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OPERATORS WITH THE DISJOINT SUPPORT PROPERTY

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0. INTRODUCTION

Let K be a compact Hausdorff space and X a real or complex Banach space. Let $C(K, X)$ be the Banach space of all continuous X-valued functions on K under the supremum norm. For a Banach space X, we write X^* for its Banach dual, $L(X)$ the space of all bounded linear operators on X and B_X the closed unit ball. A bounded linear operator T on $C(K, X)$ is said to have the *disjoint support property* if for every f, g in $C(K, X)$ such that $\|f(k)\|\|g(k)\| = 0$ for every $k \in K$ we always have $\|Tf(k)\|\,\|Tg(k)\| = 0$ for every $k \in K$. Recently Jamison and Rajagopalan [3] proved that such operators are precisely those of the form

$$Tf(k) = \Phi_k f(\varphi(k))$$

where φ is a selfmap of K and for each k, Φ_k is a bounded linear operator on X. They also gave a necessary and sufficient condition for these operators to be compact. In the following sections we prove that operators on a more general setting of a function module having the disjoint support property also assume the form described above. We shall show that some of the conditions in [3, Theorem 2] are redundant and deduce the theorem from a generalized Arzela-Ascoli theorem. Following [6], criteria for weak compactness is also discussed. In what follows K is not always assumed to be compact, it may be locally compact or completely regular. In these cases, we denote by $C_0(K, X)$ the spaces of all continuous X-valued functions vanishing at infinity on K and $C_p(K, X)$ the spaces of all continuous X-valued functions having relatively compact range in X, respectively.

1. THE DISJOINT SUPPORT PROPERTY

DEFINITION [2, Definition 4.1]. A *function module* is a triple $(K, (X_k)_{k\in K}, X)$, where K is compact, $(X_k)_{k\in K}$ a family of Banach spaces and X a closed subspace of

$\prod_{k \in K} X_k$ such that the following conditions are satisfied:

(i) $hf \in X$ for every $f \in X$ and $h \in C(K)$,

(ii) $k \mapsto \|f(k)\|$ is upper semicontinuous for every $f \in X$,

(iii) $X_k = \{f(k) : f \in X\}$ for every $k \in K$, and

(iv) $\{k : X_k \neq 0\}$ is dense in K.

For further properties of a function module, we refer to Behrends [2]. We mention in passing that if K is locally compact, $C_0(K, X)$ can be regarded as a function module in $\prod_{k \in \beta K} X_k$, where βK is the Stone-Čech compactification of K $X_k = X$ for $k \in K$ and $X_k = 0$ for $k \in \beta K \setminus K$.

THEOREM 1.1. *Let X be a function module in $\prod_{k \in K} X_k$ and let T be a bounded linear operator on X. Then T has the disjoint support property if and only if there is a selfmap φ of X and a family $(\Phi_k)_{k \in K}$ of bounded linear operators with $\Phi_k \in L(X_k)$ such that*

$$Tf(k) = \Phi_k(f(\varphi(k)))$$

or every f in

Proof. Sufficiency is clear.

To establish the necessity we shall first construct the selfmap φ. Since no continuity is required, it suffices to define $\varphi(k)$ for those k such that $Tf_0(k) \neq 0$ for some f_0. Let $\mathscr{F}$ be the the set of all non-empty compact subset F of K with the property:

$$f(l) = 0 \text{ for every } l \in F \text{ implies } Tf(k) = 0.$$

The above property is equivalent to:

For every $\varepsilon > 0$, $\|f(l)\| < \varepsilon$ for every $l \in F$ implies $\|Tf(k)\| < \varepsilon\|T\|$. We claim that $\mathscr{F}$ has a minimal element (in the sense of set inclusion). To this end take any chain (F_α) from $\mathscr{F}$. We contend that $\bigcap_\alpha F_\alpha \neq \emptyset$, as (F_α) has the finite intersection property. To show that $F = \bigcap_\alpha F_\alpha \in \mathscr{F}$, consider any $f \in X$ such that $\|f|F\| < \varepsilon$, where $\varepsilon > 0$. The set $u = \{l \in K : \|f(l)\| < \varepsilon\}$ is an open set containing F, by the upper semicontinuity of the function $k \mapsto \|f(k)\|$. Since (F_α) is totally ordered, the chain lies eventually inside u. Hence $\|Tf(k)\| < \varepsilon\|T\|$ and consequently $F \in \mathscr{F}$. Next we shall show that F is a singleton. If $f \in X$ satisfies $Tf(k) \neq 0$, take $p \in X_k^*$, such that $(Tf(k)) \neq 0$. Consider the linear functional on $C(F)$ given by

$$h \mapsto p(T(\widetilde{h}f)(k)) \quad \text{where } \widetilde{h} \text{ is any extension of } h \text{ to } K.$$

By the Riesz representation theorem, there is a Radon measure μ on F such that $\int Fhd\mu = p(T(\tilde{hf})(k))$. Arguing as in [1, Example 2.2], it can be shown that the support of $|\mu|$ is a singleton. The fact that this singleton is indeed independent of the choice of f and p follows from a standard argument (cf. [3, p. 310]). We omit the detail. Now let that singleton set be $\{l\}$. Then for every $f \in X$, $f(l) = 0$, we have $Tf(k) = 0$. For assuming the contrary, there is a $p \in X_k^*$ such that $\alpha = p(Tf(k)) > 0$. The set $u = \{l' \in K : \|f(l')\| < \alpha\}$ is open and we may find $h \in C(K)$ with $0 \leqslant h \leqslant 1, h(k) = 0$ and $h(l') = 1$ for every $l' \in K/u$. Then $\|hf - f\| < \alpha$. But $p(T(hf)(k)) = h(l) = 0$, which is a contradiction. We conclude that $F = \{l\}$ from the minimality of F. We still have to show that there is only one such l. Note that l has the property that of $h \in C(K)$, $h(l) = 1$ implies $T(hf)(k) = Tf(k)$. The uniqueness now follows from a Urysohn argument. What we have proved so far is that to every $h \in K$ such that $Tf(k) \neq 0$ for some f in X, there corresponds a unique l with

$$f(l) = 0 \text{ implies } Tf(k) = 0.$$

So, define $\varphi(k) = l$ and $\Phi_k(x) = Tf(k)$ where f is any element of X such that $f(l) = x$. Then Φ_k is clearly well-defined. For other k, put $\Phi_k = 0$ and $\varphi(k)$ can be defined arbitrarily. Then we have

$$Tf(k) = \Phi_k f(\varphi(k)) \quad \text{for every } f \text{ in } X.$$ ▨

Let $N = \{k \in K : \Phi_k = 0\}$. Then in contrast to the case of continuous functions [3, Theorem 1], the selfmap φ need not be continuous on $K \setminus N$.

Example. Let $X = c_0[0, 1]$ and let T be given by

$$Tf(0) = f(1), \quad Tf(1) = f(0) \text{ and } Tf(k) = f(k) \quad \text{for } k \neq 0, 1.$$

Then T is a bounded linear operator with the disjoint support property. The corresponding Φ_k is the identity on $\mathbf{R}$ and φ is given by

$$\varphi(0) = 1, \quad \varphi(1) = 0 \text{ and } \varphi(k) = k, \quad \text{for } k \neq 0, 1.$$

Clearly φ is not continuous at 0 and 1.

However if X is a function module in which $k \mapsto \|f(k)\|$ is continuous, then φ is continuous on $K \setminus N$. For if $\Phi_k \neq 0$ and $\mathscr{U}$ is a neighbourhood of $\varphi(k)$, there is an $l \in X$ such that $\Phi_k f(\varphi(k)) \neq 0$ and f vanishes outside $\mathscr{U}$. Since $Tf(k) = \Phi_k f(\varphi(k)) \neq 0$, and is continuous, neigbourhood V of k such that $Tf(l) \neq 0$ for every $l \in V$. It follows $\varphi(l) \in$

COROLLARY 1.2. *Let K be a locally compact Hausdorff space and X a Banach space. Let T be a bounded operator on $C_0(K, X)$ with the disjoint support property. Then there is selfmap φ of K and a bounded $L(X)$-valued function Φ on K such that*

$$Tf(k) = \Phi_k f(\varphi(k)) \quad \textit{for every } f \in C_0(K, X).$$

Let $N = \{k : \Phi_k \neq 0\}$, then φ is continuous on $K \setminus N$ and Φ is strongly continuous on $K \setminus N$. If K is compact, then Φ is strongly continuous on K.

Proof. Viewing $C_0(K, X)$ as a function module over βK, it follows from Theorem 1 that there exist a selfmap φ of βK and a family $(\Phi_k)_{k \in K}$ of bounded operators on X suhch that

$$Tf(k) = \Phi_k f(\varphi(k)) \quad \text{for every } f \in C_0(K, X).$$

The boundedness of Φ is immediate. Now the Stone extension of any $f \in C_0(K, X)$ satisfies $f(k) = 0$ whenever $k \in \beta K \setminus K$, therefore $\varphi(k) \in K$ for every k such that $\Phi_k \neq 0$. We may redefine, if necessary, the value of φ at other points so that φ corresponds to a selfmap on K. The continuity requirement for φ follows from our discussion above and that of Φ follows by considering a function vanishing outside some neighbourhood of $\varphi(k)$. If K is compact, we may just consider the constante functions. ∎

Note that in general Φ may not be continuous on K.

EXAMPLE. Let $K = \{0, 1, 2, 3, \ldots\} \cup \{1/2, 1/3, \ldots\}$ with the usual subspace topology and X be the Hilbert space ℓ^2. Let (e_i) denote the canonical basis and define φ by

$$\Phi_0 := 0 \quad \text{and} \quad \Phi_n x = \Phi_{1/n} x = (x, e_1)e_n.$$

Put $\varphi(0) := 0$, $\varphi(n) := \varphi(1/n) = n$. Then for every $f \in C_0(K, X)$, $Tf(k) := \Phi_k f(\varphi(k))$ is also in $C_0(K, X)$. Therefore Φ and φ define a bounded linear operator on $C_0(K, X)$ with the disjoint support property, while Φ fails to be continuous at $k = 0$.

We note also that given Φ and φ continuous at every k for which $\Phi_k \neq 0$, they may not define a bounded linear operator on $C_0(K, X)$. To see this just take Φ as in the above example and φ to be the identity map.

Now suppose that K is only completely regular. We know that every $f \in C_p(K, X)$ has a unique extension $\tilde{f}$ to βK and indeed $C_p(K, X)$ is isometrically isomorphic to $C(\beta K, X)$. A linear operator $T: C_p(K, X) \to C_p(K, X)$ has the disjoint support property if and only if it, regarded as an operator on $C(\beta K, X)$, also has the disjoint support property. We conclude the following

COROLLARY 1.3. *A bounded linear operator $T : C_p(K, X) \to C_p(K, X)$ has the disjoint support property if and only if there exist $\varphi: K \to \beta K$ and a strongly conti-*

nuous function $\Phi: K \to L(X)$ *with relatively strongly compact range in* $L(X)$ *such that* φ *is continuous on* $K \setminus N$, *where* $N = \{k \in K : \Phi_k = 0\}$ *and*

$$Tf(k) = \Phi_k(\tilde{f}(\varphi(k))) \quad \text{for every } f \in C_p(K, X),$$

where $\tilde{f}$ *is the Stone extension of* f *to* βK.

Note that in general we do not have $\varphi(K) \subseteq K$. For suppose K is non-compact take any $k_0 \in \beta K \setminus K$ and define $\varphi(k) = k_0$ for every $k \in K$. Then T given by $Tf(k) = \tilde{f}(\varphi(k)) \otimes 1$, the constant function with value $\tilde{f}(k_0)$, has the disjoint support property, but we cannot redefine φ to have $\varphi(k) \in K$.

2. A COMPACTNESS CRITERIA

In [3] Jamison and Rajagopalan stated and proved the following theorem (rewritten in our notations).

THEOREM [3, Theorem 2]. *The following conditions are necessary and sufficient for the weighted composition operator*

$$Tf(k) = \Phi_k f(\varphi(k))$$

to be a compact operator on $C(K, X)$.

(2.1) $\varphi : K \to K$ *and* φ *is continuous on* $X \setminus N$.

(2.2) $k \mapsto \Phi_k$ *is continuous in the uniform operator topology.*

(2.3) *If* F *is a compact subset of* $K \setminus N$, *then* $\varphi(F)$ *is finite.*

(2.3') *If* F *is a connected component of* $K \setminus N$, *there exists an open subset* U *of* K, *such that*

$$U \subseteq K \setminus N, \quad F \subseteq U, \quad \text{and} \quad \varphi(\mathscr{U}) \text{ is finite.}$$

(2.4) *If* (x_n) *is a sequence in* X, $\varepsilon > 0$ *and* F *a compact subset of* $K \setminus N$, *then there exists a subsequence* (x_{n_i}) *such that* $(\Phi_k x_{n_i})$ *is* ε*-uniformly Cauchy on* F.

(2.5) *Given a bounded sequence* (x_n) *in* $C(K, X)$, *let* $Z = \{x : Tx_n(k) = \Phi_k x_n = 0$ *for every* $n\}$. *If* $\varepsilon > 0$ *there exists a subsequence* (x_{n_i}) *and a neighbourhood* $\mathscr{U}_\varepsilon \supseteq z$ *such that*

$$\|Tx_{n_i}(k)\| < \varepsilon \quad \text{for every } k \in \mathscr{U}_\varepsilon.$$

They remarked that (2.4) implies Φ_k is a compact operator for each $k \in K$. But if we assume that every Φ_k is a compact operator and F a compact subset of

$K \setminus N$, then by (2.2) there exists $k_1, \ldots, k_n \in F$ and open neighbourhoods $U_{k_1}, \ldots, U_{k_n}$ such that for every $l \in U_{k_i}$, we have $\|\Phi_l - \Phi_{k_i}\| < \varepsilon/4$. Let (x_n) be any sequence in X. Since Φ_{k_1} is compact, there is a subsequence (x_{n_j}) for which $(\Phi_k x_{n_j})$ converges. Repeat the process for (x_{n_j}) and Φ_{k_2} and so on, we obtain a ε-uniformly Cauchy subsequence on F. Therefore (2.4) follows. For (2.5) we may just take $k_1, \ldots, k_n \in Z$ and corresponding open neighbourhoods $U_{k_1}, \ldots, U_{k_n}$ for which $\|\Phi_l - \Phi_{k_i}\| < \varepsilon$ whenever $l \in U_{k_i}$. Put $U = \bigcup_{i=1}^{n} U_{k_i}$. Then (2.5) holds on U and it is not necessary to choose a subsequence. We shall now give a characterisation of compact operators on $C_0(K, X)$ with the disjoint support property for locally compact K. Our main tool is the following generalised Arzelà-Ascoli theorem. This theorem is probably well-known. A further generalization can be found in [4, Theorem 2.1].

THEOREM A. *A subset H of $C_0(K, X)$ is relatively compact if and only if the following conditions are satisfied:*

(i) *H is equicontinuous,*

(ii) *$H(k) = \{f(k) : f \in H\}$ is relatively compact for every $k \in K$, and*

(iii) *H vanishes at infinity uniformly*, i.e. *for every $\varepsilon > 0$ there exists a compact subset D of K such that $\|f(k)\| < \varepsilon$ for every $f \in H$ and $k \in K \setminus D$.*

THEOREM 2.1. *The bounded linear operator on $C_0(K, X)$ given by*

$$Tf(k) = \Phi_k f(\varphi(k))$$

is compact if and only if the following conditions are satisfied:

(i) *Each Φ_k is a compact operator on X.*

(ii) *Φ is continuous in the uniform operator topology and the scalar function $k \mapsto \|\Phi(k)\|$ vanishes at infinity on K.*

(iii) *φ is locally constant on $K \setminus N$.*

REMARK. Condition (iii) is equivalent to condition (2.3) or (2.3′) in [3].

Proof. Necessity. We shall prove only the second part of (ii), the rest being the same as in the proof of [3, Theorem 2]. Now the compactness of T is equivalent to the relative compactness of the set $T\mathbf{B}_{C_0(K,X)}$ in $C_0(K, X)$. By the Arzelà-Ascoli theorem this set vanishes at infinity uniformly. Hence for every $\varepsilon > 0$, there is a compact subset D of K such that $\|\Phi_k f(\varphi(k))\| < \varepsilon$ for all $k \in K \setminus D$ and for all $f \in \mathbf{B}_{C_0(K,X)}$ which amounts to $\|\Phi_k\| < \varepsilon$.

Sufficiency. We need to show $T\mathbf{B}_{C_0(K,X)}$ satisfies conditions (i)–(iii) of the Arzelà-Ascoli theorem. For $k \in K$ with $\Phi_k = 0$, $T\mathbf{B}_{C_0(K,X)}$ is equicontinuous at k, by the norm-continuity of Φ. It is also equicontinuous at other points, since then φ is locally constant. Thus (i) is fulfilled. Condition (ii) follows from the fact that each Φ_k is compact. Finally (iii) is already dealt with in the necessity part. ▨

When K is completely regular, condition (iii) above cannot be carried over to βK. Namely if φ is locally constant on K, its Stone extension $\tilde{\varphi}$ need not be locally constant on βK. (We may just consider an infinite set and a function defined on it, which takes distinct values at distinct points.) Following Singh and Summers [6], we put

$$N(\Phi, \varepsilon) = \{k \in K : \|\Phi_k\| \geq e\} \quad \text{for every } \varepsilon > 0.$$

It is easy to see that Singh and Summers' condition:

$$\varphi(N(\Phi, \varepsilon)) \text{ is finite for every } \varepsilon > 0$$

is equivalent to (iii) above, when K is compact.

COROLLARY 2.2. *The bounded linear operator on $C_p(K, X)$ given by*

$$Tf(k) = \Phi_k f(\varphi(k))$$

is compact if and only if the following conditions are satisfied:

(i) *Each Φ_k is a compact operator on X.*

(ii) *Φ is continuous in the uniform operator topology.*

(iii) *$\varphi(N(\Phi, \varepsilon))$ is finite for every $\varepsilon > 0$.*

3. A WEAK COMPACTNESS CRITERIA

We shall restrict our discussion of weak compactness to the case K is compact and deduce as a corollary for $C_p(K, X)$, when K is completely regular. We shall also assume that for the operator $Tf(k) = \Phi_k f(\varphi(k))$, Φ is continuous in the uniform operator topology, a property which is not shared by all weakly compact operators of this form.

EXAMPLE. Again let $K = \{0, 1, 1/2, 1/3, \ldots\}$ with the usual subspace topology and let $X = \ell^2$ with canonical basis (e_i). Define Φ by

$$\Phi_0 = 0 \quad \text{and} \quad \Phi_{1/n}x = (x, e_n)e_1.$$

Take $\varphi(0) = \varphi(1/n) = 0$. Then for every $f \in C(K, X)$

$$Tf\left(\frac{1}{n}\right) = (f(0), e_n)e_1.$$

The range of $C(K, X)$ under T is isometric to a subspace of C_0 and the image of $\mathbf{B}_{C(K, X)}$ is the set $\left\{c_0 \ni (\xi_i) = \sum_{i=1}^{\infty} |\xi_i|^2 < \infty\right\}$, which is relatively weakly compact in c_0. Therefore T is weakly compact, while Φ is not continuous in the uniformly operator topology.

We shall resort to a theorem of Ruess and Summers [5, Theorem 2.2]. Since we do not need the theorem in its generality, we state it as follows:

THEOREM B. *A subset H of $C(K, X)$ is relatively weakly compact if and only if the following conditions are satisfied:*

(i) *H is bounded,*

(ii) *$H(k) = \{f(k) : f \in H\}$ is relatively weakly compact for every $k \in K$, and*

(iii) *if $f : K \to X$ is the pointwise-weak limit of a net in H, then $f \in C(K, X)$.*

THEOREM 3.1. *The bounded linear operator on $C(K, X)$ given by*

$$Tf(k) = \Phi_k f(\varphi(k))$$

is weakly compact if and only if the following conditions are satisfied:

(i) *Each Φ_k is a weakly compact operator on X, and*

(ii) *φ is locally constant on $K \setminus N$.*

Proof. Necessity. Condition (i) is clearly necessary. Condition (ii) can be proved similarly as in the proof of [3, Theorem 2].

Sufficiency. We need only show that if (f_α) is a net of functions in $\mathbf{B}_{C(K, X)}$ such that $\Phi_k f(\varphi(k))$ converges weakly to $g(k)$, then $g \in C(K, X)$. If k is such that $\Phi_k = 0$, then there is a neigbourhood u of k on which $\|\Phi\| < \varepsilon$, for arbitrarily chosen ε. It follows that $\|g\| < \varepsilon$ on u. If $\Phi_k \neq 0$, take a neighbourhood $\mathscr{U}$ of k on which φ is constant and that $\|\Phi_l - \Phi_k\| < \varepsilon$ for every $l \in \mathscr{U}$. Then we have

$$\|g(l) - g(k)\| \leqslant \liminf \|\Phi_l f_\alpha(\varphi(k)) - \Phi_k f_\alpha(\varphi(k))\| \leqslant \varepsilon.$$

For both cases we have g continuous at k.

In the case when K is completely regular, we have the following

COROLLARY 3.2. *The bounded operator on $C_p(K, X)$ given by*

$$Tf(k) = \Phi_k f(\varphi(k))$$

is weakly compact if and only if the following condition is satisfied:

(i) *Each Φ_k is weakly compact operator on X, and*

(ii) *$\varphi(N(\Phi, \varepsilon))$ is finite for every $\varepsilon > 0$.*

REMARK. Corollary 2.2 and Corollary 3.2 generalised part (1) and (2) of [6 Theorem 2.1] respectively.

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