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APPROXIMATION IN THE CLASS OF QUASIDIAGONAL OPERATORS

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1. INTRODUCTION

1.0. Let $\mathscr{H}$ denote a complex, infinite dimensional, separable Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the set of bounded linear operators acting on $\mathscr{H}$. An operator T in $\mathscr{B}(\mathscr{H})$ is said to be *block-diagonal* (resp. *quasidiagonal)* if there is a sequence $\{P_n\}_{n=1}^{\infty} \subseteq \mathscr{P}\mathscr{F}(\mathscr{H})$, the set of finite rank projections, converging strongly to the identity such that $TP_n = P_nT$ for all $n \geqslant 1$ (resp. $\|TP_n - P_nT\| \to 0$ as $n \to \infty$). The set of block-diagonal (resp. quasidiagonal) operators is denoted by (BD) (resp. (QD)). It is well known that $\overline{(\mathrm{BD})} = (\mathrm{QD})$ and that moreover, given $T \in (\mathrm{QD})$ and $\varepsilon > 0$, there exists $T_0 \in (\mathrm{BD})$ and $K \in \mathscr{K}(\mathscr{H})$, the compact operators on $\mathscr{H}$, such that $\|K\| < \varepsilon$ and $T = T_0 + K$ (cf. [5]).

We shall also be considering the sets $(\mathrm{Nil}) = \{M \in \mathscr{B}(\mathscr{H})\colon M^k = 0$ for some $k \geqslant 1\}$, (BDN) = (Nil) ∩ (BD), and (QDN) = (Nil) ∩ (QD). It was in response to a question of P. R. Halmos [5, Problem 7] that the closure $\overline{(\mathrm{Nil})}$ of the set of nilpotent operators on $\mathscr{H}$ was given a spectral characterization in a series of papers culminating in [1]. In particular, it was shown that if $T \in \mathscr{B}(\mathscr{H})$ is quasinilpotent, then $T \in \overline{(\mathrm{Nil})}$. Recall that T is said to be *quasinilpotent* if its spectrum $\sigma(T) = \{0\}$; we denote the quasinilpotents by (Qnil).

In his dissertation [18], L. R. Williams posed the question: "Is $\overline{(\mathrm{BDN})} = (\mathrm{QD}) \cap \overline{(\mathrm{Nil})}$?" (It has been shown by Herrero that $\overline{(\mathrm{QDN})} = \overline{(\mathrm{BDN})}$, [8]). Clearly $\overline{(\mathrm{BDN})} \subseteq (\mathrm{QD}) \cap \overline{(\mathrm{Nil})}$. That there exists an operator L in $(\mathrm{QD}) \cap \overline{(\mathrm{Nil})}$ which cannot be approximated by elements of (BDN) is also a result of Herrero [7] where he exhibits a trace obstruction to approximations of this type. Investigation of this question led him to define a new class (ZTR) of "zero-trace" quasidiagonal operators as follows:

1.1. DEFINITION. A quasidiagonal operator T is in (ZTR) if $T \cong \bigoplus_{n=1}^{\infty} T_n$ where T_n is a finite dimensional operator such that $\mathrm{tr}(T_n) = 0$ for each $n \geqslant 1$, where $\mathrm{tr}(T_n)$ denotes the trace of T_n, and $\cong$ denotes unitary equivalence.

1.2. In this oaper we further investigate the class $\overline{(\mathrm{ZTR})}$, answering some of the questions raised in [7]. The question of whether or not quasidiagonal quasinilpotent operators are limits of block-diagonal nilpotents is still open, as is the greater question of completely characterizing $\overline{(\mathrm{BDN})}$. In § 3 we offer some partial results in this direction.

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2. ZERO TRACE QUASIDIAGONAL OPERATORS

2.0. As was mentioned above, the class (ZTR) was defined by D. A. Herrero in [7]. A useful tool in the investigation of this class is his function $q_{\mathrm{ztr}}\colon (\mathrm{QD}) \to \mathbf{R}^+$ which, in analogy with Apostol's modulus of quasitriangularity, we shall refer to as a trace modulus for quasidiagonals. We note that the class $\mathscr{P}\mathscr{F}(\mathscr{H})$ of finite rank projections is endowed with a natural partial order, i.e. $F_1 \leqslant F_2$ if and only if $\operatorname{ran} F_1 \subseteq \operatorname{ran} F_2$, thereby allowing us to consider limits in this class.

2.1. Definition. Given $T \in (\mathrm{QD})$, we define

$$q_{\mathrm{ztr}}(T) = \lim_{\varepsilon\to 0} \lim_{\substack{Q \in \mathscr{P}\mathscr{F}(\mathscr{H}) \\ Q \to I}} \left(\inf \left\{ \frac{|\operatorname{tr} PTP|}{\operatorname{rank} P} : P \in \mathscr{P}\mathscr{F}(\mathscr{H}),\ P \geqslant Q,\ \|PT - TP\| < \varepsilon \right\} \right).$$

2.2. It is not difficult to see that $q_{\mathrm{ztr}}(T) \leqslant \|T\|$. Using the function $q_{\mathrm{ztr}}(\cdot)$, Herrero obtained the following characterization of $\overline{(\mathrm{ZTR})}$.

2.3. Proposition [7, Proposition 2.5]. *The following are equivalent for* $T \in (\mathrm{QD})$:

(i) $T \in \overline{(\mathrm{ZTR})}$;

(ii) $T = A + K$ *for some* A *in* (ZTR) *and some compact* K;

(iii) *given* $\varepsilon > 0$, *there exists* $K_\varepsilon \in \mathscr{K}(\mathscr{H})$, *with* $\|K_\varepsilon\| < \varepsilon$ *such that* $A_\varepsilon = T - K_\varepsilon \in (\mathrm{ZTR})$;

(iv) $q_{\mathrm{ztr}}(T) = 0$.

2.4. The class $\overline{(\mathrm{ZTR})}$ is reasonably rich and contains the compact operators, any normal operator with 0 in the closure of its essential numerical range, and of course $(\mathrm{BDN}) \subseteq (\mathrm{ZTR})$ implies $\overline{(\mathrm{BDN})} \subseteq \overline{(\mathrm{ZTR})}$. Using the techniques of [7], the next lemma follows without much difficulty. We include the proof in order to make the paper as self-contained as possible.

2.5. LEMMA. *Let* $T \in (\mathrm{QD})$ *and* $K \in \mathcal{K}(\mathcal{H})$. *Then* $q_{ztr}(T + K) = q_{ztr}(T)$.

Proof. We shall demonstrate that $q_{ztr}(T + K) \leqslant q_{ztr}(T)$ for all $T \in (\mathrm{QD})$, $K \in \mathcal{K}(\mathcal{H})$. Then $q_{ztr}((T + K) - K) \leqslant q_{ztr}(T + K)$ and the result follows. To this end, choose T and K as above, let $\varepsilon > 0$, $\delta > 0$ and choose $Q \geqslant 0$, $Q \in \mathcal{PF}(\mathcal{H})$. We wish to find $P \in \mathcal{PF}(\mathcal{H})$, $P \geqslant Q$, such that $\|P(T + K) - (T + K)P\| < \varepsilon$ and

$$\frac{|\operatorname{tr} P(T + K)P|}{\operatorname{rank} P} < q_{ztr}(T) + \delta.$$

Let $0 < \varepsilon_1 < \min\left\{\frac{\varepsilon}{2}, \frac{\delta}{4}\right\}$. Since K is compact, we may first choose $Q_0 \geqslant Q$, $Q_0 \in \mathcal{PF}(\mathcal{H})$ such that $\|K - Q_0KQ_0\| < \varepsilon_1$ and then $Q_1 \geqslant Q_0$, $Q_1 \in \mathcal{PF}(\mathcal{H})$ such that

$$\frac{|\operatorname{tr} Q_0KQ_0|}{\operatorname{rank} Q_1} < \varepsilon_1.$$

By definition of $q_{ztr}(T)$, there now exists $P \geqslant Q_1$, $P \in \mathcal{PF}(\mathcal{H})$ with $\|PT - TP\| < \varepsilon_1$ and

$$\frac{|\operatorname{tr} PTP|}{\operatorname{rank} P} < q_{ztr}(T) + \frac{\delta}{2}.$$

Then

$$\|P(T + K) - (T + K)P\| \leqslant \|PT - TP\| + \|PK - KP\| \leqslant \varepsilon_1 + \varepsilon_1 < \varepsilon.$$

Also,

$$\frac{|\operatorname{tr} P(T + K)P|}{\operatorname{rank} P} \leqslant \frac{|\operatorname{tr} PTP|}{\operatorname{rank} P} + \frac{|\operatorname{tr} PKP|}{\operatorname{rank} P} \leqslant q_{ztr}(T) + \frac{\delta}{2} + \frac{|\operatorname{tr} PKP|}{\operatorname{rank} P}.$$

But

$$\frac{|\operatorname{tr} PKP|}{\operatorname{rank} P} \leqslant \frac{|\operatorname{tr} P(Q_0KQ_0)P|}{\operatorname{rank} P} + \frac{|\operatorname{tr} P(K - Q_0KQ_0)P|}{\operatorname{rank} P} \leqslant$$

$$\leqslant \frac{|\operatorname{tr} Q_0KQ_0|}{\operatorname{rank} P} + \varepsilon_1 \leqslant 2\varepsilon_1 < \frac{\delta}{2}.$$

Thus $\frac{|\operatorname{tr} P(T + K)P|}{\operatorname{rank} P} \leqslant q_{ztr}(T) + \delta$. Since $\delta > 0$, $\varepsilon > 0$ and $Q \geqslant 0$, $Q \in \mathcal{PF}(\mathcal{H})$ were arbitrary, we conclude that $q_{ztr}(T + K) \leqslant q_{ztr}(T)$, which we showed to be sufficient to prove our claim. ▨

2.6. It is known [7] that $(\mathrm{QD}) = \overline{(\mathrm{ZTR})} + \mathbf{C}I$. However, given $T \in (\mathrm{QD})$, the scalar $\lambda \in \mathbf{C}$ and $R \in \overline{(\mathrm{ZTR})}$ in the equation $T = R + \lambda I$ need not be unique. For example, if N is normal, then $N - \lambda I$ is in $\overline{(\mathrm{ZTR})}$ whenever λ is an element of the convex hull of the essential spectrum $\sigma(N)$ of N. We are thus led to the following definitions.

2.7. DEFINITION. For $T \in (\mathrm{QD})$, we define $\tau(T) = \{\lambda \in \mathbf{C} : T - \lambda I \in \overline{(\mathrm{ZTR})}\}$. Also, we define $\eta(T) = \inf\{|\lambda| : \lambda \in \tau(T)\}$.

2.8. Let $\lambda \in \tau(T)$. Then $T - \lambda I \in \overline{(\mathrm{ZTR})}$ and so by Proposition 2.3, given $\varepsilon > 0$ we can find K_ε compact such that $\|K_\varepsilon\| \leqslant \varepsilon$ and $Z_\varepsilon = (T - \lambda I) - K_\varepsilon \in (\mathrm{ZTR})$. Thus $Z_\varepsilon = \bigoplus_{n=1}^{\infty} (Z_\varepsilon)_n$ with $\mathrm{tr}(Z_\varepsilon)_n = 0$ for all $n \geqslant 1$. Since $T = (Z_\varepsilon + \lambda I) + K_\varepsilon$, we can write $T = R_\varepsilon + K_\varepsilon$, where $R_\varepsilon = \bigoplus_{n=1}^{\infty} (R_\varepsilon)_n$, $(R_\varepsilon)_n$ acting on $\mathscr{H}_n$, $\dim \mathscr{H}_n < \infty$ and $\dfrac{\mathrm{tr}(R_\varepsilon)_n}{\dim \mathscr{H}_n} = \lambda$ for all $n \geqslant 1$. Thus it is not unreasonable to think of $\tau(T)$ as the set of all possible "normalized traces" of T. Note that since $\overline{(\mathrm{ZTR})}$ is obviously closed, it follows that $\tau(T)$ is as well.

Problem 2.7 of [7] asks whether $\mathrm{dist}[T, (\mathrm{ZTR})] = q_{\mathrm{ztr}}(T)$ for all T in (QD). The answer is no. We shall provide a counterexample (§ 2.12) and demonstrate that in fact, $q_{\mathrm{ztr}}(T) = \eta(T)$. Thus $q_{\mathrm{ztr}}(T)$ measures the norm of the smallest *scalar* perturbation of T which lies in $\overline{(\mathrm{ZTR})}$, or again, measures the modulus of the smallest normalized trace of T. As such, we cannot expect $q_{\mathrm{ztr}}(\cdot)$ to be linear. However,, the following result, while easy to prove, is nonetheless useful.

2.9. LEMMA. *For* $T \in (\mathrm{QD})$ *and* $\lambda \in \mathbf{C}$ *we have*

$$q_{\mathrm{ztr}}(T) - |\lambda| \leqslant q_{\mathrm{ztr}}(T + \lambda I) \leqslant q_{\mathrm{ztr}}(T) + |\lambda|.$$

Proof. Simply use the same projections P which implement the value of $q_{\mathrm{ztr}}(T)$ in the estimate of $\dfrac{|\mathrm{tr}\, P(T + \lambda I)P|}{\mathrm{rank}\, P}$. ∎

2.10. LEMMA. *For* $T \in (\mathrm{QD})$, $\mathrm{dist}(T, (\mathrm{ZTR})) \leqslant q_{\mathrm{ztr}}(T) \leqslant \eta(T)$.

Proof. Assume $\lambda \in \tau(T)$. Then by Proposition 2.3 and the above lemma, $q_{\mathrm{ztr}}(T) - |\lambda| \leqslant q_{\mathrm{ztr}}(T - \lambda I) = 0$. Thus $q_{\mathrm{ztr}}(T) \leqslant |\lambda|$, and so $q_{\mathrm{ztr}}(T) \leqslant \eta(T)$. That $\mathrm{dist}(T, (\mathrm{ZTR})) \leqslant q_{\mathrm{ztr}}(T)$ is [7, Lemma 2.3]. ∎

2.11. THEOREM. *For* $T \in (\mathrm{QD})$, $q_{\mathrm{ztr}}(T) = \eta(T)$.

Proof. Let

$$q = q_{\mathrm{ztr}}(T) = \lim_{\varepsilon \to 0} \lim_{\substack{Q \in \mathscr{PF}(\mathscr{H}) \\ Q \to I}} \left(\inf \left\{ \frac{|\mathrm{tr}\, PTP|}{\mathrm{rank}\, P} : P \in \mathscr{PF}(\mathscr{H}), P \geqslant Q, \|PT - TP\| < \varepsilon \right\} \right).$$

For $\varepsilon > 0$ and $Q \in \mathscr{PF}(\mathscr{H})$, $Q \geqslant 0$, we define

$$\Gamma(T, \varepsilon, Q) = \left\{\lambda : |\lambda| = q_{\mathrm{ztr}}(T) \text{ and } \inf\left\{\left|\frac{\operatorname{tr} PTP}{\operatorname{rank} P} - \lambda\right| : P \geqslant Q, \|PT - TP\| < \varepsilon\right\} = 0\right\}.$$

By definition of $q_{\mathrm{ztr}}(T)$, $\Gamma(T, \varepsilon, Q) \neq \emptyset$ for all such choices of ε and Q. Moreover, a simple diagonal argument shows that $\Gamma(T, \varepsilon, Q)$ is in fact closed and hence compact for all ε and Q. We may partially order these sets by inclusion. Note that the now directed family $\{\Gamma(T, \varepsilon, Q) : \varepsilon > 0, 0 \leqslant Q \in \mathscr{PF}(\mathscr{H})\}$ is easily seen to possess the Finite Intersection Property, and so

$$\Gamma(T) = \bigcap_{\substack{\varepsilon > 0 \\ Q \geqslant 0}} \Gamma(T, \varepsilon, Q) \neq \emptyset.$$

Let $e^{i\gamma}q \in \Gamma(T)$ for the appropriate $0 \leqslant \gamma < 2\pi$ and consider $R = T - (e^{i\gamma}q)I$. Let $\varepsilon > 0$ and choose $0 \leqslant Q \in \mathscr{PF}(\mathscr{H})$. From the definition of $\Gamma(T, \varepsilon, Q)$ we can obtain a sequence $\{P_n\}_{n=1}^{\infty}$ such that $P_n \geqslant Q$, $\|P_nT - TP_n\| < \varepsilon$ and $\left|\frac{\operatorname{tr} P_nTP_n}{\operatorname{rank} P_n} - e^{i\gamma}q\right| \leqslant \frac{1}{n}$. Then note that $\|P_nR - RP_n\| < \varepsilon$ and

$$\frac{|\operatorname{tr} P_nRP_n|}{\operatorname{rank} P_n} = \left|\frac{\operatorname{tr} P_nTP_n}{\operatorname{rank} P_n} - \frac{\operatorname{tr} P_n(e^{i\gamma}qI)P_n}{\operatorname{rank} P_n}\right| = \left|\frac{\operatorname{tr} P_nTP_n}{\operatorname{rank} P_n} - e^{i\gamma}q\right| \leqslant \frac{1}{n}.$$

Thus $q_{\mathrm{ztr}}(R) = 0$, and so $e^{i\gamma}q \in \tau(T)$; that is, $\eta(T) \leqslant |e^{i\gamma}q| = q = q_{\mathrm{ztr}}(T)$. Combining this with the above lemma, $\eta(T) = q_{\mathrm{ztr}}(T)$. ▨

2.12. THE COUNTEREXAMPLE. We now provide an example of an operator T in (QD) for which dist(T, (ZTR)) is strictly less than $q_{\mathrm{ztr}}(T)$.

Consider the binormal operator $T = \begin{bmatrix} I & \frac{1}{2}I \\ 0 & 0 \end{bmatrix}$ acting in $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. We first wish to determine $q_{\mathrm{ztr}}(T)$. Let $0 < \varepsilon < 1/70\sqrt{5}$ and let $Q \geqslant 0$ be a finite rank projection. If $P \geqslant Q$ is a finite rank projection such that $\|TP - PT\| < \varepsilon$, then precisely the same kind of analysis as in [7, § 3] reveals that there exists a projection X_0 on $\mathscr{H}$ such that

$$P_0 = \begin{bmatrix} X_0 & 0 \\ 0 & X_0 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$$

satisfies $\|P - P_0\| < 7\varepsilon < 1$. Thus $\operatorname{rank} P = \operatorname{rank} P_0 = 2 \operatorname{rank} X_0$. Then

$$\frac{\big|\,|\operatorname{tr} PTP| - |\operatorname{tr} P_0TP_0|\,\big|}{\operatorname{rank} P} \leqslant \frac{|\operatorname{tr} PTP - \operatorname{tr} P_0TP_0|}{\operatorname{rank} P} \leqslant$$

$$\leqslant \frac{|\operatorname{tr}(P - P_0)TP| + |\operatorname{tr} P_0T(P - P_0)|}{\operatorname{rank} P} \leqslant 7\|T\|\varepsilon + 7\|T\|\varepsilon.$$

(*Note.* This last step follows from the fact that both $(P - P_0)TP$ and $P_0T(P - P_0)$ have rank less than or equal to $\operatorname{rank} P$ and norm less than or equal to $7\|T\|\varepsilon$.) Thus

$$\frac{\big|\,|\operatorname{tr} PTP| - |\operatorname{tr} P_0TP_0|\,\big|}{\operatorname{rank} P} \leqslant 7\sqrt{5}\,\varepsilon.$$

Moreover, $P_0TP_0 = \begin{bmatrix} X_0 & \frac{1}{2} X_0 \\ 0 & 0 \end{bmatrix}$. Thus $\dfrac{|\operatorname{tr} P_0TP_0|}{\operatorname{rank} P} = \dfrac{\operatorname{rank} X_0}{\operatorname{rank} P} = \dfrac{1}{2}$. In particular then, $\dfrac{|\operatorname{tr} PTP|}{\operatorname{rank} P} \geqslant \dfrac{1}{2} - (7\sqrt{5})\varepsilon \geqslant \dfrac{2}{5}$. It follows that $q_{ztr}(T) \geqslant \dfrac{2}{5}$. In fact, it is obvious that by letting ε approach zero we obtain $q_{ztr}(T) \geqslant \dfrac{1}{2}$. Since $\left(T - \dfrac{1}{2} I\right) =$

$$= \begin{bmatrix} \frac{1}{2} I & \frac{1}{2} I \\ 0 & -\frac{1}{2} I \end{bmatrix}$$

is obviously in (ZTR), $q(T) \leqslant \dfrac{1}{2}$ so that $q_{ztr}(T) = \dfrac{1}{2}$.

As for dist$(T$, (ZTR)), let $R = \begin{bmatrix} 0 & \frac{1}{4} I \\ -\frac{1}{4} I & 0 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. Then $N =$

$$= T - R = \begin{bmatrix} I & \frac{1}{4} I \\ \frac{1}{4} I & 0 \end{bmatrix}$$

is self-adjoint with spectrum $\sigma(N) = \sigma_e(N) =$ $= \{(1 + \sqrt{2})/2, (1 - \sqrt{2})/2\}$. From the comments before Definition 2.6 we conclude that $N \in \overline{(\mathrm{ZTR})}$ as 0 is in the convex hull of $\sigma_e(N)$. But then dist$(T, (\mathrm{ZTR})) \leqslant$ $\leqslant \|T - N\| = \|R\| = 1/4 < q_{ztr}(T)$.

As pointed out by the referee, if we fix $0 < \delta < 1$ and set $T = \begin{bmatrix} I & \delta I \\ 0 & 0 \end{bmatrix}$, then the same analysis with $\varepsilon < \dfrac{\delta}{4}$ and $R = \begin{bmatrix} 0 & \frac{\delta}{2} I \\ -\frac{\delta}{2} I & 0 \end{bmatrix}$ yields $q_{ztr}(T) = \dfrac{1}{2}$ but dist$(T, (\mathrm{ZTR})) \leqslant \delta/2$. This shows that the two quantities are not comparable.

2.13. Still another question from [7] asks whether or not $\tau(T)$ is convex for all quasidiagonal operators T. No counterexample is known. However, note that if $q_{ztr}(T) = q \neq 0$, then either there exists a unique $0 \leqslant \theta < 2\pi$ such that $\Gamma(T) = \{e^{i\theta}q\}$, or $\tau(T)$ is not convex. This is because the average of any two such $e^{i\theta_1}q$ and $e^{i\theta_2}q$ has modulus less than $q = \eta(T)$. Proposition 2.16 provides an example of a large class of quasidiagonal operators for which the set of normalized traces is convex. First we need a definition.

2.14. DEFINITION (cf. [4]). We define the class $(\mathrm{QD})_\infty = \{0_m \oplus B^{(\infty)} + K : 0_m$ is the zero operator on a space of dimension m $(0 \leqslant m < \infty)$, $B \in (\mathrm{QD})$, $K \in \mathscr{K}(\mathscr{H})\}^-$.

2.15. The class $(\mathrm{QD})_\infty$ is obviously invariant under compact perturbations and is known to contain the class $(\mathrm{QD})_{\mathrm{nor}}$ of quasidiagonal operators which are limits of algebraically n-normal operators (for potentially different n's); i.e. are limits of operators T which can be written as $n \times n$ operator matrices with commuting normal entries. Armed with these definitions we obtain the following..

2.16. PROPOSITION. *Let $T \in (\mathrm{QD})_\infty$. Then $\tau(T)$ is convex.*

Proof. Clearly it suffices to consider only the case where $T = 0_m \oplus B^{(\infty)} + K$ as above. In fact, since $q_{ztr}(T - \lambda I) = q_{ztr}(T - \lambda I - K)$ for all $\lambda \in \mathbf{C}$, we may assume $T = 0_m \oplus B^{(\infty)}$. Moreover, since $m < \infty$ it is easy to verify that $q_{ztr}(T - \lambda I) = q_{ztr}(B^{(\infty)} - \lambda I)$ for all $\lambda \in \mathbf{C}$. Thus we may assume that $T = B^{(\infty)} = \bigoplus_{n=1}^{\infty} B_n$, where $B_n \cong B$ acts on $\mathscr{H}_n \cong \mathscr{H}$ for each $n \geqslant 1$. For each $k \geqslant 1$, we let R_k be the orthogonal projection onto the space $\bigoplus_{n=1}^{k} \mathscr{H}_n$.

Now assume that λ_1 and λ_2 lie in $\tau(T)$. Since this latter set is closed, it will suffice to show that for all integers $0 < a < b$, $\lambda = \left(\frac{a}{b}\right)\lambda_1 + \left(\frac{b-a}{b}\right)\lambda_2 \in \tau(T)$. To this end, let $\varepsilon > 0$, $\delta > 0$ and choose $Q \geqslant 0$ in $\mathscr{P}\mathscr{F}(\mathscr{H})$. (As in [7], it suffices to consider the case where $\operatorname{ran} Q \subseteq \operatorname{ran} R_N$ for some $N > 0$.) Then for $i = 1, 2$ we can find $P_i \geqslant Q$ in $\mathscr{P}\mathscr{F}(\mathscr{H})$ such that $\operatorname{ran} P_i \subseteq \operatorname{ran} R_{M_i}$ (for some sufficiently large $M_i > 0$), $\|P_i T - T P_i\| < \delta$ and $\left| \frac{\operatorname{tr} P_i T P_i}{\operatorname{rank} P_i} - \lambda_i \right| < \varepsilon$.

Let $M = \max(M_1, M_2)$ and set $E_i = \bigoplus_{j=(i-1)M+1}^{iM} \mathscr{H}_j$, $i \geqslant 1$. Let $p_1 = \operatorname{rank} P_1$, $p_2 = \operatorname{rank} P_2$ and set $k_1 = a p_2$, $k_2 = (b - a) p_1$. Then

$$\frac{k_1 p_1}{k_1 p_1 + k_2 p_2} = \frac{a}{b} \quad \text{and} \quad \frac{k_2 p_2}{k_1 p_1 + k_2 p_2} = \frac{(b-a)}{b}.$$

Take k_1 copies of P_1 acting on the spaces $E_1, E_2, \ldots, E_{k_1}$ and k_2 copies of P_2 acting on the spaces $E_{k_1+1}, \ldots, E_{k_1+k_2}$. Finally, let $P = P_1^{(k_1)} \oplus P_2^{(k_2)}$ with respect to $\bigoplus_{i=1}^{k_1+k_2} E_i$. Then $P \geqslant Q$ and $\|PT - TP\| < \delta$. Furthermore,

$$\left|\frac{\operatorname{tr} PTP}{\operatorname{rank} P} - \lambda\right| = \left|\frac{k_1 \operatorname{tr}(P_1TP_1) + k_2 \operatorname{tr}(P_2TP_2)}{k_1p_1 + k_2p_2} - \left(\frac{a}{b}\right)\lambda_1 - \left(\frac{b-a}{b}\right)\lambda_2\right| \leqslant$$

$$\leqslant \frac{k_1p_1}{k_1p_1 + k_2p_2}\left|\frac{\operatorname{tr} P_1TP_1}{p_1} - \lambda_1\right| + \frac{k_2p_2}{k_1p_1 + k_2p_2}\left|\frac{\operatorname{tr} P_2TP_2}{p_2} - \lambda_2\right| \leqslant$$

$$\leqslant \left(\frac{a}{b}\right)\varepsilon + \left(\frac{b-a}{b}\right)\varepsilon = \varepsilon.$$

We conclude that $\lambda \in \tau(T)$, and hence that $\tau(T)$ is convex. ∎

2.17. A potential candidate for a quasidiagonal operator T for which $\tau(T)$ is not convex is based on the following result of Herrero and Szarek [11]. They showed that there exists a universal constant $c > 0$ such that for all dimensions $n \geqslant 2$, one can find nilpotents M_n (of order at most 5, in fact) such that $\operatorname{dist}[M_n, \operatorname{Red}(L(\mathbf{C}^n))] > c$, where $\operatorname{Red}(L(\mathbf{C}^n))$ denotes the set of orthogonally reducing matrices in $L(\mathbf{C}^n)$. Clearly it is possible to inductively choose a sequence $\{k(n)\}_{n=1}^{\infty}$ of positive integers satisfying $k(n) > 2\sum_{i=1}^{n-1} k(i)$. Consider the Herrero-Szarek nilpotents $M_{k(n)} \in L(\mathbf{C}^{k(n)})$ and let $Q_n = (-1)^n I_{k(n)} + M_{k(n)}$, for $n \geqslant 1$. Finally, let $Q = \bigoplus_{n=1}^{\infty} Q_n$. It is not hard to see that $\{1, -1\} \subseteq \tau(Q)$. To show that $0 \notin \tau(Q)$ (i.e. $q_{z\mathrm{tr}}(Q) \neq 0$ and $\tau(Q)$ is not convex), it would suffice to prove that any finite rank projection P "almost commuting" with $Q_0 = \bigoplus_{n=1}^{\infty} Q_{2n-1}$ (i.e. $\|PQ_0 - Q_0P\|$ is small) must have $\operatorname{rank} P \in \{\sum k(2j-1) : j \in F \subset \mathbf{N},\ F \text{ finite}\}$, and that an analogous statement holds true for $Q_e = \bigoplus_{n=1}^{\infty} Q_{2n}$. On the one hand, it can be shown that there do exist $P \in \mathscr{P}\mathscr{F}(\mathscr{H})$ almost commuting with Q_0 which are far away from the obvious commuting projections, these latter being the ones that project onto finitely many of the $\mathbf{C}^{k(n)}$'s. On the other hand, the ones so far constructed have "nice" rank. Moreover, Szarek [13] has exhibited a nilpotent operator $T = \bigoplus_{n=1}^{\infty} T_n$ of a form similar to that of Q and a constant $\varepsilon > 0$ for which $\sup_m \|P_mT - TP_m\| > \varepsilon$ for all sequences $\{P_m\}_{m=1}^{\infty} \subseteq \mathscr{P}\mathscr{F}(\mathscr{H})$ satisfying $P_m \uparrow I$ strongly and $\sup_m \operatorname{rank}(P_m - P_{m-1})$ is bounded. This nilpotent is therefore in $(\mathrm{QD})\setminus(\mathrm{QD})_{\mathrm{nor}}$. However it is not known at present whether $T \in (\mathrm{QD})_{\infty}$.

A proof that a counterexample of this form to the question of the convexity of $\tau(T)$, $T \in$ (QD) does exist, or indeed, that one does not exist, would also shed light on an interesting problem of approximation by block-diagonal nilpotents, which forms the subject of the next section. The interested reader is referred to [11], [13], [4] and their references for more details.

3. BLOCK-DIAGONAL NILPOTENTS

3.0. As mentioned in § 1, the question of determining whether $Q \in (\mathrm{QD}) \cap \cap (\mathrm{Qnil})$ implies $Q \in \overline{(\mathrm{BDN})}$ is open, as is the question of characterizing $\overline{(\mathrm{BDN})}$. One direction taken in [8] and [4] is to ask whether given $Q \in (\mathrm{QD}) \cap (\mathrm{Qnil})$ and the canonical map $\pi\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, does $C^*(\pi(Q))$ admit a faithful unital representation ρ onto a space $\mathscr{H}_\rho$ such that $\rho(\pi(Q))$ is quasidiagonal in $\mathscr{B}(\mathscr{H}_\rho)$? If this were always the case, then by using Voiculescu's Theorem [14], one could show that given $\varepsilon > 0$, there exists a finite dimensional nilpotent operator M and a unitary operator U (M and U depending on ε) such that $\|Q - U^*(Q \oplus \oplus M^{(\infty)})U\| < \varepsilon$. Herrero [8, Theorem 2.5] has shown that this latter condition is sufficient to deduce that $Q \in \overline{(\mathrm{BDN})}$.

However, Szarek's example $T = \bigoplus_{n=1}^{\infty} T_n$ of a block-diagonal nilpotent which is not in $(\mathrm{QD})_{\mathrm{nor}}$ suggests that such a representation is not absolutely necessary to the proof that $Q \in \overline{(\mathrm{BDN})}$, and also serves as a candidate for a $Q \in (\mathrm{QD}) \cap (\mathrm{Qnil})$ for which such a representation might not exist! The results of this section provide more evidence that such representations might not always exist, as they are not necessary to any of our proofs. We begin with the following.

3.1. DEFINITION. An operator T in (QD) is said to belong to the class (DSN) if $T \cong \bigoplus_{n=1}^{\infty} T_n$, where each T_n is a finite dimensional nilpotent operator. T is said to be *a direct sum of nilpotents*.

3.2. REMARK. (BDN) $\not\subseteq$ (DSN) $\not\subseteq$ (ZTR). Moreover, given $Q \in (\mathrm{QD}) \cap \cap (\mathrm{Qnil})$, Herrero [8] has shown that given $\varepsilon > 0$, there exists $Q' \in (\mathrm{DSN}) \cap \cap (\mathrm{Qnil})$ with $\|Q - Q'\| < \varepsilon$.

3.3. For use in the next two propositions, we record here the following result which is essentialy Lemma 2.3 of [8]. We require a mildly stronger formulation which falls out immediately from the proof of that lemma. The proof, while not included in that paper, in turn follows directly from Lemma 5.10 of [9] and Lemma 2.2 of [8].

3.4. LEMMA. *Let $T \in L(\mathbf{C}^d)$, $0 < d < \infty$ and let $k > 1$. Assume that $R \in L(\mathbf{C}^a)$, $0 < a < \infty$ is a nilpotent of order m such that* $\operatorname{nul} R > (k-1)d$. *Let* $\alpha > 0$ *and* $\beta > \operatorname{spr}(T)$, *the spectral radius of* T. *Then there exists a nilpotent* $M \in L(\mathbf{C}^d \oplus \mathbf{C}^a)$ *of order at most* $k + m - 1$ *such that*

$$\|T \oplus R - M\| \leqslant 2\alpha\sqrt{2}\,\|T\| + (1 + 2\alpha\sqrt{2})\left\{\alpha\|T\| + \beta + \frac{\|T^k\|}{\alpha\beta^{k-1}}\right\}.$$

3.5. REMARK. As noted in [8], for a proper choice of α and β, we obtain

$$\|T \oplus R - M\| \leqslant 4\,[\|T \oplus R\|\,\|T^k\|]^{1/k+1}.$$

3.6. PROPOSITION. *Let* $Q \in (\mathrm{QD}) \cap (\mathrm{Qnil})$ *and* $B \in \overline{(\mathrm{BDN})}$. *Then* $Q \oplus B \in \in \overline{(\mathrm{BDN})}$.

Proof. By Remark 3.2 we may assume $Q = \bigoplus_{n=1}^{\infty} Q_n \in (\mathrm{DSN}) \cap (\mathrm{Qnil})$ and $B = \bigoplus_{n=1}^{\infty} B_n \in (\mathrm{BDN})$, $B^m = 0$ for some $m \geqslant 1$. We assume Q_n acts on $\mathscr{H}_n \simeq \mathbf{C}^{d(n)}$ for each $n \geqslant 1$. From the spectral radius formula, we know that $\lim_{k\to\infty} \|Q^k\|^{1/k} = 0$. In what follows, it may help to think of k as arbitrary but temporarily fixed with respect to n.

For each $k \geqslant 2$, we define a function $p_k : \mathbf{N} \to \mathbf{N}$ such that $p_k(n) = (k-1)\,d(n)$, $p_k(0) = 0$ and let $R_{(n,k)} = \bigoplus_{i=p_k(n-1)+1}^{p_k(n-1)+p_k(n)} B_i$ for $n \geqslant 1$. Then we may rewrite $Q \oplus B \simeq \bigoplus_{n=1}^{\infty} (Q_n R_{(n,k)})$ for each $k \geqslant 2$. By Lemma 3.4 and Remark 3.5 we deduce that $\|Q_n \oplus R_{(n,k)} - M_{(n,k)}\| \leqslant 4(\|Q \oplus B\|\,\|Q^k\|)^{1/k+1}$ for a suitably chosen $M_{(n,k)}$ nilpotent of order at most $(k + m - 1)$ acting upon the same space that $Q_n \oplus R_{(n,k)}$ acts upon.

Let $M_k = \bigoplus_{n=1}^{\infty} M_{(n,k)}$. Then M_k is clearly a block-diagonal nilpotent of order at most $k + m - 1$ and

$$\|Q \oplus B - M_k\| \leqslant \sup_n \|Q_n \oplus R_{(n,k)} - M_{(n,k)}\| \leqslant 4(\|Q \oplus B\|\,\|Q^k\|)^{1/k+1}.$$

Since the right hand side of this equation converges to zero as k tends to infinity, $Q \oplus B \in \overline{(\mathrm{BDN})}$. ∎

3.7. *Note*. Although the proof of this proposition is very in flavour to that of Lemma 2.5 of [8], the conclusion is very different. For here one no longer needs infinitely many blocks B_n of B as above to look the same, and perhaps most significantly, *the size of the blocks B_n no longer needs to have an upper bound*! In particular, if $Q \in (\mathrm{QD}) \cap (\mathrm{Qnil})$ and T is Szarek's nilpotent sitting in $(\mathrm{QD})\setminus(\mathrm{QD})_{\mathrm{nor}}$, then $Q \oplus T \in \overline{(\mathrm{BDN})}$. We shall see another occurence of this type of phenomenon in Proposition 3.12.

3.8. Notation. Let $(\mathrm{Nil})_k = \{M \in (\mathrm{Nil}) : M^k = 0\}$ and $(\mathrm{BDN})_k = (\mathrm{BD}) \cap (\mathrm{Nil})_k$.

3.9. Proposition. *Let* $B = \bigoplus_{n=1}^{\infty} B_n \in (\mathrm{DSN})$ *and fix* $k \geqslant 1$ *an integer. Assume* $\lim_{n\to\infty} \|B_n^k\|^{1/k} = 0$. *Then* $B \in \overline{(\mathrm{BDN})}$.

Proof. The condition implies that B is essentially nilpotent of order k. A quantitative version of Olsen's Theorem [12] as found in [6, Corollary 7.6] shows that given $\varepsilon > 0$ we can find K compact, $\|K\| < \varepsilon$ such that $(B + K)$ is nilpotent of order k, and as such is in $(\mathrm{QD}) \cap (\mathrm{Nil})_k$. From [8] we deduce that $(B + K) \in \overline{(\mathrm{BDN})_k}$. Since $\varepsilon > 0$ was arbitrary, $B \in \overline{(\mathrm{BDN})}$. ▨

3.10. Corollary. *Let* $Q = \bigoplus_{n=1}^{\infty} Q_n \in (\mathrm{DSN}) \cap (\mathrm{Qnil})$ *and assume that for some fixed* $k \geqslant 1$, $\liminf_n \|Q_n^k\|^{1/k} = 0$. *Then* $Q \in \overline{(\mathrm{BDN})}$.

Proof. Let $\{n_j\}_{j=1}^{\infty}$ be a subsequence of $\mathbf{N}$ such that $\lim_{j\to\infty} \|Q_{n_j}^k\|^{1/k} = 0$. Apply the above proposition to $B = \bigoplus_{n=1}^{\infty} Q_{n_j}$ to get that $B \in \overline{(\mathrm{BDN})}$. Then $Q = Q_0 \oplus B$, where $Q_0 = \bigoplus_{n \notin \{n_j\}_{j=1}^{\infty}} Q_n$, and $Q_0 \in (\mathrm{DSN}) \cap (\mathrm{Qnil})$. Proposition 3.6 now applies to show that $Q = Q_0 \oplus B \in \overline{(\mathrm{BDN})}$. ▨

3.11. The following observation would also seem to suggest that for $Q \in (\mathrm{QD})$, neither the size of the blocks in the equation $Q = \bigoplus_{n=1}^{\infty} Q_n + K_\varepsilon$, $\|K\|_\varepsilon < \varepsilon$, $K_\varepsilon \in \mathscr{K}(\mathscr{H})$ nor the possible non-existence of faithful representations ρ of $C^*(\pi(Q))$ for which $\rho(\pi(Q))$ is quasidiagonal may provide the full picture in the characterization of $\overline{(\mathrm{BDN})}$.

Recall that for $T \in \mathscr{B}(\mathscr{H})$, T is said to be *algebraic* if there exists a non-zero polynomial p such that $p(T) = 0$. We denote the set of algebraic operators by (Alg). Following the notation of [4], we shall write (AlgQD) for $(\mathrm{Alg}) \cap (\mathrm{QD})$. We mention that Voiculescu's description of $\overline{(\mathrm{Alg})}$ as the set of biquasitriangular operators [15] readily implies that $\overline{(\mathrm{Alg})} \supseteq (\mathrm{QD})$, and it is an interesting open problem to determine whether $\overline{(\mathrm{AlgQD})} = \overline{(\mathrm{Alg}) \cap (\mathrm{QD})}$ is equal to $\overline{(\mathrm{Alg})} \cap \overline{(\mathrm{QD})} = (\mathrm{QD})$ (cf. [4] for partial results).

The observation mentioned above is that the proof of Proposition 5.4 in [7] goes through with only slight notational modifications to prove the following:

3.12. PROPOSITION. *If* $T \in (\overline{\mathrm{AlgQD}})$ *and* N *is a normal operator such that* $\sigma(N)$ *is connected and includes* $\{0\} \cup \sigma(T)$, *then* $N \oplus T \in (\overline{\mathrm{BDN}})$.

3.13. The statement in [7] required that $T \in (\mathrm{QD})_{\mathrm{nor}}$. Since $(\mathrm{QD})_{\mathrm{nor}}$ is known to be a proper subset of $(\overline{\mathrm{AlgQD}})$, this result is genuinely stronger.

3.14. A few other results concerning whether $Q \in (\mathrm{QD}) \cap (\mathrm{Qnil})$ implies $Q \in (\overline{\mathrm{BDN}})$ are known (cf. [7], [8], [9]). For example, it is known that if Q has $\{0\}$ as an essential reducing eigenvalue, then $Q \in (\overline{\mathrm{BDN}})$. (This follows from Proposition 3.6 as well.) The problem may very well possess a finite dimensional solution. A sufficient condition for such a Q to belong to $(\overline{\mathrm{BDN}})$ would be to show that if $T \in L(\mathbf{C}^n)$ and $\|T^k\|^{1/k} < \varepsilon$, then $\|T - M\| = \delta$ for some $M \in L(\mathbf{C}^n)$ satisfying $M^k = 0$ and $\delta = O(\varepsilon)$ is independent of n. Campell and Gellar [3], [6] have partial results in this direction.

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