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POSITIVE SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS ON LIE GROUPS

WOLFGANG ARENDT, CHARLES J. K. BATTY and DEREK W. ROBINSON

1. INTRODUCTION

The general theory of elliptic operators affiliated with representations of Lie groups originated in 1960 with the unpublished thesis of Langlands [14]. (For a statement of the principal results see [15].) Although second-order elliptic operators, and in particular the Laplacian, were subsequently used by various authors (see, for example, [7], [12], [21], [22], [24]) in analyzing the differential structure of representations the general theory remained undeveloped until recently [26], [27], [1]. Our purpose is to continue this development in two different directions. First we discuss the characterization of second-order elliptic operators by positivity or dispersivity properties. Then we examine second-order operators with variable coefficients on function spaces and relate differentiability of the coefficients to smoothness of the action of the operator resolvents, or the corresponding semigroups. This emphasis on operators with non-constant coefficients is the principal difference between the current investigation and earlier work.

The motivation for the examination of elliptic operators affiliated with continuous representations of a general Lie group G comes from various sources. Langlands 1960 work established that the closure of each strongly elliptic operator generates a continuous holomorphic semigroup and hence he deduced that each representation has a dense set of analytic elements. This answered a question first raised by Harish-Chandra [9] and provided the basis for the application of analytic arguments to the analysis of representations. Nelson [21] independently obtained a similar result but in a slightly more restricted framework. Langlands analyzed the semigroups generated by general order strongly elliptic operators whilst Nelson only examined the simplest examples, second order Laplacians. On the other hand Nelson also demonstrated that the integrability of a skew-adjoint representation of a Lie algebra could be characterized in terms of generator properties of the Laplacian. These results were developed later by various authors (see, for example, [7], [12], [1], [28]).

The Laplacian and the corresponding heat semigroup also played an important role in the extension of Littlewood-Paley theory to Lie groups [31]. This theory is the basis of many classical results in harmonic analysis and it is useful for a variety of purposes in representation theory. More recent developments occur in the work of Varopoulos [33] which refers to other eariler work in this area.

Elliptic operators and their semigroups are also of interest for the description of geometric properties of manifolds principally through the analysis of spectral properties of the associated semigroup kernels (see, for example, the recent book by Davies [3]). Similarly in the Lie group setting one can relate properties of the semigroups to the geometry of the group. For example, if the group is nilpotent its dimension at infinity characterizes the rate of decay of the heat semigroup for large times [34].

Other applications arise in the description of evolution phenomena for systems with a Lie group symmetry. Such systems are generally described by partial differential equations formulated in a representation and stability, or conservation, properties can be expressed by ellipticity conditions. Although we principally examine strongly elliptic operators some of these applications naturally concern elliptic, or subelliptic, operators. The totality, and variety, of these applications indicate the interest in developing a general theory of elliptic operators on Lie groups and the current paper is a modest contribution to the theory which concentrates on characterizing second-order operators by positivity properties of the corresponding semigroup and initiating the basic theory of second-order operators with smooth coefficients. In order to be more precise we must introduce some notation, and recall some definitions.

Let $\mathscr{X}$ be a Banach space, G a Lie group, and U a strongly, or weakly*, continuous representation of G by bounded linear operators $U(g)$, $g \in G$, acting on $\mathscr{X}$. We fix a basis $a_1, \ldots, a_d$ of the Lie algebra $\mathfrak{g}$ and define A_i $(= dU(a_i))$ to be the generator of the one-parameter subgroup $t \in \mathbf{R} \mapsto U(e^{-ta_i})$. Then for each $n = 1, 2, \ldots$ we introduce the C^n-subspace $\mathscr{X}_n$ of $\mathscr{X}$ as

$$\mathscr{X}_n = \bigcap_{1 \leq i_1, \ldots, i_n \leq d} D(A_{i_1} \ldots A_{i_n}).$$

The corresponding C^n-seminorms ρ_n are then defined inductively by $\rho_0(x) = \|x\|$ and

$$x \in \mathscr{X}_n \mapsto \rho_n(x) = \sup_{1 \leq i \leq d} \rho_{n-1}(A_i x)$$

for $n = 1, 2, \ldots$. The C^n-norm is given by

$$x \in \mathscr{X}_n \to \|x\|_n = \|x\| + \rho_n(x)$$

and $\mathcal{X}_n$ is a Banach space with respect to $\|\cdot\|_n$. The C^∞-subspace

$$\mathcal{X}_\infty = \bigcap_{n\geq 1} \mathcal{X}_n$$

is a Fréchet space with respect to the topology defined by the family of norms $\{\|\cdot\|_n\,;\, n\geq 1\}$.

Next, if $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with α_i non-negative integers and if $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$ we use the standard notation $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$. Then the m-th order form

$$\xi \in \mathbf{R}^d \mapsto C_m(\xi) = \sum_{\alpha;\, |\alpha| \leq m} c_\alpha \xi^\alpha$$

with coefficients $c_\alpha \in \mathbf{C}$ is defined to be *strongly elliptic* if

$$\mathrm{Re}((-1)^{m/2} \sum_{\alpha;\, |\alpha| = m} c_\alpha \xi^\alpha) \geq p_m |\xi|^m \tag{1.1}$$

for some $p_m > 0$ and all $\xi \in \mathbf{R}^d$. The largest value of p_m for which this inequality holds is called the *ellipticity constant* of the form C_m. Now we define the operator

$$A_m = \sum_{\alpha;\, |\alpha| \leq m} c_\alpha A^\alpha \tag{1.2}$$

where $A^\alpha = A_1^{\alpha_1} \ldots A_d^{\alpha_d}$ and $\mathrm{D}(A_m) = \mathcal{X}_m$. If C_m is strongly elliptic then A_m is said to be a *strongly elliptic differential operator* with constant coefficients.

In addition to this general structure we need some specific notation for function spaces. First, $C(G)$ denotes the continuous functions over G, $C_b(G)$ the uniformly bounded continuous functions, $C_0(G)$ the continuous functions which vanish at infinity, and $C_c(G)$ the continuous functions with compact support. Second we need the usual L_p-spaces $L_p(G;\, dg)$, for $p \in [1, \infty]$, with respect to the left-invariant Haar measure dg. At the risk of some confusion with the foregoing C^n-norm we use $\|\cdot\|_p$ to denote the L_p-norm

$$\|f\|_p = \left(\int_G dg\, |f(g)|^p\right)^{1/p}$$

if $p \in [1, \infty)$ and $\|\cdot\|_\infty$ the supremum norm on L_∞, or C_b. On each of these spaces G is represented by left translations L where

$$(L(g)f)(h) = f(g^{-1}h)$$

and the action is strongly continuous of C_0 and on L_p for $p \in [1, \infty)$. On L_∞ the representation is weak*-continuous. Finally $L_{p;n}$, and $C_{0;n}$, denote the C^n-subspaces of L_p, and C_0, respectively, corresponding to the isometric representations (L_p, G, L) etc., and we set

$$C_{b;1}(G) = \{f \in L_{\infty;1} \cap C_b\,;\ A_i f \in C_b,\ i = 1, \ldots, d\}$$

$$C_{b;n+1}(g) = \{f \in C_{b;1}\,;\ A_i f \in C_{b;n},\ i = 1, \ldots, d\}$$

where $A_i = dL(a_i)$ on $L_{\infty;1}$. We note that the spaces $L_{p;n}$ correspond to the usual Sobolev spaces if $G = \mathbf{R}^d$ and the notations $H_{p;n}$ and $W_{p;n}$ are conventionally used, but in the present context $L_{p;n}$ appears more appropriate.

Next we consider operators with non-constant coefficients associated with left translations L on $\mathcal{X} = L_p(G; dg)$, $p \in [1, \infty]$, or $\mathcal{X} = C_0(G)$. Let $A_i = dL(a_i)$ denote the corresponding generators and define

$$A_m = \sum_{|\alpha| \leq m} c_\alpha A^\alpha$$

with domain $D(A_m) = \mathcal{X}_m$ where $c_\alpha \in L_\infty(G; dg)$ if $\mathcal{X} = L_p$, or $c_\alpha \in C_b(G)$ if $\mathcal{X} = C_0(G)$. Then A_m is said to be *uniformly strongly elliptic* if there is a $p_m > 0$ such that

$$\inf_{g \in G} \operatorname{Re}\Big((-1)^{m/2} \sum_{|\alpha| = m} c_\alpha(g)\xi^\alpha\Big) \geq p_m |\xi|^m$$

for all $\xi \in \mathbf{R}^d$. The analysis of such operators, particularly if $m = 2$, is the main focus of the following sections.

2. POSITIVITY AND DISPERSIVITY

In this section we principally examine strongly elliptic differential operators with bounded coefficients on $C_0(G)$ and in particular on $C_0(\mathbf{R}^d)$, or $C_0(\Omega)$ where Ω is a bounded open subset of $\mathbf{R}^d$. Our interest is to analyze the implication of positivity properties. Although this appears to be a rather restricted range for investigation the results have an impact for general strongly elliptic operators. We begin with a statement of the relevant conclusion.

First recall that if A_m is an m-th order operator associated with the strongly elliptic form C_m, with constant coefficients, in the representation $(\mathcal{X}, G, U)$ then

1. A_m is closable and its closure $\bar{A}_m$ generates a holomorphic semigroup S,
2. the action of S is determined by a representation independent kernel K

$$S_t = \int_G dg K_t(g) U(g),$$

which is jointly analytic in t and g;

3. there exist a, b, and $\omega > 0$ such that

$$|K_t(g)| \leqslant at^{-d/m} e^{-b(|g|^m/t)^{1/(m-1)}} e^{\omega t}$$

for all $t > 0$ and $g \in G$.

The first two statements were proved by Langlands [14] and the third is established in [27].

Now we derive a precise characterization of positive K and also lower bounds.

THEOREM 2.1. *The following conditions are equivalent:*

1. *the kernel K is pointwise positive*, i.e.

$$K_t(g) \geqslant 0$$

for all $t > 0$ and $g \in G$,

2. *$m = 2$ and the coefficients of the strongly elliptic form C_2 are real.*

Moreover, if G is connected and these conditions are satisfied then there exist $a, b, \omega > 0$ such that

$$K_t(g) \geqslant at^{-d/2} e^{-b|g|^2/t} e^{-\omega t}$$

for all $t > 0$ and $g \in G$.

The theorem follows from a sequence of more detailed results on elliptic operators acting on $C_0(G)$. The equivalence of Conditions 1 and 2 of the theorem is established at the end of Subsection 2.2 and the lower bound on K is derived in Subsection 2.3. Lower bounds of this type are already known for a more restricted class of operators [33] but we give an independent proof by different techniques. We begin the discussion by considering properties of partial differential operators on $\mathbf{R}^d$.

2.1. DIFFERENTIAL OPERATORS ON CONTINUOUS FUNCTIONS. Let $\Omega \subset \mathbf{R}^d$ be an open set and $\mathcal{B}$ a Banach space of bounded continuous functions with the supremum norm over $\tilde{\Omega} \supset \Omega$. Consider the partial differential operator

$$A_m = \sum_{\alpha;\, |\alpha| \leqslant m} c_\alpha \partial^\alpha$$

with domain $D(A_m) = C_c^\infty(\Omega)$ where $\partial^\alpha = \partial^{\alpha_1} / \partial \xi_1^{\alpha_1} \dots \partial^{\alpha_d} / \partial \xi_d^{\alpha_d}$, and the c_α are continuous functions over Ω.

PROPOSITION 2.2. *If A_m has an extension which generates a positive semigroup on $\mathcal{B}$ then the coefficients of A_m are real, $c_\alpha = 0$ for $|\alpha| > 2$, and*

$$-\sum_{\alpha;\, |\alpha|=2} c_\alpha(\eta)\xi^\alpha \geqslant 0$$

for all $\eta \in \Omega$ and $\xi \in \mathbf{R}^d$.

REMARK. An analogous result has been obtained by Miyajima and Okazawa [16] (see also [17]). They consider $L_p(\Omega)$, $p \in [1, \infty)$ instead of $C_0(\Omega)$ and use Kato's inequality as a necessary condition for positivity. Their result also requires that $C_c^\infty(\Omega)$ is a core of A_m.

Proof. The key to the proof is the observation that A_m must satisfy the *positive maximum principle* on Ω, i.e. if $f \in D(A_m)$, $f \geq 0$, and $f(\xi) = 0$ for some $\xi \in \Omega$ then $(A_m f)(\xi) \leq 0$. In order to establish this we note that if $\tilde{A}_m \supseteq A_m$ and $\tilde{A}_m$ generates the positive semigroup T then

$$(A_m f)(\xi) = (\tilde{A}_m f)(\xi) = \lim_{t \to 0} -(T_t f)(\xi)/t \leq 0.$$

Next observe that

$$\partial^\beta P_\alpha(0) = \alpha! \, \delta_{\alpha,\beta}$$

where $P_\alpha(\xi) = \xi^\alpha$, $\alpha! = \alpha_1! \ldots \alpha_d!$, and $\delta_{\alpha,\beta}$ denotes the Kronecker delta function. Moreover, if $c \in (0, 1)$ and $|\alpha| > 2$ then

$$|\xi^\alpha| < c|\xi|^2$$

if $|\xi| \in (0, c)$ where $|\xi|^2 = \xi_1^2 + \ldots + \xi_d^2$. This follows because

$$|\xi^\alpha|/|\xi|^2 \leq |\xi|^{\alpha_1} \ldots |\xi|^{\alpha_d}/|\xi|^2 = |\xi|^{|\alpha|-2} < c^{|\alpha|-2} \leq c.$$

Now we can prove the proposition with the following three formal observations.

OBSERVATION 1. If A_m is real, i.e. if $\overline{A_m f} = A_m \bar{f}$ for all $f \in D(A_m)$, then the coefficients c_α are real.

Proof. For each multi-index α and $\eta \in \Omega$ let $f \in C_c^\infty(\Omega)$ be a real-valued function such that $f(\xi) = P_\alpha(\xi - \eta)$ in a neighbourhood of η. Then

$$c_\alpha(\eta) = (\alpha!)^{-1}(A_m f)(\eta) \in \mathbf{R}.$$

OBSERVATION 2. If A_m is real and satisfies the positive maximum principle on Ω then $c_\alpha = 0$ whenever $|\alpha| > 2$.

Proof. Let $|\alpha| > 2$ and assume $c_\alpha(\eta) \neq 0$. Then choose $c > 0$ sufficiently small that

$$\alpha! \, |c_\alpha(\eta)| + 2c\{c_{(2,0,\ldots,0)}(\eta) + \ldots + c_{(0,\ldots,0,2)}(\eta)\} > 0.$$

Next for $|\eta - \xi| < c$ define g by

$$g(\xi) = (\operatorname{sign} c_\alpha(\eta))(\xi - \eta)^\alpha + c|\xi - \eta|^2.$$

So $g(\xi) > 0$ for $0 < |\eta - \xi| < c$ by the estimate preceding Observation 1. Now choose a positive $\varphi \in C_c^\infty(\Omega)$ with $\varphi = 1$ in a neighbourhood of η and such that $\varphi(\xi) = 0$ if $|\eta - \xi| \geqslant c/2$. Then $f = \varphi g$ is a positive function in $C_c^\infty(\Omega)$ and $f(\eta) = 0$. But

$$(A_m f)(\eta) = \alpha!\,|c_\alpha(\eta)| + 2c\{c_{(2,0,\dots,0)}(\eta) + \dots + c_{(0,\dots,0,2)}(\eta)\} > 0$$

which is a contradiction. Hence $c_\alpha = 0$ for $|\alpha| > 2$.

OBSERVATION 3. If A_2 is a real second-order operator satisfying the positive maximum principle on Ω then A_2 satisfies the ellipticity property of Proposition 2.2.

Proof. Let $\eta \in \Omega$, $\xi \in \mathbf{R}^d$, and define φ by

$$\varphi(\zeta) = \frac{1}{2} \sum_{i,j=1}^{d} \xi_i \xi_j (\eta_i - \zeta_i)(\eta_j - \zeta_j).$$

Then choose $f \in C_c^\infty(\Omega)$ such that f is positive, and $f = \varphi$ in a neighbourhood of η. Then $f(\eta) = 0$ and

$$(A_2 f)(\eta) = \sum_{\alpha;|\alpha|=2} c_\alpha(\eta)\xi^\alpha \leqslant 0$$

by the positive maximum principle.

This completes the proof of Proposition 2.2 and to complete this subsection we make several remarks on possible extensions of this result.

An operator B on $\mathscr{B}$ is defined to be *resolvent-positive* if there exists a $\mu \in \mathbf{R}$ such that $(\lambda I + B)^{-1}$ exists and is positive for $\lambda > \mu$, e.g. the generator of a positive semigroup is resolvent-positive. If B is resolvent-positive, densely-defined and

$$\limsup_{\lambda\to\infty} \|\lambda(\lambda I + B)^{-1}\| < \infty, \tag{2.1}$$

it follows that B satisfies the positive maximum principle. Moreover, the conclusion of Proposition 2.2 is valid if we assume that A_m has a densely-defined resolvent-positive extension satisfying (2.1). In general a resolvent-positive operator does not satisfy the positive maximum principle (see Example 2.4 below) but it does satisfy a weakened form of this principle. In particular if B is resolvent positive, $0 \leqslant f \in D(B^3)$, and $f(\xi) = 0$, then $(Bf)(\xi) \leqslant 0$. This follows by noting that if $R(\lambda) = (\lambda I + B)^{-1}$ then $R(\lambda_1) - R(\lambda_2) = (\lambda_2 - \lambda_1)R(\lambda_1)R(\lambda_2)$ and hence $\lambda \mapsto R(\lambda)$ is decreasing and in particular bounded for $\lambda \geqslant \mu + 1$. But for $f \in D(B)$ one has

$$\lambda R(\lambda) f = f - R(\lambda) B f$$

and consequently $R(\lambda)f \to 0$ as $\lambda \to \infty$. Then if $f \in D(B^2)$ it follows that $\lambda R(\lambda)f \to f$ as $\lambda \to \infty$. Finally if $f \in D(B^3)$

$$\lim_{\lambda \to \infty} \lambda^2 R(\lambda)f - \lambda f = \lim_{\lambda \to \infty} - \lambda R(\lambda)Bf = - Bf.$$

Thus if in addition $f \geqslant 0$ and $f(\xi) = 0$ then

$$(Bf)(\xi) = \lim_{\lambda \to \infty} - \lambda^2 (R(\lambda)f)(\xi) \leqslant 0.$$

As a consequence of these remarks we have an alternative version of Proposition 2.2.

COROLLARY 2.3. *Assume A_m has coefficients $c_\alpha \in C^\infty(\Omega)$. If A_m has a resolvent positive extension then the conclusions of Proposition 2.2 are valid.*

Proof. The assumption on the c_α ensures that $C_c^\infty(\Omega) \subseteq D(A_m^3)$. Hence the proof of Proposition 2.2 applies with the weakened form of the positive maximum principle.

We also remark that if A_m has an extension which generates a positive group then the arguments of Proposition 2.2 can be applied to both $\pm A_m$. Then one is forced to the conclusion that $c_\alpha = 0$ if $|\alpha| > 1$.

Finally we give an example of an operator which is resolvent-positive but does not satisfy the unrestricted form of the positive maximum principle.

EXAMPLE 2.4. Let $\mathscr{B} = C([-1,0]) \times \mathbf{R}$ and define B by $D(B) = C_1([-1,0]) \times \{0\}$ and

$$B(f, 0) = (-f', f(0)).$$

Then the resolvent set equals $\mathbf{C}$ and $(\lambda I + B)^{-1}(f, c) = (g, 0)$ with

$$g(x) = e^{\lambda x} \left\{ c + \int_x^0 dy\, e^{-\lambda y} f(y) \right\}.$$

Therefore B is resolvent positive. Now let $f(x) = -x$. Then $0 \leqslant (f, 0) \in D(B)$ and $f(0) = 0$. But $-f'(0) = 1 > 0$.

2.2. LIE GROUPS. Let G be a connected Lie group and $a_1, \ldots, a_d$ a basis of the Lie algebra $\mathfrak{g}$. We will use the exponential map $s \in \mathbf{R}^d \mapsto e^{-s \cdot a} \in G$, where $s \cdot a = s_1 a_1 + \cdots + s_d a_d$, to lift the results of the last subsection to give statements about differential operators associated with left translations L acting on $C_0(G)$.

First recall that there exists open sets $\Omega \subset G$, and $\hat{\Omega} \subset \mathbf{R}^d$, such that $e \in \Omega$, and $0 \in \hat{\Omega}$, and the exponential map is a C^∞-diffeomorphism of $\hat{\Omega}$ onto Ω. We assume that the boundary of $\hat{\Omega}$ is smooth. For example, we could choose $\hat{\Omega}$ to be a ball centred at the origin. Now if f is a function on G we denote by $\hat{f}$ the function on $\mathbf{R}^d$ given by

$$\hat{f}(s) = f(e^{-s \cdot a}).$$

Then the generators $A_1, \ldots, A_d$ of left translations on $C_0(G)$ introduced in Section 1 correspond to the infinitesimal generators of the positive $*$-automorphism groups L_i where

$$(L_i(t)f)(g) = (L(e^{-ta_i})f)(g) = f(e^{ta_i} g).$$

This identification leads to the following standard result.

LEMMA 2.5. *For each $i = 1, \ldots, d$ there exist $b_{ij} \in C^\infty(\hat{\Omega})$ such that*

$$\widehat{(A_i f)}(s) = \sum_{j=1}^{d} b_{ij}(s) \partial \hat{f}(s) / \partial s_j$$

for all $s \in \hat{\Omega}$ and $f \in C_c^\infty(\Omega)$, and in addition

$$b_{ij}(0) = \delta_{ij}.$$

Using this identification we can now prove a Lie group version of Proposition 2.2.

PROPOSITION 2.6. *Let A_m be an m-th order differential operator on $C_0(G)$ with domain $C_c^\infty(G)$ given by*

$$A_m f = \sum_{|\alpha| \leqslant m} c_\alpha A^\alpha f$$

where $c_\alpha \in C(G)$. If A_m has an extension which generates a positive semigroup then $c_\alpha = 0$ for $|\alpha| > 2$, the c_α with $|\alpha| \leqslant 2$ are real, and in addition

$$(2.2) \qquad - \sum_{\alpha;\, |\alpha| = 2} c_\alpha(g) \xi^\alpha \geqslant 0$$

for all $g \in G$ and $\xi \in \mathbf{R}^d$.

Proof. Define $\hat{A}_m$ on $C_0(\mathbf{R}^d)$ by

$$D(\hat{A}_m) = C_c^\infty(\hat{\Omega})$$

$$(\hat{A}_m \hat{f}) \circ \sigma = A_m(\hat{f} \circ \sigma)$$

where $\sigma: \Omega \mapsto \hat{\Omega}$ is the inverse of the exponential map. Since A_m has an extension which generates a positive semigroup it follows that A_m satisfies the positive maximum principle on Ω. Consequently $\hat{A}_m$ satisfies the principle on $\hat{\Omega}$.

Next it follows from Lemma 2.5 that

$$(\hat{A}_m \hat{f})(s) = (\widehat{A_m f})(s) = \sum_{\alpha; |\alpha| \leq m} \hat{c}_\alpha(s)(\partial^\alpha \hat{f})(s)$$

where $\hat{c}_\alpha(0) = c_\alpha(e)$ if $|\alpha| = m$. Thus applying Proposition 2.2 to $\hat{A}_m$ one concludes that $c_\alpha(e) = 0$ if $|\alpha| = m > 2$. Then by repetition of the argument $c_\alpha(e) = 0$ if $|\alpha| >$ > 2. In addition it follows from this same proposition that $c_\alpha(e)$ is real if $|\alpha| > 2$. In addition it follows from this same proposition that $c_\alpha(e)$ is real if $|\alpha| \leq 2$ and (2.2) is satisfied for $g = e$.

Finally if R denotes right translations on $C_0(G)$, then one can repeat the foregoing arguments with A_m replaced by $R(g)A_mR(g)^{-1}$. This effectively replaces e by g and hence one obtains the general statement of the proposition.

Next we establish a result in the converse direction.

PROPOSITION 2.7. *Let A_2 be a second-order differential operator on $C_0(G)$ with domain $D(A_2) \subseteq C_{0;2}(G)$ given by*

$$A_2 f = \sum_{|\alpha| \leq 2} c_\alpha A^\alpha f$$

where $c_\alpha \in C_b(G)$ are real coefficients which satisfy the ellipticity condition (2.2). Assume A_2 is closable and its closure generates a semigroup T.

Then T is positive, and T is contractive if, and only if, $c_0 \geq 0$.

Proof. Let $\omega = \inf_{g \in G} c_0(g)$ and $B_2 = A_2 - \omega I$. Then the closure $\bar{B}_2$ of B_2 generates the semigroup U; $U_t = T_t e^{\omega t}$. But U is positive and contractive if, and only if, the operator B_2 is *dispersive* on $C_0(G)$, (see, for example, [19], page 249), i.e. if $f \in D(B_2) = D(A_2)$ and

$$f(g) = \sup_{h \in G} f(h)$$

then $(B_2 f)(g) \geq 0$. (Note that our definition of the generator A of the semigroup T corresponds to the formal relation $T_t = e^{-tA}$. Other authors, for example [19], [35], adopt a convention which replaces A by $-A$ and is consistent with the relation $T_t = e^{tA}$. This must be borne in mind when making comparisons with the literature.)

Now replacing f by $R(g)f$ we may effectively assume the maximum is attained at the identity e. Then

$$(B_2 f)(e) = (\widehat{A_2 f})(0) - c_0(e)f(e) = \sum_{\alpha; 1 \leq |\alpha| \leq 2} \hat{c}_\alpha(0)(\partial^\alpha \hat{f})(0)$$

where $\hat{c}_\alpha(0) = c_\alpha(e)$ if $|\alpha| = 2$. Since zero is a local maximum of $\hat{f}$ it follows that $(\partial^\alpha \hat{f})(0) = 0$ for $|\alpha| = 1$ and the matrix $(\partial^2 \hat{f}(0)/\partial_{s_i}\partial_{s_j})$ is negative-definite. But the ellipticity condition (2.2) implies that the matrix associated with the coefficients $\hat{c}_\alpha(0)$ with $|\alpha| = 2$ is negative-definite. Therefore the Hadamard (pointwise) product of these two matrices is positive-definite. In particular

$$\sum_{\alpha;|\alpha|=2} \hat{c}_\alpha(0)(\partial^\alpha \hat{f})(0) \geqslant 0$$

and one has $(B_2 f)(e) \geqslant 0$.

Therefore we have concluded that U is positive and contractive. Since $T_t = U_t e^{-\omega t}$ it follows that T is positive and if $c_0 \geqslant 0$ it is also contractive. If, however, T is positive and contractive then A_2 is dispersive. But choosing $f \in D(A_2)$ with $0 \leqslant f \leqslant 1$ and $f = 1$ in a neighbourhood of $g \in G$ then

$$c_0(g) = c_0(g)f(g) = (A_2 f)(g) \geqslant 0.$$

We are now in a position to prove the first statement of Theorem 2.1. Recall that the theorem concerns a strongly elliptic operator A_m, with constant coefficients c_α, and the closure $\bar{A}_m$ of A_m generates a continuous semigroup S with kernel K on $C_0(G)$ by the work of Langlands [14]. But positivity of K ensures that S is positive. Hence $c_\alpha = 0$ for $|\alpha| > 2$ and the c_α with $|\alpha| \leqslant 2$ are real by Proposition 2.6. Thus $1 \Rightarrow 2$ in Theorem 2.1. Conversely, if one has a second-order operator then reality of the c_α together with strong ellipticity implies that S is positive by Proposition 2.7. Therefore the kernel K must be positive and $2 \Rightarrow 1$.

Next we turn to examination of lower bounds on the kernel associated with a second-order operator.

2.3. Lower bounds. Let A_2 be a second-order strongly elliptic operator associated with left translations on $C_0(G)$ with real coefficients. Then A_2 can be written in the form

$$A_2 = -\sum_{i,j=1}^{d} c_{ij}A_iA_j + \sum_{i=1}^{d} c_iA_i + c_0 I$$

where $c = (c_{ij})$ is a real-valued strictly positive-definite matrix and the c_i are real. Now let K denote the positive semigroup kernel corresponding to K. We use standard techniques of diffusion processes [18] and partial differential equations to obtain lower bounds on K. It is convenient for this purpose to assume $c_0 = 0$. The general case can then be recovered by simply multiplying K_t by $e^{c_0 t}$.

First, using the exponential map from $\hat{\Omega} \subset \mathbf{R}^d$ to $\Omega \subset G$ as in Subsection 2.2 and Lemma 2.5 the operator A_2 on $C_c^\infty(G)$ corresponds to a partial differential operator $\hat{A}_2$ on $C_c^\infty(\hat{\Omega})$ of the form

$$\hat{A}_2 = -\sum_{i,j=1}^{d} c_{ij}(s) \frac{\partial^2}{\partial s_i \partial s_j} + \sum_{i=1}^{d} c_i(s) \frac{\partial}{\partial s_i} \tag{2.3}$$

where the coefficients are smooth functions and $c_{ij}(0) = (c_{ij})$, and we have set $c_0 = 0$. Therefore by reducing the size of Ω if necessary we may assume that $c(s) = (c_{ij}(s))$ is uniformly positive definite, i.e.

$$\sum_{i,j=1}^{d} c_{ij}(s)\xi_i\xi_j \geqslant p|\xi|^2 \tag{2.4}$$

for some $p > 0$ and all $\xi \in \mathbf{R}^d$, $s \in \hat{\Omega}$.

Next we may extend c_{ij}, c_i, to smooth bounded functions on $\mathbf{R}^d$ with bounded derivatives of first-order in such a way that (2.4) holds throughout $\mathbf{R}^d$. We then regard $\hat{A}_2$ as extended to a differential operator on $\mathbf{R}^d$ by (2.3). Moreover we note that the left-invariant Haar measure dg on G is given in the local co-ordinates by $dg = ds\varphi(s)$ where φ is a strictly positive smooth function on $\hat{\Omega}$. Thus

$$\int_G dg\, f(g) = \int ds\, \varphi(s)\hat{f}(s)$$

for all $f \in C_c(\Omega)$.

The closure $\bar{A}_2$ of the operator A_2 generates a positive semigroup S on $C_0(G)$, with the kernel K_t. The semigroup S is also contractive, because we are assuming $c_0 = 0$, and consequently there exists a Markov process X on G corresponding to S with $X_0 = e$ and with the transition density function $p_t(g, h) = K_t(g^{-1}h)$ (see, for example, [5], Chapter 4). There is also a process Y on $\mathbf{R}^d$ with $Y_0 = 0$ governed by the differential operator $\hat{A}_2$ (see [5], Chapter 8). The transition density function $q_t(s, u)$ corresponding to Y is the fundamental solution of the parabolic equation

$$\frac{\partial}{\partial t} q_t = \hat{A}_2 q_t \tag{2.5}$$

on $\mathbf{R}^d$ (see [4], page 162 or [5], page 370).

Let X^0 and Y^0 denote the processes obtained by truncating X and Y at their first exit times from Ω and $\hat{\Omega}$ respectively. Further let p^0 and q^0 denote the corresponding transition functions. (The generators of the corresponding semigroups are given by the operators A_2, and $\hat{A}_2$, on Ω, and $\hat{\Omega}$, with Dirichlet boundary conditions ([5], page 368).) It is clear that

$$p_t^0(g, h) \leqslant p_t(g, h) = K_t(g^{-1}h) \tag{2.6}$$

for all $g, h \in \Omega$. Moreover X^0 and $\exp Y^0$ have the same finite-dimensional distributions so

$$q_t^0(s, u) = p_t^0(e^{-s \cdot a}, e^{-u \cdot a})\varphi(u). \tag{2.7}$$

Furthermore, it follows from the strict Markov property for Y that

$$q_t(s, u) \leqslant q_t^0(s, u) + \sup_{t' \leqslant t,\, v \in \partial\hat{\Omega}} q_{t'}(v, u) \tag{2.8}$$

(see, for example, [4] or [18]).

The fundamental solution q_t of the parabolic equation (2.5) may be constructed by Levi's parametrix method ([6], Chapter 1). This gives the solution in the form

$$q_t(s, u) = \sigma_t(s, u) + \int_0^t d\tau \int_{\mathbf{R}^d} dv\, \sigma_\tau(s, v)\Phi(t - \tau, v, u)$$

where

$$\sigma_t(s, u) = (4\pi t)^{-d/2}(\det c(u))^{1/2}\exp\{-(s - u)^T c(u)^{-1}(s - u)/4t\}$$

and

$$|\Phi(t, v, u)| \leqslant c(p')\, t^{-(d+1)/2}\exp\{-p'|v - u|^2/4t\}$$

for all $p' < p$, where p is the ellipticity constant occurring in (2.4) and $c(p') > 0$. Clearly

$$|\sigma_t(s, u)| \leqslant ct^{-d/2}\exp\{-\rho|s - u|^2/4t\}$$

where $\rho = (\sup\|c(u)\|)^{-1}$. Thus if $\rho' = \min(\rho, p')$ then

$$|q_t(s, u) - \sigma_t(s, u)| \leqslant c'(\rho')\int_0^t d\tau \int_{\mathbf{R}^d} dv\, \tau^{-d/2}(t - \tau)^{-(d+1)/2}\exp\{-\rho'|s - v|^2/4t\}\cdot$$

$$\cdot\exp\{-\rho'|v - u|^2/4(t - \tau)\}\cdot$$

Now separating the integral over τ into a part over $(0, t/2)$ and a part over $(t/2, t)$ and then estimating in a strightforward manner one obtains an estimate

$$|q_t(s, u) - \sigma_t(s, u)| \leqslant a(\lambda)t^{-(d-1)/2}\exp\{-\lambda|s - u|^2/4t\}$$

valid for each $\lambda < \min(\rho, p)$.

LEMMA 2.8. [32]. *For each* $\rho < (\sup\|c(s)\|)^{-2} \wedge (\sup\|c(s)^{-1}\|)^{-1}$ *there exist* $M_1, M_2, M_3 > 0$ *such that the fundamental solution* q_t *of the parabolic equation* (2.5) *satisfies*

$$M_1 t^{-d/2}\exp\{-\rho|s - u|^2/4t\} \geqslant q_t(s, u) \geqslant$$

$$\geqslant M_2 t^{-d/2}\exp\{-\lambda|s - u|^2/4t\} - M_3 t^{-(d-1)/2}\exp\{-\rho|s - u|^2/4t\},$$

with $\lambda = (\sup\|c(s)\|)$, *uniformly for* $t \in (0, 1]$.

This is a direct consequence of the foregoing estimates. It is the key to the subsequent lower bounds on the kernel K. We note in passing that the $t^{-(d-1)/2}$ factor in the correction term is consequent on the smoothness of the coefficients of $\hat{A}_2$. Similar estimates hold if the coefficients are only Hölder continuous but then one has a factor $t^{-(d-\varepsilon)/2}$ with the value of $\varepsilon>0$ dependent on the degree of continuity.

LEMMA 2.9. *There exist a, r, $\tau > 0$ such that*

$$K_t(g) \geqslant at^{-d/2}$$

for $t \in (0, \tau]$ and all $g \in G$ with $|g|^2 \leqslant rt$.

Proof. It follows from (2.8) and Lemma 2.8 that

$$q_t^0(s, u) \geqslant M_2 t^{-d/2}\exp\{-\lambda|s-u|^2/4t\} - M_3 t^{-(d-1)/2}\exp\{-\rho|s-u|^2/4t\} - \\ - \sup_{\tau\leqslant t, v\in\partial\hat{\Omega}} (M_1\tau^{-d/2}\exp\{-\rho|v-u|^2/4\tau\}).$$

Now suppose $\hat{\Omega}$ is the ball of radius 2ε centred at the origin and that $|u| \leqslant \varepsilon$, $|s-u|^2 < bt$. Then

$$q_t^0(s, u) \geqslant M_2 t^{-d/2}\, e^{-\lambda b/4} - M_3 t^{-(d-1)/2} - \sup_{\tau\leqslant t}(M_1\tau^{-d/2}e^{-\rho\varepsilon^2/4\tau}).$$

Since $\tau^{-d/2}e^{-\rho\varepsilon^2/4\tau} \to 0$ as $\tau \to 0$ it follows that there exist $a, \tau > 0$ such that

$$q_t^0(s, u) \geqslant at^{-d/2} \tag{2.9}$$

for $|s-u|^2 \leqslant bt$, $|u| < \varepsilon$, and $0 < t \leqslant \tau$. Moreover, there is a $c > 0$ such that $|s| \leqslant \varepsilon$, $|u| \leqslant \varepsilon$, implies

$$c^{-1}|s-u| \leqslant |e^{-s\cdot a}\, e^{u\cdot a}| \leqslant c|s-u|.$$

Therefore it follows from (2.6), (2.7), and (2.9), that there exist a', $r > 0$ such that

$$K_t(g^{-1}h) \geqslant a't^{-d/2}$$

for $|g^{-1}h|^2 \leqslant rt$ and $t \in (0, \tau]$, which immediately gives the desired conclusion.

Now we are prepared to prove the second statement of Theorem 2.1. It is a consequence of Lemma 2.9 and a standard convolution argument [10].

Take $g \in G$ and $t > 0$ and let k be the smallest positive integer such that $4|g|^2 < rkt$, $t < k\tau$. Since G is connected there is a sequence of points $g_0 = e$, $g_1, \ldots, g_k = g$ such that $|g_i^{-1}g_{i+1}| = |g|/k$. Then by the convolution formula

$$K_t(g) = \int_G dh_1 \ldots \int_G dh_{k-1} K_{t/k}(h_1)K_{t/k}(h_1^{-1}h_2) \ldots K_{t/k}(h_{k-1}^{-1}g) \geqslant$$

$$\geqslant \int_{B_1} dh_1 \ldots \int_{B_{k-1}} dh_{k-1} K_{t/k}(h_1)K_{t/k}(h_1^{-1}h_2) \ldots K_{t/k}(h_{k-1}^{-1}g)$$

where $B_i = \{h \in G;\ |h^{-1}g_i|^2 < rt/16k\}$. But if $h_i \in B_i$ and $h_{i+1} \in B_{i+1}$ then

$$|h_i^{-1}h_{i+1}|^2 \leqslant (|h_i^{-1}g_i| + |g_i^{-1}g_{i+1}| + |g_{i+1}^{-1}h_{i+1}|)^2 \leqslant rt/k.$$

It follows that

$$K_{t/k}(h_i^{-1}h_{i+1}) \geqslant ak^{d/2}t^{-d/2}$$

so

$$K_t(g) \geqslant (ak^{d/2}t^{-d/2})^k V((rt/16k)^{1/2})^{k-1}$$

where $V(\theta)$ is the volume of the ball $\{h \in G;\ |h| \leqslant \theta\}$. One estimates, however, with the aid of the exponential map that $V(\theta) \geqslant c\theta^d$ for some positive constant c and all small θ. Therefore

$$K_t(g) \geqslant a^k(c(r/16)^{d/2})^{k-1}k^{d/2}t^{-d/2}.$$

But $k - 1 \leqslant 4|g|^2/rt + t/\tau$ so if we set

$$b = \max(-4r^{-1}\log(ac(r/16)^{d/2}),\ 0)$$

$$\omega' = \max(-\tau^{-1}\log(ac(r/16)^{d/2}),\ 0)$$

then

$$K_t(g) \geqslant at^{-d/2}e^{-b|g|^2/t}e^{-\omega' t}.$$

Now at the outset we assume $c_0 = 0$ but if this is not the case then it is necessary to multiply the lower bound by $e^{-c_0 t}$. Thus setting $\omega = \omega' + c_0$ one obtains the bounds of Theorem 2.1.

3. SECOND-ORDER ELLIPTIC OPERATORS

Next we discuss properties of second-order elliptic operators, with variable coefficients, associated with left translations on the function spaces $L_p(G;\ dg)$ and $C_0(G)$. Our approach is fairly standard. We begin by examining the operators on $L_2(G;\ dg)$ using Hilbert space techniques. Then we prove a Sobolev embedding lemma which can be exploited to transform the L_2-results onto $C_0(G)$. Finally we argue with duality and interpolation techniques that many of the properties lift to the L_p-spaces with $p \neq 2$.

3.1. SELF-ADJOINT ELLIPTIC OPERATORS. In this subsection we examine second--order symmetric elliptic operators associated with left translations on $L_2(G\,;\ dg)$. But as a preliminary we discuss the definition of such operators through quadratic forms in a broader setting.

Let $A_1, \ldots, A_d$ be closed densely-defined operators on the Hilbert space $\mathscr{H} = L_2(\Omega; \mu)$, where (Ω, μ) is a measure space, and assume that the subspace

$$\mathscr{H}_1 = \bigcap_{i=1}^{d} \mathrm{D}(A_i) \tag{3.1}$$

is dense in $\mathscr{H}$.

Next let $C = (c_{ij})$ be a $d \times d$-matrix with elements $c_{ij} \in L_\infty(\Omega; \mu)$ satisfying $\overline{c_{ij}(x)} = c_{ji}(x)$. Define

$$\varkappa_m = \inf_{x \in \Omega} \inf_{\substack{\xi \in \mathbf{R}^d \\ |\xi| = 1}} - \sum_{i,j=1}^{d} c_{ij}(x)\xi_i\xi_j \tag{3.2}$$

and

$$\varkappa_M = \sup_{x \in \Omega} \sup_{\substack{\xi \in \mathbf{R}^d \\ |\xi| = 1}} - \sum_{i,j=1}^{d} c_{ij}(x)\xi_i\xi_j . \tag{3.3}$$

We assume throughout the ellipticity condition $\varkappa_m > 0$. Our aim is to analyze the elliptic operator A which is formally given by

$$A = - \sum_{i,j=1}^{d} A_i^* c_{ij} A_j . \tag{3.4}$$

Define the form $\varphi, \psi \in \mathscr{H}_1 \times \mathscr{H}_1 \to (\varphi, \psi)_A \in \mathbf{C}$ by

$$(\varphi, \psi)_A = - \sum_{i,j=1}^{d} (A_i\varphi, c_{ij}A_j\psi). \tag{3.5}$$

It then follows from the symmetry property of the c_{ij} and the ellipticity condition that

$$(\varphi, \varphi)_A \geqslant \varkappa_m \sum_{i=1}^{d} (A_i\varphi, A_i\varphi) \geqslant 0 . \tag{3.6}$$

In particular the form $(\cdot, \cdot)_A$ is positive. But as the A_i are closed this bound also implies that the form is closed. Therefore by the basic representation theorem for quadratic forms ([13], Chapter VI) there exists a unique positive self-adjoint operator A such that $\mathscr{H}_1 = \mathrm{D}(A^{1/2})$ and

$$(\varphi, \psi)_A = (A^{1/2}\varphi, A^{1/2}\psi) .$$

The domain $\mathrm{D}(A)$ of A consists of those $\psi \in \mathscr{H}_1$ for which there exist a $\xi \in \mathscr{H}$ satisfying

$$(\varphi, \psi)_A = (\varphi, \xi)$$

for all $\varphi \in \mathscr{H}_1$ and the $A\psi = \xi$.

Although this approach allows one to give a precise meaning to the elliptic operator A under very general circumstances it fails to provide any useful characterization of the domain of the self-adjoint A. This weakness can only be overcome by adopting more stringent assumptions on the A_i. If, for example, the A_i are generators of a Lie group representation one can then use elliptic regularity techniques to identify $\mathrm{D}(A)$. Before considering this question in detail we make four remarks on the general framework.

First, one easily derives the bound

$$(\varphi, \varphi)_A \leqslant \varkappa_M \sum_{i=1}^{d} (A_i\varphi, A_i\varphi)$$

for all $\varphi \in \mathscr{H}_1$. Therefore on the subspace $\mathscr{H}_1 = \mathrm{D}(A^{1/2})$ the form norm

$$\varphi \mapsto \|\varphi\|_A = ((\varphi, \varphi)_A + (\varphi, \varphi))^{1/2} \tag{3.7}$$

is equivalent to the Sobolev norm

$$\varphi \mapsto \|\varphi\|_1 = \left(\sum_{i=1}^{d} (A_i\varphi, A_i\varphi) + (\varphi, \varphi) \right)^{1/2} \tag{3.8}$$

and $\mathscr{H}_1$ is complete with respect to these norms.

Second,

$$\|A_i\varphi\|^2 \leqslant \sum_{i=1}^{d} (A_i\varphi, A_i\varphi) \leqslant \varkappa_m^{-1}(\varphi, \varphi)_A$$

for all $\varphi \in \mathscr{H}_1$ by (3.6). But if $\varphi \in \mathrm{D}(A)$ then

$$(\varphi, \varphi)_A = (\varphi, A\varphi) \leqslant \varepsilon(A\varphi, A\varphi) + (4\varepsilon)^{-1}(\varphi, \varphi)$$

for all $\varepsilon > 0$. Therefore

$$\|A_i\varphi\|^2 \leqslant \varepsilon\varkappa_m^{-1}\|A\varphi\|^2 + (4\varepsilon\varkappa_m)^{-1}\|\varphi\|^2 \tag{3.9}$$

for all $\varphi \in \mathrm{D}(A)$ and $\varepsilon > 0$. In particular the A_i are relatively bounded by A with relative bound zero. Consequently one can use perturbation theory to define and discuss operators such as

$$A + \sum_{i=1}^{d} c_iA_i + c_0$$

with the $c_i \in L_\infty(\Omega; \mu)$.

Third, if the A_i generate positive groups on $L_2(\Omega;\mu)$ then the contraction semigroup generated by A is also positive. This follows because by a result of Nagel and Uhlig [20] each A_i automatically satisfies Kato's equality on $L_2(\Omega;\mu)$, i.e. $\varphi \in \mathrm{D}(A_i)$ implies $|\varphi| \in \mathrm{D}(A_i)$ and $A_i|\varphi| = (\operatorname{sign}\varphi)A_i\varphi$ where $\operatorname{sign}\varphi(x) = \varphi(x)/|\varphi(x)|$ if $\varphi(x) \neq 0$ and $\operatorname{sign}\varphi(x) = 0$ if $\varphi(x) = 0$ (see, for example [19], Chapter C–II). But this implies that

$$(\varphi, \varphi)_A = -\sum_{i,j=1}^{d} (A_i\varphi, c_{ij}A_j\varphi) \geqslant (|\varphi|, |\varphi|)_A .$$

Therefore A satisfies the first Beurling-Deny criterion (see [25], Theorem XIII.50) and the semigroup generated by A is positive.

Fourth, if the groups generated by the A_i are positive and isometric for the norm $\|\cdot\|_\infty$ then one can also show that the second Beurling-Deny criterion (see [25], Theorem XIII.51(c)) is satisfied. This implies that the semigroup generated by A interpolates on the whole range of L_p-spaces. We omit the details, since we will give a different proof of a similar result in Theorem 3.13.

After these preliminaries we return to the Lie group setting. Therefore we now take $\mathscr{H} = L_2(G; \mathrm{d}g)$ and let A_i denote the skew-adjoint generators of the one-parameter groups $t \mapsto L(\mathrm{e}^{-ta_i})$ where L denotes left translations. We also use A_i to denote the generator of the corresponding one-parameter group on $L_\infty(G; \mathrm{d}g)$. Then we define the Sobolev spaces $\mathscr{H}_m = L_{2;m}(G; \mathrm{d}g)$ as in the introduction,

$$\mathscr{H}_m = L_{2;m}(G; \mathrm{d}g) = \bigcap_{1 \leqslant i_1, \ldots, i_m \leqslant d} \mathrm{D}(A_{i_1} \ldots A_{i_m}).$$

Similarly we introduce the subspaces $L_{p;m}(G; \mathrm{d}g)$ of $L_p(G; \mathrm{d}g)$ for each $p \in [1, \infty]$. Now we can derive a precise characterization of the domain $\mathrm{D}(A)$ of the self-adjoint operator associated with the form $(\cdot,\cdot)_A$ on $\mathscr{H}_1 \times \mathscr{H}_1$ if the coefficients $c_{ij} \in L_{\infty;1}$.

Note that since left translations L are weak* continuous on L_∞ it follows that $f \in \mathrm{D}(A_i)$ if, and only if,

$$\sup_{0<t\leqslant 1} \|f - L(\mathrm{e}^{-ta_i})f\|_\infty/t < \infty$$

i.e. if, and only if,

$$\sup_{0<t\leqslant 1} \sup_{g\in G} |f(\mathrm{e}^{ta_i}g) - f(g)|/t < \infty$$

(see, for example, [2], Proposition 3.1.23). Therefore one concludes that $f \in L_{\infty;1}$ if, and only if,

$$\sup\{|f(h^{-1}g) - f(g)|/|h|;\ g \in G,\ |h| \leqslant 1\} < \infty.$$

Thus the condition $f \in L_{\infty;1}$ corresponds to a uniform Lipschitz condition. Similarly $f \in L_{\infty;n+1}$ if, and only if, $f \in L_{\infty;n}$ and the derivatives $A^\alpha f$ with $|\alpha| = n$ satisfy this Lipschitz property.

THEOREM 3.1. *If $c_{ij} \in L_{\infty;1}$ then the positive self-adjoint operator canonically associated with the form*

$$\varphi \in L_{2;1}(G; dg) \mapsto (\varphi, \varphi)_A = -\sum_{i,j=1}^{d} (A_i\varphi, c_{ij}A_j\varphi)$$

has domain $D(A) = L_{2;2}(G; dg)$.

Proof. The basis of the proof is provided by the following elementary observations.

1. If U is a continuous one-parameter group on the Hilbert space $\mathscr{H}$ with generator A then $\varphi \in D(A)$ if, and only if,

$$\sup_{0<t<1} \|(I - U(t))\varphi\|/t < \infty.$$

2. If $c \in L_{\infty;1}$ then $cL_{2;1}(G; dg) \subseteq L_{2;1}(G; dg)$ and $A_i(cf) = (A_i c)f + cA_i f$ for all $f \in L_{2;1}(G; dg)$.

The first of these is another version of the result cited above ([2], Proposition 3.1.23) and the second follows because

$$(I - L(h))(cf) = ((I - L(h))c)f + (L(h)c)(I - L(h))f,$$

for all $c \in L_\infty$ and $f \in L_2$.

Now if $\varphi \in \mathscr{H}_2$ then $A_j\varphi \in \mathscr{H}_1$ and $c_{ij}A_j\varphi \in \mathscr{H}_1$ because $c_{ij} \in L_{\infty;1}$. Therefore

$$\psi \in \mathscr{H}_1 \mapsto (\psi, \varphi)_A = \sum_{i,j=1}^{d} (\psi, A_i c_{ij} A_j \varphi)$$

is continuous and $\varphi \in D(A)$ by definition. Thus $\mathscr{H}_2 \subseteq D(A)$.

In order to prove the converse inclusion $D(A) \subseteq \mathscr{H}_2$ it is convenient to define U_i as the one-parameter group generated by A_i on $\mathscr{H} = L_2(G; dg)$. Then $U_i\mathscr{H}_1 = \mathscr{H}_1$ and U_i restricted to H_1 is continuous with respect to the norm (3.8) on $\mathscr{H}_1$. This follows because

$$(\text{ad } A_j)(U_i(t))\varphi = U_i(t) \sum_{k=1}^{d} b^i_{jk}(t) A_k \varphi \tag{3.10}$$

for all $\varphi \in \mathscr{H}_1$ and $t \in \mathbf{R}$, where the coefficients b^i_{jk} are analytic and $b^i_{jk}(0) = 0$ (see, for example, [11]). Now if A_{1i} denotes the generator of U_i on $\mathscr{H}_1$ then

$$\mathscr{H}_2 = \bigcap_{i=1}^{d} D(A_{1i}).$$

This states that the first Sobolev space for left translations L on $L_{2;1}$ is equal to the second Sobolev space for L on L_2. But since $\mathscr{H}_1$ is a Hilbert space one has $\varphi \in D(A_{1i})$ if, and only if,

$$\sup_{0<t<1} \|(I - U_i(t))\varphi\|_1/t < \infty.$$

Now $\|\cdot\|_1$ is equivalent to the norm (3.7) and hence we conclude from these remarks that $\varphi \in \mathscr{H}_2$ if, and only if,

$$\sup_{0<t<1} \|(I - U_i(t))\varphi\|_A/t < \infty$$

for $i = 1, \ldots, d$.

The groups U_i are unitary on L_2 but not on $L_{2;1}$. Nevertheless one has an approximate unitarity expressed as follows.

LEMMA 3.2. *If* $c_{ij} \in L_{\infty;1}$ *then*

$$(3.11) \qquad |(U_i(t)\varphi, \psi)_A - (\varphi, U_i(-t)\psi)_A| \leqslant c|t| \cdot \|\varphi\|_A \cdot \|\psi\|_A$$

for some $c \geqslant 0$, *and all* $\varphi, \psi \in \mathscr{H}_1$, $0 < |t| < 1$, *and* $i, j = 1, \ldots, d$.

Proof. The left hand side of (3.11) is the sum of d^2 terms of the form

$$(3.12) \qquad \begin{aligned} (cA_jU_i(t)\varphi, A_k\psi) - (cA_j\varphi, A_kU_i(-t)\psi) &= (c(\operatorname{ad} A_j)(U_i(t))\varphi, A_k\psi) + \\ &+ ((\operatorname{ad} c)(U_i(t))A_j\varphi, A_k\psi) - (cA_j\varphi, (\operatorname{ad} A_k)(U_i(-t))\psi), \end{aligned}$$

with $c \in L_{\infty;1}$. We estimate the three terms on the right separately.

a. Since the coefficients b^i_{jk} in (3.10) are analytic one has $|b^i_{jk}(t)| \leqslant a_1|t|$ for $|t| < 1$ and a suitable $a_1 > 0$. Therefore

$$\begin{aligned} |(c(\operatorname{ad} A_j)(U_i(t))\varphi, A_k\psi)| &\leqslant \|c\|_\infty \cdot \|(\operatorname{ad} A_j)(U_i(t))\varphi\| \cdot \|A_k\psi\| \leqslant \\ &\leqslant a_1'|t| \cdot \|\varphi\|_1 \cdot \|\psi\|_1 \end{aligned}$$

for $|t| < 1$, $\varphi, \psi \in \mathscr{H}_1$, and $i, j = 1, \ldots, d$. Since $\|\cdot\|_1$ is equivalent to $\|\cdot\|_A$ this gives the appropriate bound.

b. If $\xi \in \mathscr{H}$ then

$$U_i(-t)((\operatorname{ad} c)(U_i(t))\xi)(g) = (c(e^{ta_i}g) - c(g))\xi(g).$$

Therefore since $c \in L_{\infty;1}$ one has an estimate

$$\|(\operatorname{ad} c)(U_i(t))A_j\varphi\| \leqslant a_2|t| \cdot \|\varphi\|_1.$$

Hence the second term again has the appropriate bound.

c. The third term on the right of (3.12) is estimated by repetition of the argument used for the first term

Now assume $\psi \in D(A)$ then

$$(\varphi, \psi)_A = (\varphi, A\psi)$$

for all $\varphi \in \mathscr{H}_1$. Therefore setting $\Delta_i(t) = (I - U_i(t))$ one has

$$\|\Delta_i(t)\psi\|_A^2 = (\Delta_i(t)\psi, \Delta_i(t)\psi)_A + (\Delta_i(t)\psi, \Delta_i(t)\psi) =$$

$$= (\Delta_i(-t)\Delta_i(t)\psi, \psi)_A + (\Delta_i(-t)\Delta_i(t)\psi, \psi) +$$

$$+ (\Delta_i(t)\psi, \Delta_i(t)\psi)_A - (\Delta_i(-t)\Delta_i(t)\psi, \psi)_A \leqslant$$

$$\leqslant \|\Delta_i(-t)\Delta_i(t)\psi\|\{\|A\psi\| + \|\psi\|\} + c|t| \cdot \|\Delta_i(t)\psi\|_A \cdot \|\psi\|_A$$

where we have used Lemma 3.2. But a Duhamel estimate gives

$$\|\Delta_i(-t)\Delta_i(t)\psi\| \leqslant |t| \cdot \|\Delta_i(t)\psi\|_1$$

and since $\|\cdot\|_1$ is equivalent to $\|\cdot\|_A$ this means

$$\|\Delta_i(-t)\Delta_i(t)\psi\| \leqslant c'|t| \cdot \|\Delta_i(t)\psi\|_A$$

for a suitable $c' > 0$. Hence

$$\|\Delta_i(t)\psi\|_A^2 \leqslant |t|\{c'(\|A\psi\| + \|\psi\|) + c\|\psi\|_A\}\|\Delta_i(t)\psi\|_A.$$

This then implies that

$$\|(I - U_i(t))\psi\|_A/|t| \leqslant c'(\|A\psi\| + \|\psi\|) + c\|\psi\|_A < \infty \tag{3.13}$$

for $0 < |t| < 1$. Consequently $\psi \in \mathscr{H}_2$ which means $D(A) \subseteq \mathscr{H}_2$.

This not only proves Theorem 3.1 but also proves the following.

Corollary 3.3. *If $c_{ij} \in L_{\infty;1}$ then the norm $\|\cdot\|_2$ and the graph norm $\psi \in D(A) \mapsto \|A\psi\| + \|\psi\|$ are equivalent norms on $\mathscr{H}_2 = D(A)$.*

Proof. Set

$$|c|_\infty = \sup\{|c_{ij}(g)|;\ g \in G,\quad 1 \leqslant i, j \leqslant d\}$$

$$|c|_{\infty;1} = \sup\{|c_{ij}(h^{-1}g) - c_{ij}(g)|/|h|;\ g \in G,\ 0 < |h| < 1,\quad 1 \leqslant i, j \leqslant d\}.$$

Then if $\psi \in D(A)$ one has

$$\|A\psi\| + \|\psi\| \leqslant \sum_{i,j=1}^{d} \|A_i(c_{ij}A_j\psi)\| + \|\psi\| \leqslant$$

$$\leqslant |c|_{\infty;1}\|\psi\|_1 + |c|_{\infty}\|\psi\|_2 + \|\psi\| \leqslant$$

$$\leqslant (1 + |c|_{\infty;1} + |c|_{\infty})\|\psi\|_2$$

and the graph norm is dominated by $\|\cdot\|_2$. Equivalence of the norms then follows from the closed graph theorem (see, for example, [30], Chapter 10.4). Alternatively (3.13) shows that $\|\cdot\|_2$ is dominated by the graph norm. For this one needs that $\|\cdot\|_A$ is equivalent to $\|\cdot\|_1$ and $\|\cdot\|_1$ is dominated by the graph norm. But these properties were established in the discussion prior to Theorem 3.1.

COROLLARY 3.4. *The subspace $C_c^\infty(G)$ is a core of A.*

Proof. Since $D(A) = \mathscr{H}_2$ and the graph norm of A is equivalent to $\|\cdot\|_2$ the statement of the corollary is equivalent to the statement that C_c^∞ is $\|\cdot\|_2$-dense in $\mathscr{H}_2$. But this latter property follows by standard approximation and regularization procedures. In fact C_c^∞ is $\|\cdot\|_n$-dense in $\mathscr{H}_n = L_{2;n}$ for all $n = 1, 2, \ldots$.

COROLLARY 3.5. *Let $c_i \in L_\infty$ and define B on $\mathscr{H}_1$ by*

$$B = \sum_{i=1}^{d} c_i A_i + c_0 I.$$

Then the operator $C = A + B$ generates a holomorphic semigroup. Corollaries 3.3 and 3.4 are valid with A replaced by C.

Proof. If $\psi \in \mathscr{H}_1$ then

$$\|B\psi\| \leqslant |c|_\infty \|\psi\|_1$$

where $|c|_\infty$ now denotes the sum of the L_∞-norms of the c_i. Thus B is relatively bounded by A with relative bound zero by (3.9). Therefore the statement of the corollary follows from perturbation theory.

It follows from the discussion of quadratic forms that $\mathscr{H}_1 = D(A^{1/2})$ and if the coefficients $c_{ij} \in L_{\infty;1}$ then Theorem 3.1 establishes that $\mathscr{H}_2 = D(A)$. We next demonstrate that increased smoothness properties of the coefficients allows more detailed comparison of the C^n-subspaces $\mathscr{H}_n = L_{2;n}(G; dg)$ and the domains $D(A^{n/2})$. The proof uses arguments developed earlier to discuss the differential structure of general continuous representations of Lie groups [29]. Similar ideas occur for unitary representations in Nelson's work [21]. The new features in the following discussion arise because A has non-constant coefficients, and G is not necessarily abelian.

THEOREM 3.6. *If* $c_{ij} \in L_{\infty;n}$ *then*

$$D(A^{m/2}) = \mathscr{H}_m = L_{2;m}(G; dg) \tag{3.14}$$

for $m = 1,2, \ldots, n+1$, *or, equivalently,*

$$(I + A)^{1/2}\mathscr{H}_m = \mathscr{H}_{m-1} \tag{3.15}$$

for $m = 1,2, \ldots, n+1$, *where* $\mathscr{H}_0 = \mathscr{H} = L_2(G; dg)$.

Proof. If (3.14) holds then

$$\mathscr{H}_m = D(A^{m/2}) = (I + A)^{-1/2}D(A^{(m-1)/2}) = (I + A)^{-1/2}\mathscr{H}_{m-1}$$

for $m = 1, \ldots, n+1$. But this immediately gives (3.15). Conversely if (3.15) holds then

$$\mathscr{H}_m = (I + A)^{-1/2}\mathscr{H}_{m-1} = (I + A)^{-m/2}\mathscr{H} = D(A^{m/2})$$

for $m = 1, \ldots, n+1$. But (3.15) can be reexpressed without the square root. Since $c_{ij} \in L_{\infty;n} \subseteq L_{\infty;1}$ it follows from the foregoing that $D(A^{1/2}) = \mathscr{H}_1$ and $D(A) = \mathscr{H}_2$ or, equivalently, $(I + A)^{1/2}\mathscr{H}_m = \mathscr{H}_{m-1}$ for $m = 1, 2$. But this implies that (3.15) is equivalent to the conditions

$$(I + A)\mathscr{H}_{p+1} = \mathscr{H}_{p-1} \tag{3.16}$$

for $p = 1, \ldots, n$. Clearly (3.15) implies (3.16) but conversely (3.16) implies

$$(I + A)^{1/2}\mathscr{H}_3 = (I + A)^{-1/2}\mathscr{H}_1 = \mathscr{H}_2$$

and then

$$(I + A)^{1/2}\mathscr{H}_4 = (I + A)^{-1/2}\mathscr{H}_2 = \mathscr{H}_3$$

etc. Now we prove (3.16).

The key to establishing (3.16) is to prove first the special case $p = 2$. This can be achieved with the aid of an approximate skew-symmetry of the A_i with respect to the scalar product $(\cdot, \cdot)_A$.

LEMMA 3.7. *If* $c_{ij} \in L_{\infty;2}$ *then there is a* $C > 0$ *such that*

$$|(A_i\varphi, \psi)_A + (\varphi, A_i\psi)_A| \leqslant C\|\varphi\| \cdot \|\psi\|_2 \tag{3.17}$$

for all $\varphi, \psi \in \mathscr{H}_2$ *and* $i = 1, \ldots, d$.

Proof. The left hand side of (3.17) is a sum of d^2 terms of the form

$$(cA_jA_i\varphi,\ A_k\psi) + (cA_j\varphi,\ A_kA_i\psi) =$$

$$= (c(\operatorname{ad} A_j)(A_i)\varphi,\ A_k\psi) + ((\operatorname{ad} c)(A_i)A_j\varphi,\ A_k\psi) -$$

$$- (cA_j\varphi, (\operatorname{ad} A_i)(A_k)\psi),$$

with $c \in L_{\infty;2}$. We estimate the three terms on the right separately.

a. Using the structure relations relations of $\mathfrak{g}$ one has

$$|(c(\operatorname{ad} A_j)(A_i)\varphi,\ A_k\psi)| \leqslant \sum_{l=1}^{d} |c_{ij}^l| \cdot |(cA_l\varphi,\ A_k\psi)| \leqslant$$

$$\leqslant \sum_{l=1}^{d} |c_{ij}^l| \cdot \|\varphi\| \cdot \|A_l\bar{c}A_k\psi\| \leqslant \sum_{l=1}^{d} |c_{ij}^l| \cdot \|\varphi\| \cdot \|(A_l\bar{c})A_k\psi + \bar{c}A_lA_k\psi\|$$

where (A_lc) denotes the left derivative of $c \in L_\infty$. Thus

$$|(c(\operatorname{ad} A_j)(A_i)\varphi,\ A_k\psi)| \leqslant C\|\varphi\| \cdot \|\psi\|_2$$

for a suitable $C > 0$.

b.

$$|((\operatorname{ad} c)\,(A_i)A_j\varphi,\ A_k\psi)| = |((A_ic)A_j\varphi,\ A_k\psi)| = |(A_j\varphi,\ (A_ic)A_k\psi)|.$$

But since $c \in L_{\infty;2}$ one has $A_i\bar{c} \in L_{\infty;1}$ and so $(A_i\bar{c})\mathscr{H}_1 \subseteq \mathscr{H}_1$. Moreover,

$$|(A_j\varphi, \overline{(A_ic)}A_k\psi)| \leqslant \|\varphi\| \cdot \|A_ic\|_{\infty;1} \cdot \|\psi\|_2$$

which gives a bound of the appropriate type.

c. This term is bounded in the same manner as the first.

Returning to the proof of Theorem 3.6 we must prove $(I + A)\mathscr{H}_{p+1} = \mathscr{H}_{p-1}$ for $2 \leqslant p \leqslant n$ or equivalently $(I + A)^{-1}\mathscr{H}_{p-1} = \mathscr{H}_{p+1}$.

Suppose $p = 2$. Take $\xi \in \mathscr{H}_1$ and set $\psi = (I + A)^{-1}\xi$. Then $\psi \in \mathrm{D}(A) = \mathscr{H}_2$ and we must show that $A_i\psi \in \mathscr{H}_2$ for each $i = 1, \ldots, d$. But if $\varphi \in \mathscr{H}_2$ then

$$|(\varphi,\ A_i\psi)_A| \leqslant |(A_i\varphi,\ \psi)_A| + |(A_i\varphi,\ \psi)_A + (\varphi,\ A_i\psi)_A| \leqslant$$

$$\leqslant |(A_i\varphi,\ A(I + A)^{-1}\xi)| + C\|\varphi\| \cdot \|\psi\|_2$$

by Lemma 3.7 and the form definition of A. Now

$$|(A_i\varphi, A(I + A)^{-1}\xi)| \leqslant |(\varphi, A_i\xi)| + |(\varphi, A_i(I + A)^{-1}\xi)| \leqslant$$

$$\leqslant C'\|\varphi\| \cdot \|\xi\|_1$$

for a suitable $C' > 0$. Here we have used the fact that $(I + A)^{-1}$ is a bounded operator on $\mathscr{H}_1$ with respect to the norm $\|\cdot\|_1$. This is an easy consequence of the equivalence of the norm $\|\cdot\|_1$ and $\|\cdot\|_A$ discussed at the beginning of the subsection. For example, if $\varphi \in D(A^2) \subseteq \mathscr{H}_1$ then

$$\|A_i\varphi\|^2 \leqslant \varkappa_m^{-1}(\varphi, \varphi)_A = \varkappa_m^{-1}(\varphi, (I + A)\varphi) \leqslant$$

$$\leqslant \varkappa_m^{-1}((I + A)\varphi, (I + A)\varphi)_A \leqslant \varkappa_m^{-1}\varkappa_M\|(I + A)\varphi\|_1^2.$$

Hence

$$\|A_i(I + A)^{-1}\psi\|^2 \leqslant \varkappa_m^{-1}\varkappa_M\|\psi\|_1^2$$

for all $\psi \in D(A) = \mathscr{H}_2$ and then for all $\psi \in \mathscr{H}_1$ by continuity.

Therefore we have established that $\varphi \in \mathscr{H}_2 \mapsto (\varphi, A_i\psi)_A$ is $\|\cdot\|$-continuous. Since $\mathscr{H}_2$ is dense in $\mathscr{H}$ there exists an $\eta \in \mathscr{H}$ such that

$$(\varphi, A_i\psi)_A = (\varphi, \eta) \tag{3.18}$$

for all $\varphi \in \mathscr{H}_2$ by the Riesz representation theorem. Now as $\mathscr{H}_2$ is $\|\cdot\|_1$-dense in $\mathscr{H}_1$ the identity (3.18) extends to all $\varphi \in \mathscr{H}_1$. But this then implies that $A_i\psi \in D(A) = \mathscr{H}_2$. Therefore we have proved that $(I + A)^{-1}\mathscr{H}_1 \subseteq \mathscr{H}_3$ or $\mathscr{H}_1 \subseteq (I + A)\mathscr{H}_3$. Since $c_{ij} \subseteq L_{\infty;n}$, with $n \geqslant 2$, it follows however that $c_{ij}\mathscr{H}_2 \subset \mathscr{H}_2$ and hence $(I + A)\mathscr{H}_3 \subseteq \mathscr{H}_1$. Thus we have $(I + A)\mathscr{H}_3 = \mathscr{H}_1$.

Now we argue by induction. Suppose (3.16) is valid for $p = 1, 2, \ldots, m - 1$ with $2 < m \leqslant n$. Take $\xi \in \mathscr{H}_{m-1}$ and set $\psi = (I + A)^{-1}\xi \in \mathscr{H}_m \subseteq \mathscr{H}_3$. In order to deduce that $(I + A)\mathscr{H}_{m+1} \supseteq \mathscr{H}_{m-1}$ it is necessary to prove that $A_i\psi \in \mathscr{H}_m$ for each $i = 1, \ldots, d$. But since $\psi \in \mathscr{H}_3$ we have the identity

$$A_i\psi = A_i(I + A)^{-1}\xi =$$

$$= (I + A)^{-1}A_i\xi + (I + A)^{-1}(\operatorname{ad} A)(A_i)\psi.$$

But $A_i\xi \in \mathscr{H}_{m-2}$ and $(I + A)^{-1}A_i\xi \in \mathscr{H}_m$ by the induction hypothesis. Therefore if we can show that $(\operatorname{ad} A)(A_i)\psi \in \mathscr{H}_{m-2}$ it follows by similar reasoning that $(I +$

$+ A)^{-1}(\mathrm{ad}\, A)(A_i)\psi \in \mathscr{H}_m$ and consequently $A_i\psi \in \mathscr{H}_m$. Now $(\mathrm{ad}\, A)(A_i)\psi$ is a sum of d^2 terms of the form

$$(\mathrm{ad}\, A_j c_{jk} A_k)(A_i)\psi = (\mathrm{ad}\, A_j)(A_i) c_{jk} A_k \psi + A_j (\mathrm{ad}\, A_i)(c_{jk}) A_k \psi +$$
$$+ A_j c_{jk} (\mathrm{ad}\, A_i)(A_k) =$$
$$= \sum_{l=1}^{d} c^l_{ji} A_l c_{jk} A_k \psi + A_j (A_i c_{jk}) A_k \psi + \sum_{l=1}^{d} c^l_{ik} A_j c_{jk} A_l \psi$$

where we have used the structure relations of $\mathfrak{g}$. Since $c_{jk} \in L_{\infty;n}$ and $\psi \in \mathscr{H}_m$ one has $A_k\psi \in \mathscr{H}_{m-1}$, $c_{jk}A_k\psi \in \mathscr{H}_{m-1}$ and $A_l c_{jk} A_k \psi \in \mathscr{H}_{m-2}$. This follows because repeated use of Statement 2 of the proof of Theorem 3.1 implies that if $c \in L_{\infty;m-1}$ and $\xi \in \mathscr{H}_{m-1}$ then $c\xi \in \mathscr{H}_{m-1}$. Similarly all the other terms on the right are in $\mathscr{H}_{m-2}$. Therefore, by induction, we have established that $(I + A)\mathscr{H}_{p+1} \supseteq \mathscr{H}_{p-1}$ for $p = 1, \ldots, n$. But conversely $c_{ij} \in L_{\infty;n}$ implies $c_{ij}\mathscr{H}_p \subseteq \mathscr{H}_p$ for $p = 1, \ldots, n$ and hence $(I + A)\mathscr{H}_{p+1} \subseteq \mathscr{H}_{p-1}$ for $p = 1, \ldots, n$. Thus (3.16) is established and the proof is complete.

Theorem 3.6 is again stable under the addition of linear terms to A.

PROPOSITION 3.8. *If* $c_{ij} \in L_{\infty;n}$, $c_i \in L_{\infty;n-1}$ and

$$C = A + \sum_{i=1}^{d} c_i A_i + c_0 I$$

then for $m = 1, \ldots, n + 1$

$$\mathscr{H}_m = \mathrm{D}((\lambda I + C)^{m/2})$$

for all sufficiently large $\lambda \in \mathbf{R}$. *Moreover,* $\|\cdot\|_m$ *is equivalent to the graph norm* $\psi \mapsto \|(\lambda I + C)^{m/2}\psi\| + \|\psi\|$.

Proof. First, it follows from Corollary 3.5 that $\mathrm{D}(C) = \mathscr{H}_2$. Moreover C generates a holomorphic semigroup. Therefore if $\lambda \geq 1$ is sufficiently large $(\lambda I + C)$ generates a holomorohic semigroup T satisfying $\|T_t\| \leq M\mathrm{e}^{-t}$ for some $M \geq 1$ and all $t \geq 0$. In particular one can define $(\lambda I + C)^{1/2}$ by standard means (see, for example, [23] or [35]). Similarly $(\lambda I + A)$ generates a holomorphic semigroup S, satisfying $\|S_t\| \leq \mathrm{e}^{-\lambda t}$ and one can define $(\lambda I + A)^{1/2}$. But $\mathrm{D}((\lambda I + A)^{1/2}) = \mathrm{D}(A^{1/2})$, by spectral theory, and $\mathrm{D}(A^{1/2}) = \mathscr{H}_1$, by Theorem 3.6. Therefore if we can establish that $\mathrm{D}((\lambda I + C)^{1/2}) = \mathrm{D}((\lambda I + A)^{1/2})$ then it follows that $\mathrm{D}((\lambda I + C)^{1/2}) = \mathscr{H}_1$.

Now $\varphi \in \mathrm{D}((\lambda I + C)^{1/2})$ if, and only if,

$$\lim_{\delta \to 0} \int_{\delta}^{\infty} \mathrm{d}t\, t^{-3/2}(I - T_t)\varphi$$

exists and $\varphi \in D((\lambda I + A)^{1/2})$ if, and only if

$$\lim_{\delta\to 0}\int_{\delta}^{\infty} dt\, t^{-3/2}(I - S_t)\varphi$$

exists [35]. Next suppose $\varphi \in D((\lambda I + A)^{1/2})$ then

$$\left\|\int_{\delta_1}^{\delta_2} dt\, t^{-3/2}(I - T_t)\varphi\right\| \leqslant \left\|\int_{\delta_1}^{\delta_2} dt\, t^{-3/2}(I - S_t)\varphi\right\| + \int_{\delta_1}^{\delta_2} dt\, t^{-3/2}\|(S_t - T_t)\varphi\|.$$

Hence to deduce that $\varphi \in D((\lambda I + C)^{1/2})$ it suffices to show that

$$\lim_{\delta_1,\,\delta_2\to 0}\int_{\delta_1}^{\delta_2} dt\, t^{-3/2}\|(S_t - T_t)\varphi\| = 0.$$

But

$$\|(S_t - T_t)\varphi\| \leqslant M\int_0^t ds\, e^{-(t-s)}\|BS_s\varphi\|$$

where $B = C - A$. Now the estimate $\|B\psi\| \leqslant |c|_\infty \|\psi\|_1$ used in Corollary 3.5 together with (3.9) gives bounds

$$\|B\psi\| \leqslant \varepsilon\|A\psi\| + k\varepsilon^{-1}\|\psi\|$$

for all $\psi \in D(A)$ and $\varepsilon \in (0, 1)$. Setting $\psi = S_s\varphi$, $\varepsilon = s^{1/4}$, and noting that one has bounds $\|S_s\varphi\| \leqslant e^{-\lambda s}\|\varphi\|$, $\|AS_s\varphi\| \leqslant cs^{-1/2}e^{-\lambda s}\|A^{1/2}\varphi\|$, one obtains estimate

$$\|BS_s\varphi\| \leqslant c's^{-1/4}e^{-\lambda s}(\|A^{1/2}\varphi\| + \|\varphi\|)$$

for all $s \in (0, 1)$. Therefore

$$\lim_{\delta_1,\,\delta_2\to 0}\int_{\delta_1}^{\delta_2} dt\, t^{-3/2}\|(S_t - T_t)\varphi\| \leqslant M' \lim_{\delta_1,\,\delta_2\to 0}\int_{\delta_1}^{\delta_2} dt\, t^{-3/4} = 0.$$

Thus we have established that $D((\lambda I + A)^{1/2}) \subseteq D((\lambda I + C)^{1/2})$. But the converse inclusion is established by a similar argument in which the roles of A and C, S and T, interchanged. Then one needs the observation that

$$\|B\psi\| \leqslant \varepsilon(\|C\psi\| + \|B\psi\|) + k\varepsilon^{-1}\|\psi\|,$$

because $A = C - B$, and hence

$$\|B\psi\| \leqslant \varepsilon(1-\varepsilon)^{-1}\|C\psi\| + k\varepsilon^{-1}(1-\varepsilon)^{-1}\|\psi\|$$

for all $\psi \in D(C) = \mathscr{H}_2$ and $\varepsilon \in (0, 1)$. In addition one needs an estimate $\|(\lambda I + C)T_t\| \leqslant ct^{-1/2}$ for $t \in (0, 1)$. But this follows from holomorphy of T ([23], Chapter 2, Theorem 6.13).

Next one has $(\lambda I + C)\mathscr{H}_2 = \mathscr{H}$ for large λ. Therefore suppose $(\lambda I + C)\mathscr{H}_{m+2} = \mathscr{H}_m$ for $m = 0, 1, \ldots, n-2$ with $\mathscr{H}_0 = \mathscr{H}$. Let $\xi \in \mathscr{H}_{m+1} \subseteq \mathscr{H}_m$ then $\psi = (\lambda I + C)^{-1}\xi \in \mathscr{H}_{m+2}$. Hence if $B = C - A$ one has $B\psi \in \mathscr{H}_{m+1}$ because $c_i \in L_{\infty; n-1}$. Therefore

$$(\lambda I + A)\psi = \xi - B\psi \in \mathscr{H}_{m+1}$$

and one concludes that

$$(\lambda I + C)^{-1}\xi \in (\lambda I + A)^{-1}\mathscr{H}_{m+1} = \mathscr{H}_{m+3}.$$

Thus $\mathscr{H}_{m+1} \subseteq (\lambda I + C)\mathscr{H}_{m+3}$. But the converse inclusion is straightforward and hence $(\lambda I + C)\mathscr{H}_{m+1} = \mathscr{H}_{m-1}$ for $m = 1, \ldots, n$. One then concludes that $\mathscr{H}_m = D((\lambda I + C)^{m/2})$ for $m = 1, \ldots, n+1$ by the argument used at the beginning of the proof of Theorem 3.6.

Finally the graph norm is dominated by $\|\cdot\|_m$ and $\mathscr{H}_m = D((\lambda I + C)^{m/2})$ is closed with respect to both norms. Hence the norms are equivalent by the closed graph theorem ([30], Chapter 10.4).

Note that since the semigroup S generated by C is holomorphic

$$S_t\mathscr{H} \subseteq \bigcap_{n \geqslant 1} D(C^n)$$

for $t > 0$. Consequently the smoothness conditions on the coefficients of C imply the smoothness property

$$S_t\mathscr{H} \subseteq \mathscr{H}_{n+1}$$

of the action of S, for $t > 0$.

3.2. C_0- AND L_p-SEMIGROUPS. Next we examine properties of the semigroup S on $L_2(G; dg)$ generated by the elliptic operators discussed in the previous subsection with respect to the spaces $C_0(G)$ and $L_p(G; dg)$. The starting point for the discussion is a Sobolev embedding result of a type which is quite standard in differential geometry. This result can be proved by classical methods or by semigroup techniques. Since these latter techniques are discussed at length in [27] we describe the classical method.

PROPOSITION 3.9. *Let* $d\hat{g}$ *denote right Haar measure on* G *and* $L_{\hat{p};m} = L_{p;m}(G; d\hat{g})$ *the* C^m*-subspaces for* G *acting by left translations on* $L_p(G; d\hat{g})$. *Then*

$$L_{p;m}(G;\ d\hat{g}) \subseteq C_{0;n}(G)$$

for $m > d/p + n$, *where* d *is the dimension of* G. *Moreover, the embedding is continuous.*

Proof. We use the exponential map as in Subsections 2.2 and 2.3. In local coordinates $d\hat{g}$, on Ω is given by $d\hat{g} = ds\,\varphi(s)$ where φ is a smooth function on $\hat{\Omega}$ satisfying $c_1 \leqslant \varphi(s) \leqslant c_2$ where $0 < c_1 \leqslant c_2 < \infty$. Thus

$$\int_{\Omega} d\hat{g} f(g) = \int_{\hat{\Omega}} ds\, \varphi(s)\, f(e^{-s\cdot a})$$

for $f \in C_c(\Omega)$. Moreover, for any $h \in G$ local coordinates at h are given by $s \mapsto e^{-s\cdot a}h$, mapping $\hat{\Omega}$ onto Ωh, and

$$\int_{\Omega h} d\hat{g}\, f(g) = \int_{\Omega} d\hat{g} (R(h)f)(g) = \int_{\hat{\Omega}} ds\, \varphi(s) f(e^{-s\cdot a}h)$$

if f has support in Ωh. Now take $\delta > 0$ such that $\{s; |s| \leqslant \delta\} \subseteq \hat{\Omega}$. Let $\psi \in C_c^\infty(\mathbf{R})$ be such that $\psi(0) = 1$, and $\psi(t) = 0$, if $|t| \geqslant \delta$. For $f \in C_c^\infty(G)$, $h \in G$, and $\theta \in \mathbf{S}_{d-1} = \{\theta \in \mathbf{R}^d; |\theta| = 1\}$ one has

$$f(h) = (-1)^m \int_0^\delta dr\, \frac{r^{m-1}}{(m-1)!} \frac{\partial^m}{\partial r^m} (\psi(r) f(e^{-r\theta\cdot a}h)).$$

Taking $d\theta$ to be Lebesgue measure on $\mathbf{S}_{d-1}$ and ω_{d-1} the surface area of $\mathbf{S}_{d-1}$

$$f(h) = \frac{(-1)^m}{(m-1)!} \frac{1}{\omega_{d-1}} \int_{\mathbf{S}_{d-1}} d\theta \int_0^\delta dr\, r^{m-1} \frac{\partial^m}{\partial r^m} (\psi(r) f(e^{-r\theta\cdot a}h)).$$

Thus $f(h)$ is a finite sum of terms of which a typical one is

$$\frac{(-1)^{m+|\alpha|}}{(m-1)!} \frac{1}{\omega_{d-1}} \int_{\mathbf{S}_{d-1}} d\theta \int_0^\delta dr\, r^{m-1} \psi^{(j)}(r) \theta^\alpha (A^\alpha f)(e^{-r\theta\cdot a}h)$$

where α is a multi-index, A^α the differential operator associated with left translations, and $|\alpha| + j = m$. Thus this term is bounded by

$$C_1 \int_{S_{d-1}} d\theta \int_0^\delta dr\, r^{d-1} |(A^\alpha f)(e^{-r\theta \cdot a}h)| r^{m-d} \leqslant$$

$$\leqslant C_2 \Big(\int_{S_{d-1}} d\theta \int_0^\delta dr\, r^{d-1} r^{q(m-d)} \Big)^{1/q} \Big(\int_{|s| \leqslant \delta} ds |(A^\alpha f)(e^{-s \cdot a}h)|^p \Big)^{1/p} \leqslant$$

$$\leqslant C_3 \Big(\int_{|s| \leqslant \delta} ds\, \varphi(s)\, |(A^\alpha f)(e^{-s \cdot a}h)|^p \Big)^{1/p} \leqslant$$

$$\leqslant C_4 \Big(\int_G d\hat{g}\, |(A^\alpha f)(g)|^p \Big)^{1/p} \leqslant C_4 \|f\|_{\hat{p};m}$$

for suitable $C_i > 0$, where q is conjugate to p, i.e. $q^{-1} + p^{-1} = 1$. The first estimate is the Hölder inequality and for a finite bound one requires that $r^{q(m-d)+d-1}$ is integrable at the origin, i.e. $m > d/p$. This establishes the proposition for $n = 0$. The general case is obtained by replacing f by $A^\beta f$ with $|\beta| \leqslant n$ and estimating $\|f\|_{\infty;n}$ as above.

Since $d\hat{g} = dg \Delta(g)^{-1}$, where Δ is the modular function, there is an isometric isomorphism from $L_{\hat{p}}$ to L_p given by $f \mapsto \Delta^{-1/p} f$. Therefore $\tilde{A}_i = \Delta^{1/p} A_i \Delta^{-1/p}$ generates the group $t \mapsto \tilde{L}_i(t) = \Delta^{1/p} L(e^{-ta_i}) \Delta^{-1/p}$ on $L_{\hat{p}}$. But if $\hat{L}$ denotes left translations on $L_{\hat{p}}$, and $\hat{A}_i$ the corresponding generators one has

$$\tilde{L}_i(t) = \Delta^{1/p}(e^{-ta_i}) \hat{L}(e^{-ta_i}).$$

Therefore

$$\tilde{A}_i = \hat{A}_i + (\beta_i/p)I$$

where $\beta_i = (A_i \Delta)(e)$. But if $f \in C_c^\infty$

$$\int_G dg |(A^\alpha f)(g)|^p = \int_G dg \Delta(g)^{-1} |(\tilde{A}^\alpha \Delta^{1/p} f)(g)|^p.$$

Consequently there is a $C \geqslant 1$ such that

$$\|f\|_{p;n} \leqslant C^n \|\Delta^{1/p} f\|_{\hat{p};n}$$

for all $n = 1, 2, \ldots$. Similarly one obtains bounds

$$C^{-n}\|\Delta^{1/p}f\|_{\hat{p};n} \leqslant \|f\|_{p;n}.$$

It follows immediately that $\Delta^{1/p}L_{p;n} = L_{\hat{p};n}$ for all $n = 1, 2, \ldots$. But Proposition 3.9 gives an estimate

$$\|f\|_{0;n} \leqslant C_4\|f\|_{\hat{p};m}$$

whenever $m > d/p + n$. Hence replacing f by $\Delta^{1/p}f$ one finds bounds

$$\|\Delta^{1/p}f\|_{0;n} \leqslant C_4'\|f\|_{p;m}. \tag{3.20}$$

Therefore one has the following:

COROLLARY 3.10. *If $C_0^{(p)}(G)$ denotes the closure of $C_c^\infty(G)$ with respect to the norm*

$$\|f\|_\infty^{(p)} = \sup\{|(\Delta^{1/p}f)(g)|\,;\, g \in G\}$$

and $C_{0;n}^{(p)}$ the C^n-subspaces corresponding to left translations then

$$L_{p;m}(G;\, dg) \subseteq C_{0;n}^{(p)}(G)$$

for $m > d/p + n$, and the embedding is continuous.

Proof. If $n = 0$ the result follows immediately from (3.20). But if $n \geqslant 1$ it also follows from (3.20) because a calculation similar to the above establishes that $\Delta^{1/p}C_{0;n} = C_{0;n}^{(p)}$ and

$$\|f\|_{\infty;n}^{(p)} \leqslant C^n\|\Delta^{1/p}f\|_{\infty;n}$$

for a suitable $C \geqslant 1$.

REMARK. It is not generally true that $L_{p;m} \subset C_{0;n}$. If G is the two-dimensional $(ax + b)$-group one can construct an $f \in L_{2;2}$ such that $f \notin C_0$, but $\Delta^{1/2}f \in C_0$ in agreement with Corollary 3.10. Explicitly, this group G can be described as the matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \in (0, \infty)$, $b \in (-\infty, \infty)$ equipped with the usual product. Then if G is topologized as a subset of $\mathbf{R}^2$ it is a Lie group and $dg = da\,db\,a^{-2}$, $d\ = da\,db\,a^{-1}$ (see, for example, [8], § 58, Example 4). Now fix $\varphi_1 \in C^\infty(\mathbf{R}_+)$ such that φ_1 vanishes near the origin and increases to one at infinity. Further fix $\varphi_2 \in C_c^\infty(\mathbf{R})$. Then defining φ by $\varphi(a, b) = \varphi_1(a)\varphi_2(b)$ one has $\varphi \in L_{p;\infty}(G;\, dg)$ for all $p \in [1, \infty]$ but $\varphi \notin C_0(G)$. Nevertheless $\Delta^\alpha\varphi \in C_0(G)$ for all $\alpha > 0$ because $\Delta(a, b) = a^{-1}$.

Now we return to the discussion of the semigroups S on L_2 generated by elliptic operators and their properties with respect to the spaces C_0 and L_p. To avoid confusion with the L_2-formalism we let A_i^0 denote the generators of the groups $t \mapsto L(e^{-ta_i})$ of left translations on $C_0(G)$. Then $C_{0;1}(G) = \bigcap_i D(A_i^0)$, and $C_{0;m} = \{\varphi \in C_{0;1};\ A_j^0\varphi \in C_{0;m-1}\}$. Then we define the analogue C^0 of the elliptic operator C considered in Proposition 3.8 by $D(C^0) = C_{0;2}(G)$ and

$$C^0 = \sum_{i,j=1}^{d} A_i^0 c_{ij} A_j^0 + \sum_{i=1}^{d} c_i A_i^0 + c_0 I$$

where $c_i \in C_b(G)$ and $c_{ij} \in C_{b;1}(G)$. We assume the hermitian matrix (c_{ij}) satisfies the standard ellipticity assumption

$$\inf\left\{ -\sum_{i,j=1}^{d} c_{ij}(g)\xi_i\xi_j;\ g \in G,\ \xi \in \mathbf{R}^d,\ |\xi| = 1\right\} > 0 \tag{3.19}$$

and note that C^0 can be written in the form

$$C^0 = \sum_{i,j=1}^{d} c_{ij} A_i^0 A_j^0 + \sum_{i=1}^{d} c_i' A_i^0 + c_0 I.$$

This follows because A_i^∞ is a derivation and A_i^0 is the part of A_i^∞ in C_0 so if $c \in C_{b;1}$ and $\varphi \in C_{0;1}$ then $c\varphi \in C_{0;1}$ and $A_i^0(c\varphi) = A_i(c)\varphi + cA_i^0\varphi$. Since L is not strongly continuous on C_b this is not immediate but it can be first established for φ of compact support, using the uniform continuity of c and $A_i(c)$ on compacts, and then by approximation.

PROPOSITION 3.11. *If $c_{ij} \in C_{b;1}$ then $C^0 - c_0 I$ is dissipative. Moreover if the coefficients c_i, c_{ij}, $i,j = 1, \ldots, d$ are real then $C^0 - c_0 I$ is dispersive.*

Proof. If $\varphi \in C_{0;2}$ and

$$\varphi(g) = \sup_{h \in G} |\varphi(h)|$$

then for dissipativity we must prove that

$$\mathrm{Re}((C^0 - c_0 I)\varphi)(g) \geqslant 0.$$

But since φ attains its maximum at g one has $(A_i^0\varphi)(g) = 0$ for $i = 1, \ldots, d$ and

$$\sum_{i,j=1}^{d} a_{ij}(A_i^0 A_j^0 \varphi)(g) \leqslant 0$$

for each positive-definite matrix (a_{ij}). Therefore

$$\mathrm{Re}((C^0 - c_0 I)\varphi)(g) = \sum_{i,j=1}^{d} \mathrm{Re}(c_{ij}(g))(A_i^0 A_j^0 \varphi)(g) +$$

$$+ \sum_{i=1}^{d} (\mathrm{Re}\, c_i'(g))(A_i^0 \varphi)(g) \geqslant 0$$

by the ellipticity assumption.

Dispersivity follows by a similar argument. If $\varphi \in C_{0;2}$, $\varphi \geqslant 0$, and

$$\varphi(g) = \sup_{h \in G} \varphi(h)$$

one must prove that

$$((C^0 - c_0 I)\varphi)(g) \geqslant 0 .$$

But if the coefficients of $C^0 - c_0 I$ are real this again follows from the ellipticity assumption.

Note that dissipativity of $C^0 - c_0 I$ implies closability of C^0 because C^0 is densely defined.

THEOREM 3.12. *Assume $c_i \in C_{\mathrm{b};n}$, $c_{ij} \in C_{\mathrm{b};n+1}$ and $c_0 \in C_{\mathrm{b}}$, where $n > d/2$. The closure $\bar{C}^0$ of C^0 generates a strongly continuous semigroup T^0 on $C_0(G)$ satisfying*

$$\|T_t^0\| \leqslant e^{-\omega t}$$

where $\omega = \inf\{\mathrm{Re}\, c_0(g);\, g \in G\}$. If $c_0 \geqslant 0$, and the coefficients c_i, c_{ij} are real then T^0 is positive and contractive.

Moreover, if T denotes the semigroup generated by C on $L_2(G;\, \mathrm{d}g)$ then $T_t\varphi = T_t^0\varphi$ for all $\varphi \in L_2 \cap C_0$ and $t \geqslant 0$.

Proof. It suffices to consider the case $c_0 = 0$. The general case is then obtained by bounded perturbation theory. Now as C^0 is dissipative its closure $\bar{C}^0$ is automatically dissipative and it generates a semigroup if, and only if, the range $\mathrm{R}(\lambda I + \bar{C}^0)$ of $\lambda I + \bar{C}^0$ is dense in $C_0(G)$ for all large $\lambda \in \mathbf{R}$.

Let C denote the elliptic operator on L_2 determined by the coefficients c_i, c_{ij}, and S the semigroup generated by C. Then $\tilde{S} = \Delta^{1/2} S \Delta^{-1/2}$, where Δ denotes multiplication with the modular function, is a holomorphic semigroup on $L_{\hat{2}}$ with generator $\tilde{C} = \Delta^{1/2} C \Delta^{-1/2}$ where $\mathrm{D}(\tilde{C}) = \Delta^{1/2}\mathrm{D}(C) = \Delta^{1/2} L_{2;2}$. Now if $\tilde{A}_i = \Delta^{1/2} A_i \Delta^{-1/2}$ we verified prior to Corollary 3.10 that $\tilde{A}_i = \hat{A}_i + (\beta_i/2) I$ where $\hat{A}_i$, $i = 1, \ldots, d$, denote the generators of left translations on $L_{\hat{2}}$ and $\beta_i = (A_i \Delta)(e)$. Moreover, $\Delta^{1/2} L_{2;2} = L_{\hat{2};2}$.

Next if

$$B = \sum_{i,j=1}^{d} \tilde{A}_i c_{ij} \tilde{A}_j + \sum_{i=1}^{d} c_i \tilde{A}_i$$

is defined on $L_{\hat{2}}$ by the method of Subsection 3.1, i.e. via a form, then one immediately verifies that $B = \tilde{C}$. In particular $D(B) = D(\tilde{C}) = \Delta^{1/2} L_{2;2} = L_{\hat{2};2}$ and $D((\lambda I + B)^{m/2}) = \Delta^{1/2} D((\lambda I + C)^{m/2}) = \Delta^{1/2} L_{2;m} = L_{\hat{2};m}$ for $m = 1, 2, \ldots, n+2$, and λ sufficiently large.

Next since $c_{ij} \in L_{\infty;1}$ it follows that $c_{ij} L_{\hat{2};1} \subset L_{\hat{2};1}$ and one can define $\hat{C}$ on $L_{\hat{2};2}$ by

$$\hat{C} = \sum_{i,j=1}^{d} \hat{A}_i c_{ij} \hat{A}_j + \sum_{i=1}^{d} c_i \hat{A}_i .$$

One then computes that $\hat{C} = B + P$ where

$$P = \sum_{i=1}^{d} \hat{c}_i \hat{A}_i + \hat{c}_0$$

is linear in the $\hat{A}_i$ and the coefficients $\hat{c}_i$ are completely determined by c_{ij}, c_i, and β_i. But estimates similar to those made at the beginning of Subsection 3.1 show that P is relatively bounded with respect to the $\|\cdot\|_{\hat{2}}$-norm by B with relative bound zero. Since $B = \tilde{C}$ generates the holomorphic semigroup $\tilde{S} = \Delta^{1/2} S \Delta^{-1/2}$ it follows from perturbation theory that $\hat{C}$ generates a holomorphic semigroup on $L_{\hat{2}}$. In particular $(\lambda I + \hat{C})^{-1}$ exists for all sufficiently large positive λ, and $D(\hat{C}) = L_{\hat{2};2}$.

If $\xi \in C_c^\infty(G) \in L_{\hat{2};n}$ then $\psi = (\lambda I + \hat{C})^{-1} \xi \subseteq L_{\hat{2};n+2}$ by Proposition 3.8. But $L_{\hat{2};n+2} \subseteq C_{0;2}$ by Proposition 3.9. Thus $\psi \in C_{0;2}$. It follows immediately that

$$\xi = (\lambda I + \hat{C})\psi = (\lambda I + C^0)\psi$$

because $\hat{A}_i \varphi = A_i^0 \varphi$ and $\hat{A}_i \hat{A}_j \varphi = A_i^0 A_j^0 \varphi$ for all $\varphi \in L_{\hat{2};2} \cap C_{0;2}$. Therefore we have deduced that $C_c^\infty \subseteq R(\lambda I + C^0)$. Hence the range of $\lambda I + C^0$ is dense in C_0 and C^0 generates a continuous contraction semigroup T. If the coefficients c_i, and c_{ij}, are real then C^0 is dispersive, by Proposition 3.11, and T is positive.

Finally it follows from the foregoing that

$$(\lambda I + \bar{C}^0)^{-1} \xi = (\lambda I + \hat{C})^{-1} \xi$$

for all $\xi \in C_c^\infty$ if λ is sufficiently large. Moreover, $(\lambda I + \bar{C}^0)^{-1}$ is a contraction on C_0 and $(\lambda I + \hat{C})^{-1}$ a contraction on $L_{\hat{2}}$. Let $\varphi \in C_0 \cap L_{\hat{2}}$ then there exist $\xi_n \in C_c^\infty$

such that $\xi_n \to \varphi$ in C_0 and in $L_{\hat{2}}$. Since $(\lambda I + \bar{C}^0)^{-1}\xi_n = (\lambda I + \hat{C})^{-1}\xi_n$ it follows that

$$(\lambda I + \bar{C}^0)^{-1}\varphi = (\lambda I + \hat{C})^{-1}\varphi$$

for all $\varphi \in C_0 \cap L_{\hat{2}}$ and all sufficiently large λ. But if $\hat{T}$ denotes the semigroup generated by C on $L_{\hat{2}}$ then

$$\hat{T}_t\varphi = \lim_{n\to\infty}(I + (t/n)\hat{C})^{-n}\varphi$$

in $L_{\hat{2}}$ and

$$T_t^0\varphi = \lim_{n\to\infty}(I + (t/n)\,\bar{C}^0)^{-n}\varphi$$

in C_0. Therefore one must have

$$\hat{T}_t\varphi = T_t^0\varphi$$

for all $\varphi \in C_0 \cap L_{\hat{2}}$. Now if $\varphi \in L_2 \cap L_{\hat{2}}$ we want to argue that $\hat{T}_t\varphi = T_t\varphi$. For this it suffices to show that

$$(\lambda I + \hat{C})^{-1}\varphi = (\lambda I + C)^{-1}\varphi$$

for all $\varphi \in L_2 \cap L_{\hat{2}}$ and all sufficiently large λ. But if $\varphi \in L_2 \cap L_{\hat{2}}$ then $(I + \Delta^{-1/2})\cdot\varphi \in L_2$ and we next construct smooth L_2-approximations to $(I + \Delta^{-1/2})\varphi$.

Introduce the operator X by $D(X) = C_c^\infty(G)$ and

$$X = (I + \Delta^{-1/2})C(I + \Delta^{-1/2})^{-1}.$$

Since

$$(I + \Delta^{-1/2})A_i(I + \Delta^{1/2})^{-1}\psi = (A_i + (\beta_i/2)I)\psi - (\beta_i/2)(I + \Delta^{-1/2})^{-1}\psi$$

for all $\psi \in C_c^\infty(G)$ one has

$$X = C + P_1 + (I + \Delta^{-1/2})^{-1}P_2 + (I + \Delta^{-1/2})^{-2}P_3$$

where P_1, P_2, and P_3 are linear in the generators A_i. Thus X is a small perturbation of the generator C on L_2. Then it follows from Corollary 3.5 and perturbation theory that $(\lambda I + X)C_c^\infty(G)$ is dense in L_2 for all sufficiently large λ. Therefore one may choose a sequence $\chi_n \in C_c^\infty(G)$ such that

$$\lim_{n\to 0}\|(\lambda I + X)\chi_n - (1 + \Delta^{-1/2})\varphi\|_2 = 0$$

but setting $\psi_n = (I + \Delta^{-1/2})\chi_n$ one has $\psi_n \in C_c^\infty(G)$ and

$$\lim_{n\to\infty} \|(I + \Delta^{-1/2})((\lambda I + C)\psi_n - \varphi)\|_2 = 0.$$

This implies that

$$\lim_{n\to\infty} \|(\lambda I + C)\psi_n - \varphi\|_2 = 0$$

and

$$\lim_{n\to\infty} \|(\lambda I + C)\psi_n - \varphi\|_{\hat{2}} = 0.$$

Since $C\psi_n = \hat{C}\psi_n$ it follows that

$$(\lambda I + \hat{C})^{-1}\varphi = \lim_{n\to\infty} (\lambda I + \hat{C})^{-1}(\lambda I + C)\psi_n = \lim_{n\to\infty} \psi_n$$

in $L_{\hat{2}}$. Similarly

$$(\lambda I + C)^{-1}\varphi = \lim_{n\to\infty} \psi_n$$

in L_2. Hence $(\lambda I + C)^{-1}\varphi = (\lambda I + \hat{C})^{-1}\varphi$ for all $\varphi \in L_2 \cap L_{\hat{2}}$, and $\hat{T}_t\varphi = T_t\varphi$ for all $\varphi \in L_2 \cap L_{\hat{2}}$.

In conclusion one has

$$T_t\varphi = \hat{T}_t\varphi = T_t^0\varphi$$

for all $\varphi \in C_c$. Since C_c is dense in $C_0 \cap L_2$ it follows that

$$T_t\varphi = T_t^0\varphi$$

for all $\varphi \in C_c \cap L_2$.

Finally we argue that the semigroup T interpolates to all the L^p-spaces.

THEOREM 3.13. *Assume $c_i \in C_{b;n}$, $c_{ij} \in C_{b;n+1}$ and $c_0 \in C_b$, where $n > d/2$. Further let T denote the semigroup generated by the corresponding elliptic operator C on $L_2(G; dg)$. Then there exist strongly continuous semigroups T^p on $L_p(G; dg)$, $p \in [1, \infty)$, T^0 on $C_0(G)$, and a weak*-continuous semigroup T^∞ on $L_\infty(G; dg)$ with the following properties:*

1. $T^2 = T$,

2. *$T_t^p\varphi = T_t^q\varphi$ for all $\varphi \in L_p \cap L_q$ and all $p, q \in [1, \infty]$.*

3. $T_t^0\varphi = T_t^p\varphi$ *for all* $\varphi \in C_0 \cap L_p$ *and all* $p \in [1, \infty]$,

4. $\|T_t^p\|_{p\to p} \leqslant e^{-(\omega_*/p+\omega/q)t}$ *for all* $t \geqslant 0$ *and* $p \in [1, \infty]$, *where* $\omega = \inf\{\mathrm{Re}\, c_0(g); g \in G\}$, $\omega_* = \inf\left\{\mathrm{Re}\left(c_0(g) - \sum_{i=1}^{d} A_i c_i\right)(g);\ g \in G\right\}$ *and* q *is conjugate to* p. *If the coefficients* c_i, c_{ij}, c_0 *are real the semigroups are positive.*

Proof. It follows from Theorem 3.12 that

$$\|T_t\varphi\|_\infty \leqslant e^{-\omega t}\|\varphi\|_\infty$$

for all $\varphi \in C_0(G)$. Next consider the operator

$$C^* = \sum_{i,j=1}^{d} A_j c_{ij} A_i - \sum_{i=1}^{d} c_i A_i + c_0^* I$$

where $c_0^* = \bar{c}_0 - \sum_i (A_i\bar{c}_i)$. Then C^* with domain $D(C^*) = D(C) = L_{2,2}$ generates a semigroup T^* on L_2. Since $(\varphi, C\psi) = (C^*\varphi, \psi)$ for all $\varphi, \psi \in D(C) = D(C^*)$ the operator C^* is the adjoint of C, and T^* is the adjoint of T. It then follows from the foregoing estimate applied to T^* that

$$\|T_t^*\varphi\|_\infty \leqslant e^{-\omega_* t}\|\varphi\|_\infty$$

for all $\varphi \in C_c$ where $\omega_* = \inf \mathrm{Re}\{c_0(g) - \sum_i (A_i c_i)(g);\ g \in G\}$. So by duality

$$\|T_t\varphi\|_1 \leqslant e^{-\omega_* t}\|\varphi\|_1$$

for all $\varphi \in L_2 \cap L_1$. Then by the Riesz-Thorin interpolation theorem there exist bounded operators T^p on L_p such that $T_t^p\varphi = T_t^q\varphi$ for all $\varphi \in L_p \cap L_q$, and $T_t^p\varphi = T_t\varphi$ for $\varphi \in L_2 \cap L_p$. Moreover

$$\|T_t^p\varphi_p\|^2 \leqslant e^{-(\omega_*/p+\omega/q)t}\,\|\varphi\|_p$$

for all $\varphi \in L_p$ where q is the conjugate variable to p.

Since T is a semigroup each T^p is a semigroup and strong continuity for $p \in [1,\infty)$ follows from strong continuity on the dense subspace $L_p \cap L_2$.

Finally if the coefficients are real then T is positive on C_0 by Theorem 3.1. But positivity on the L_p-spaces follows by another density argument.

In conclusion we remark that the semigroups T^p on the L_p-spaces, $p \in (1, \infty)$, constructed in the above manner are automatically holomorphic.This follows by a slight variation of a general argument of Stein (see, for example, [25], Theorem X.55(c)).

The generator C^p of T^p on L_p is automatically an extension of the elliptic operator associated with the coefficients c_i, c_{ij}, and c_0 on $L_{p;2}$ but we do not know whether C^p is closed on $L_{p;\,2}$ for $p \in (1, \infty)$. This is true for $p = 2$ by Theorem 3.1 but for $p = 1, \infty$ it is certainly false because the domain of the Laplacian on $L_1(\mathbf{R}^d;\ dx)$ $(L_\infty(\mathbf{R}^d;\ dx))$ strictly contains $L_{1;\,2}(\mathbf{R}^d;\ dx)$ $(L_{\infty;\,2}(\mathbf{R}^d;\ dx))$.

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WOLFGANG ARENDT
Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra,
Australia.

Permanent address:

Laboratoire de Mathématiques,
Université de Franche—Comté,
25030 Besançon — Cedex,
France.

CHARLES J. K. BATTY
Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra,
Australia.

Permanent address:

St. John's College,
Oxford OX1 3JP,
England.

DEREK W. ROBINSON
Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra,
Australia.

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