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A RIESZ DECOMPOSITION PROPERTY AND IDEAL STRUCTURE OF MULTIPLIER ALGEBRAS

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INTRODUCTION

In this paper, we prove a Riesz decomposition property of the local semigroups consisting of Murray-von Neumann equivalence classes of projections in a C^*-algebra $\mathscr{A}$ with the FS property (or equivalently, of real rank zero [12]) in the associated multiplier algebra $M(\mathscr{A})$ and in the corona algebra $M(\mathscr{A})/\mathscr{A}$. Isolating of this property makes it possible to generalize some known results for separable AF algebras concerning the structure of projections and closed ideals of the associated multiplier and corona algebras to a much larger class of C^*-algebras, i.e., the class of C^*-algebras of real rank zero.

We first fix some notations and state some related facts. Let $\mathscr{A}$ be a C^*-algebra and p, q two projections in $\mathscr{A}$. Throughout, we say that p and q are equivalent, denoted by $p \sim q$, if there exists a partial isometry v in $\mathscr{A}$ such that $vv^* = p$ and $v^*v = q$. We denote the set of all equivalence classes of projections by $D(\mathscr{A})$ and denote the element in $D(\mathscr{A})$ with representative projection r by $[r]$. The notation '$[p] \leqslant [q]$' means that p is equivalent to a subprojection of q, while '$[p] < [q]$' means that p is equivalent to a proper subprojection of q. Obviously, '$\leqslant$' defines a natural (partial) order on $D(\mathscr{A})$. A local addition is defined on $D(\mathscr{A})$ as follows: For two projections p and q in $\mathscr{A}$, $[p] + [q]$ is defined if and only if there are projections p' and q' in $\mathscr{A}$ such that $p' \sim p$, $q' \sim q$ and $p'q' = 0$, and $[p] + [q]$ is defined to be $[p' + q']$ whenever such p' and q' exist. Then $D(\mathscr{A})$ is an abelian local semigroup.

A subset $\mathscr{I}$ of $D(\mathscr{A})$ is said to be an ideal of $D(\mathscr{A})$ if $[q] \leqslant [p_1] + [p_2] + \ldots + [p_n]$ for some $[p_i]$ in $\mathscr{I}$ implies that $[q]$ is in $\mathscr{I}$. Obviously, an ideal is closed under the local addition. If $\mathscr{I}$ and $\mathscr{J}$ are two ideals of $D(\mathscr{A})$, then in a natural way with respect to the local addition on $D(\mathscr{A})$ we define the 'sum' $\mathscr{I} + \mathscr{J}$ of $\mathscr{I}$ and $\mathscr{J}$ to be the union of all elements in $\mathscr{I} \cup \mathscr{J}$ and all possible sums of them. Clearly, $\mathscr{I} + \mathscr{J}$ is again an ideal of $D(\mathscr{A})$ provided $D(\mathscr{A})$ has the *Riesz decomposition property* in the following sense: If x, y and z are three elements in $D(\mathscr{A})$ such that $x \leqslant y + z$, then there exist elements x_1 and x_2 in $D(\mathscr{A})$ such that $x = x_1 + x_2$, $x_1 \leqslant y$ and $x_2 \leqslant z$. It is well known that $D(\mathscr{A})$ has the Riesz decomposition property if $\mathscr{A}$ is an AF algebra (see [20], for example). It is easily verified that the set of all ideals of $D(\mathscr{A})$ forms a lattice with respect to the addition "+" and the intersection '$\cap$' as long as $D(\mathscr{A})$ has the Riesz decomposition property.

A C^*-algebra $\mathscr{A}$ is said to have the 'FS' property, or briefly to have FS, if the set of all self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements of $\mathscr{A}$. It is known that FS is equivalent to real rank zero and to the 'HP' property: Every hereditary C^*-subalgebra of $\mathscr{A}$ has an approximate identity consisting of projections ([3, 2.7] and [30]). The class of C^*-algebras with FS contains a number of interesting subclasses. Trivial examples include AF algebras, von Neumann algebras, AW*-algebras and the Calkin algebra. Nontrivial exapmles include Bunce-Deddens algebras ([2] and [7]). Recently, more C^*-algebras have been proved to have the FS property. For example, all purely infinite simple C^*-algebras have FS ([34] and [37, Part I]); in particular, the Cuntz algebras $\mathscr{O}_n$ $(2 \leqslant n \leqslant \infty)$, the Cuntz-Krieger algebras $\mathscr{O}_A$, if A is an irreducible matrix, and the corona algebras of many C^*-algebras have FS. Many multiplier algebras have FS ([12] and [37, Part I and IV]). The irrational rotation C^*-algebras corresponding to a dense subset of irrational numbers have FS ([14]). Moreover, this class is closed under the inductive limit and strong Morita equivalence ([12]).

In § 1 we shall prove the Riesz decomposition property for C^*-algebras with FS and associated multiplier and corona algebras (Theorem 1.1). In § 2, we shall prove some consequences of the Riesz decomposition property. One consequence is that if $\{e_n\}$ is a fixed increasing sequential approximate identity of $\mathscr{A}$ consisting of projections, then every projection in $M(\mathscr{A})$ is equivalent to a diagonal projection with the form: $\sum_{i=1}^{\infty} r_i$, where $[r_i] \leqslant [e_i - e_{i-1}]$ for all i. Another consequence is that the closed ideal lattice of $M(\mathscr{A})$ is isomorphic to the ideal lattice of $D[M(\mathscr{A})]$, which generalizes a recent result of G. A. Elliott [23] for separable AF algebras by a different proof. We also prove that every closed ideal of $M(\mathscr{A})$ is the closed linear span of its projections.

In § 3, we prove that if $\mathscr{A}$ is a σ-unital C^*-algebra with FS, then the generalized Calkin algebra $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple if and only if $\mathscr{A}$ is either elementary or $\mathscr{A}$ is simple and every nonzero projection of $\mathscr{A}$ is infinite (in other words, $\mathscr{A}$ is purely infinite and simple), where $\mathscr{K}$ is the C^*-algebra consisting of all compact operators on a separable Hilbert space. This result proves to be important for the further investigation on the structure of $M(\mathscr{A} \otimes \mathscr{K})$ in our subsequent papers ([37, Part, I, II, III, IV]). In addition, we shall consider the ideal structure of $M(\mathscr{A})$ for certain C^*-algebras with FS by means of states on $K_0(\mathscr{A})$.

The main body of this paper is one of four independent parts of the author's Ph. D. thesis. The author sincerely thanks Professor Lawrance G. Brown for his supervision. He is very grateful to L. G. Brown and G. K. Pedersen for showing him their manuscript of [12], in which their Theorem (2.6) and Corollary (3.3) form a foundation of the author's recent papers [35]—[38]. He also wishes to thank the referee and Professor Ronald G. Douglas for their very helpful suggestions and corrections.

1. A RIESZ DECOMPOSITION PROPERTY

1.1. THEOREM. *If $\mathscr{A}$ is a C^*-algebra with* FS, *then $D(\mathscr{A})$ has the Riesz decomposition property. If $\mathscr{A}$ is a σ-unital C^*-algebra with* FS, *then $D[M(\mathscr{A})]$ has the Riesz decomposition property. If, in addition, every projection in $M(\mathscr{A})/\mathscr{A}$ is the image of a projection of $M(\mathscr{A})$ (in particular if $K_1(\mathscr{A}) = 0$ [32, § 2]), then $D[M(\mathscr{A})/\mathscr{A}]$ has the same property.*

We will carry out the proof of Theorem 1.1 using the following lemmas and corollaries. From now on, we denote the hereditary C^*-subalgebra of $\mathscr{A}$ generated by an element x of $M(\mathscr{A})$ by her(x).

1.2. SPLITTING LEMMA. *If $\mathscr{A}$ is a C^*-algebra with* FS, *and if p and q are projections in $\mathscr{A}$, then there are projections e and f in $\mathscr{A}$ such that $[f] \leqslant [p]$, $[e] \leqslant [1-p]$ and $q = e + f$. Here 1 is the identity of $M(\mathscr{A})$.*

Proof. If we write $q = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ as the decomposition of q with respect to $1 = p + (1-p)$, then $a = pqp$, $b = pq(1-p)$ and $c = (1-p)q(1-p)$. Hence $a - a^2 = bb^*$, $c - c^2 = b^*b$ and $ab + bc = b$. By the spectral mapping theorem, $\|a\| \leqslant 1$ implies

$$\|b\|^2 = \|bb^*\| = \|a - a^2\| \leqslant \sup_{0 \leqslant t \leqslant 1} |t - t^2| = 1/4,$$

and hence $\|b\| = \|pq(1-p)\| \leqslant 1/2$. If $\|q(1-p)\| < 1$, then $\|q - qpq\| < 1$. It is well-known that this implies $[q] \leqslant [p]$ (the partial isometry used here is $v = (qpq)^{-1/2}(qp)$). Set $f = q$ and $e = 0$, as desired. If $\|q(1-p)\| = 1$, set $x = q(1-p)q$. Then $\|x\| = 1$. Since her(q) has FS, we can find a positive element h in her(q) with finite spectrum such that $\|x - h\| < \delta < 1/6$. Write $h = \sum_{j=1}^{n} t_j e_j$, where $\{t_j\}$ is a finite set of positive numbers and $\{e_i\}$ is a set of mutually orthogonal projections in $q\mathscr{A}q$. Choose ε such that $1/2 + \delta < \varepsilon < 1 - 2\delta$ (such ε exists since $\delta < 1/6$). Set $e = \sum_{\{j:t_j \geqslant \varepsilon\}} e_j$. Then $e \leqslant q$ and

$$\|ph\| = \|ph^2p\|^{1/2} \geqslant \|p[\sum_{\{j:t_j \geqslant \varepsilon\}} t_j^2 e_j]p\|^{1/2} \geqslant \varepsilon\|pep\|^{1/2} = \varepsilon\|pe\|,$$

and so

$$\|pe\| \leqslant (1/\varepsilon)\|ph\| \leqslant (1/\varepsilon)[\|p(h-x)\| + \|px\|] \leqslant$$
$$\leqslant (1/\varepsilon)[\|h-x\| + \|pq(1-p)q\|] \leqslant$$
$$\leqslant (1/\varepsilon)[\delta + \|b\|] \leqslant (1/\varepsilon)[\delta + 1/2] < 1.$$

Then $\|e - e(1-p)e\| = \|epe\| = \|pe\|^2 < 1$. Hence $[e] \leqslant [1-p]$. Let $f = q - e$. Then

$$\|f(1-p)f\| = \|fxf\| \leqslant \|f(x-h)f\| + \|fhf\| \leqslant \|x-h\| + \|\sum_{\{j: t_j < \varepsilon\}} t_j e_j\| < \delta + \varepsilon < 1$$

and so $[f] \leqslant [p]$. Thus $q = e + f$ is as desired.

1.3. COROLLARY. *Suppose that $\mathscr{A}$ is a C^*-algebra with FS. If p_1, p_2 and q are projections in $\mathscr{A}$ such that $q \leqslant p_1 + p_2$ and $p_1 p_2 = 0$, then there are projections e and f in $\mathscr{A}$ such that $[e] \leqslant [p_1]$, $[f] \leqslant [p_2]$ and $q = e + f$.*

Proof. Lemma 1.2 applies to $\mathrm{her}(p_1 + p_2) \subset \mathscr{A}$.

The proof of the following lemma was due G. A. Elliot ([23]) for separable AF algebras. The Riesz decomposition property for $D(\mathscr{A})$ in Corollary 1.2 allows us to extend his result to σ-unital C^*-algebras with FS property by the same proof. Since Elliott's proof in [23] is perhaps too brief, we give more details here for the reader's convenience.

1.4. LEMMA. *If $\mathscr{A}$ is a σ-unital C^*-algebra with FS, and P_1, P_2 and Q are projections in $M(\mathscr{A})$ such that $Q \leqslant P_1 + P_2$ and $P_1 P_2 = 0$, then there are projections Q_1 and Q_2 in $M(\mathscr{A})$ such that $[Q_1] \leqslant [P_1]$, $[Q_2] \leqslant [P_2]$ and $Q = Q_1 + Q_2$.*

Proof. Since $(P_1 + P_2)M(\mathscr{A})(P_1 + P_2) \cong M((P_1 + P_2)\mathscr{A}(P_1 + P_2))$, we can assume that $P_1 + P_2 = 1$. First, $\mathrm{her}(P_1)$, $\mathrm{her}(P_2)$, $\mathrm{her}(Q)$ and $\mathrm{her}(1-Q)$ all have increasing approximate identities consisting of increasing sequences of projections ([11, 3.34] and [32, 1.2]), say $\{p_n(1)\} \subset \mathrm{her}(P_1)$, $\{p_n(2)\} \subset \mathrm{her}(P_2)$, $\{q_n\} \subset \mathrm{her}(Q)$ and $\{q'_n\} \subset \mathrm{her}(1-Q)$, respectively. Set $e_n = p_n(1) + p_n(2)$ and $f_n = q_n + q'_n$ for each n. Then $\{e_n\}$ and $\{f_n\}$ are approximate identities of $\mathscr{A}$. By the proof of [21, 2.4] (change notation if necessary), there is a unitary u in $M(\mathscr{A})$ such that

$$e_1 \leqslant uf_1u^* \leqslant e_2 \leqslant uf_2u^* \leqslant e_3 \leqslant \ldots \leqslant e_n \leqslant uf_nu^* \leqslant e_{n+1} \leqslant uf_{n+1}u^* \leqslant \ldots .$$

Since $uQu^* = \sum_{n=1}^{\infty} u(q_n - q_{n-1})u^*$ and $u(q_n - q_{n-1})u^* \leqslant (uf_nu^* - e_n) + (e_n - uf_{n-1}u^*)$ (where $q_0 = 0$ and $f_0 = 0$), then by Corollary 1.3 there is a partial isometry v_n in $\mathscr{A}$ such that $v_n v_n^* = u(q_n - q_{n-1})u^*$ and $v_n^* v_n = r_n + r'_n$ for some projections $r_n \leqslant$

$\leqslant uf_n u^* - e_n$ and $r'_n \leqslant e_n - uf_{n-1}u^*$. Set $\sum_{n=1}^{\infty} v_n = v$. Then $v \in M(\mathscr{A})$, $vv^* = uQu^*$ and $v^*v = \sum_{n=1}^{\infty} (r_n + r'_n)$. By construction, $r_n + r'_{n+1} \leqslant e_{n+1} - e_n = [p_{n+1}(1) - p_n(1)] + [p_{n+1}(2) - p_n(2)]$ for all $n \geqslant 0$. Applying Corollary 1.3 again, we find partial isometries w_n in $\mathscr{A}$ and some projections $r_n(1) \leqslant p_{n+1}(1) - p_n(1)$ and $r_n(2) \leqslant p_{n+1}(2) - p_n(2)$ such that $w_n w_n^* = r_n + r'_{n+1}$ and $w_n^* w_n = r_n(1) + r_n(2)$ for each n. Set $w = \sum_{n=0}^{\infty} w_n$. Then $w \in M(\mathscr{A})$, $ww^* = \sum_{n=0}^{\infty} (r_n + r'_{n+1})$ and $w^*w = \sum_{n=0}^{\infty} [r_n(1) + r_n(2)]$, where $r_0 = 0$.

Set $Q'_1 = \sum_{n=0}^{\infty} r_n(1)$ and $Q'_2 = \sum_{n=0}^{\infty} r_n(2)$. Then $Q'_1 \leqslant P_1$ and $Q'_2 \leqslant 1 - P_1$. Set $Q_1 = u^*vwQ'_1w^*v^*u$ and $Q_2 = u^*vwQ'_2w^*v^*u$, as desired.

1.5. Corollary. *Assume that $\mathscr{A}$ is a σ-unital C^*-algebra with* FS, *and every projection in $M(\mathscr{A})/\mathscr{A}$ lifts. If $\bar{p}_1$, $\bar{p}_2$ and $\bar{q}$ are three projections in $M(\mathscr{A})/\mathscr{A}$ such that $\bar{p}_1\bar{p}_2 = 0$ and $\bar{q} \leqslant \bar{p}_1 + \bar{p}_2$, then there are projections $\bar{q}_1$ and $\bar{q}_2$ in $M(\mathscr{A})/\mathscr{A}$ such that $[\bar{q}_1] \leqslant [\bar{p}_1]$, $[\bar{q}_2] \leqslant [\bar{p}_2]$ and $\bar{q} = \bar{q}_1 + \bar{q}_2$.*

Proof. By [32, 2.5], there are three projections P_1, P_2 and Q in $M(\mathscr{A})$ such that $P_1P_2 = 0$, $Q \leqslant P_1 + P_2$, $\pi(P_1) = \bar{p}_1$, $\pi(P_2) = \bar{p}_2$ and $\pi(Q) = \bar{q}$. Then Lemma 1.4 applies.

We have finished the proof of Theorem 1.1. The following easy corollary generalizes a well known result for AF algebras.

1.6. Corollary. *If $\mathscr{A}$ is a C^*-algebra with* FS *and cancellation, then* $K_0(\mathscr{A})$ *has the Riesz interpolation property in the following sense:*

$$a_i \leqslant b_j \ (i, j = 1, 2) \text{ in } K_0(\mathscr{A}) \Rightarrow a_i \leqslant c \leqslant b_j \text{ for some } c \in K_0(\mathscr{A}).$$

Here $x \leqslant y$ if and only if $y - x \neq 0$. (See [24, § 2].*)*

Proof. Since $\mathscr{A}$ has FS, $\mathscr{A}$ has an approximate identity consisting of projections. Formally, $K_0(\mathscr{A}) \cong K_0(\mathscr{A})^+ - K_0(\mathscr{A})^+$ by [4, 5.5.5 and 6.3.1]. Since $\mathscr{A}$ has cancellation, $K_0(\mathscr{A})^+ \cong D(\mathscr{A} \otimes \mathscr{K})$. This means that $K_0(\mathscr{A})$ is directed in the terminology of Fuchs who initiated the study of Riesz groups [24, § 2]. Then the Riesz decomposition property and the Riesz interpolation property are equivalent for directed partially ordered groups ([24, 2.3]). Now Theorem 1.1 aplies.

2. APPLICATIONS

2.1. COROLLARY. *If $\mathscr{A}$ is a σ-unital (non-unital) C^*-algebra with FS, and if $\{e_n\}$ is a fixed approximate identity of $\mathscr{A}$ consisting of an increasing sequence of projections ($e_0 = 0$), then every projection Q in $M(\mathscr{A})$ is equivalent to a diagonal form: $D_Q = \sum_{j=1}^{\infty} p_j$, where $\{p_j\}$ is a sequence of projections in $\mathscr{A}$ such that $p_j \leqslant e_j - e_{j-1}$ for all j. Moreover, Q is in $M(\mathscr{A}) \setminus \mathscr{A}$ if and only if p_j is non-zero for infinitely many j's.*

Proof. The proof consists of two steps. First, by the same argument as in [21, 2.4] we can find a unitary V in $M(\mathscr{A})$ such that $VQV^* = \sum_{i=1}^{\infty} q_i$, where q_i is a projection in $(e_{m_i} - e_{m_{i-1}})\mathscr{A}(e_{m_i} - e_{m_{i-1}})$ for each $i \geqslant 1$ (here $e_{m_0} = 0$). Then use the Riesz decomposition property of $(e_{m_i} - e_{m_{i-1}})\mathscr{A}(e_{m_i} - e_{m_{i-1}})$ for each $i \geqslant 1$, we can construct a partial isometry W in $M(\mathscr{A})$ such that $WVQV^*W^*$ has the diagonal form as desired. The reader is referred to the proof of [38, 4.1] for more details.

If $\mathscr{A}$ is a separable AF algebra, the proof of G. A. Elliott in [21, 3.1] yields that every closed ideal of $M(\mathscr{A})$ is generated by its projections. Using the idea of G. A. Elliot in [21, 3.1] and combining Theorem 1.1, we can go further to reach the following stronger conclusion, which reduces the study of closed ideals of $M(\mathscr{A})$ to the comparison of projections with diagonal forms. Here we give the construction in detail for the purpose of future reference in a subsequent paper [33].

2.2. THEOREM. *If $\mathscr{A}$ is a σ-unital C^*-algebra with FS and $\{e_m\}$ is a fixed increasing sequential approximate identity of $\mathscr{A}$ consisting of projections, then every closed ideal $\mathscr{I}$ of $M(\mathscr{A})$ is the closed linear span of projections with the form: $\sum_{i=1}^{\infty} p_i$, where $[p_i] \leqslant [e_i - e_{i-1}]$ for each $i \geqslant 1$.*

Proof. We can assume that $\mathscr{I}$ is not contained in $\mathscr{A}$. Let X be any positive element in $\mathscr{I} \setminus \mathscr{A}$ and ε be any positive number. Set $X_1 = X^{1/2}$. By [32, 1.6] we can assume that

$$h = \sum_{n=2}^{\infty} [(1 - e_{m_n})X_1(e_{m_{n-1}} - e_{m_{n-2}}) + (e_{m_{n-1}} - e_{m_{n-2}})X_1(1 - e_{m_n})]$$

is an element in $\mathscr{A} \cap \mathscr{I}$, where $e_0 = 0$. Let $H = X_1 - h$. Then H is in $\mathscr{I}$. We can choose $\{e_{m_n}\}$ such that $\|h\|$ is small enough, and so $\|H^2 - X\| < \varepsilon/2$. Set $f_n = e_{m_n} - e_{m_{n-1}}$ for each $n \geqslant 1$. Then

$$H^2 = H\Big[\sum_{n=1}^{\infty} f_{m_{3n}}\Big]H + H\Big[\sum_{n=1}^{\infty} f_{m_{3n-1}}\Big]H + H\Big[\sum_{n=1}^{\infty} f_{m_{3n-2}}\Big]H.$$

It is easy to see that the above three sums converge in the strict topology.

Set $H_0 = H\left[\sum_{n=1}^{\infty} f_{m_{3n}}\right]H$, which of course is in $\mathscr{I}$. By computation, $H_0 = \sum_{n=1}^{\infty} x_n$, where x_n is a positive element of $\text{her}(e_{m_{3n}} - e_{m_{3n-3}}) \cap \mathscr{I}$ for each $n \geqslant 1$. Choose a positive element h_n in $\text{her}(x_n)$ with finite spectrum such that $\|x_n - h_n\| < \varepsilon/2^{n+4}$ for each $n \geqslant 1$. Then $\sum_{n=1}^{\infty} (x_n - h_n)$ is an element in $\mathscr{A} \cap \mathscr{I}$, and hence $\sum_{n=1}^{\infty} h_n$ is in $\mathscr{I}$. It is obvious that $\left\| H_0 - \sum_{n=1}^{\infty} h_n \right\| < \varepsilon/12$. Set $h_n = \sum_{i=1}^{N_n} t_n p_{ni}$, where $t_{ni} \geqslant 0$ are numbers and $\{p_{ni}\}$ is a set of mutually orthogonal subprojections of $e_{m_{3n}} - e_{m_{3n-3}}$ for each $n \geqslant 1$. Set $\mu = \sup\{t_{ni} : 1 \leqslant i \leqslant N_n$ and $n \geqslant 1\}$. Choose an integer $N_0 \geqslant 1$ such that $\mu/N_0 < \varepsilon/24$, and then set $P_j = \sum_{n=1}^{\infty} \sum_{\left\{i: \frac{(j-1)\mu}{N_0} < t_{ni} \leqslant \frac{j\mu}{N_0}\right\}} p_{ni}$ for each $1 \leqslant j \leqslant N_0$. It is easily verified that the $\{P_j\}$ is a set of mutually orthogonal projections in $\mathscr{I}$, since $\mathscr{I}$ is hereditary and

$$P_j \leqslant \left[\frac{N_0}{(j-1)\mu}\right] \sum_{n=1}^{\infty} \sum_{\left\{i: \frac{(j-1)\mu}{N_0} < t_{ni} \leqslant \frac{j\mu}{N_0}\right\}} t_{ni} p_{ni} \leqslant \sum_{n=1}^{\infty} h_n \quad \text{for } 2 \leqslant j \leqslant N_0.$$

It also follows that $\left\| \sum_{n=1}^{\infty} h_n - \sum_{j=1}^{N_0-1} \frac{(j+1)\mu}{N_0} P_{j+1} \right\| < \varepsilon/12$, and hence

$$\left\| H_0 - \sum_{j=1}^{N_0} \frac{(j+1)\mu}{N_0} P_{j+1} \right\| < \frac{\varepsilon}{6}.$$

Since $r_n = \sum_{\left\{i: \frac{(j-1)\mu}{N_0} < t_{ni} \leqslant \frac{j\mu}{N_0}\right\}} p_{ni} \leqslant e_{m_{3n}} - e_{m_{3n-3}}$ for $n \geqslant 1$, we can apply the Riesz decomposition property to write r_n as a finite sum $\sum p_i$ of projections satisfying $[p_i] \leqslant [e_i - e_{i-1}]$ for each $m_{3n-2} \leqslant i \leqslant m_{3n}$. Then P_j has the required form.

Similarly, we can find two linear combinations of projections with the required form in $\mathscr{I}$ close to $H\left[\sum_{n=1}^{\infty} f_{m_{3n-1}}\right]H$ and $H\left[\sum_{n=1}^{\infty} f_{m_{3n-2}}\right]H$ in norm, respectively. Then the sum L_ε of these three linear combinations of projections in $\mathscr{I}$ satisfies $\|H^2 - L_\varepsilon\| < \varepsilon/2$, and hence $\|X - L_\varepsilon\| \leqslant \|X - H^2\| + \|H^2 - L_\varepsilon\| < \varepsilon$.

The third consequence of the Riesz decomposition property is that the lattice of ideals of $D[M(\mathscr{A})]$ is isomorphic to the lattice of closed ideals of $M(\mathscr{A})$.

This generalizes the recent result of G. A. Elliott [23] for separable AF algebras. With the aid of Theorem 1.1, G. A. Elliott's proof for separable AF algebras works for σ-unital C^*-algebras with FS. We give the following different proof which seems to be simpler.

2.3. THEOREM. *If $\mathscr{A}$ is a σ-unital C^*-algebra with* FS, *then the lattice of ideals of $D[M(\mathscr{A})]$ is isomorphic to the lattice of closed ideals of $M(\mathscr{A})$.*

Proof. Define two maps as G. A. Elliot did in [23]: $D(\cdot)$ is a map from the lattice of closed ideals of $M(\mathscr{A})$ to the lattice of ideals of $D[M(\mathscr{A})]$, where for any closed ideal $\mathscr{I}$ of $M(\mathscr{A})$, $D(\mathscr{I})$ is defined to be the set of equivalence classes of projections in $\mathscr{I}$. $I(\cdot)$ is a map from the lattice of ideals of $D[M(\mathscr{A})]$ to the lattice of closed ideals of $M(\mathscr{A})$, where $I(\mathscr{L})$ is defined to be the closed ideal of $M(\mathscr{A})$ generated by projections in $M(\mathscr{A})$ with $[P]$ in $\mathscr{L}$. We shall prove that $D(\cdot)$ and $I(\cdot)$ are mutual inverses. For any closed ideal $\mathscr{I}$ of $M(\mathscr{A})$, $\mathscr{I} = I[D(\mathscr{I})]$ by Theorem 2.2. It is clear that $\mathscr{L} \subset D[I(\mathscr{L})]$. We need only to show $D[I(\mathscr{L})] \subset \mathscr{L}$.

Let $[P]$ be any element in $D[I(\mathscr{L})]$. By definition, there are projections P_i and elements X_i and Y_i in $M(\mathscr{A})$ such that $\|\sum_{i=1}^{n} X_i P_i Y_i - P\| \leqslant \varepsilon < 1$ and $[P_i] \in \mathscr{L}$, $1 \leqslant i \leqslant n$. Working in $M(\mathscr{A}) \otimes M_n \cong M(\mathscr{A} \otimes M_n)$, we identify $M(\mathscr{A})$ with $M(\mathscr{A}) \otimes e_{11}$, P with $P \otimes e_{11}$ and P_i with $P_i \otimes e_{11}$. By a slight variation of the standard argument which appeared in [15, 1.5], we can find a partial isometry U in $M(\mathscr{A} \otimes M_n)$ such that $UU^* = P \otimes e_{11}$, and $U^*U = Q \leqslant \sum_{i=1}^{n} P \otimes e_{ii}$. Since we have proved the Riesz decomposition property, we can go further to reach our goal.

Since $\mathscr{A}$ has FS, $\mathscr{A} \otimes M_n$ has FS ([12]). By Theorem 1.1 we find a partial isometry W in $M(\mathscr{A} \otimes M_n)$ such that $WW^* = Q$ and $W^*W = \sum_{i=1}^{n} (Q_i \otimes e_{ii})$, where $Q_i \leqslant P_i$ $(1 \leqslant i \leqslant n)$. Let $W_0 = UW$. Then W_0 is a partial isometry in $M(\mathscr{A} \otimes M_n)$ such that $W_0^* W_0 = \sum_{i=1}^{n} Q_i \otimes e_{ii}$ and $W_0 W_0^* = P \otimes e_{11}$. Set $W_i = W_0(Q_i \otimes e_{ii})V_i$. Then W_i is a partial isometry in $M(\mathscr{A}) \otimes e_{11}$ such that

$$W_i W_i^* = R_i \leqslant P \otimes e_{11} \text{ and } W_i^* W_i = V_i^*(Q_i \otimes e_{ii})V_i \leqslant P_i \otimes e_{11} \quad (1 \leqslant i \leqslant n).$$

Hence $[R_i] \leqslant [P_i]$ in $D[M(\mathscr{A})]$ $(1 \leqslant i \leqslant n)$. It is easily seen that $R_i R_j = 0$ if $i \neq j$ and $\sum_{i=1}^{n} R_i = P \otimes e_{11}$. Thus $\sum_{i=1}^{n} [R_i] = [P]$ in $D(M(\mathscr{A}))$. Since $\mathscr{L}$ is an ideal of $D[M(\mathscr{A})]$ and $[P_i] \in \mathscr{L}$, $[R_i] \in \mathscr{L}$ $(1 \leqslant i \leqslant n)$. Therefore $[P] \in \mathscr{L}$.

There is a connection between the ideal structure of $D(M(\mathscr{A}))$ and the ideal structure $M(\mathscr{A})$ even if $\mathscr{A}$ is a C^*-algebra not necessarily with FS. Using an argument similar to the proof of Theorem 2.3, we immediately conclude the following:

2.4. PROPOSITION. *If $\mathscr{A}$ is any C^*-algebra, then there is an injective additive mapping from the set of ideals of $D[M(\mathscr{A} \otimes \mathscr{K})]$ to the ideal lattice of $M(\mathscr{A} \otimes \mathscr{K})$.*

In $M(\mathscr{A} \otimes \mathscr{K})$, some projections generate proper closed ideals but some projections do not. A natural question is: for which projections P in $M(\mathscr{A} \otimes \mathscr{K})$ is the closed ideal generated by P proper? Let us point out the following matters concerning this question. We denote the ideal of $D[M(\mathscr{A} \otimes \mathscr{K})]$ generated by $[P]$ by $D([P])$. It is easy to see that $D([P]) = \{[Q] \in D[M(\mathscr{A} \otimes \mathscr{K})] : [Q] \leqslant m[P]$ for some $m \in N\}$.

2.5. DEFINITION. An element $[P] \in D[M(\mathscr{A} \otimes \mathscr{K})]$ is said to have degree $n \in N$ if $n[P] \geqslant [1]$ and $(n - 1)[P] \not\geqslant [1]$. The degree of $[P]$ is denoted by $d([P]) = n$. We say that $[P]$ has infinite degree, denoted by $d([P]) = \infty$, if such n does not exist.

2.6. POPOSITION. *If $\mathscr{A}$ is a C^*-algebra and P is a projection in $M(\mathscr{A} \otimes \mathscr{K})$, then $n[P] \geqslant [1]$ if and only if $n[P] = 1$.*

Proof. It suffices to show that $[1] \leqslant [Q]$ if and only if $Q \sim 1$. A proof of this fact by using K-theory can be found in [17, 3.6], and another direct proof without using K-theory was given in [39].

2.7. PROPOSITION. *If $\mathscr{A}$ is any C^*-algebra, then*

(a) *$d([P]) = \infty$ if and only if the closed ideal $\mathscr{I}(P)$ generated by P is not equal to $M(\mathscr{A} \otimes \mathscr{K})$. Consequently, if there is a projection P in $M(\mathscr{A} \otimes \mathscr{K})$ but not in $\mathscr{A} \otimes \mathscr{K}$ with $d([P]) = \infty$, then $\mathscr{I}(P)$ strictly contains $\mathscr{A} \otimes \mathscr{K}$, and hence $M(\mathscr{A} \otimes \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is not simple.*

(b) *If $d([1 \otimes e_{11}]) = \infty$ and $\mathscr{A}$ is nonunital, then the ideal generated by $1 \otimes e_{11}$ is a proper ideal of $M(\mathscr{A} \otimes \mathscr{K})$ containing $M(\mathscr{A}) \otimes \mathscr{K}$.*

Consequently, if $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple, then every element in $D[M(\mathscr{A} \otimes \otimes \mathscr{K})]$ not in $D[A \otimes \mathscr{K}]$ has a finite degree.

Proof. (a) By the definition of the degree of $[P]$, $d([P]) = \infty$ if and only if $[1] \notin D([P])$ if and only if $D([P])$ is a proper ideal of $D[M(\mathscr{A} \otimes \mathscr{K})]$. Proposition 2.4 applies.

(b) Since $\mathscr{A}$ is nonunital, $1 \otimes e_{11} \in M(\mathscr{A} \otimes \mathscr{K}) \setminus \mathscr{A} \otimes \mathscr{K}$. Since $d(1 \otimes e_{11}) = \infty$, by (a) the ideal $\mathscr{J} = I(D([1 \otimes e_{11}]))$ is not equal to $M(\mathscr{A} \otimes \mathscr{K})$. Clearly, all matrix units $\{1 \otimes e_{ij}\}$ are in $\mathscr{J}$, since $1 \otimes e_{ij} = (1 \otimes e_{ij})(1 \otimes e_{jj})$ and $[1 \otimes e_{11}] = [1 \otimes e_{jj}]$. It follows that $M(\mathscr{A}) \otimes \mathscr{K}$ is a subset of $\mathscr{J}$.

3. IDEALS OF $M(\mathscr{A})$ AND $M(\mathscr{A})/\mathscr{A}$

3.1. PROPOSITION. *If $\mathscr{A}$ is a C^*-algebra, then*

(a) *There is an injective mapping φ from the ideal lattice of $\mathscr{A}$ to the ideal lattice of $M(\mathscr{A} \otimes \mathscr{K})$ such that $\mathscr{I} \otimes \mathscr{K}$ is an essential ideal (see [1]) of $\varphi(\mathscr{I})$ and $\varphi(\mathscr{I}) \not\subset \mathscr{A} \otimes \mathscr{K}$ for any ideal $\mathscr{I}$ of $\mathscr{A}$.*

(b) *There is an injective mapping ω from the ideal lattice of $\mathscr{A}$ to the ideal lattice of $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$. Consequently, if $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple, then $\mathscr{A}$ must be simple. If $\mathscr{A}$ is σ-unital, then the above φ and ω are lattice isomorphisms (not necessarily onto).*

Proof. (a) Define $\varphi(\mathscr{I}) = M(\mathscr{A} \otimes \mathscr{K}, \mathscr{I} \otimes \mathscr{K})$ for each ideal $\mathscr{I}$ of $\mathscr{A}$, where $M(\mathscr{A} \otimes \mathscr{K}, \mathscr{I} \otimes \mathscr{K}) = M(\mathscr{A} \otimes \mathscr{K}) \cap (\mathscr{I} \otimes \mathscr{K})^{**}$. It is easily seen that for any two ideals $\mathscr{I}$ and $\mathscr{J}$ of $\mathscr{A}$, $\varphi(\mathscr{I} \cap \mathscr{J}) = \varphi(\mathscr{I}) \cap \varphi(\mathscr{J})$. If $\mathscr{I} \neq \mathscr{J}$ are two ideals of $\mathscr{A}$ then either there exists x in $\mathscr{I} \setminus \mathscr{J}$ or there exists x in $\mathscr{J} \setminus \mathscr{I}$. Correspondingly, either $x \otimes 1$ is in $\varphi(\mathscr{I}) \setminus \varphi(\mathscr{J})$ or $x \otimes 1$ is in $\varphi(\mathscr{J}) \setminus \varphi(\mathscr{I})$. If $\mathscr{A}$ is σ-unital, then $\varphi(\mathscr{I}) + \varphi(\mathscr{J})$ (see [11, 3.48, 3.49]). Hence, φ is a lattice isomorphism.

(b) Define $\psi(I) = \mathscr{A} \otimes \mathscr{K} + \varphi(\mathscr{I})$ for each ideal of $\mathscr{A}$. Then $\omega = \pi \circ \psi$, as desired, where π is the canonical map from $M(\mathscr{A} \otimes \mathscr{K})$ to the quotient algebra.

The following lemma is another useful consequence of the Riesz decomposition property:

3.2. LEMMA. *If $\mathscr{A}$ is a σ-unital simple non-elementary C^*-algebra with FS, and if p and q are nonzero projections in $\mathscr{A}$, then for any integer $n \geq 1$ there exists a nonzero subprojection r of q such that $2^n[r] \leq [p]$.*

Proof. First, using the same argument as in the proof for Theorem 2.3, where the Riesz decomposition property is the key point, we can write $p = \sum_{i=1}^{m} q_i$: here all q_i are nonzero projections and such that $[q_i] \leq [q]$. Considering q_1 instead of p, we can assume that $p \leq q$.

Since $\mathscr{A}$ is σ-unital, simple and non-elementary, $\mathscr{A}$ does not have a minimal projection. Let p_1 be a subprojection of p such that $0 < p_1 < p$. Using the proof for Theorem 2.3 again, we can write $p - p_1$ into a direct sum of nonzero subprojections such that each of the summands is equivalent to a subprojection of p_1. Hence, we can find a nonzero subprojection p_{11} of p_1 and a nonzero subprojection p_{12} of $p - p_1$ such that $p_{11} \sim p_{12}$. Repeating the same argument n times, we can find a nonzero projection r in $\mathscr{A}$ as desired.

The following theorem proves to be important for studying the structure of $M(\mathscr{A} \otimes \mathscr{K})$ from various viewpoints in our subsequent papers ([35], [36] and [37]).

3.3. THEOREM. *If $\mathscr{A}$ is a σ-unital C^*-algebra with FS, then $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple if and only if either $\mathscr{A}$ is elementary or $\mathscr{A}$ is simple and every non-*

zero projection in $\mathscr{A}$ is infinite (in other words, $\mathscr{A}$ is purely infinite). Moreover, every projection in $M(\mathscr{A} \otimes \mathscr{K})$ not in $\mathscr{A} \otimes \mathscr{K}$ is equivalent to the identity of $M(\mathscr{A} \otimes \mathscr{K})$ in case $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple.

Proof. If $\mathscr{A}$ is elementary, then $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is the Calkin algebra. We can assume that $\mathscr{A}$ is non-elementary and every nonzero projection of $\mathscr{A}$ is infinite. Let P be any projection in $M(\mathscr{A} \otimes \mathscr{K})$ but not in $\mathscr{A} \otimes \mathscr{K}$. By Proposition 2.1, $P \sim \sum_{i=1}^{\infty} p_i \otimes e_{ii}$, where p_i is a nonzero projection in $\mathscr{A}$ for $i \geqslant 1$. Fix a nonzero projection q in $\mathscr{A}$. Since p_i is infinite for each $i \geqslant 1$, by a standard argument for infinite projections in a simple C^*-algebra (see [15, 1.5] or [3, 3.12.1]), for each $i \geqslant 1$ we can find a subprojection r_i of p_i such that $r_i \sim q$. Set $Q = \sum_{i=1}^{\infty} r_i \otimes e_{ii}$. Then Q is a projection in $M(\mathscr{A} \otimes \mathscr{K})$ such that $\sum_{i=1}^{\infty} q \otimes e_{ii} \sim Q$ and $[Q] \leqslant [P]$. On the other hand, $\sum_{i=1}^{\infty} q \otimes e_{ii}$ is equivalent to the identity by [10, 2.5]. Thus, the identity of $M(\mathscr{A} \otimes \mathscr{K})$ is equivalent to a subprojection of P. It follows from Proposition 2.6 that P is equivalent to the identity. By Theorem 2.3, $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple.

Now we prove the converse. By Proposition 3.1 we see that $\mathscr{A}$ must be simple if $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple. If $\mathscr{A}$ is non-elementary, and if there exists a nonzero finite projection p_0 in $\mathscr{A}$, we show that $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is not simple. By Lemma 3.2, there is a nonzero projection p_1 in $p_0\mathscr{A}p_0$ such that $2[p_1] < [p_0]$. By Lemma 3.2 again, there is a nonzero projection p_2 in $p_1\mathscr{A}p_1$ such that

$$2^2[p_2] < [p_1], \quad \text{and hence} \quad 2^2[p_2] + [p_1] < [p_0].$$

Repeating by induction, for any $n \geqslant 2$ we can find a nonzero projection p_n in $p_{n-1}\mathscr{A}p_{n-1}$ such that

$$2^n[p_n] < [p_{n-1}], \text{ and hence } 2^n[p_n] + \sum_{n=1}^{n-1} 2^{i-1}[p_i] < [p_0].$$

Set $Q = \sum_{i=1}^{\infty} p_i \otimes e_{ii}$. Then Q is a projection of $M(\mathscr{A} \otimes \mathscr{K})$ but not in $\mathscr{A} \otimes \mathscr{K}$. We show that the closed ideal of $M(\mathscr{A} \otimes \mathscr{K})$ generated by Q is proper, and the conclusion will follow.

If the closed ideal generated by Q is $M(\mathscr{A} \otimes \mathscr{K})$, then $[1] = m[Q]$ for some $m \geqslant 1$ by Proposition 2.6 and Proposition 2.7. Let $e'_{k+1} = e_{mk+1,mk+1} + e_{mk+2,mk+2} + \ldots + e_{m(k+1),m(k+1)}$ for $k \geqslant 0$. Set $Q_0 = \sum_{k=1}^{\infty} p_k \otimes e'_k$. Then Q_0 is a representative of $m[Q]$, and hence $Q_0 \sim 1$. It is easily verified that if $n_2 > n_1 > m$, we have

$$\left[\sum_{k=n_1}^{n_2} p_k \otimes e'_k\right] = \sum_{k=n_1}^{n_2} m[p_k] < \sum_{k=n_1}^{n_2} 2^{k-1}[p_k] < [p_0],$$

where $\mathscr{A}$ is naturally identified with $\mathscr{A} \otimes e_{11}$. Set $f'_n = \sum_{i=1}^{n} p_0 \otimes e_{ii}$ for $n \geq 1$. By [10, 2.8] there is a sequence $\{f_n\}$ of projections of $\mathscr{A} \otimes \mathscr{K}$ such that $f'_n \sim f_n$ for $n \geqslant 1$ and $\{f_n\}$ constitudes an approximate identity of $\mathscr{A} \otimes \mathscr{K}$ consisting of projections. Set $p' = \sum_{k=1}^{m} p_k \otimes e'_k$. Then there is an $n_0 \geqslant 1$ such that $\|(1 - f_{n_0})p'\|$ is small enough. It follows from [21, 2.1] that there exists a unitary U in $M(\mathscr{A} \otimes \mathscr{K})$ such that $Up'U^* \leqslant f_{n_0}$. It follows that $1 \sim 1 - f_{n_0} \leqslant \leqslant U(1 - p')U^*$. Thus, $[p_0] < \sum_{k=m+1}^{m_0} m[p_k]$ for some $m_0 > m + 1$. But $\sum_{k=m+1}^{m_0} m[p_k] < [p_0]$, which contradicts the hypothesis that p_0 is a nonzero finite projection of $\mathscr{A}$.

3.4. Remarks. (i) A C^*-algebra $\mathscr{A}$ is said to be purely infinite if for any nonzero positive element x in $\mathscr{A}$ there exists an infinite projection in $(x\mathscr{A}x)^-$ ([15] and [33]). In [34] and [37, Part I], the author has proved that if $\mathscr{A}$ is a simple C^*-algebra, then $\mathscr{A}$ is purely infinite if and only if $\mathscr{A}$ has FS and every nonzero projection in $\mathscr{A}$ is infinite. Moreover, a σ-unital purely infinite simple C^*-algebra is either unital or stable. Many examples of purely infinite simple C^*-algebras can be found in [15], [37, Part I]. By the way, Theorem 3.3 provides a positive anwer for the conjecture [39, 6.18].

(ii) H. Lin proved ([28]) that $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is simple if $\mathscr{A}$ is σ--unital and satisfies an apparently different purely infinite condition. Recently, it was proved ([29]) that these and some other apparently different purely infinite conditions on a simple C^*-algebra are all equivalent. Thus, combining [28] and [29], we have another proof for the direction 'if' in Theorem 3.3.

(iii) Very recently, M. Rørdam has proved ([31]) the direction 'only if' of Theorem 3.3 without assuming that $\mathscr{A}$ has FS. He gives a new proof for both directions via comparison of positive diagonal elements rather than comparison of diagonal projections.

For the remaining part of this paper, we will take a look at the relation between the state space of $K_0(\mathscr{A})$ and the closed ideals of $M(\mathscr{A})$.

Let $\mathscr{A}$ be a simple C^*-algebra with FS. If $[p_0]$ in $D[\mathscr{A} \otimes \mathscr{K}]$ is a fixed nonzero element, we denote by $S([p_0])$ the state space of $K_0(\mathscr{A})$ with respect to the order unit $[p_0]$. $S([p_0])$ is a compact convex subset of the product $R^{D[\mathscr{A}\otimes\mathscr{K}}$ which is non-empty if and only if $\mathscr{A}$ is stably finite ([20, Chapter 4] and [4, 6.8]). It turns out that the structure of $S([p_0])$ is closely related to the structure of closed ideals of $M(\mathscr{A})$.

If $\mathscr{A}$ is a σ-unital C^*-algebra with FS, for each τ in $S([p_0])$, a mapping $\bar{\tau}$ from the set of all projections in $M(\mathscr{A})$ to $\mathbf{R}^+ \cup \{\infty\}$ is defined as follows: For any projection P in $M(\mathscr{A})$, by [32, 1.2] we can write $P = \sum_{i=1}^{\infty} p_i$ for some mutually orthogonal projections p_i in $\mathscr{A}$, where the sum converges in the strict topology. Then we define $\bar{\tau}(P) = \sum_{i=1}^{\infty} \tau(p_i) \in [0, \infty]$. It is easily verified that $\bar{\tau}(P) = \sum_{i=1}^{\infty} \tau(p_i)$ is independent of $\{p_i\}$ and satisfies:

(i) $\bar{\tau}(P + Q) = \bar{\tau}(P) + \bar{\tau}(Q)$ whenever $PQ = 0$ and

(ii) $\bar{\tau}(P) \leqslant \bar{\tau}(Q)$ if $[P] \leqslant [Q]$.

For each τ in $S([p_0])$, let $\mathscr{J}_\tau$ be the closed ideal of $M(\mathscr{A})$ generated (as a C^*-algebra) by the set of projections $\{P \in M(\mathscr{A}) : \bar{\tau}(P) < +\infty\}$.

If $\mathscr{A}$ is a separable nonunital matroid algebra, G. A. Elliott proved ([21, § 3]) that $M(\mathscr{A})/\mathscr{A}$ is simple if $\mathscr{A}$ is finite (since the unique $\mathscr{J}_\tau$ is equal to $M(\mathscr{A})$), and $M(\mathscr{A})/\mathscr{A}$ has a unique nontrivial closed ideal $\bar{\mathscr{J}}$ if $\mathscr{A}$ is infinite (since the unique $\mathscr{J}_\tau$ is strictly between $\mathscr{A}$ and $M(\mathscr{A})$). H. Lin considered the closed ideals of $M(\mathscr{A})/\mathscr{A}$ if $\mathscr{A}$ is a simple separable nonunital AF algebra. He proved ([27]) that $M(\mathscr{A})/\mathscr{A}$ is simple if and only if $\mathscr{A}$ has a continuous scale, i.e., $\hat{1}(\tau) = \bar{\tau}(1)$ is bounded and continuous on $S([p_0])$. Assuming that there are only finitely many extremal traces on $\mathscr{A}$, he proved ([27]) that the closed ideal lattice of $M(\mathscr{A})/\mathscr{A}$ is isomorphic to the lattice of subsets of those extremal traces on $\mathscr{A}$ such that $\bar{\tau}(1) = +\infty$. We shall consider the ideal structure of $M(\mathscr{A})$ for certain non-AF algebras by means of states.

With the aid of Theorem 2.2, we need only consider projections of $M(\mathscr{A})$ with diagonal form. In this way, the Riesz decomposition property yields simpler arguments for technical points. The following theorem includes a slight generalization of [27, 1.2].

3.5. THEOREM. *Suppose that $\mathscr{A}$ is a σ-unital, simple, stably finite and non-elementary C^*-algebra with* FS. *Then the following hold:*

(i) $\mathscr{J}_0 = \bigcap_{\tau \in S([p_0])} \mathscr{J}_\tau$ *is a closed ideal of $M(\mathscr{A})$ strictly containing $\mathscr{A}$. If, in addition, $\mathscr{A}$ has cancellation, then $\mathscr{J}_0$ is the smallest closed ideal of $M(\mathscr{A})$ strictly containing $\mathscr{A}$,* i.e., *the intersection of all closed ideals of $M(\mathscr{A})$ strictly containing $\mathscr{A}$.*

(ii) *In case $\mathscr{A}$ has cancellation, $M(\mathscr{A})/\mathscr{A}$ is simple if and only if $\bar{\tau}(P) < \infty$ for every projection P in $M(\mathscr{A})$ and each τ in $S([p_0])$ (this condition is equivalent in Lin's terminology to "$\mathscr{A}$ has a continuous scale", see* [27]*).*

(iii) *In case τ is any state in $S([p_0])$, $\mathscr{A}$ is a proper closed ideal of $\mathscr{I}_\tau$. If, in addition, $\bar{\tau}(1) = +\infty$, then $\mathscr{I}_\tau$ is a proper closed ideal of $M(\mathscr{A})$ strictly containing $\mathscr{A}$; consequently, $M(\mathscr{A} \otimes \mathscr{K})/\mathscr{A} \otimes \mathscr{K}$ is always non-simple.*

Proof. First, there exist nonzero states on $K_0(\mathscr{A})$ since $\mathscr{A}$ is stably finite. Hence $S([p_0])$ is not empty.

(i) Let $\{e_n\}$ be an approximate identity of $\mathscr{A}$ consisting of an increasing sequence of projections. For each $n \geqslant 1$ applying Lemma 3.2 to p_0 and $f_n = e_n - e_{n-1}$, we can choose projections $r_n < f_n$ such that $2^n[r_n] \leqslant [p_0]$. Then $\tau(r_n) \leqslant 2^{-n}\tau(p_0) = 2^{-n}$ for $n \geqslant 1$. Set $P = \sum_{n=1}^{\infty} r_n$. Then it is routine to show that P is a projection in $M(\mathscr{A}) \setminus \mathscr{A}$. Moreover, $\bar{\tau}(P) < +\infty$ for all τ in $S([p_0])$. Hence, $\mathscr{I}_0$ strictly contains $\mathscr{A}$. If, in addition, $\mathscr{A}$ has cancellation, then $S([p_0])$ has the property: For any two projections p, q in $\mathscr{A} \otimes \mathscr{K}$, $[p] \leqslant [q]$ if $\tau(p) < \tau(q)$ ([4] or [6]). Now we can apply Lin's argument in [27, 2]. Here we give an outline as follows.

If $\mathscr{I}$ is any closed ideal of $M(\mathscr{A})$ strictly containing $\mathscr{A}$, we show that $\mathscr{I}_0 \subset \mathscr{I}$. Let $P_0 = \sum_{i=1}^{\infty} r_i$ be any projection of $\mathscr{I}_0$ such that $r_i \leqslant f_i$ for all $i \geqslant 1$. Fix a projection $P = \sum_{i=1}^{\infty} p_i$ in $\mathscr{I} \setminus \mathscr{A}$. Choose $m_j \nearrow \infty$ such that $\sum_{i=m_{j-1}}^{m_j} p_i \neq 0$ for each $j \geqslant 1$. Since $S([p_0])$ is compact, we have $\delta_j = \inf_{\tau \in S([p_0])} \tau\left(\sum_{i=m_{j-1}}^{m_j} p_i\right) > 0$ for each $j \geqslant 1$. Recursively, we can choose $n_j \nearrow \infty$ such that

$$\sup_{\tau \in S([p_0])} \tau\left[\sum_{i=n_{j-1}}^{n_j} r_i\right] < \delta_j \quad \text{for each } j \geqslant 1.$$

Hence, there is a partial isometry v_j in $\mathscr{A}$ for each $j \geqslant 1$ such that

$$v_j v_j^* = \sum_{i=n_{j-1}}^{n_j} r_i \quad \text{and} \quad v_j^* v_j \leqslant \sum_{i=m_{j-1}}^{m_j} p_i .$$

Define $V = \sum_{i=1}^{\infty} v_i$. Then V is a partial isometry $M(\mathscr{A})$ such that $VV^* = \sum_{i=n_0}^{\infty} r_i$ and $V^*V \leqslant \sum_{i=1}^{\infty} p_i = P$. Since J is hereditary, V^*V is in $\mathscr{I}$ and so is VV^*. Since $\mathscr{A} \subset \mathscr{I}$ and $\sum_{i=1}^{n_0-1} r_i \in \mathscr{A}$, $P_0 \in \mathscr{I}$. Hence, $\mathscr{I}_0 \subset \mathscr{I}$. (ii) clearly follows from (i).

(iii) If $\bar{\tau}(1) = +\infty$, the identity of $M(\mathscr{A})$ is not in $\mathscr{J}_\tau$. If $1 \in \mathscr{J}_\tau$, by the proof of Theorem 2.3, we would write $1 = \sum_{i=1}^{m} P_i$, where $\{P_i\}$ is a set of mutually orthogonal projections in $\mathscr{J}_\tau$. It would follow that $\bar{\tau}(1) = \sum_{i=1}^{m} \bar{\tau}(P_i) < +\infty$. This is a contradiction.

3.6. PROPOSITION. *Assume that $\mathscr{A}$ is a σ-unital simple non-elementary C^*-algebra with* FS. *If $\mathscr{A}$ has a quasitrace τ such that $\bar{\tau}(1) = +\infty$ and $[p] \leqslant [q]$ whenever $\tau(p) < \tau(q)$, where p and q are projections in $\mathscr{A}$, then any projection P in $M(\mathscr{A})$ not in $\mathscr{J}_\tau$ is equivalent to the identity of $M(\mathscr{A})$.*

Proof. Since P is not in $\mathscr{J}_\tau$, then $\bar{\tau}(P) = +\infty$. By [32, 1.2], we can write $P = \sum_{i=1}^{\infty} p_i$, where $\{p_i\}$ is a set of mutually orthogonal projections of $\mathscr{A}$. There exists $n_1 \geqslant 1$ such that $\tau\left(\sum_{i=1}^{n_1} p_i\right) > \tau(f_1)$. Then we can find a partial isometry v_1 in $\mathscr{A}$ such that $v_1 v_1^* = f_1$ and $v_1^* v_1 = f_1' < \sum_{i=1}^{n_1} p_i$. There exists $m_1 \geqslant 2$ such that $\tau\left(\sum_{i=1}^{n_1} p_i - f_1'\right) < \tau(f_{m_1} - f_1)$. We can find a partial isometry v_2 in $\mathscr{A}$ such that $v_2 v_2^* = g_1 < f_{m_1} - f_1$ and $v_2^* v_2 = \sum_{i=1}^{n_1} p_i - f_1'$. Proceeding in this way we can find a sequence of partial isometries $\{v_i\}$ in $\mathscr{A}$ with mutually orthogonal initial projections and mutually orthogonal final projections such that $\left(\sum_{i=1}^{\infty} v_i\right)\left(\sum_{i=1}^{\infty} v_i\right)^* = 1$ and $\left(\sum_{i=1}^{\infty} v_i\right)^*\left(\sum_{i=1}^{\infty} v_i\right) = P$. It is routine to check that $\sum_{i=1}^{\infty} v_i$ is a partial isometry in $M(\mathscr{A})$.

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