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CERTAIN FULL GROUP C^*-ALGEBRAS WITHOUT PROPER PROJECTIONS

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The basic problem studied in the present paper is when the full group C^*-algebra $C^*(G)$ of a discrete torsion free group does not have a proper projection. This problem was originally motivated as an approach [4] to the Kadison conjecture [18] which states that "there is no proper projection in the reduced group C^*-algebra $C_r^*(F_2)$," where F_2 is the free group of two generators. Even though the Kadison conjecture was finally settled by Pimsner and Voiculescu [17], the question of when a group C^*-algebra, full or reduced, has a proper projection has been an interesting one ever since.

In [5] Cuntz introduced a notion of K-amenability of a given group. It turns out that the group F_2 being not amenable is actually K-amenable. Combining this result with Choi's construction [3] of a faithful tracial state on $C^*(F_2)$, one sees that $C_r^*(F_2)$ has no proper projection implies $C^*(F_2)$ has no proper projection. In general, a consequence of the Baum-Connes conjecture [2] is the so-called generalized Kadison conjecture, that is, "There is no proper projection in the reduced C^*-algebra $C_r^*(G)$ for any torsion free, countable, and discrete group G". It has been proved for a large class of groups that the Baum-Connes conjecture holds [2; 15]. Moreover, those groups are olso K-amenable. One might be tempted to conclude that for those groups, there is no proper projection in their full group C^*-algebras either. But after a moment's thought, one will find that this is not that obvious due to the fact we do not know if the full group C^*-algebra possesses a faithful tracial state.

In the present article, inspired by the method of Choi [3] and a construction of the authors of [11], we define so-called C-groups and MC-groups. We can build up a large class of torsion free groups with these properties from smaller ones. It will be shown in Section 1 that C-groups and MC-groups have no proper projections in their full group C^*-algebras. Therefore, any discrete groups of torsion or of Kazdan's property T are neither C-groups nor MC-groups. We will also remark at the end of Section 1 that if we weaken the definitions of C-groups and MC-groups to define C′-groups and MC′-groups respectively, we can actually show that C′-groups and MC′-groups are the same by applying the Voiculescu's non-commutative Weyl--von Neumann theorem [23]. This is suggested by the referee. In Section 2, we start with free abelian groups using free product and amalgamated free product to build up the C-groups. The main theorem is Theorem 5 which states that any free product of finitely generated free abelian groups with isomorphic subgroups amalgamated is a C-group. In Section 3, we study MC-groups. The main result there is Theorem 6: the restricted Wreath product of two amenable MC-groups is again an MC-group. Section 4 is a by-pass to the generalized Kadison conjecture for torsion free nilpotent groups. For these groups the generalized Kadison conjecture can also be proved by proving the Baum-Connes conjecture using the Pimsner-Voiculescu exact sequence [16]. But we proceed to give a purely analytical proof based on the work of Anderson and Paschke [1]. In Section 5, we observe that using inductive limits, one can obtain more groups from smaller groups which satisfy the above projectionless property. At the end we conclude this paper by presenting a concrete example of a finitely generated torsion free K-amenable group which is neither a C-group nor an MC-group.

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Throughout this paper all groups are assumed to be countable and discrete.

1. THE BASIC CONSTRUCTIONS

Let $\mathscr{H}$ be an infinite dimensional Hilbert space, G be a countable discrete group, and $D(G, \mathscr{H})$ be the space of all unitary representations of G on $\mathscr{H}$. We endow $D(G, \mathscr{H})$ with the topology defined by the following: A net $\{\pi_\lambda\}_{\lambda\in\Lambda}$ converges to π in $D(G, \mathscr{H})$ if $\{\pi_\lambda(g)\}_{\lambda\in\Lambda}$ converges to $\pi(g)$ in norm for each g in G. It can be easily verified that $D(G, \mathscr{H})$ equipped with this topology is Hausdorff. Moreover, $D(G, \mathscr{H})$ is homeomorphic to $D(G, \mathscr{H}')$ if $\mathscr{H}$ and $\mathscr{H}'$ both have the same dimension. Therefore, in the sequel, we will simply write $D(G)$ when there is no danger of confusion.

The basic problem about the space $D(G)$ we are going to study is a kind of "connectedness."

Definition 1. (Cf. [11]). An element in $D(G)$ is *maximal* if it extends to a faithful representation of the full group C^*-algebra $C^*(G)$.

REMARKS 1. Since G is countable and $\mathscr{H}$ is infinite dimensional, $D(G)$ always has a maximal element.

DEFINITION 2. We say that G is a C-*group* if for any infinite dimensional Hilbert space $\mathscr{H}$ and any π in $D(G, \mathscr{H})$, there is a Hilbert space $\mathscr{H}'$ containing $\mathscr{H}$, and a π' in $D(G, \mathscr{H}')$, such that $\pi' \mid \mathscr{H} = \pi$ and there is a path $\{\pi_t'\}$ in $D(G, \mathscr{H}')$ connecting π' and 1_G the trivial representation of G on $\mathscr{H}'$. We say that G is an MC-*group* if there is a Hilbert space $\mathscr{H}$ such that $D(G, \mathscr{H})$ contains a maximal element which is connected to 1_G on $\mathscr{H}$ by a path $\{\pi_t\}$ in $D(G, \mathscr{H})$.

A group G being a C-group is equivalent to the following property that for any representation π of G on an infinite dimensional Hilbert space $\mathscr{H}$, there is a representation π' of G so that $\pi \oplus \pi'$ is connected to $1_G \oplus 1_G$ in $D(G, \mathscr{H} \oplus \mathscr{H})$. We would like to thank M.-D. Choi for pointing this out to us. Obviously, a C-group is an MC-group. But we do not know if the converse is true.

The following construction is a variation of that in [11]. The only useful fact is Proposition 1 which can be obtained without using the more general statement (see Lemma 1 below).

Definition 3. For any x in $C^*(G)$, let $f_x : D(G, \mathscr{H}) \to \mathscr{L}(\mathscr{H})$ be the evaluation at x. Let $C(D) = \{f_x \mid x \in C^*(G)\}$. We define a *normed $*$-algebra structure on* $C(D)$ as follows:

(i) $f_x^*(\pi) = \pi(x)^*$,

(ii) $(f_x + \lambda f_y)(\pi) = \pi(x) + \lambda\pi(y)$

(iii) $f_x \cdot f_y(\pi) = \pi(x) \cdot \pi(y) = f_x(\pi) \cdot f_y(\pi)$

(iv) $\|f_x\|_0 = \sup_{\pi \in D(G,\mathscr{H})} \|f_x(\pi)\| = \sup_{\pi \in D(G,\mathscr{H})} \|\pi(x)\|$.

Since $D(G, \mathscr{H})$ always has a maximal element, (iv) becomes $\|f_x\|_0 = \|x\|$.

LEMMA 1. *$C(D)$ is a C^*-algebra. Moreover, $C(D)$ is isomorphic to $C^*(G)$.*

Proof. Since for x in $C^*(G)$, $\|x\| = \|f_x\|_0$, the map $\varphi : x \mapsto f_x$ is a $*$-preserving isomorphism from $C^*(G)$ onto $C(D)$.

REMARK 2. From the above lemma we do not gain anything. However, the notion $C(D)$ suggests that each f_x is actually a continuous function.

PROPOSITION 1. *For any x in $C^*(G)$, the function f_x in $C(D)$ is continuous in the operator norm.*

Proof. Let $\{\pi_\lambda\}$ be a net converging to π in $D(G)$. Since $C^*(G)$ is generated by a unitary group $U_G = \{U_g \in C^*(G) \mid g \in G\}$ [14], we find that the element x can be approximated by finite linear combinations of elements in U_G. Now for any $\varepsilon > 0$, there is a finite sum $\sum A_g U_g$ in $C^*(G)$, where the A_g's are complex numbers, such that $\|x - \sum A_g U_g\| < \varepsilon$. Therefore,

$$\|f_x(\pi_\lambda) - f_x(\pi)\| = \|\pi_\lambda(x) - \pi(x)\| \leqslant$$

$$\leqslant \|\pi_\lambda(x - \sum A_g U_g)\| + \|\pi_\lambda(\sum A_g U_g) - \pi(\sum A_g U_g)\| + \|\pi(\sum A_g U_g - x)\| \leqslant$$

$$\leqslant 2\|x - \sum A_g U_g\| + \sum |A_g| \|\pi_\lambda(U_g) - \pi(U_g)\| \leqslant$$

$$\leqslant 2\varepsilon + \sum |A_g| \|\pi_\lambda(U_\lambda) - \pi(U_g)\|.$$

Therefore, $\lim_{n\to\infty} \|f_x(\pi_n) - f_x(\pi)\| \leqslant 2\varepsilon$. This shows the desired continuity.

THEOREM 1. *If G is an* MC-*group or a* C-*group, then there is no proper projection in* $C^*(G)$.

Proof. We need only to prove the theorem for G being an MC-group. Let π be a maximal element in $D(G, \mathscr{H})$ so that there is a path $\{\pi_t\}$ in $D(G, \mathscr{H})$, with $\pi_1 = \pi$ and $\pi_0 = 1_G$. Suppose P is a projection in $C^*(G)$. Then $\pi_0(P) = 0$ or 1. We may assume $\pi_0(P) = 0$, for otherwise, take $1 - P$ for P in the following argument. Now f_P is in $C(D)$. By the previous proposition $f_P(\pi_t) = \pi_t(P)$ is a norm continuous path of projections with $\pi_0(P) = 0$. This can happen only if $\pi_t(P) \equiv 0$ for all t in [0, 1]. Since π_1 is maximal, $P = 0$. This shows P is a trivial projection.

REMARK 3. The idea of this theorem originated from Cohen [4]. The proof we give here is similar to that of Choi [3].

We say that G is a C′-group (resp. MC′-group) if the corresponding "path connectedness" of 1_G and π' in the definition of a C-group (resp. MC-group) is replaced by "in the same connected component."

OBSERVATION. C′-groups and MC′-groups are the same.

Proof. One direction is obvious. We only prove that an MC′-group is a C′-group. Before showing this, we remark that Theorem 1 is still true for MC′-and C′-groups. This can be proved by a similar argument.

Suppose π is a maximal element in $D(G, \mathscr{H})$ which is in the same connected component of 1_G. Let σ be in $D(G, \mathscr{H})$. We show that $\sigma \oplus \pi$ is in the same connected component of 1_G in $D(G, \mathscr{H} \oplus \mathscr{H})$. In fact, since π is faithful on $C^*(G)$ and there is no proper projection in $C^*(G)$, the image of π contains no compact operators. Therefore, there are no compact operators in the images of $\sigma \oplus \pi$ and of $\pi \oplus \pi$. By Voiculescu's non-commutative Weyl-von Neumann Theorem $\pi \oplus \pi$ and $\sigma \oplus \pi$ are approximately equivalent. Consequently, $\pi \oplus \pi$ and $\sigma \oplus \pi$ are in the same connected component. Therefore, $\sigma \oplus \pi$ and 1_G are in the same connected component of $D(G, \mathscr{H} \oplus \mathscr{H})$. This shows the observation.

2. THE C-GROUPS

THEOREM 2. *If G is a* C-*group (or an* MC-*group), then its subgroups are* C-*groups (or* MC-*groups respectively).*

We only give the proof for G being a C-group. When G is an MC-group, the corresponding theorem can be proved similarly.

LEMMA 2. (Cf. [6]). *Let A be a C^*-subalgebra of the C^*-algebra B, and let π be a representation of A on $\mathscr{H}$. Then there is a representation π' of B on $\mathscr{H}'$ such that $\mathscr{H}'$ contains $\mathscr{H}$ and $(\pi' | A) | \mathscr{H} = \pi$. Moreover,* $\ker(\pi' | A) \subseteq \ker \pi$.

Proof. The first part of the lemma can be found in [6], and the second part is obvious.

LEMMA 3. (Cf. [19]). *Let H be a subgroup of G. Then $C^*(H)$ is naturally contained in $C^*(G)$. (Recall all groups considered are countable and discrete.)*

Proof of Theorem 2. Let π be a representation of $C^*(H)$. Then by Lemma 1, there is a π' in $D(G, \mathscr{H}')$ such that $(\pi' | C^*(H)) | \mathscr{H} = \pi$ *and* $\ker(\pi' | C^*(H)) \subseteq \ker \pi$. Since G is a C-group, there is a $\mathscr{H}''$ containing $\mathscr{H}'$, and a path $\{\pi'_t\}$ in $D(G, \mathscr{H}'')$ such that $\pi'_1 | \mathscr{H}' = \pi'$ and $\pi'_0 = 1_G$ on $\mathscr{H}''$.

On the other hand, $(\pi'_1 | C^*(H)) | \mathscr{H} = \pi$. Therefore, restricting $\{\pi'_t\}$ to the subgroup H, gives us the desired path. This proves the theorem.

PROPOSITION 2. *Let G be a free abelian group. Then G is a* C-*group.*

Proof. G is isomorphic to $\bigoplus_{i=1}^{\infty} \mathbf{Z} x_i$, where $\{x_i\}_{i=1}^{\infty}$ is a set of free generators of G. Let π be a representation of G on $\mathscr{H}$, and let M_π be the von Neumann algebra generated by the image of π. The unitary group U_π of M_π is connected, hence there is a path u_i^t connecting $\pi(x_i)$ and 1 in U_π for each i. Let $\pi_t(x_i) = u_i^t$, then since the x_j's are free generators, π_t extends to a representation of G for each t in $[0, 1]$, and $\{\pi_t\}$ is in fact a continuous family in $D(G, \mathscr{H})$ with $\pi_0 = 1_G$ and $\pi_1 = \pi$.

THEOREM 3. *If G and H are* C-*groups, so is the free product $G * H$ of G and H.*

Proof. Let π be a representation of $G * H$ on $\mathscr{H}$. Then $\pi | G$ and $\pi | H$ are representations of G and H, so there is $\mathscr{H}'$ containing $\mathscr{H}$, and paths $\{\sigma_t\}$ and $\{\gamma_t\}$ in $D(G, \mathscr{H}')$ and $D(H, \mathscr{H}')$ respectively, such that $\sigma_1 | \mathscr{H} = \pi | G$, $\sigma_0 = 1_G$, $\gamma_1 | \mathscr{H} = \pi | H$ and $\gamma_0 = 1_H$. Let $\sigma_t * \gamma_t$ be the representation of $G * H$ obtained by extending σ_t and γ_t. Then $\{\sigma_t * \gamma_t\}$ is a path in $D(G * H, \mathcal{H}')$ so that $\sigma_1 * \gamma_1 | \mathcal{H} = \pi$ and $\sigma_0 * \gamma_0 = 1_{G*H}$.

REMARK 4. This theorem extends Theorem 8 of [4] and Theorem 1 of [3].

THEOREM 4. *Let G be a C-group and H be a subgroup. Then the amalgamated free product $G *_H G$ is also a C-group.*

Proof. Denote $G_1 = G_2 = G$, and $H_1 = H_2 = H$. Let σ be the identity isomorphism of G_1 onto G_2, which carries H_1 onto H_2. Then $G_1 *_{H_1} G_2 = G *_H G$. Let π be in $D(G_1 *_{H_1} G_2, \mathcal{H})$, $\rho = \pi | G_1$ and $\gamma = \pi | G_2$. Let π' in $D(G_1 *_{H_1} G_2, \mathcal{H} \oplus \mathcal{H})$ be defined by

$$\pi'(g_1) = \begin{bmatrix} \rho(g_1) & 0 \\ 0 & \gamma(\sigma(g_1)) \end{bmatrix} \quad \text{for } g_1 \text{ in } G_1$$

and

$$\pi'(g_2) = \begin{bmatrix} \gamma(g_2) & 0 \\ 0 & \rho(\sigma^{-1}(g_2)) \end{bmatrix} \quad \text{for } g_2 \text{ in } G_2.$$

Then one checks that if h_1 is in H_1, then

$$\pi'(h_1) = \begin{bmatrix} \rho(h_1) & 0 \\ 0 & \gamma(\sigma(h_1)) \end{bmatrix} = \begin{bmatrix} \pi(h_1) & 0 \\ 0 & \pi(h_1) \end{bmatrix} = \pi'(\sigma(h_1))$$

since h_1 and $\sigma(h_1)$ are identified in $G_1 *_{H_1} G_2$ and $\gamma = \rho$ on $G_1 \cap G_2$. Therefore, π' extends to a representaton of $G_1 *_{H_1} G_2$. Let π'_t be defined as follows:

$$\pi'_t(g_1) = \pi'(g_1) \quad \text{for } g_1 \text{ in } G_1,$$

and

$$\pi'_t(g_2) = U_t^* \pi'(g_2) U_t \quad \text{for } g_2 \text{ in } G_2,$$

where $U_t = \begin{bmatrix} \cos\frac{\pi}{2} t & \sin\frac{\pi}{2} \cdot t \\ -\sin\frac{\pi}{2} t & \cos\frac{\pi}{2} \cdot t \end{bmatrix}$. One can easily check that for any $h_1 \in H_1$, $\pi'_t(h_1) = \pi'_t(\sigma(h_1))$, so that π'_t extends to a representation of $G_1 *_{H_1} G_2$ for each t in $[0,1]$. Moreover, $\{\pi'_t\}$ is a continuous path in $D(G_1 *_{H_1} G_2, \mathcal{H} \oplus \mathcal{H})$. Now $\pi'_1 = \pi'$ and

$$\pi'_0(g_1) = \begin{bmatrix} \rho(g_1) & 0 \\ 0 & \gamma(\sigma(g_1)) \end{bmatrix} \quad \text{for } g_1 \text{ in } G,$$

and

$$\pi_0'(g_2) = \begin{bmatrix} \rho(\sigma^{-1}(g_2)) & 0 \\ 0 & \gamma(g_2) \end{bmatrix} \quad \text{for } g_2 \text{ in } G.$$

Since G is a C-group, there is a Hilbert space $\mathscr{H}'$ containing $\mathscr{H} \oplus \mathscr{H}$, and a path $\{\alpha_t\}$ in $D(G, \mathscr{H}')$ such that $\alpha_1 | \mathscr{H} \oplus \mathscr{H} = \pi_0' | G_1$ and $\alpha_0 = 1_{G_1}$ on $\mathscr{H}'$. Now we define a path $\{\alpha_t'\}$ in $D(G_1 *_{H_1} G_2, \mathscr{H}')$ by

$$\alpha_t'(g_1) = \alpha_t(g_1) \quad \text{for } g_1 \text{ in } G_1$$

and

$$\alpha_t'(g_2) = \alpha_t(\sigma^{-1}(g_2)) \quad \text{for } g_2 \text{ in } G_2.$$

Since for $h_2 \in H_2$, $\alpha_t'(h_2) = \alpha_t(\sigma^{-1}(h_2)) = \alpha_t'(\sigma^{-1}(h_2))$, each α_t' extends to an element in $D(G_1 *_{H_1} G_2, \mathscr{H}')$. We then have $\alpha_1' | \mathscr{H} \oplus \mathscr{H} = \pi_0'$ and $\alpha_0' = 1_{G_1 *_{H_1} G_2}$ on $\mathscr{H}'$. The desired path in $D(G_1 *_{H_1} G_2, \mathscr{H}')$ for the original representation π in $D(G_1 *_{H_1} G_2, \mathscr{H})$ can be obtained by connecting the paths constructed above. This shows $G *_H G$ is a C-group.

COROLLARY 1. *W th the hypothesis in Theorem 4. The amalgamated free produc* $G *_H G *_H G *_H \ldots *_H G$ *is a C-group.*

The proof of this corollary can be made in one of two ways. One way is to repeat the procedure of that for Theorem 4. The other is to use induction and Theorem 2, while observing that $G *_H G *_H G$ is a subgroup of

$$(G *_H G) *_H (G *_H G) = G *_H G *_H G *_H G.$$

THEOREM 5. *Let* $\{G_i\}_{i=1}^n$ *be a finite collection of finitely generated free abelian groups,* $\{H_i\}_{i=1}^n$ *be corresponding isomorphic subgroups. Suppose there exist isomorphisms* $\rho_{ij}: H_i \to H_j$, *such that* $\rho_{ij}\rho_{jk} = \rho_{ik}$ *for all* i, j & k. *Then the amalgamated free product* $G_1 *_{H_1} G_2 *_{H_2} \ldots *_{H_{n-1}} G_n$ *is a C-group.*

To prove this theorem, we need a lemma, which is a simple exercise in abelian group theory.

LEMMA 4. *Let* G_i, H_i *and* ρ_{ij} *be as in Theorem 5 for* $i, j = 1, \ldots, n$. *Then there are decompositions* $H_i = \bigoplus_{j=1}^{m} \mathbf{Z}x_j^i$ *for some* $\{x_j^i\}_{j=1}^m$ *in* G_i *so that* ρ_{ik} *takes* x_j^i *to* x_j^k, *and* $G_i = \bigoplus_{j=1}^{m} \mathbf{Z}y_j^i \oplus N_i$, *with* $n_j^i y_j^i = x_j^i$ *for some integers* n_j^i *and for* $i = 1, \ldots, n$; $j = 1, \ldots, m$.

Proof of the Theorem. Let $G_i = \bigoplus_{j=1}^{m} \mathbf{Z}y_j^i \oplus N_i$, so that $H_i = \bigoplus_{j=1}^{m} \mathbf{Z}(n_j^i y_j^i)$, and $\rho_{ik}: H_i \to H_k$ via $\rho_{ik}: n_j^i y_j^i \to n_j^k y_j^k$. Let

$$G = \bigoplus_{k=1}^{m} \mathbf{Z}x_k \oplus N_1 \oplus N_2 \oplus \ldots \oplus N_n.$$

Then the G_i's can be considered as subgroups of G in the following way

$$G_i \simeq \bigoplus_{k=1}^{m} \mathbf{Z}(n_k^1 \ldots n_k^{i-1} \ldots n_k^{i+1} n_k^n x_k) \oplus N_i, \quad i = 1, 2, \ldots, n.$$

Then $H_i \cong \bigoplus_{k=1}^{m} \mathbf{Z}(n_k^1 \ldots n_k^n x_k) = H$ for all i. Moreover, ρ_{ij} becomes the identity map. Therefore, $G_1 *_{H_1} G_2 *_{H_2} \ldots *_{H_{n-1}} G_n$ is a subgroup of $G *_H G *_H \ldots *_H G$ here G appears n times. Combining Corollary 1 and Theorem 2 completes the proof of Theorem 5.

3. THE MC-GROUPS

PROPOSITION 3. *Let G and H be* MC-*groups, and let H be amenable. Then $G \oplus H$ is also an* MC-*group.*

Proof. Since H is amenable, $C^*(G \oplus H) \simeq C^*(G) \otimes_{\min} C^*(H)$ [12; 20]. Let π and σ be maximal elements of $D(G, \mathscr{H})$ and $D(H, \mathscr{H})$ respectively. Let $\{\pi_t\}$ and $\{\sigma_t\}$ be the corresponding paths for π and σ. Then $\{\pi_t \otimes \sigma_t\}$ is a path in $D(G \oplus H, \mathscr{H} \otimes \mathscr{H})$, so that $\pi_1 \otimes \sigma_1 = \pi \otimes \sigma$ and $\pi_0 \otimes \sigma_0 = 1_{G \oplus H}$. Moreover, $\pi \otimes \sigma$ is faithful [20]. This shows $G \oplus H$ is an MC-group.

PROPOSITION 4. *Let G be an* MC-*group, and φ de a periodic automorphism of G. Then the semidirect product $G \times_\varphi \mathbf{Z}$ is also an* MC-*group.*

Proof. Let π in $D(G, \mathscr{H})$ be a maximal element such that there is a path $\{\pi_t\}$ in $D(G, \mathscr{H})$ with $\pi_1 = \pi$ and $\pi_0 = 1_G$. Since $C^*(G \times_\varphi \mathbf{Z}) = C^*(G) \times_\varphi \mathbf{Z}$, the regular representation of $C^*(G) \times_\varphi \mathbf{Z}$ defined by

$$R_t(y)(\xi)(n) = \sum_{s=-\infty}^{\infty} \pi_t(\varphi_{-n}(y(s)))(\xi(n-s)),$$

where y is in $\ell^1(\mathbf{Z}, C^*(G))$ and ξ is in $\ell^2(\mathbf{Z}, \mathscr{H})$, extends to a representation of $C^*(G) \times_\varphi \mathbf{Z}$ for each t in $[0, 1]$. Moreover, R_1 is faithful [14]. Clearly, the induced representation R_t of $G \times_\varphi \mathbf{Z}$ is $R_t(g)\xi(n) = \pi_t(\varphi_{-n}(g))(\xi(n))$ for g in G, and $R_t(\varphi)\xi(n) = \xi(n-1)$

for φ corresponding to the generator of **Z**. Since $\{\varphi_{-n}(g)\}$ for a fixed g is a finite set, we see that $R_t(g)$ is actually continuous in norm. Now R_1 is maximal in $D(G \rtimes_\varphi \mathbf{Z}, \ell^2(\mathbf{Z}, \mathscr{H}))$ and R_0 is the representation pulled back from the regular representation λ of **Z** on $\ell^2(\mathbf{Z}, \mathscr{H}) = \ell^2(\mathbf{Z}) \otimes \mathscr{H}$. Choose a path $\{\sigma_t\}$ in $D(\mathbf{Z}, \ell^2(\mathbf{Z}))$ which connects λ to $1_\mathbf{Z}$, we then have a path $\{\sigma_t \otimes 1\}$ in $D(G \rtimes_\varphi \mathbf{Z}, \ell^2(\mathbf{Z}, \mathscr{H}))$ connecting R_0 to $1_{G \rtimes_\varphi \mathbf{Z}}$. Now combining the two paths $\{R_t\}$ and $\{\sigma' \otimes 1\}$ gives a desired path in $D(G \rtimes_\varphi \mathbf{Z}, \ell^2(\mathbf{Z}, \mathscr{H}))$.

REMARK 5. The above proof does not build up a large class of amenable groups. For instance, we cannot even obtain a similar result for all finitely generated torsion free nilpotent groups. However, the method of the proof can help us to build up a new class of groups.

DEFINITION 4. (Cf. [21]). Let G and N be two groups. The *restricted Wreath product* G wr N is the semidirect product $(\bigoplus_{n \in N} G_n) \times_\alpha N$, where $G_n = G$, for each n in N and $\alpha(n)$ acts on $\bigoplus_{n \in N} G_n \simeq G'$ as an automorphism defined by $\alpha(n)(\{g_m\}) = \{g_{m \cdot n}\}_{m \in N}$, for each n in N. In other words, $\alpha(n)$ translates the coordinates of an element in G'.

THEOREM 6. *If G and N are amenable* MC-*groups, so is G* wr *N.*

The proof of this theorem is based on the following construction.

Let $\mathscr{H}$ be a Hilbert space of infinite dimension, and $\mathscr{H}^0 = \mathscr{H} \oplus \mathbf{C}$. Let $\{N_i\}_{i=1}^\infty$ be increasing finite subsets of N so that $\bigcup N_i = N$. Let $V_i = \bigotimes_{n \in N_i} \mathscr{H}_n^0$ be the algebraic tensor product of the vector spaces $\{\mathscr{H}_n^0 = \mathscr{H}^0 \mid n \in N_i\}$ over **C**. We view V_i to be contained in V_j for $i < j$ as follows. Let $v = \bigotimes_{n \in N_i} (v_n, \lambda_n)$ in V_i, then $v \otimes (\bigotimes_{n \in N_j \setminus N_i} (0,1))$ is in V_j. Let $\mathscr{H}' = \bigcup_{i=1}^\infty V_i$ equipped with the inner product defined by

$$\langle v \mid w \rangle = \langle \bigotimes_{n \in N_i} (v_n, \lambda_n) \mid \bigotimes_{n \in N_i} (w_n, \beta_n) \rangle = \prod_{n \in N_i} (\langle v_n \mid w_n \rangle_{\mathscr{H}} + \lambda_n \bar{\beta}_n),$$

for v and w in V_i. It is clear that the inner product is well defined. We still denote $\mathscr{H}'$ for the completion of $\mathscr{H}'$ under the norm defined by the above inner product.

DEFINITION 5. $\mathscr{H}'$ obtained by the above construction is called the *restricted infinite tensor product* of $\mathscr{H}$.

Proof of Theorem 6. Let $\{\pi_t\}$ be a path in $D(G, \mathscr{H})$, such that π_1 is maximal and $\pi_0 = 1_G$ on $\mathscr{H}$. Extending π_t to $\mathscr{H} \oplus \mathbf{C}$ in the obvious way, we see that $(0,1)$ is an invariant vector for all $\{\pi_t(g) \mid t \in [0, 1] \text{ and } g \in G\}$. For each t in $[0, 1]$, let

π'_t be the representation of $G' = \bigoplus_{n \in N} G_n$ on $\mathscr{H}'$, the restricted infinite tensor product of $\mathscr{H}$, defined first by

$$\pi'_t(\{g_n\}_{n \in N})(\bigotimes_{n \in N_i}(v_n, \lambda_n)) = \bigotimes_{n \in N_i}(\pi_t(g_n)(v_n), \lambda_n)$$

on V_i then extend to $\mathscr{H}'$. It is clear that π'_t is a unitary representation of G' on $\mathscr{H}'$, for each t. Moreover, $\{\pi'_t\}$ is a continuous path in $D(G, \mathscr{H}')$, and π'_1 extends to a faithful representation of $C^*(G')$. This can be seen by an inductive argument on the subalgebra $C^*(G'_i)$, where $G'_i = \{\{g_n\}_{n \in N} = g' \in G' \text{ if } g_n \neq e, \text{ then } n \in N_i\}$. Since G'_i is contained in G', $C^*(G'_i)$ is naturally contained in $C^*(G')$, and $C^*(G')$ is the inductive limit of $\{C^*(G'_i)\}_{i=1}^{\infty}$. Note also that since G is amenable, the tensor products involved are unique.

Now let y be in $\ell^1(N, C^*(G'))$ and ξ be in $\ell^2(N, \mathscr{H}')$. The regular representation ρ_t of $\ell^1(N, C^*(G'))$ is given by $\rho_t(y)\xi(n) = \sum_{s \in N} \pi'_t(\alpha_{n^{-1}}(y(s)))(\xi(s^{-1}n))$. And ρ_1 extends to a faithful representation of $C^*(G') \times_\alpha N = C^*(G' \times_\alpha N)$ since π'_1 is faithful and N is amenable.

Let $(g'_k, s_0) \in G' \times_\alpha N$, where $g'_k = \{g_m\}_{m \in N}$ with $g_m = \begin{cases} e & \text{if } m \neq k \\ g & \text{if } m = k \end{cases}$, for a fixed k in N, and let $y_{g,s_0,k} \in \ell^1(N, C^*(G'))$ so that $y_{g,s_0,k}(s) = \begin{cases} 0 & \text{if } s \neq s_0 \\ g_k & \text{if } s = s_0 \end{cases}$. Then the corresponding unitary representation of $G \times_\alpha N$ is given by

$$\begin{aligned}
\tilde{\rho}_t(g'_k, s_0)\xi(n) &= \rho_t(y_{g,s_0,k})\xi(n) = \\
&= \sum_{s \in N_i} \pi'_t(\alpha_{n^{-1}}(y_{g,s_0,k}(s)))(\xi(s^{-1}n)) = \\
&= \pi'_t(\alpha_{n^{-1}}(\{g_m\}_{m \in N}))(\xi(s_0^{-1}n)) = \\
&= \pi'_t(\{g_{mn^{-1}}\}_{m \in N})(\xi(s_0^{-1}n))
\end{aligned}$$

Since $G' \times_\alpha N$ is generated by those (g'_k, s_0) described above, we need only to check the norm continuity of $\tilde{\rho}_t(g', s)$ for those (g'_k, s_0). In fact, if $\xi(n) = \bigotimes_{m \in N_i}(v^n_m, \lambda^n_m)$, we consider it to be in V_j for some sufficiently large $j > i$ so that $kn \in N_j$ and $s_0^{-1}N_i \subset \subset N_j$. Then

$$\begin{aligned}
\rho_t(y_{g,s_0,k})\xi(n) &= \pi'_t(\{g_{mn^{-1}}\}_{m \in N})(\xi(s_0^{-1}n)) = \\
&= \pi'_t(\{g_{mn^{-1}}\}_{m \in N})(\bigotimes_{m \in N_j}(v^{s_0^{-1}n}_m, \lambda^{s_0^{-1}n}_m)) = \\
&= \bigotimes_{m \in N_j}(\pi_t(g_{mn^{-1}})(v^{s_0^{-1}n}_m), \lambda^{s_0^{-1}n}_m) = \\
&= \bigotimes_{\substack{m \in N_j \\ m \neq kn}}(v^{s_0^{-1}n}_m, \lambda^{s_0^{-1}n}_m) \otimes (\pi_t(g)(v^{s_0^{-1}n}_{kn}), \lambda^{s_0^{-1}n}_{kn}).
\end{aligned}$$

Therefore,

$$\|\rho_t(y_{g,s_0,k})(\xi) - \rho_{t_0}(y_{g,s_0,k}(\xi))\|_{\ell^2}^2 =$$

$$= \sum_{n\in N} \|\pi_t'(\{g_{mn^{-1}}\}_{m\in N})(\xi(s_0^{-1}n)) - \pi_{t_0}'(\{g_{mn^{-1}}\}_{m\in N})(\xi(s_0^{-1}n))\|_{\mathscr{H}'}^2 =$$

$$= \sum_{n\in N} \prod_{\substack{m\in N_{j_n} \\ m\neq kn}} (\langle v_m^{s_0^{-1}n} \mid v_m^{s_0^{-1}n}\rangle_{\mathscr{H}} + |\lambda_m^{s_0^{-1}n}|^2)(\|(\pi_t(g) - \pi_{t_0}(g))(v_{kn}^{s_0^{-1}n})\|_{\mathscr{H}}^2) \leqslant$$

$$\leqslant \|\pi_t(g) - \pi_{t_0}(g)\|_{\mathscr{H}}^2 \cdot \|\xi\|_{\ell^2}^2.$$

This shows the desired continuity of ρ_t on $C^*(G') \times_\alpha N$. This continuity automatically gives the continuity of $\tilde{\rho}_t$ on $G' \times_\alpha N$.

Now arguing as the last paragraph in the proof of Theorem 6, we complete the proof.

4. THE NILPOTENT GROUP

It seems to us that the conditions (C) and (MC) for a group G is sometimes too strong to be obtained. For instance, the 2-step nilpotent groups already creates a difficulty for us to determine if they are C-groups or MC-groups. Therefore, a weaker version of the conditions (C) and (MC) is expected.

Inspired by the work of Anderson and Paschke [1], we are able to prove the generalized Kadison conjecture for any torsion free nilpotent groups without appealing the property (C) or (MC). For these groups, the Baum-Connes conjecture can also be proved, by the Pimsner-Voiculescu sequence [16], so that the gneralized Kadison conjecture follows to be true. However, the method we employed here is purely analytic, and is in the same spirit as Theorem 1. Moreover, more structure of the group C^*-algebras of these nilpotent groups can be seen through the course we are pursuing.

The following theorem is due to Anderson and Paschke for 2-step nilpotent groups [1]. We believe it is also known to them in general cases. However we will present a proof here for our purpose.

THEOREM 7. *Let* $0 \to C \to G \to N \to 0$ *be a central extension of groups. Suppose* N *is amenable. Then* $C^*(G)$ *is a continuous field of* C^**-algebras over the Pontrjagin dual* $\hat{C}$ *of* C.

Proof. Let τ be a character on C, t_τ be the maximal ideal of $C(\hat{C})$ at τ, and I_τ be the ideal of $C^*(G)$ generated by t_τ. Then the quotient C^*-algebra $C^*(G)/I_\tau$ is isomorphic to the twisted group C^*-algebra $C^*(N, \tau \circ \sigma)$, where σ is the normalized 2-cocycle determined by the groups extension [9; 22]. Let π_τ be the quotient map from $C^*(G)$ onto $C^*(N, \pi, \tau \circ \sigma)$. We will show

LEMMA 5. (Cf. [1]). i) *For any* $x \in C^*(G)$, $\|\pi_\tau(x)\|$ *is continuous in* τ.
ii) $\|x\| = \max_{\tau \in \hat{C}} \|\pi_\tau(x)\|$.

Proof of the Lemma. To prove i), we argue as in [1], but with some necessary changes. We need only to prove that for fixed x in $C^*(G)$, $O_x = \{\lambda \in \hat{C} \mid \|\pi_\lambda(x)\| < r\}$ is open for each $r > 0$; and $T_x = \{\lambda \in \hat{C} \mid \|\pi_\lambda(x)\| \leqslant r\}$ is closed for each $r \geqslant 0$. In fact, for $\varepsilon > 0$. If $\|\pi_{\lambda_0}(x)\| \leqslant r$, we let k_{λ_0} in I_{λ_0}, be such that $\|\pi_{\lambda_0}(x)\| \geqslant \geqslant \|x + k_{\lambda_0}\| - \varepsilon$. Suppose $k_{\lambda_0} = \sum_{i=1}^{l} Ag_i Ug_i + \delta_\varepsilon$, where $Ag_i \in t_{\lambda_0}$, and where $\|\delta_\varepsilon\| < \varepsilon$. Let $M > 0$ be such that $|\lambda - \lambda_0| < \varepsilon/M$, then $|Ag_i(\lambda) - Ag_i(\lambda_0)| < \varepsilon/l$ for $i = 1, 2, \ldots, l$. (Note: $\hat{C}$ is a compact metrizable space.) Now define χ_ε to be a function in $C(\hat{C})$ satisfying

$$\chi_\varepsilon(\rho) = \begin{cases} 1 & \text{if } |\rho - \lambda_0| \geqslant \varepsilon/M \\ 0 & \text{if } |\rho - \lambda_0| \geqslant \varepsilon/2M \end{cases} \quad \text{and} \quad \sup_{\rho \in \hat{C}} |\chi_\varepsilon(\rho)| = 1.$$

Then

$$\|\pi_{\lambda_0}(x)\| \geqslant \|x + k_{\lambda_0}\| - \varepsilon \geqslant \left\| x + \sum_{i=1}^{l} Ag_i Ug_i \right\| - 2\varepsilon =$$

$$= \left\| x + \sum_{i=1}^{l} \chi_\varepsilon Ag_i Ug_i + \sum_{i=1}^{l} (1 - \chi_\varepsilon) Ag_i Ug_i \right\| - 2\varepsilon \geqslant$$

$$\geqslant \left\| x + \sum_{i=1}^{l} x_\varepsilon Ag_i Ug_i \right\| - 3\varepsilon \geqslant$$

$$\geqslant \inf_{j_\lambda \in I_\lambda} \|x + j_\lambda\| - 3\varepsilon =$$

$$= \|\pi_\tau(x)\| - 3\varepsilon, \quad \text{for any } \lambda, \text{ such that } |\lambda - \lambda_0| < \varepsilon/2M.$$

Therefore, if ε was chosen so that $\|\pi_{\lambda_0}(x)\| + 3\varepsilon < r$, then $\|\pi_\lambda(x)\| < r$ for all λ in C such that $|\lambda - \lambda_0| < \varepsilon/2M$. This shows O_x in $\hat{C}$ is open. To show T_x is closed, we define a family of representations of $C^*(G)$. Let $s: N \to G$ be the normalized cross section (which defines the 2-cocycle) for the quotient map $q: G \to N$, and let $[g] =$ $= q(g)$ for g in G. For each τ in $\hat{C}$, τ defines a representation of C on $\mathbf{C}$. The induced representation [10; 13] $\mathrm{ind}_C^G\tau: G \to \ell^2(N)$ is defined as $\mathrm{ind}_C^G\tau(g)(\xi)(n) = \tau(s(n)\cdot g\cdot s([s(n)\cdot g])^{-1})\xi([g]n)$, here ξ is in $\ell^2(N)$. Clearly if g is in C, then $\mathrm{ind}_C^G\tau(g) = \tau(g)$. Hence, it factors through a representation π'_τ of $C^*(N, \tau\circ\sigma)$ on $\ell^2(N)$. It is well-known that π'_τ is faithful [9; 22]. Now one observes that $\mathrm{ind}_C^G\tau(x)$ is continuous in the strong operator topology for each x in $C^*(G)$. Since the set of operators of norm $\leqslant r$ is strong operator closed, T_x is closed. The second assertion (ii) in the lemma now can be proved exactly the same way as that for Theorem 1, part C) in [1].

Theorem 7 can now be proved by a straightforward verification.

COROLLARY 2. *Let $0 \to C \to G \to N \to 0$ be a central extension of groups. Suppose N is amenable and C is torsion free. If N satisfies the generalized Kadison conjecture, so does G.*

Proof: Let p be a projection in $C^*(G)$, and τ_0 be the trivial character of C. Then $\pi_{\tau_0}(p)$ is either 0 or 1. We may assume $\pi_{\tau_0}(p)=0$ as argued in the proof of Theorem 1. By Theorem 8, $\|\pi_\tau(p)\|$ is continuous. Since $\hat{C}$ is connected and $\|\pi_\tau(p)\| = 1$ or 0, we conclude $\|\pi_\tau(p)\| \equiv 0$ for all τ in $\hat{C}$. But since $\|p\| = \max\|\pi_\tau(p)\| = 0$, we must have $p = 0$. This shows the corollary.

THEOREM 8. *A torsion free nilpotent group satisfies the generalized Kadison conjecture.*

Proof. Let C be the center of G. It is well known that G/C is also torsion free Therefore, the theorem follows from an induction on the length of upper central series of G.

5. THE INDUCTIVE LIMITS AND AN EXAMPLE

Let $A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$ be a system of C^*-algebras, such that φ_i's are injective $*$-homomorphisms. It is well-known that if each A_i has no proper projection, neither does $A = \lim\limits_{n\to\infty} A_n$, the inductive limit of $\{A_n, \varphi_n\}$. Let $G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} \dots$ be a system of groups, such that φ_i's are injective homomorphisms. If each $C^*(G_i)$ (resp. $C^*_r(G_i)$) has no proper projection, neither does $\lim\limits_{n\to\infty} C^*(G_n) = C^*(\lim\limits_{n\to\infty} G_n)$ (resp. $\lim\limits_{n\to\infty} C^*_r(G_n) = C^*_r(\lim\limits_{n\to\infty} G_n)$). One may use this fact to build up a larger class of groups whose full group C^*-algebras have no proper projection. This class includes all locally nilpotent torsion free groups, and all free groups.

As we indicated in the beginning of § 4, the conditions (C) and (MC) are strong properties of a given group. Any group of torsion or of Kazdan's property (T) satisfy neither the property (C) nor (MC). We now produce an example of a finitely generated K-amenable torsion free group of no such properties.

Let G be the universal group generated by u and v, so that u and v satisfy the relation $uv = v^2u$. Then an easy spectral argument shows that G is not a C-group nor an MC-group, while G is still a finitely generated torsion free group. In fact, G is an HNN group and is K-amenable [15].

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