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STABILITY AND INSTABILITY OF L^p-SPECTRA OF DIFFERENTIAL OPERATORS

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A differential operator $A = \sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha$ defined on a minimal domain $\mathscr{D}_0$ (e.g., $C_0^\infty(\mathbf{R}^n)$) can be extended to a closed operator in L^2 (L^p-spaces), by taking the closure $\mathscr{D}_p$ of $\mathscr{D}_0$ in the graph-norm $\|f\|_A = \|f\|_p + \|Af\|_p$. Moreover, such extension is often shown to be unique, in the sense that the dual operator $(A \mid \mathscr{D}_p)^*$ in the dual space $L^{p'}$, $p' = p/(p-1)$ has the same domain as the formal adjoint $A^* = \sum D^\alpha(a_\alpha(x)\ldots)$, or $(A \mid \mathscr{D}_p)^* = A \mid \mathscr{D}_{p'}$ in case of formally symmetric A.

There are numerous results of this sort for fairly general classes of differential or pseudodifferential operators A (see for instance [7], [8], [13], [10], [11], [2], [3]).

It is natural then to ask what happens to L^p-spectra of such operators A, when p varies from 1 to ∞. For Schrödinger operators this problem was raised and partly answered by B. Simon [10], [11].

In most known cases the prevailing phenomenon is spectral stability: "L^p-spectrum" = "L^2-spectrum" over the whole range of p or some interval $|1/2 - 1/p| < \varepsilon$, after which spectrum may change abruptly, typically filling in the whole plane.

The following examples will ilustrate this.

1. *Constant coefficient* elliptic or subelliptic operators A are spectrally stable over the whole range ($1 \leqslant p < \infty$) in the first (elliptic) case, or some interval $\{|1/p - 1/2| < \varepsilon < 1/2\}$ in the second (subelliptic) (see [5], [8]).

The same holds for relatively small perturbations $A + B$ ([8], [14], [2]).

2. *Schrödinger operators*: $-\Delta + V(x)$ on $\mathbf{R}^n$. If $V(x)$ is relatively compact L^p-spectral stability follows from the above discussion (see also [11]).

Recently this result was extended to fairly large classes of "noncompact" potentials, including N-body $V = \sum_{i<j} V(x_i - x_j)$ ([9]), periodic V, and yet more general classes $\hat{K}_n$ ([4]),which incorporate both.

3. *Degenerate-singular elliptic operators.* One such class of operators was studied in [2] (see also [13]). It was modeled after Schrödinger-type operators: $-\nabla\cdot\rho\nabla + V$, where metric $\rho(x)$ is allowed to "degenerate" and potential V to "blow up" to certain degrees on a "small" closed subset $\Sigma \subset \mathbf{R}^n$. The degeneracy and singularity can be measured in terms of the distance function $d(x) = \mathrm{dist}(x; \Sigma)$, as

$$\rho(x) \approx d(x)^{\alpha}; \quad V(x) \approx d(x)^{-\beta}.$$

The classical examples include "Legendre", "Tricomi" and "Laguerre" operators: $A = \partial(1 - x^2)\partial$; $A = \partial_x^2 + x\partial_y^2$ and $A = \partial x\partial + \dfrac{x^2 + \alpha^2}{4x}$.

Among other results of [3] we established L^p-stability of the discrete part of spectra, $\sigma_d(A)$ = "isolated eigenvalues". In particular, operators with purely discrete spectrum (like the classical Laguerre) were shown to be spectrally stable in the whole range of $L^p (1 < p < \infty)$.

The behavior of continuous spectra however remained open. In the present note we shall demonstrate the phenomena of continuous spectral stability and instability for operators of the Laguerre-type: $L = -\partial x^{\alpha}\partial + bx^{-\beta}$ on $[0, \infty)$, $\alpha, \beta > 0$ with proper "Dirichlet" boundary conditions at $\{0\}$ and ∞.

Specifically we shall study two examples:

$$L = -\partial x^{2\alpha}\partial + bx^{-2\beta}; \quad \text{with } \alpha + \beta = 1,\ \alpha, \beta > 0,$$

$$M = -\partial x^2\partial + bx^{-2\beta}; \quad \beta > 0.$$

The common feature of both is their relation to the Bessel's equation. In both cases the fundamental solution, resp. Green's function, can be explicitly written in terms of Bessel functions. However, spectral properties of L and M are quite different. Namely,

THEOREM. (i) *Operator L is L^p-stable,* i.e. *L^p-spectrum = L^2-spectrum = $[0, +\infty)$ for all $1 \leqslant p < \infty$.*

(ii) *Operator M is unstable, its L^p-spectrum coincides with the parabolic region Ω_p depending on p,*

$$(1)\quad \Omega_p = \left\{\mu + i\nu : \nu^2 \leqslant 2C_p\left(\mu - \frac{1}{4}\right) + C_p^2\right\} \textit{ with constant } C_p = 2\left(\frac{1}{2} - \frac{1}{p}\right)^2.$$

Notice that the family of confocal paiabolae $\{\Omega_p\}_p$ varies continuously from the smallest $\Omega_2 = [1/4; \infty)$ = "L^2-spectrum" to the largest $\Omega_1 = \{\nu^2 \leqslant \mu\} =$ = "L^1-spectrum".

Let us outline the proof.

We need to construct the Green's function (resolvent kernel) $G(x, \xi, \lambda)$ of operators L, M and to analyse its L^p-operator norm.

The standard construction involves two linearly independent solutions of the differential equation $L[y] = \lambda y$. Namely,

$$(2) \qquad G(x, \xi, \lambda) = \frac{1}{W} \begin{cases} -y_2(x)y_1(\xi); & x < \xi \\ y_1(x)y_2(\xi); & x > \xi \end{cases}$$

where W = Wronskian of $\{y_1 ; y_2\}$, and functions y_1, y_2 satisfy suitable boundary conditions at $\{0\}$ and $\{\infty\}$ respectively.

In both cases, (L and M), solutions $\{y_1 ; y_2\}$ are expressed in terms of Bessel functions $\{J_{\pm s}(z)\}$ (see [15], Chapter 17). For the operator L these are

$$(L1) \quad y_{1,2}(x; \lambda) = \frac{x^{\frac{1}{2}-\alpha}}{\sqrt{1-\alpha}} J_{\pm s}\left(\frac{\sqrt{\lambda}}{1-\alpha} x^{1-\alpha}\right); \text{ order } s = \frac{1}{1-\alpha} \sqrt{\left(\frac{1}{2} - \alpha\right)^2 - b}.$$

Here we assumed $(1/2 - \alpha)^2 - b > 0$, otherwise one takes imaginary values of $\{x\}$, i.e. Bessel functions of the 2nd kind.

Similarly in the second case, M,

$$(M1) \qquad y_{1,2}(x, \lambda) = \frac{1}{\sqrt{x}} J_{\pm s}\left(\frac{\sqrt{b}}{\beta} x - \beta\right); \text{ order } s = \frac{1}{\beta} \sqrt{\frac{1}{4} - \lambda}.$$

The reader will notice spectral parameter λ to appear in two different places; namely the argument of J_s in ($L1$) and its order $s = s(\lambda)$ in ($M1$).

Next we use the well known asymptotics of Bessel functions at large and small values of z ([15], Chapter 17).

$$(3) \qquad J_s(z) \sim \begin{cases} \dfrac{1}{\Gamma(s+1)} (z/2)^s; & z \ll 1 \\[2ex] \sqrt{2/\pi z} \cos\left(z - \dfrac{\pi s}{2} - \dfrac{\pi}{4}\right); & z \gg 1. \end{cases}$$

Hence it follows in the first case, L,

$$(L2) \qquad y_{1,2} \sim \begin{cases} C_1 x^{\tau \pm \sqrt{\tau^2 - b}}; \ \tau = \frac{1}{2} - \alpha; & x \ll 1 \\ C_2 x^{-\alpha/2} \cos\left(\frac{\sqrt{\lambda}}{1-\alpha} \cdot x^{1-\alpha} \mp \frac{\pi s}{2} - \frac{\pi}{4}\right); & x \gg 1 \end{cases}$$

while in the second case

$$(M2) \qquad y_{1,2} \sim \begin{cases} C_1 x^{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda}}: & \text{small } x \\ C_2 x^{(\beta-1)/2} \cos\left(\frac{\sqrt{b}}{\beta} x^{-\beta} \mp \frac{\pi s}{2} - \frac{\pi}{4}\right); & \text{large } x. \end{cases}$$

To show that the L^p-spectrum of the operator L is $[0, \infty)$ for all $1 \leqslant p \leqslant \infty$, we take $\lambda \in \mathbf{C} \setminus \mathbf{R}_+$ and call $\sqrt{\lambda}/(1-\alpha) = \mu + i\nu$. Next we choose two linear combinations of y_1, y_2 of $(L1)$, one of which exponentially decays at ∞, i.e. $\sim x^{-\alpha/2} \exp[(-\nu + i\mu) x^{1-\alpha}]$, and the other vanishes at $x = 0$. It is easy to see that the corresponding Green's kernel is asymptotic

$$G(x, \xi; \lambda) \sim \begin{cases} (x\xi)^\sigma; \ \sigma = \tau + \sqrt{\tau^2 - b}: & 0 < x, \ \xi < 1, \\ x^\sigma \exp(-\nu \xi^{1-\alpha}); & x < 1; \ \xi > 1, \\ \quad \dots \quad ; & x > 1; \ \xi < 1. \end{cases}$$

Hence G maps boundedly any L^p space into itself. So any $\lambda \in \mathbf{C} \setminus \mathbf{R}_+$ belongs in the resolvent set of L, i.e. L^p-spec $L \subseteq [0, \infty)$. But L^p-spectrum can only increase with $p \geqslant 2$ (or $p \leqslant 2$). So all L^p-spectra are equal to L^2-spec $= [0, \infty)$ (stability!).

In the second case (operator M) we estimate $G(x; \xi; \lambda)$ in four different regions of (x, ξ)

$$(4) \qquad |G(x, \xi; \lambda)| < \text{Const} \begin{cases} (x\xi)^{\frac{\beta-1}{2}}; & x < 1; \xi < 1, \\ x^{\frac{\beta-1}{2}} \xi^{-\mu}; & x < 1; \xi > 1, \\ \xi^{\frac{\beta-1}{2}} x^{-\nu}; & x > 1; \xi < 1, \\ \xi^{-\mu} x^{-\nu}; & x > 1; \xi > 1, \end{cases}$$

where μ and ν denote $(1/2) - \operatorname{Re}\sqrt{(1/4) - \lambda}$; and $(1/2) + \operatorname{Re}\sqrt{(1/4) - \lambda}$ respectively. Writing λ in polar coordinates $\lambda = (1/4) + re^{i\theta}$, respectively $\mu = (1/2) - \sqrt{r}\,|\sin\theta/2|$;

$\nu = (1/2) + \sqrt{r}\,|\sin\theta/2|$, we observe that operator (4) is L^p-bounded iff

$$(5) \qquad p\nu > 1 \ \text{ or } \ \nu = \frac{1}{2} + \sqrt{r}\sin\theta/2 > \frac{1}{p} \Leftrightarrow \lambda = \frac{1}{4} + re^{i\theta} \in \Omega_p.$$

Indeed, condition (5) is necessary for all functions $\{f(x) = R_\lambda(x, \xi)\}_\xi$ to belong to $L^p(\mathbf{R}, dx)$, equivalently $R_\lambda[f] \in L^p$ for all f in $C_0(\mathbf{R})$. Hence follows the inclusion: $\Omega_p \subset$ "L^p-spectrum".

It remains to show that for each λ not in Ω_p the integral kernel G_λ is L^p-bounded. The latter reduces via (4) to L^p-estimates of the Hardy integral

$$f(x) \to \int_1^\infty K(x, \xi) f(\xi) d\xi \tag{6}$$

with the kernel $K(x, \xi) = x^{-\mu}\xi^{-\nu}$ homogeneous of degree -1. Hardy integral (6) is well known to be L^p, provided $\int_1^\infty K(1, \xi)\xi^{-1/p} d\xi < \infty$ (see [12]), which in our case amounts to $\nu + 1/p > 1$. The latter holds by (5) for all $p \leqslant 2$.

Thus we establish the equality of L^p-spectrum to Ω_p for each $1 \leqslant p \leqslant 2$ and by duality also for $2 \leqslant p < \infty$. Q.E.D.

REMARK 1. The above examples suggest that spectral stability or instability for the "Laguerre-type" operators $L = -\partial x^\alpha \partial + bx^{-\beta}$ depends on the exponent $\alpha + \beta$ = "degree of degeneracy" + "degree of singularity", which appeared in a different context in our earlier work [3].

We established stability of L for $\alpha + \beta = 2$ and instability for $\alpha = 2$ and $\beta > 0$ i.e. $\alpha + \beta > 2$. It is natural to ask what happens with other pairs (α, β). The answer is not known. It is tempting to conjecture that $\alpha + \beta \geqslant 2$ are instable, whereas $\alpha + \beta < 2$ stable. However, looking at somewhat more general classes $L = -\partial\rho\partial + V$ with "local singularities" of the type $\rho(x) \approx |x|^\alpha$; $V(x) \approx |x|^{-\beta}$ (cf. [3]) one finds a counterexample given by the well known Legendre operator $Q = -\partial(x^2 - 1)\partial$ on the half-line $[1, \infty)$.

Indeed the method of our Theorem (case M) applies to show that Q is unstable, its L^p-spectra "blowing up" the same way as for M. The role of Bessel functions $\frac{1}{\sqrt{x}} J_s(cx^{-\beta})$ will be now played by the Legendre functions of the 1st and 2nd kind.

This result also follows from the recent work [1], where the L^p-instability phenomenon was demonstrated in the more general context of Laplace operators on hyperbolic spaces $\mathbf{H}_n$ and on fundamental regions $\mathbf{H}_n/\Gamma$ (modulo Kleinian subgroup). Let us notice that the Legendre operator M represents the radial part of the Laplace operator $-\Delta$ on $\mathbf{H}_2$.

REMARK 2. The simplest case of continuous instability was observed earlier [6] for the operator $L = \partial x^2 \partial$ on $L(\mathbf{R}^+; dx)$, i.e. $\beta = 0$ in our notation. Indeed, change of the variable, $t \to x = e^t$, and conjugation with $e^{t/p}$, transform L into a constant coefficient operator $\tilde{L} = \partial_t^2 + (1 - (2/p))\partial_t + (p^{-2} - p^{-1})$ on $L^2(\mathbf{R})$. The L^p-spectrum of the latter is easily seen to coincide with the parabola $\Gamma_p = \{x = y^2/(1-(2/p))^2 + (p - 1)/2\}$, depending on p. In our case, however, the deviation from stability is more drastic, with L^p-spectrum filling in the whole parabolic region Ω_p.

The above examples and the results of [1] suggest that unlike the regular elliptic case (e.g. Schrödinger) continuous instability prevails for degenerate-singular elliptic operators.

REFERENCES

1. Davies, E. B.; Simon, B.; Taylor, M., L^p-spectral theory of Kleinian groups, Preprint 1987
2. Gurarie, D., On L^p-spectrum of elliptic operators, *J. Math. Anal. Appl.*, **108** (1985), 223—229.
3. Gurarie, D., L^p and spectral theory for a class of global elliptic operators, *J. Operator Theory*, **19**(1988), 243—274.
4. Hempel, R.; Voigt, J., The spectrum of a Schrödinger operator in $L^p(\mathbf{R}^n)$ is p-independent, *Comm. Math. Phys.*, **104** (1986).
5. Hörmander, L., *Linear partial differential operators*, Springer, 1964.
6. Jörgens, K., Lineare integral operatoren, B. G. Teubner, Stuttgart, 1970.
7. Reed, M.; Simon, B., *Methods of modern mathematical physics. II, IV*, Academic Press, 1978.
8. Schechter, M., *Spectra of partial differential operators*, North Holland, 1971.
9. Sigal, I., A generalized Weyl theorem and L^p-spectra of Schrödinger operators, *J. Operator Theory*, **13**(1985), 119—130.
10. Simon, B., Schrödonger semigroups, *Bull. Amer. Math. Soc.*, **7** (1983), 447—526.
11. Simon, B., Brownian motion; L^p properties of differential operators and localization of binding, *J, Funct. Anal.*, 35(1980).
12. Stein, E., *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.
13. Triebel, H., *Interpolation theory, function spaces, differential operators*, Berlin, 1978.
14. Weder, R., The unified approach to spectral analysis. I, *Comm. Math. Phys.*, **60** (1980); II, *Proc. Amer. Math. Soc.*, 75(1979), 81—82.
15. Wittaker, E. T., Watson, G. N., *A course of modern analysis*, Cambridge University Press, 1927.

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