(1 - e^{2\pi i m\alpha})^{-1}e^{2\pi i m x}$$

so

$$|D(\psi, n)| \leqslant 2n^{-1}\sum_{m=1}^{\infty} |a_m| \min(n, 2|1 - e^{2\pi i m\alpha}|^{-1})$$

and

$$\sum_{n=1}^{\infty} n^{-1}|D(\psi, n)| \leqslant 2\sum_{m=1}^{\infty}|a_m|\sum_{n=1}^{\infty} n^{-1}\min(n, 2|1 - e^{2\pi i m\alpha}|^{-1}) \leqslant$$

$$\leqslant \text{const}\sum_{m=1}^{\infty}|a_m|(1 + \log^+|1 - e^{2\pi i m\alpha}|^{-1}) < \infty$$

by hypothesis, where we have used the inequality

$$\sum_{n=1}^{\infty} n^{-2}\min(n, \lambda) = \sum_{n\leqslant\lambda} n^{-1} + \lambda\sum_{n>\lambda} n^{-2} \leqslant \log\lambda + \text{const.} \quad (\lambda \geqslant 1).$$

It then follows as before from Wermer's theorem that T has a hyperinvariant subspace,

(b) Let

$$u(x) = \sum_{m \neq 0} a_m (1 - e^{2\pi i m\alpha})^{-1} e^{2\pi i m x};$$

the series converges absolutely by hypothesis, so u is continuous on the circle, moreover $u(x) - u(x+\alpha) = \psi(x)$. Now let $v = e^u$, and write $\varphi(x) = \eta(x)|\varphi(x)| = \eta(x)e^{\psi(x)}$ where $|\eta(x)| = 1$. Then $M_v M_\eta S_\alpha M_v^{-1} = T$, so that T is similar to the unitary operator $M_\eta S_\alpha$. ▨

COROLLARY 1. *If ψ has an absolutely convergent Fourier series then for almost all choices of α, T has a hyperinvariant subspace.*

Proof. $\int_0^1 \log^+|1 - e^{2\pi i m\alpha}|^{-1} d\alpha$ is a finite constant independent of m, so by integrating we find that if $\sum |a_m| < \infty$ then the hypothesis of (a) holds for almost all α. ▨

COROLLARY 2. *If ψ is infinitely differentiable on the circle and α is not a Liouville number then T is similar to a scalar multiple of a unitary operator.*

Proof. In this case $|a_m| = O(m^{-k})$ for all k and $|1 - e^{2\pi i m\alpha}|^{-1} = O(m^k)$ for some k, so the hypothesis of (b) holds. ▨

Finally we note that if α is not a Liouville number and $\sum_1^\infty |a_m| \log m < \infty$ then by (a), T has a hyperinvariant subspace. However, since $\omega(t) \leqslant \sum |a_m| \min(2, 2\pi m t)$ and $\int t^{-1} \min(2, 2\pi m t) <$ const. $+ \log m$, the condition $\sum |a_m| \log m < \infty$ implies $\int t^{-1}\omega(t) < \infty$, so this result also follows from the theorem.

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Received October 18, 1988.

Note added in proof. G. W. MacDonald ("Invariant subspaces for generalised Bishop operators", to appear) has obtained results which overlap with those of the present paper and also cover many non-invertible operators.