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ON CERTAIN AUTOMORPHISMS OF REDUCED CROSSED PRODUCTS WITH DISCRETE GROUPS

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For a unital C^*-algebra A and a discrete group G acting on A by automorphisms, we let $\mathrm{Aut}_A(A \underset{\alpha,r}{\rtimes} G)$ denote the topological group of all automorphisms β of the reduced crossed product $A \underset{\alpha,r}{\rtimes} G$, such that $\beta(A) = A$.

The analogue group in the framework of the von Neumann algebras was studied by I. M. Singer [13], H. Behncke [1] and G. Zeller-Meyer [17]. Inspired by their work we give a description of $\mathrm{Aut}_A(A \underset{\alpha,r}{\rtimes} G)$, under certain assumptions on the dynamical system (A, G, α) (see Theorems 2.6. and 2.8.). This is done in the first part of the paper.

In the second part, we analyse the topological group $\mathrm{Aut}_A(A \underset{\alpha,r}{\rtimes} G)$ from the homotopy point of view, in the case when $A = C(K)$, where K is a compact connected topological group and G is a dense subgroup of K acting on K by left translations. Using some facts of cohomology of groups we compute the homotopy groups of $\mathrm{Aut}_{C(K)}(C(K) \underset{\alpha,r}{\rtimes} G)$ in terms of the homotopy groups of K, $\mathrm{Aut}_G(K) = \{\sigma \in \mathrm{Aut}(K) : \sigma(G) = G\}$ and some manageable algebraic objects built from G and K (see Theorem 3.11). The computations are more precise when the abelianized of G is either free or a torsion group. These situations include the cases of the irrational rotation algebras and of the Bunce-Deddens algebras (see 3.14).

Automorphisms of the above type have been recently considered by O. Bratteli, G. A. Elliott, D. E. Evans, A. Kishimoto [2], B. Brenken [3] and A. Kumjian [7].

§ 1

Throughout this paper G will be a group with neutral element e, endowed with the *discrete topology*. We stress here that the induced topologies on G coming from the embedings (in the algebraic sense) of G in some topological groups will be never considered.

If K is a group, then K^0 will denote the opposite group.

Recall that for a compact connected commutative topological group. its Pontrjagin dual is torsion free ([6]).

If K is a locally compact group, there is a natural structure of topological group on $\mathrm{Aut}(K) :=$ the group of all continuous automorphisms of K (see [6], § 26). If G is a subgroup of K, we let $\mathrm{Aut}_G(K)$ denote $\{\sigma \in \mathrm{Aut}(K): \sigma(G) = G\}$. We endow $\mathrm{Aut}_G(K)$ with the topology given by: $\sigma_i \to \sigma$ in $\mathrm{Aut}_G(K)$ iff $\sigma_i \to \sigma$ in $\mathrm{Aut}(K)$ and $(\sigma_i|G) \to (\sigma|G)$ in $\mathrm{Aut}(G)$.

If L is a topological group we let L_0 denote the path connected component of the identity. The homotopy groups of L are denoted by $\pi_n(L)$, where the base point is the identity of L.

For a unital C^*-algebra A we let $U(A)$ denote the unitary group of A and $Z(A)$ the center of A. $\mathrm{Aut}(A) :=$ the group of all $*$-automorphisms of A is considered with the topology of pointwise norm convergence. Given an action $\alpha: G \to \mathrm{Aut}(A)$, $Z^1(G, U(A))$ is by definition the space of all maps $m: G \to U(A)$ satisfying the identity:

$$m(g \cdot h) = m(g) \cdot \alpha_g(m(h)); \quad g, h \in G.$$

We let $Z^1(G, U(A))$ have the product topology induced from $\prod_{g \in G} U(A)$. We denote by $k(G; A)$ the set of all the maps from G to A having finite support.

§ 2

Under certain assumptions we shall give a description of the topological group $\mathrm{Aut}_A(A \rtimes_{\alpha,r} G)$.

Consider an injective $*$-representation $\pi: A \to B(H)$. We shall identify A with its image by π and $A \rtimes_{\alpha,r} G$ with the norm closure of $(\tilde{\pi} \times U)(\ell^1(G, A))$ in $B(\ell^2(G, H))$, where:

$$(\tilde{\pi}(a)\xi)(g) = \pi(\alpha_{g^{-1}}(a))\xi(g)$$

$$(U_h\xi)(g) = \xi(h^{-1}g)$$

for any $a \in A$, $g, h \in G$ and $\xi \in \ell^2(G, H)$ (see [8], Theorem 7.7.5).

We shall need the following:

2.1. LEMMA. *There is a unique linear, injective and contractive map $x \to (x_g)_{g \in G}$ from $A \rtimes_{\alpha,r} G$ to $\ell^\infty(G, A)$ which extends the natural inclusion $\ell^1(G, A) \to \ell^\infty(G, A)$. For any $x, y \in A \rtimes_{\alpha,r} G$ we have:*

$$(x^*)_g = \alpha_g(x^*_{g^{-1}}) \quad \text{and} \quad (x \cdot y)_g = \sum_{h \in G} x_h \cdot \alpha_h(y_{h^{-1}g})$$

(strong convergence in $B(H)$). Moreover, the map $E : A \rtimes_{\alpha,r} G \to A$, given by $E(x) := x_e$ is a faithful conditional expectation and for any $x \in A \rtimes_{\alpha,r} G$ and $g \in G$, $x_g = E(xU_g^*)$.

Proof. See ([17], Theorem 4.12).

2.2. DEFINITION. We say that *G acts properly outer on A*, if each α_g, $g \neq e$, has the property: if $a \in A$ and $a\alpha_g(x) = xa$ for all x in A, then $a = 0$.

2.3. REMARK. a) Let $G \times X \to X$, $(g, x) \to g \cdot x$, be a continuous action of G on the compact space X. The corresponding action $G \to \mathrm{Aut}(C(X))$ is properly outer iff for each $g \neq e$, $\{x \in X : g \cdot x = x\}$ has no interior points.

b) Let L be a discrete ICC-group and $G \to \mathrm{Aut}(L)$ an action of G on L by outer automorphisms. Then the induced action $\alpha : G \to \mathrm{Aut}(C^*_{\mathrm{red}}(L))$ is properly outer and $Z(C^*_{\mathrm{red}}(L)) \simeq \mathbf{C}$ (see [16], 22.12).

2.4. LEMMA. *Suppose that G acts properly outer on A or on $Z(A)$ and that the spectrum of $Z(A)$ is connected. Consider $v \in U(A \rtimes_{\alpha,r} G)$ such that $vAv^* = A$. Then, there are $a \in U(A)$ and $h \in G$, unique with the property that $v = aU_h$.*

Proof. Let $(v_g)_{g \in G}$ the map associated with v, by Lemma 2.1. Fix $a \in A$. Then $b := vav^* \in A$ and $vx = bv$. Hence, for any $g \in G$: $v_g\alpha_g(a) = bv_g \Leftrightarrow v_g\alpha_g(a) = vav^*v_g \Leftrightarrow v^*v_g\alpha_g(a) = av^*v_g \Leftrightarrow \alpha_g(a^*)v_g^*v = v_g^*va^* \Leftrightarrow \alpha_g(a^*)v_g^*v_h = v_g^*v_h\alpha_h(a^*)$ for any $h \in G$. Hence:

$$\text{(1)} \qquad cv_g^*v_h = v_g^*v_h\alpha_{hg^{-1}}(c), \quad c \in A,\ g, h \in G.$$

Since $v^*Av = A$, by (1) and Lemma 2.1 it follows that:

$$\text{(2)} \qquad v_gv_g^* \in Z(A), \quad g \in G.$$

From (1) and (2), we deduce:

$$\text{(3)} \qquad cv_gv_g^*v_hv_h^* = v_gv_g^*v_hv_h^*\alpha_{hg^{-1}}(c), \quad c \in Z(A),\ g, h \in G.$$

By the hypothesis, (1) and (3), it follows that:

$$\text{(4)} \qquad v_gv_g^*v_hv_h^* = 0, \quad g \neq h \in G.$$

But:

$$\text{(5)} \qquad \sum_{g \in G} v_gv_g^* = 1 \ \text{(strong convergence in } B(H))$$

since $vv^* = 1$ (see Lemma 2.1). Using (4) we deduce that each $v_gv_g^*$ is a projection in $Z(A)$ and hence it must be trivial; but by (5), only one is nonzero, say $v_hv_h^*$ $(=1)$, which implies $v = v_hU_h$ (see Lemma 2.1).

2.5. Assume the hypothesis of Lemma 2.4. Let $B := A \times_{\alpha,r} G$ and consider $\beta \in \mathrm{Aut}_A(B)$. We shall describe all such automorphisms.

By Lemma 2.4, there are $b_g \in U(A)$ and a map $\sigma : G \to G$ such that $\beta(U_g) = b_g \cdot U_{\sigma(g)}$, $g \in G$.

We shall need the following:

LEMMA. $\sigma \in \mathrm{Aut}(G)$.

Proof. Let $N(A) = \{v \in U(B) : vAv^* = A\}$. It is clear that $U(A)$ is a normal subgroup of $N(A)$. Lemma 2.4 says that $N(A)/U(A)$ is canonically isomorphic to G. Consequently any $\beta \in \mathrm{Aut}_A(B)$ induces an automorphism σ of G. Further we have

$$a_{gh} = a_g \cdot \alpha_g(a_h), \quad \text{for } g, h \in G$$

where, by definition, $a_g = b_{\sigma^{-1}(g)} \in U(A)$.

Notice that for any $g \in G$ and $a \in A$:

$$\beta(\alpha_g(a)) = \beta(U_g)\beta(a)\beta(U_g)^* = a_{\sigma(g)}U_{\sigma(g)}\beta(a)U^*_{\sigma(g)}a^*_{\sigma(g)} =$$

$$= \mathrm{ad}\, a_{\sigma(g)}(\alpha_{\sigma(g)}(\beta(a)).$$

Hence we can define a map:

$$\Phi : \mathrm{Aut}_A(B) \to \mathscr{G} \subset Z^1(G, U(A)) \times \mathrm{Aut}(A) \times \mathrm{Aut}(G)$$

where $\mathscr{G} := \{((c_g), \rho, \sigma) : \rho \circ \alpha_g = \mathrm{ad}\, c_{\sigma(g)} \circ \alpha_{\sigma(g)} \circ \rho,\ g \in G\}$ by:

$$\Phi(\beta) := ((a_g), \beta|A, \sigma).$$

We have the following:

2.6. THEOREM. *Φ is a homeomorphism.*

Proof. By the above computations it is clear that Φ is well defined and injective ($B = C^*(A \cup U_G) \subset B(\ell^2(G, H))$). To prove the surjectivity of Φ, take $((c_g), \rho, \sigma) \in \mathscr{G}$. Define

$$\beta : \ell^1(G, A) \to B(\ell^2(G, H)), \text{ by } \beta((x_g)_{g\in G}) := \sum_{g\in G} \rho(x_g) \cdot c_{\sigma(g)} \cdot U_{\sigma(g)}.$$

Since

$$\sum_{g\in G} \|\rho(x_g)c_{\sigma(g)}U_{\sigma(g)}\| \leqslant \sum_{g\in G} \|\rho(x_g)\| \cdot \|c_{\sigma(g)}U_{\sigma(g)}\| =$$

$$= \sum_{g\in G} \|x_g\| = \|(x_g)_{g\in G}\|_1 < +\infty,$$

it follows that $\beta(\ell^1(G, A)) \subset B$. We have $C^*(\beta(\ell^1(G, A))) = B$. By ([17], Proposition 2.7), β is a unital representation of $\ell^1(G, A)$. Since $E : B \to A$ $(= \rho(A))$, $E(b) = b_e$, is linear, positive and faithful, and since $E \circ \beta|k(G, A) = \beta \circ E|k(G, A)$, we can apply ([17], Theorem 4.22, equivalence (i) $\Leftrightarrow$ (iii), with $\theta = E$) to deduce that β induces an injective $*$-homomorphism $\beta : B \to B$. Hence $\beta \in \mathrm{Aut}_A(B)$ and $\Phi(\beta) = ((c_g), \rho, \sigma)$.

Φ^{-1} is obviously continuous and for the continuity of Φ, observe that

$$\|E(\beta'(U_g)\beta(U_g)^*)\| = \begin{cases} 1, & \sigma(g) = \sigma'(g) \\ 0, & \sigma(g) \neq \sigma'(g) \end{cases}$$

where β (resp. β') belongs to $\mathrm{Aut}_A(B)$ and σ (resp. σ') is the associated automorphism of G.

2.7. Remarks. a) Assume the hypothesis of the above theorem. If $E : B \to A$ is the conditional expectation from Lemma 2.1, and if $\beta \in \mathrm{Aut}(B)$, then:

$$\beta \in \mathrm{Aut}_A(B) \Leftrightarrow \beta \circ E = E \circ \beta.$$

b) Assume that G acts properly outer on A and that $Z(A)$ has connected spectrum. The above theorem easily implies that the topological group of the automorphisms of $A \underset{\alpha, r}{\rtimes} G$ which leave A pointwise fixed is isomorphic to $Z^1(G, UZ(A))$.

2.8. We have found a more precise description of the topological group $\mathrm{Aut}_A(B)$ in the case when $A = C(K)$, where K is a compact connected topological group and G is a dense subgroup of K acting on K by translations (the induced action on $C(K)$ is given by $\alpha_g(a) = a(g^{-1} \cdot))$.

In fact, this example is "generic" (at least) for the case when $A = C(X)$, with X compact metrizable and G countable and commutative, acting freely and minimally on X, such that the action is equicontinuous relative to some metric d (i.e. $(\forall)\ \varepsilon > 0$, $(\exists)\ \delta = \delta(\varepsilon) > 0$ such that $d(x, y) < \delta \Rightarrow d(g \cdot x, g \cdot y) < \varepsilon$, $g \in G$).

For $\beta \in \mathrm{Aut}_{C(K)}(C(K) \underset{\alpha, r}{\rtimes} G)$, Theorem 2.6 gives that $\beta(U_g) = a_{\sigma(g)} U_{\sigma(g)}$, $g \in G$, where $\sigma \in \mathrm{Aut}(G)$ and $(\exists)\ \varphi \in \mathrm{Homeo}(K)$ such that $\beta(a) = a \circ \varphi^{-1}$, $a \in C(K)$. The relation $\beta \circ \alpha_g = \mathrm{ad}\, a_{\sigma(g)} \circ \alpha_{\sigma(g)} \circ \beta|A$, $g \in G$, is equivalent with the condition:

$$\sigma(g)\varphi(k) = \varphi(gk), \quad (\forall)\, g \in G,\ (\forall) k \in K$$

which exactly says that σ extends to a map (also denoted by σ) belonging to $\mathrm{Aut}_G(K)$ and $\varphi(\cdot) = \sigma(\cdot)\varphi(e)$.

Define the homomorphism of groups $J : K^0 \underset{I}{\rtimes} \mathrm{Aut}_G(K) \to \mathrm{Aut}(Z^1(G, U(A)))$ by $J(k, \sigma)(a_g)_g = (a_{\sigma^{-1}(g)} \circ \sigma^{-1}(\cdot\, k^{-1}))_g$ where the homomorphism $I: \mathrm{Aut}_G(K) \to$

$\to \mathrm{Aut}(K^0)$ is the inclusion map. If $\Psi : \mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G) \to Z^1(G, UC(K)) \rtimes_J (K^0 \rtimes_I \rtimes_I \mathrm{Aut}_G(K))$ is given by $\Psi(\beta) := ((a_g), \varphi(e), \sigma)$, we obtain the following:

THEOREM: *Ψ is an isomorphism of topological groups.*

2. REMARK. Note that since in the above case, A is masa in B (see [17], Proposition 4.14), we have $\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G) = \{\beta \in \mathrm{Aut}(C(K) \rtimes_{\alpha,r} G) : \beta(C(K)) \subset \subset C(K)\}$.

§ 3

In this section we shall analyse $\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)$ from the homotopy point of view. The hypothesis is the same as in 2.8, namely G is a dense subgroup of K (= compact connected topological group) acting on K by left translations. As a consequence of Theorem 2.8 we have:

$$\pi_n(\mathrm{Aut}_{C(K)}(C(K) \rtimes_{\alpha,r} G)) = \pi_n(Z^1(G, \ UC(K))) \rtimes (\pi_n(K) \times \pi_n(\mathrm{Aut}_G(K))).$$

The semidirect product structure of $\pi_n(\mathrm{Aut}_{C(K)}(C(K) \times_{\alpha,r} G))$ comes from 2.8. For $n \geqslant 1$, this coincides with the ordinary direct product (see [14], Chapter 1, §6, Corollary 10).

The analysis of $Z^1(G, UC(K))$ requires some elements of cohomology of groups. We shall recall some definitions and standard notations.

Let M be an abelian group (written additively). We say that M is a G-module if we are given a homomorphism $G \to \mathrm{Aut}(M)$. We let ${}^g x$ denote the image of $x \in M$ under the automorphism given by $g \in G$. If M and N are G-modules, a map $f: M \to N$ is called a G-homomorphism if it is a group homomorphism which preserves the action of G or equivalently, if it is a homomorphism of G-modules. For a G-module M, we denote by M^G the submodule consisting of the elements fixed by G.

Cohomology groups (of low dimension) are easily described using standard n-cocycles $Z^n(G, M)$ and standard n-coboundaries $B^n(G, M)$, for we have $H^n(G, M) = Z^n(G, M)/B^n(G, M)$.

$n = 0$: $Z^0(G, M) = M^G$, $B^0(G, M) = 0$.

$n = 1$: $Z^1(G, M)$ consists of all the maps $g \to m_g$ from G to M satisfying $m_{gh} = m_g + {}^g m_h$; $B^1(G, M)$ consists of the maps $g \to m_g$ in $Z^1(G, M)$ of the form $m_g = {}^g x - x$ for some $x \in M$.

$n = 2$: $Z^2(G, M)$ consists of all the maps $(g, h) \to m_{g,h}$ from $G \times G$ to M satisfying

$${}^g m_{h,k} - m_{gh,k} + m_{g,hk} - m_{g,h} = 0;$$

$B^2(G, M)$ consists of all the maps in $Z^2(G, M)$ having the form

$$m_{g,h} = {}^g t_h - t_{gh} + t_g$$

for some map $g \to t_g$ from G to M.

Note that if G acts trivially on M then $H^1(G, M) = Z^1(G, M) = \mathrm{Hom}(G, M)$. For every exact sequence of G-modules:

$$0 \to M \xrightarrow{j} N \xrightarrow{q} P \to 0$$

there is a connecting homomorphism $\delta\colon Z^1(G, P) \to H^2(G, M)$ such that the sequence:

$$0 \to Z^1(G, M) \xrightarrow{j_*} Z^1(G, N) \xrightarrow{q_*} Z^1(G, P) \xrightarrow{\delta} H^2(G, M) \xrightarrow{j_*} H^2(G, N)$$

is exact. Moreover the connecting homomorphism depends functorialy on the given exact sequence. Let us briefly recall the definition of δ. Let $g \to p_g$ be an 1-cocycle in $Z^1(G, P)$. Since q is onto there is a map $g \to n_g$ from G to N such that $q(n_g) = p_g$ for all g in G. As $g \to p_g$ is an 1-cocycle we must have $q(n_g + {}^g n_h - n_{gh}) = 0$ and so for any $g, h \in G$ there is a unique $m_{g,h} \in M$ such that

$$j(m_{g,h}) = n_g + {}^g n_h - n_{gh}.$$

It is easy to check that the map $(g, h) \to m_{g,h}$ defines a 2-cocycle and moreover its class of cohomology $[(m_{g,h})]$ in $H^2(G, M)$ depends only on the given 1-cocycle $g \to p_g$. By definition $\delta((p_g)) = [(m_{g,h})]$.

Let $[G, G]$ be the commutator subgroup of G and let $G_{\mathrm{ab}} = G/[G, G]$ the abelianized of G. Let

$$0 \to [G, G] \to G \xrightarrow{\psi} G_{\mathrm{ab}} \to 0$$

be the corresponding exact sequence. It is clear that for any abelian group M the quotient map ψ induces an isomorphism of groups $\mathrm{Hom}(G_{\mathrm{ab}}, M) \to \mathrm{Hom}(G, M)$. If $g \in G$ we shall write many times $\dot{g}$ instead of $\psi(g)$.

We shall regard the exponential sequence

$$0 \to \mathbf{Z} \to \mathbf{R} \xrightarrow{\exp} \mathbf{T} \to 0 \quad (\exp(x) := e^{2\pi i x})$$

as an exact sequence of G-modules (with trivial G-actions). Therefore we have an exact sequence

$$0 \to Z^1(G, \mathbf{Z}) \to Z^1(G, \mathbf{R}) \to Z^1(G, \mathbf{T}) \xrightarrow{\delta_1} H^2(G, \mathbf{Z}) \to H^2(G, \mathbf{R}).$$

We want to find the image of δ_1. This is an easy question, however we record the answer in a lemma for later use.

3.1. LEMMA. *The image of the connecting homomorphism* δ_1 *in the above sequence is naturally isomorphic to* $\mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z})$.

Proof. The exponential sequence may be seen as an injective presentation of $\mathbf{Z}$, hence there is an exact sequence

$$0 \to \mathrm{Hom}(G_{ab}, \mathbf{Z}) \to \mathrm{Hom}(G_{ab}, \mathbf{R}) \to \mathrm{Hom}(G_{ab}, \mathbf{T}) \to \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z}) \to 0.$$

On the other hand we have the obvious identifications:

$$\begin{array}{ccccc} Z^1(G, \mathbf{Z}) & \longrightarrow & Z^1(G, \mathbf{R}) & \xrightarrow{\exp_*} & Z^1(G, \mathbf{T}) \\ \wr & & \wr & & \wr \\ \mathrm{Hom}(G_{ab}, \mathbf{Z}) & \to & \mathrm{Hom}(G_{ab}, \mathbf{R}) & \to & \mathrm{Hom}(G_{ab}, \mathbf{T}) \end{array}$$

whence image $\delta_1 \simeq \mathrm{coker}(\exp_*) \simeq \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z})$. ▨

Let $\Lambda^2 G_{ab}$ denote the second exterior power of G_{ab}.

3.2. LEMMA. *There is a natural isomorphism*

$$\mu : H^2(G_{ab}, \mathbf{T}) \to \mathrm{Hom}(\Lambda^2 G_{ab}, \mathbf{T})$$

which takes the class of the 2-cocycle $(m_{\dot{g},\dot{h}})$ *to the homomorphism* $\dot{g} \wedge \dot{h} \to m_{\dot{g},\dot{h}} - m_{\dot{h},\dot{g}}$.

Proof. Let M be an abelian group considered as G-module with trivial G action. By ([4], Chapter V, § 6, Exercise 5) there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, M) \to H^2(G_{ab}, M) \xrightarrow{\mu} \mathrm{Hom}(\Lambda^2 G_{ab}, M) \to 0.$$

The statement of the lemma is obtained by taking M to be the divisible group $\mathbf{T}$. ▨

If L is a normal subgroup of G the cohomology groups of G, L and G/L are connected by the exact sequence of Hochschild-Serre. We need the following case:

$$0 \to H^1(G/L, M^L) \xrightarrow{\eta_1^*} H^1(G, M) \xrightarrow{\rho} H^1(L, M)^{G/L} \xrightarrow{\tau} H^2(G/L, M^L) \xrightarrow{\eta^*} H^2(G, M)$$

(see [12], p. 118). The maps η_1^*, η^* are the inflation homomorphisms, ρ is the restriction homomorphism and τ is the transgression homomorphism. For a very concrete definition of τ see ([15], p. 215). The action of G/L on $H^1(L, M)$ is induced by the following action of G on $Z^1(L, M)$:

$$(g \cdot m)_l = {}^g m_{g^{-1}lg}, \quad \text{for every 1-cocycle } \mathbf{m} = (m_l)_l \text{ in } Z^1(L, M).$$

Let us consider the above exact sequence in the case $L = [G, G]$ and $M = \mathbf{T}$ with trivial G-actions.

It is clear that $\eta_1^* : \mathrm{Hom}(G_{\mathrm{ab}}, \mathbf{T}) \to \mathrm{Hom}(G, \mathbf{T})$ is an isomorphism hence we get the exact sequence

$$0 \to \mathrm{Hom}([G, G], \mathbf{T})^G \xrightarrow{\tau} H^2(G_{\mathrm{ab}}, \mathbf{T}) \xrightarrow{\eta^*} H^2(G, \mathbf{T}).$$

We are interested to describe the image of the transgression homomorphism τ. This can be better done via the isomorphism exhibited in Lemma 3.2.

3.3. LEMMA. *Let* $\varphi \in \mathrm{Hom}([G, G], \mathbf{T})^G$ and let $\hat{\varphi} \in \mathrm{Hom}(\Lambda^2 G_{\mathrm{ab}}, \mathbf{T})$ *be the image of* φ *under the homomorphism* $\mu\tau$. *Then* $\hat{\varphi}$ *is given by the following formula*

$$\hat{\varphi}(\dot{g} \wedge \dot{h}) = \varphi(ghg^{-1}h^{-1}) \quad \textit{for all } g, h \in G.$$

Proof. Using the fact that $\varphi(g^{-1}tg) = \varphi(t)$ for all $g \in G$, $t \in [G, G]$, it is easily seen that the formula

$$\varphi'(\dot{g} \wedge \dot{h}) = \varphi(ghg^{-1}h^{-1})$$

gives a well defined homomorphism $\varphi' \in \mathrm{Hom}(\Lambda^2 G_{\mathrm{ab}}, \mathbf{T})$. Choose a map $s\colon G_{\mathrm{ab}} \to G$ such that $\dot{\widehat{s(u)}} = u$ for each $u \in G_{\mathrm{ab}}$ and $s(1) = 1$. The description of the transgression homomorphism given in ([15], p. 215) can be used to see that $\tau(\varphi) \in H^2(G_{\mathrm{ab}}, \mathbf{T})$ is given by the 2-cocycle $\lambda_{u,v} = \varphi(s(uv)^{-1}s(u)s(v))$. Consequently

$$\hat{\varphi}(u \wedge v) = \mu\tau(\varphi)(u \wedge v) = \lambda_{u,v} - \lambda_{v,u} = \varphi(s(uv)^{-1}s(u)s(v)s(u)^{-1}s(v)^{-1}s(vu)) =$$

$$= \varphi(s(u)s(v)s(u)^{-1}s(v)^{-1}) = \varphi'(u \wedge v), \quad \text{for all } u, v \in G_{\mathrm{ab}}.$$

3.4. Let $U_0C(K)$ denote the connected component of identity of the group $UC(K)$. Each element in $U_0C(K)$ has the form $\exp(2\pi i a)$ for some $a \in C(K, \mathbf{R})$. Let us denote by exp the map $a \to \exp(2\pi i a)$. Since K is connected, the kernel of exp is isomorphic to $\mathbf{Z}$. If we let G act trivially on $\mathbf{Z}$ and by left translations on $C(K, \mathbf{R})$ and $U_0C(K)$ we get an exact sequence of G-modules:

$$0 \to \mathbf{Z} \to C(K, \mathbf{R}) \xrightarrow{\exp} U_0C(K) \to 0.$$

Using the Haar integral we can relate this sequence to the exponential sequence as described in the following diagram of G-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{T} & \longrightarrow & 0 \\ & & \| & & \downarrow i & & \downarrow i' & & \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & C(K,\mathbf{R}) & \longrightarrow & U_0C(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow p & & \downarrow p' & & \\ 0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{T} & \longrightarrow & 0. \end{array}$$

Here i and i' are the natural inclusions, $p(a) = \int_K a(x)\mathrm{d}x$ for $a \in C(K,\mathbf{R})$ and $p'(\exp(a)) = \exp(p(a))$. Note that $pi = \mathrm{id}_{\mathbf{R}}$ and $p'i' = \mathrm{id}_{\mathbf{T}}$.

We denote by $\hat{K} \subset UC(K)$ the (abelian) topological group of all continuous homomorphisms $K \to \mathbf{T}$. Recall that $\hat{K}$ is discrete since K is assumed to be compact (when K is abelian, $\hat{K}$ is exactly the Pontrjagin dual of K). It is a result of Scheffer [11] that each continuous map $K \to \mathbf{T}$ is homotopic to a unique homomorphism in $\hat{K}$.

The arguments given in [11] prove that one has the following:

3.5. PROPOSITION. *Let K be a compact connected topological group. There is a split exact sequence of groups*

$$0 \to U_0C(K) \overset{j}{\hookrightarrow} UC(K) \underset{s}{\overset{q}{\rightleftarrows}} \hat{K} \to 0$$

where $q(a)(x) = p'(a(\cdot)a(x^{-1}\cdot)^*)$ for $a \in UC(K)$ and $x \in K$.

The section s is given by the natural inclusion $\hat{K} \hookrightarrow UC(K)$. If we let G act trivially on $\hat{K}$ and by left translations on $U_0C(X)$ and $UC(K)$ then the above sequence is an exact sequence of G-modules. Note however that the section s is not G-linear.

The exact sequence in 3.5 induces an exact sequence of groups

$$0 \to Z^1(G, U_0C(X)) \xrightarrow{j_*} Z^1(G, UC(K)) \xrightarrow{q_*} Z^1(G, \hat{K}) \xrightarrow{\delta} H^2(G, U_0C(K)).$$

If M is a topological group together with a G-action $G \to \mathrm{Aut}(M)$ of the discrete group G we let $Z^1(G, M)$ have the product topology induces from $\prod_{g \in G} M$.

Let $\Gamma_{K,G} := \mathrm{image}\, q_* = \ker \delta \subset Z^1(G, \hat{K}) = \mathrm{Hom}(G, \hat{K}) = \mathrm{Hom}(G_{\mathrm{ab}}, \hat{K})$.

3.6. LEMMA. *There is an exact sequence of groups*

$$0 \to Z^1(G, U_0C(K)) \xrightarrow{j_*} Z^1(G, UC(K)) \xrightarrow{q_*} \Gamma_{K,G} \to 0$$

with $\Gamma_{K,G}$ totally disconnected.

Proof. The continuity of q_* follows from Proposition 3.5. $\Gamma_{K,G}$ is totally disconnected since $\hat{K}$ is discrete. ▨

A more precise description of $\Gamma_{K,G}$ is provided by the following:

3.7. PROPOSITION. *Let $\gamma : g \to \gamma_g$ be a homomorphism from G to $\hat{K}$. Then $\gamma \in \Gamma_{K,G}$ if and only if there is $\varphi \in \mathrm{Hom}([G, G], \mathbf{T})$ satisfying $\varphi(g^{-1}tg) = \varphi(t)$ for all $g \in G$, $t \in [G, G]$, such that:*

$$\gamma_g(h) - \gamma_h(g) = \varphi(h^{-1}ghg^{-1}) \quad \textit{for all } g, h \in G.$$

In particular, if G is abelian then:

$$\Gamma_{K,G} = \{\gamma \in \mathrm{Hom}(G, \hat{K}) : \gamma_g(h) = \gamma_h(g)\}.$$

Proof. For $\gamma \in Z^1(G, \hat{K}) = \mathrm{Hom}(G, \hat{K})$ let us compute $\delta\gamma \in H^2(G, U_0C(K))$. If we regard γ as a map $G \to UC(K)$ (via the section s in 3.5) then $q(\gamma_g) = \gamma_g$ and so

$$(\delta\gamma)_{g,h} = \gamma_g(\cdot) + \gamma_h(g^{-1}\cdot) - \gamma_{gh}(\cdot) = \gamma_h(g^{-1}).$$

This computation also shows that the 2-cocycle $(g, h) \to \gamma_h(g^{-1})$ takes values in $\mathbf{T} \subset U_0C(K)$. Consequently the connecting homomorphism δ factors through the natural map $H^2(G, \mathbf{T}) \to H^2(G, U_0C(K))$. Moreover, since $\hat{K}$ is abelian we have $\mathrm{Hom}(G, \hat{K}) = \mathrm{Hom}(G_{\mathrm{ab}}, \hat{K})$ and so δ can be factorized as in the following (commutative) diagram:

$$\begin{array}{ccc} \mathrm{Hom}(G, \hat{K}) & \xrightarrow{\delta} & H^2(G, U_0C(K)) \\ \downarrow{\scriptstyle \delta'} & & \uparrow{\scriptstyle i'_*} \\ H^2(G_{\mathrm{ab}}, \mathbf{T}) & \xrightarrow{\psi^*} & H^2(G, \mathbf{T}). \end{array}$$

Here i'_* is induced by the natural embeding $i' : \mathbf{T} \to U_0C(K)$, ψ^* is induced by the quotient map $\psi: G \to G_{\mathrm{ab}}$ and $(\delta'\gamma)_{\dot{g},\dot{h}} = \gamma_h(g^{-1})$. As a consequence of 3.4 $p'_*i'_* = \mathrm{id}_{H^2(G,\mathbf{T})}$ hence i'_* is injective. In this way we find that $\ker\delta$ consists of those elements γ in $\mathrm{Hom}(G, \hat{K})$ for which $\delta'\gamma \in \ker\psi^*$ or equivalently $\mu\delta'\gamma \in \mathrm{image}(\mu\tau)$ (see Lemma 3.2 and the discussion before Lemma 3.3).

Finally, using Lemma 3.3, we deduce that $\gamma \in \Gamma_{K,G} = \ker\delta$ iff $\gamma_h(g^{-1}) - \gamma_{g^{-1}}(h) = \varphi(ghg^{-1}h^{-1})$ for some $\varphi \in \mathrm{Hom}([G, G], \mathbf{T})^G$. ▨

Having Lemma 3.6 and Proposition 3.7 we shall concentrate ourself on $Z^1(G, U_0C(K))$.

3.8. PROPOSITION. *There is an exact sequence of groups*

$$0 \to \mathrm{Hom}(G, \mathbf{Z}) \to Z^1(G, C(K, \mathbf{R})) \xrightarrow{\exp_*} Z^1(G, U_0C(K)) \xrightarrow{\delta_0} \mathrm{Ext}^1_{\mathbf{Z}}(G_{\mathrm{ab}}, \mathbf{Z}) \to 0.$$

Proof. The commutative diagram from 3.4 induces the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & Z^1(G, \mathbf{Z}) & \longrightarrow & Z^1(G, \mathbf{R}) & \longrightarrow & Z^1(G, \mathbf{T}) & \xrightarrow{\delta_1} & H^2(G, \mathbf{Z}) \\
& \| & & \downarrow i_* & & \downarrow i'_* & & \| \\
0 \longrightarrow & Z^1(G, \mathbf{Z}) & \longrightarrow & Z^1(G, C(K, \mathbf{R})) & \longrightarrow & Z^1(G, U_0C(K)) & \xrightarrow{\delta_0} & H^2(G, \mathbf{Z}) \\
& \| & & \downarrow p_* & & \downarrow p'_* & & \| \\
0 \longrightarrow & Z^1(G, \mathbf{Z}) & \longrightarrow & Z^1(G, \mathbf{R}) & \longrightarrow & Z^1(G, \mathbf{T}) & \xrightarrow{\delta_1} & H^2(G, \mathbf{Z}).
\end{array}$$

Since $p'_* i'_* = \mathrm{id}$ it follows that image δ_0 = image δ_1. By Lemma 3.1 image $\delta_1 \simeq \simeq \mathrm{Ext}^1_{\mathbf{Z}}(G_{\mathrm{ab}}, \mathbf{Z})$. ▨

3.9. LEMMA. i) *δ_0 is constant on the path components of $Z^1(G, U_0C(K))$.*

ii) $\mathrm{Hom}(G, \mathbf{Z}) \to Z^1(G, C(K, \mathbf{R})) \xrightarrow{\exp_*} Z^1(G, U_0C(K))$ *is a (Hurewicz) fibration.*

Proof. i) Let $[0,1] \ni t \to a(t) = (a_g(t))_g \in Z^1(G, U_0C(K))$ be a continuous path. Since the sequence

$$\mathbf{Z} \to C(K, \mathbf{R}) \to U_0C(K)$$

is a covering space, for each $g \in G$ we can lift the path $t \to a_g(t) \in U_0C(K)$ to a continuous path $t \to f_g(t) \in C(K, \mathbf{R})$ such that $\exp f_g(t) = a_g(t)$. For each t we have

$$(\delta_0 a(t))_{g,h} = f_g(t) + {}^g f_h(t) - f_{gh}(t) \in \mathbf{Z}.$$

Since the above expression depends continuously on t, we must have $\delta_0 a(0) = = \delta_0 a(1)$.

ii) Let X be a topological space and let $F : X \times [0,1] \to Z^1(G, U_0C(K))$, $f : X \times \{0\} \to Z^1(G, C(K, \mathbf{R}))$ be continuous maps satisfying $\exp_* f(x, 0) = F(x, 0)$ for all x in X. Our aim is to produce a continuous map $H : X \times [0,1] \to Z^1(G, C(K, \mathbf{R}))$ which extends f and lifts F. Now since the sequence $\mathbf{Z} \to C(K, \mathbf{R}) \xrightarrow{\exp} U_0C(K)$ is a covering space, for each $g \in G$ there is a continuous map $H'_g : X \times [0,1] \to C(K, \mathbf{R})$ which extends f_g and lifts F_g. Let us check that for each x, t, the map $g \to H'_g(x, t)$ belongs to $Z^1(G, C(K, \mathbf{R}))$. Define $\alpha_{g,h}(x, t) = H'_g(x, t) + {}^g H'_h(x, t) - H'_{gh}(x, t)$. It is clear that $\alpha_{g,h}(x, t) \in \mathbf{Z}$ since $\exp H'_g(x, t) = F_g(x, t)$. Also, $\alpha_{g,h}(x, 0) = 0$ since $H'_g(x, 0) = f_g(x, 0)$. As $\alpha_{g,h}(x, t)$ depends continuously on t we must have $\alpha_{g,h}(x, t) = 0$ for all x, t. The map $(x, t) \to (H'_g(x, t))_{g \in G}$ is a solution of the given lifting problem with initial data.

3.10. PROPOSITION.

$$\pi_n(Z^1(G, U_0C(K))) = \begin{cases} \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z}) & \text{for } n = 0 \\ \mathrm{Hom}(G_{ab}, \mathbf{Z}) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

Proof. Let Z_0^1 be the path component of 1 in $Z^1(G, U_0(C(K))$. Since the group $Z^1(G, C(K, \mathbf{R}))$ is contractible at the zero cocycle (use the obvious homotopy $H((a_g)_g, t) = (ta_g)_g$) it follows that in the sequence 3.8 one has $\ker\delta_0 = \text{image}(\exp_*) \subset Z_0^1$. On the other hand, by Lemma 3.9(i), $Z_0^1 \subset \ker\delta_0$. Thus $Z_0^1 = \text{image}(\exp_*)$ and $\pi_0(Z^1(G, U_0C(K))) = Z^1(G, U_0C(K))/Z_0^1 \simeq \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z})$. As a consequence of Lemma 3.9(ii) the sequence

$$0 \to \mathrm{Hom}(G, \mathbf{Z}) \to Z^1(G, C(K, \mathbf{R})) \xrightarrow{\exp_*} Z_0^1 \to 0$$

defines a fibration and we have seen that its total space is contractible and the fibre is totally disconnected. The homotopy sequence of the fibration gives us

$$\pi_n(Z_0^1) = \begin{cases} \mathrm{Hom}(G, \mathbf{Z}) & \text{for } n = 1 \\ 0 & \text{for } n \geq 2. \end{cases}$$

▨

The results of this section can be collected in the following:

3.11. THEOREM. *Let K be a compact connected topological group and let G be a dense subgroup of K acting on K by left translations. Then:*

$$\pi_n(\mathrm{Aut}_{C(K)}(C(K) \underset{\alpha,r}{\rtimes} G)) = \begin{cases} \pi_0(Z^1(G, UC(K)) \rtimes (\pi_0(K) \rtimes \pi_0(\mathrm{Aut}_G(K))) & \textit{for } n = 0. \\ \mathrm{Hom}(G, \mathbf{Z}) \times \pi_1(K) \times \pi_1(\mathrm{Aut}_G(K)) & \textit{for } n = 1 \\ \pi_n(K) \times \pi_n(\mathrm{Aut}_G(K)) & \textit{for } n \geq 2. \end{cases}$$

$\pi_0(Z^1(G, UC(K)))$ fits into the following exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(G_{ab}, \mathbf{Z}) \xrightarrow{i_*} \pi_0(Z^1(G, UC(K))) \xrightarrow{q_*} \Gamma_{K,G} \to 0$$

where $\Gamma_{K,G}$ is described by Proposition 3.7.

Proof. The theorem follows from Lemma 3.6 and Proposition 3.10.

3.12. REMARK. If G is abelian or countable, then $\mathrm{Aut}_G(K)$ is totally pathwise-disconnected, hence

$$\pi_n(\mathrm{Aut}_G(K)) = \begin{cases} \mathrm{Aut}_G(K) & \text{for } n = 0 \\ 0 & \text{for } n \geq 1. \end{cases}$$

There are explicitely formulae for the maps i_* and q_* from above: if $[\chi] \in \operatorname{Ext}^1_{\mathbf{Z}}(G_{\mathrm{ab}}, \mathbf{Z})$ is represented by $\chi \in \operatorname{Hom}(G, \mathbf{T})$ then $i_*[\chi]$ is the component of the 1-cocycle $g \to \chi(g)$; if $a = (a_g)_g \in Z^1(G, UC(K))$ then $q_*[a]$ is the homomorphism $\gamma \in \operatorname{Hom}(G, \hat{K})$ given by $\gamma_g(x) = p'(a_g(\cdot)a_g(x^{-1}\cdot)^*)$ (see 3.5).

3.13. COROLLARY. *Under the hypothesis of the above theorem we have:*
a) *if* G_{ab} *is free, then:*

$$\pi_0(\operatorname{Aut}_{C(K)}(C(K) \underset{\alpha,r}{\rtimes} G)) = \Gamma_{K,G} \rtimes (\pi_0(K) \times \pi_0(\operatorname{Aut}_G(K)));$$

b) *if* G_{ab} *is a torsion group, then*

$$\pi_0(\operatorname{Aut}_{C(K)}(C(K) \underset{\alpha,r}{\rtimes} G)) = \operatorname{Hom}(G, \mathbf{T}) \rtimes (\pi_0(K) \times \pi_0(\operatorname{Aut}_G(K))).$$

3.14. REMARK. For the case of the Bunce-Deddens algebras, i.e. $K = \mathbf{T}$ and G is an infinite torsion subgroup of $\mathbf{T}$, the above corollary gives:

$$\pi_n(\operatorname{Aut}_{C(\mathbf{T})}(C(\mathbf{T}) \underset{\alpha}{\times} G)) = \begin{cases} \operatorname{Hom}(G, \mathbf{T}) \times \mathbf{Z}_2 & \text{for } n = 0 \\ \mathbf{Z} & \text{for } n = 1 \\ 0 & \text{for } n \geqslant 2. \end{cases}$$

where $\mathbf{Z}_2$ acts on $\operatorname{Hom}(G, \mathbf{T})$ by conjugation.

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