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THE IDEAL STRUCTURE OF GRUPOID CROSSED PRODUCT C^*-ALGEBRAS

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1. INTRODUCTION

Motivated by Mackey's normal subgroup analysis in group representation theory, much work has been devoted to the study of the ideal structure of crossed product C^*-algebras. Let us recall the classical settings and give a brief history of the main results. Assume that G is a second countable locally compact group acting continuously on a separable C^*-algebra A. In their memoir [5], Effros and Hahn had conjectured that if G is amenable, every primitive ideal of the crossed product $C^*(G, A)$ is the kernel of a representation induced from the isotropy of the action of G on the primitive ideal space $\mathrm{Prim}\,A$ — in fact their conjecture was set in the case when A is abelian — . An important step was accomplished by Sauvageot who proved in [21] when A is abelian and in [22] in the general case that, if G is amenable, every primitive ideal contains an induced primitive ideal and that, if G is discrete, the reverse inclusion holds. The final solution was given by Gootman and Rosenberg who showed in [11] that every primitive ideal is contained in an induced primitive ideal without any assumption on G. Shortly after, Fack and Skandalis observed in [7] that the methods of [21], [11] could be applied to the C^*-algebra of a foliation and obtained similar results in that case. Moreover they showed that the C^*-algebra of a foliation with Hausdorff graph is simple if and only if the foliation is minimal and pointed out that, due to the possible presence of holonomy, this result is not directly attainable by the methods of [21], [11]. The object of the present work is to prove this results in the framework of groupoid crossed products, not so much to deal with new situations but mainly because groupoids offer a natural setting where the previous ideas and techniques become clearer. The main results of this paper are Theorem 3.3, which generalizes the main theorem of [11] and the Corollary 4.6, which generalizes Theorem 2.6 of [7].

The proof of Theorem 3.3 is not intrinsically different from that of Gootman and Rosenberg. The crux is Proposition 1.11 which is the topological version of the local cross section theorem of [11]. Its idea, due to P. Forrest [9], has been exploited in [8], [16] to reduce arbitrary groupoids to groupoids with discrete equivalence classes in the measure-theoretical setting (such a global reduction does not seem to exist in the topological setting but in particular cases as foliations). The theorem follows almost immediately, at least when the isotropy is continuous. Let us sketch how, in the case of the C^*-algebra $C^*(G)$ of a locally compact groupoid G. The continuity of the isotropy amounts to the existence of a continuous Haar system for the isotropy group bundle $G' = \{\gamma \in G: r(\gamma) = s(\gamma)\}$. Therefore the C^*-algebra $C^*(G')$ is well defined, acts on $C^*(G)$ as multipliers and restriction from G to G' defines a generalized conditional expectation P, in the sense of [19] from $C_c(G)$ to $C_c(G')$. The point is that, just as in Section 2 of the first chapter of [23], where Ş. Strătilă and D. Voiculescu describe the ideal structure of an AF C^*-algebra, the expectation P can be approximated by maps of the form $Q(f) = \sum_i e_i f e_i$, for $f \in C_c(G)$, and where the e_i's are functions defined on the unit space $G^{(0)}$ and $(e_i f e_i)(\gamma) = e_i \circ r(\gamma) f(\gamma) e_i \circ s(\gamma)$. The local cross section theorem is precisely what is needed to construct the e_i's. A similar device has been used, for example in [15], to construct an approximate unit for a groupoid C^*-algebra. Let now I be an ideal of $C^*(G)$ and J be its restriction to $C^*(G')$. The ideal induced by J, which consists of the a's in $C^*(G)$ such that $P(f^* a^* a f)$ is in J for each f in $C_c(G)$ clearly contains I. The framework of groupoid crossed products also gives a more precise condition under which every ideal of the crossed product $C^*(G, A)$ is an induced ideal. For this, it is necessary and sufficient that the equivalence relation associated with the action of G on $\operatorname{Prim} A$ be measurewise amenable (Theorem 3.6). This can happen without G being amenable. Let us turn now to the ideal structure of crossed product C^*-algebras. Sauvageot, Gootman and Rosenberg's theorem asserts that, in the amenable case, it is sufficient to study the ideals induced from the isotropy. This is a difficult task when the isotropy is not trivial. The second main result, Theorem 4.3, deals with the case when the primitive ideals with trivial isotropy are dense. In fact, we need a stronger hypotesis, which we call discretely trivial isotropy. Its proof is modelled after Proposition 4.4 of the second chapter of [17]. It yields as Corollary 4.6 the simplicity of the reduced crossed product $C^*_{\text{red}}(G, A)$ when G is Hausdorff and its action on $\operatorname{Prim} A$ is minimal and has points with discretely trivial isotropy.

The definition of the twisted groupoid crossed product C^*-algebra $C^*(G, \Sigma, A)$ associated with the twisted groupoid dynamical system (G, Σ, A) is that of [18]. The twist Σ has been included for completeness and further reference but does not play any role here. Let me simply recall that G is a second countable, not necesarily Hausdorff, locally compact groupoid with Haar system λ and that A is a boundle of separable C^*-algebras over the unit space $G^{(0)}$ on which G acts continuously by automorphisms. The C^*-crossed product $C^*(G, A)$ is the completion for the largest C^*-norm continuous for the inductive limit topology of the $*$-algebra $C_c(G, A)$ consisting of "continuous sections with compact support" of the pull-back bundle r^*A over G. Results and terminology of [18] will be sometimes used without further reference. Here is a brief description of the content of the paper. The first section contains the local cross section theorem and auxiliary results on topological groupoids. Although they will be applied chiefly to the semi-direct product $\operatorname{Prim} A \rtimes G$, they are written in a somewhat broader generality, because it is easier to write γ than (x, γ). The second section gives an equivariant version of Effros' decomposition theory [4] of representations over the primitive ideal space. This is a refinement of the disintegration theorem of [18]. It also contains a definition of induced and restricted representation which does not use the isotropy subgroups C^*-algebra. Section 3 compares arbitrary ideals and ideals induced from the isotropy of $\operatorname{Prim} A \rtimes G$, while Section 4 compares arbitrary ideals and ideals induced from $\operatorname{Prim} A$, when the action of G on $\operatorname{Prim} A$ is amenable and essentialy free. It also contains further results on the ideals of a crossed product and counter-examples. In an appendix, G. Skandalis gives an example of a minimal foliation with non-Hausdorff graph which has a non simple C^*-algebra.

1. CONSTRUCTION OF APPROXIMATE CROSS-SECTION

We collect in this section some topological results about groupoids and actions of groupoids which will be used in the proof of the Effros-Hahn conjecture. Given a topological groupoid G, we want to circumvent two difficulties. One of them is minor: the isotropy usually fails to be continuous. Since it is Borel, Lusin's theorem will provide sets of large measure where it is continuous. The other is more serious: the quotient map of $G^{(0)}$ onto $G^{(0)}/G$ may fail to have any reasonable section. Just as in [11], the local compactness of G will be used to get some control of the quotient.

Let us first spell out the assumptions we make on the topological groupoid G. They are prompted by the basic example of a topological transformation group (X, G) where the group G is locally compact. We require the range and source maps r, s of G onto $G^{(0)}$ to be open. We assume that $G^{(0)}$ is Hausdorff but not G. The local compactness requirement will be as follows. Let us say that a subset K of G is left [resp. right] conditionally compact if from every compact subset L of $G^{(0)}$, $LK = K \cap r^{-1}(L)$ [resp. $KL = K \cap s^{-1}(L)$] is compact. A subset which is both left and right conditionally compact will be called conditionally compact (c.c. for short). We shall assume that each point of G has a conditionally compact neighborhood. A topological groupoid satisfying all these requirements will be called locally conditionally compact (l.c.c. for short).

Let G be a topological groupoid. We denote by $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$ the isotropy group bundle of G. It is closed in G if $G^{(0)}$ is Hausdorff. We denote by R the graph of the equivalence relation on $G^{(0)}$ given by $r(\gamma) \sim s(\gamma)$. We endow it with the quotient topology from G, which is finer, and usually strictly finer, than the topology induced by $G^{(0)} \times G^{(0)}$, which we call the product topology. When G is l.c.c., G' and R are usually not l.c.c. groupoids in the above sense. For the bundle map $r|G'$ of G' may fail to be open and points of the diagonal Δ of R may not have conditionally compact neighborhoods. As we shall see, these properties are related to the continuity of the isotropy. The space $\Sigma^{(0)}$ of the closed subgroups of G is equipped with the Fell topology, where basic open sets are of the form

$$\mathfrak{U}(K; U_1, \ldots, U_n) = \left\{H \in \Sigma^{(0)}: H \cap K = \emptyset,\ H \cap U_1 \neq \emptyset, \ldots, H \cap U_n \neq \emptyset\right\}$$

where $K, U_1, \ldots, U_n$ are subsets of G with K compact and $U_1, \ldots, U_n$ open. It is Hausdorff and conditionally compact over $G^{(0)}$. The isotropy map S sends $x \in G^{(0)}$ into the isotropy subgroup $G(x) = xGx \in \Sigma^{(0)}$.

LEMMA 1.1. *Let G be a topological groupoid and $x \in G^{(0)}$. The following properties are equivalent.*

i) *The isotropy map is continuous at x.*

ii) *The restriction to G' of the range map is open at every $\gamma \in G(x)$ (i.e. sends a neighborhood of γ onto a neighborhood of x).*

iii) *For every* $\gamma \in G(x)$ *and every neighborhood* U *of* γ *in* G, $G'U$ *is a neighborhood of* $G(x)$ *in* G.

Proof. The inverse image by S of the basic open set $\mathcal{U}(K; U_1, \ldots, U_n)$ is just

$$r(K \cap G')^c \cap r(U_1 \cap G') \cap \ldots \cap r(U_n \cap G').$$

Suppose that S is continuous at x and let U be a neighborhood of $\gamma \in G(x)$. Then $\mathcal{U}(\emptyset; U)$ is a neighborhood of $G(x)$ in $\Sigma^{(0)}$ and $r(U \cap G')$ is a neighborhood of x in $G^{(0)}$. Conversely, suppose that (ii) holds. Let $\mathcal{U}(K; U_1, \ldots, U_n)$ be a neighborhood of $G(x)$. Then each U_i is a neighborhood of some $\gamma_i \in G(x)$ and $r(U_i \cap G')$ is a neighborhood of x in $G^{(0)}$. Since $r(U \cap G') = G'U \cap G^{(0)}$, the property (iii) implies (ii). Let us show that (ii) implies (iii). We form the semi-direct product $G' \ltimes G$ where G' acts by left multiplication on G and call π_2 the projection onto the second factor and π_1 the multiplication map $(\gamma', \gamma) \mapsto {\gamma'}^{-1}\gamma$. The openness of $r|G'$ on $G(x)$ implies the openness of π_2 on $G(x) \ltimes xG$. If U is a neighborhood of $\gamma \in G(x)$ in G, $\pi_1^{-1}(U)$ is a neighborhood of $\{(\gamma', \gamma'\gamma) : \gamma' \in G(x)\}$ in $G' \times G$ and $G'U = \pi_2(\pi_1^{-1}(U))$ is a neighborhood of $G(x)$ in G. ■

REMARK 1.2. This lemma provides two characterizations of the continuity of the isotropy: namely the openness of the boundle map of the isotropy group bundle and the openness of the map (r, s) of G onto R. When G is l.c.c., we shall see that these conditions are equivalent to the existence of a continuous Haar system on G'.

Let G be a l.c.c. groupoid and denote by $C_{cc}(G)$ the space of continuous functions on G with conditionally compact support — more exactly, in the non-Hausdorff case, functions which vanish outside some Hausdorff open set U and whose restriction to U is continuous with c.c. support and linear combinations of those. For each $x \in G^{(0)}$, xG is a Hausdorff locally compact space and the restriction of $f \in C_{cc}(G)$ to xG is continuous with compact support. At this point we want to make two more assumptions on G. The first one is that, for every $x \in G^{(0)}$, every function in $C_c(xG)$ extends to a function in $C_{cc}(G)$. The second one is the existence of $F \in C_{cc}(G)$ such that $0 \leqslant F \leqslant 1$ and $F(x) = 1$ for every $x \in G^{(0)}$. These properties are satisfied when G is Hausdorff and normal and also for the semi-direct product $X \rtimes G$ of a transformation groupoid (X, G) with G locally compact, which is our primary interest. We also note that if G has these properties, so does the isotropy bundle G'. From now on, they will be included in the definition of a l.c.c. groupoid. A Haar system for G consists of measures λ^x on xG, $x \in G^{(0)}$, with support xG and satisfying the left invariance property:

$$\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}, \quad \text{that is} \quad \int f(\gamma\gamma')d\lambda^{s(\gamma)}(\gamma') = \int f(\gamma')d\lambda^{r(\gamma)}(\gamma')$$

for every $f \in C_{cc}(G)$. We say that the system is continuous if for every $f \in C_{cc}(G)$, the function $\lambda(f)(x) = \int f d\lambda^x$ is continuous.

LEMMA 1.3. (cf. appendix of [10]). *Let* G *be a* l.c.c. *group bundle. The following properties are equivalent.*

i) *Its bundle map* p *is open.*

ii) *It admits a continuous Haar system* λ.

Proof. (ii) $\Rightarrow$ (i) Let U be an open space set in G, $\gamma \in U$ and $x = p(\gamma)$. There exists $f \in C_{cc}(G)$ such that $0 \leqslant f \leqslant 1$, $f(\gamma) = 1$ and f vanishes outside U. The set of y's such that $\lambda(f)(y) \neq 0$ is an open neighborhood of x contained in $r(U)$.

(i) $\Rightarrow$ (ii) Fix $F \in C_{cc}(G)$ such that $0 \leqslant f \leqslant 1$ and $F(x) = 1$ for $x \in G^{(0)}$. For each $G^{(0)}$, choose the (left) Haar measure λ^x of the locally compact group $G(x) = p^{-1}(x)$ such that $\int F \, d\lambda^x = 1$. We will show that this Haar system is continuous. We first observe that for every compact subset K of G, the function $x \mapsto \lambda^x(K)$ is bounded. Let U be the open neighborhood of $G^{(0)}$ defined by $F(\gamma) > 1/2$. We have $1 = \int F \, d\lambda^x \geqslant \int_U F \, d\lambda^x \geqslant (1/2)\lambda^x(U)$, hence $\lambda^x(U) \leqslant 2$. We can cover K by finitely many open sets $V_1, \dots, V_n$ such that $V_i^{-1}V_i \subset U$ for every i. Then $\lambda^x(V_i) = \lambda^x(\gamma\gamma^{-1}V_i) \leqslant \lambda^x(\gamma V_i^{-1}V_i) \leqslant \lambda^x(\gamma U) = \lambda^x(U) \leqslant 2$ where γ has been chosen in $G(x) \cap V_i$ (if $G(x) \cap V_i = \emptyset$, $\lambda^x(V_i) = 0$). Therefore, $\lambda^x(K) \leqslant 2n$. Let $(x_i)_{i \in I}$ be a net converging to x in $G^{(0)}$ and ω be a generalized limit for I. For every $f \in C_{cc}(G)$, the function $i \mapsto \tilde{f}(i) = \lambda(f)(x_i)$ is bounded. We set $\mu(f) = \omega(\tilde{f})$. It is a positive linear functional on $C_{cc}(G)$. Let us show that it depends only on the restriction on f to $G(x)$. Suppose that f and g coincide on $G(x)$. Let K be a compact set containing the restriction to $\{x_i, \ i \in I\} \cup \{x\}$ of $\operatorname{supp} f \cup \operatorname{supp} g$. We then have

$$\left|\tilde{f}(i) - \tilde{g}(i)\right| \leqslant \int |f(\gamma) - g(\gamma)| \, d\lambda^{x_i}(\gamma) \leqslant$$

$$\leqslant \sup_{\gamma \in G(x_i)} |f(\gamma) - g(\gamma)| \, \lambda^{x_i}(K).$$

A compactness argument shows that $\sup_{\gamma \in G(x_i)} |f(\gamma) - g(\gamma)|$ converges to 0. Hence $\mu(f) = \mu(g)$. Let us next show that μ is left invariant. Fix $\gamma \in G(x)$. Because p is open, there exist a net $(\gamma_i)_{i \in I}$ converging to γ such that $p(\gamma_i) = x_i$. Let $f \in C_{cc}(G)$ and choose $g \in C_{cc}(G)$ such that $g(\gamma\gamma') = f(\gamma')$ for every $\gamma' \in G(x)$. Then

$$\tilde{g}(i) - \tilde{f}(i) = \int (g(\gamma_i\gamma') - g(\gamma\gamma')) \, d\lambda^{x_i}(\gamma') + \int (g(\gamma\gamma') - f(\gamma')) \, d\lambda^{x_i}(\gamma').$$

Therefore there exists a finite constant C such that

$$\left|\tilde{g}(i) - \tilde{f}(i)\right| \leqslant C \left\{ \sup_{\gamma' \in G(x_i)} |g(\gamma_i\gamma') - g(\gamma\gamma')| + \sup_{\gamma \in G(x_i)} |g(\gamma\gamma') - f(\gamma')| \right\}.$$

A compactness argument shows that the right hand side converges to 0. Hence $\mu(g) = \mu(f)$. Since $\tilde{F}(i) = 1$ for every i, $\mu(F) = 1$. By uniqueness of the Haar measure on $G(x)$, we must have $\mu = \lambda^x$. This implies that $\lambda(f)(x_i)$ converges to $\lambda(f)(x)$ for every $f \in C_{cc}(G)$. ∎

COROLLARY 1.4. *Let G be a l.c.c. group bundle (we do not assume that the bundle map is open). Let $\Sigma^{(0)}$ be the space of the closed subgroups of G and*

$\Sigma = \left\{(H,\gamma): H \in \Sigma^{(0)}, \gamma \in H\right\}$ *be the canonical group-bundle over* $\Sigma^{(0)}$ *with the topology induced from* $\Sigma^{(0)} \times G$. *Then*

i) Σ *is* l.c.c.

ii) Σ *admits a continuous Haar system.*

Proof. For (i), it suffices to show that Σ is a closed subset of $\Sigma^{(0)} \times G$. Suppose that (H_i, γ_i) converges to (H,γ) and that $\gamma \notin H$. Let K be a compact subset of $G^{(0)}$ containing $p(\gamma_i)$ and $p(\gamma)$. Since $H \cap KL = \emptyset$, we have eventually $H_i \cap KL = \emptyset$. On the other hand $\gamma_i \in KL$ for i large enough, hence $\gamma_i \notin H_i$. For (ii), it suffices to show that the bundle map $p\colon \Sigma \to \Sigma^{(0)}$ is open. This is the case because

$$p(\mathcal{U}(K;U_1,\dots,U_n) \times V) = \mathcal{U}(K\,;U_1,\dots,U_n,V)$$

where K is compact and $U_1,\dots,U_n$ are open in G. ∎

We shall use this corollary to equip any l.c.c. group bundle G — with bundle map not necessarily open — with a Haar system. Let S be the map which sends $x \in G^{(0)}$ into $G(x) = p^{-1}(x) \in \Sigma^{(0)}$. Then G is the pull-back of Σ via S. Let λ be a continuous Haar system for Σ, normalized by $F \in C_{cc}(G)$. The pull-back of λ consists of the Haar measures λ^x on $G(x)$ normalized by $\int F\,\mathrm{d}\lambda^x = 1$. Under usual hypoteses, this Haar system is Borel, in the sense that for every $f \in C_{cc}(G)$, the function $\lambda(f)$ is Borel on $G^{(0)}$. It suffices to show that S is Borel.

LEMMA 1.5. *Suppose that the* l.c.c. *group bundle G is second-countable. Then the map S of $G^{(0)}$ into $\Sigma^{(0)}$ is Borel.*

Proof. Since $S^{-1}(\mathcal{U}(K;U_1,\dots,U_n)) = p(K)^c \cap p(U_1) \cap \dots \cap p(U_n)$, it suffices to show that the image by p of an open set in G is a Borel set in $G^{(0)}$. But, with our assumptions, every open set in G is a countable union of c.c. sets and the image by p of a c.c. set is closed. ∎

We return to the study of the isotropy group bundle G' of a l.c.c. groupoid G. We choose once for all $F \in C_{cc}(G)$, with $0 \leqslant F \leqslant 1$ and $F(x) = 1$ for $x \in G^{(0)}$ to normalize a Haar system (β^x) on G' and (β^H) on Σ. Now G acts by conjugation on the space of closed subgroups $\Sigma^{(0)}$ and its canonical group bundle Σ. We denote by $\gamma^{-1}\beta^H\gamma$ the image of β^H by the conjugation map $\gamma' \in H \mapsto \gamma^{-1}\gamma'\gamma \in \gamma^{-1}H\gamma$. The uniqueness of the Haar measure on $\gamma^{-1}H\gamma$ provides a 1-cocycle δ, defined on the semi-direct product $\Sigma^{(0)} \rtimes G$, such that

$$\gamma^{-1}\beta^H\gamma = \delta(H,\gamma)^{-1}\beta^{\gamma^{-1}H\gamma}.$$

LEMMA 1.6. *With above notation,*

i) *the 1-cocycle δ is continuous and*

ii) *its continuous cohomology class does not depend on the choice of β.*

Proof. Applying the equality defining δ to F, we have

$$\delta(H,\gamma)^{-1} = \int F(\gamma^{-1}\gamma'\gamma)\,\mathrm{d}\beta^H(\gamma').$$

Suppose that the net (H_i,γ_i) converges to (H,γ). Let K be a compact subset of G containing γ_i, γ. We can find a continuous function f on the compact space $K(\operatorname{supp} f)K^{-1}\times K$ such that $f(\gamma',\gamma)=F(\gamma^{-1}\gamma'\gamma)$ when $s(\gamma')=r(\gamma)$. It is then easy to estimate

$$\int f(\gamma',\gamma_i)\,\mathrm{d}\beta^{H_i}(\gamma')-\int f(\gamma',\gamma)\,\mathrm{d}\beta^{H}(\gamma')$$

to show that it tends to 0. Another continuous Haar system β' on Σ will be of the form $\beta'=h\beta$ where h is a continuous positive function on $\Sigma^{(0)}$. The 1-cocycle δ' associated with β' will be related to δ through

$$h(H)\delta'(H,\gamma)=\delta(H,\gamma)h(\gamma^{-1}H\gamma).$$

■

With a slight abuse of notation, we can call δ the modular function for G. It has the following property: for every $H\in\Sigma^{(0)}$, $\delta_H(\gamma')=\delta(H,\gamma')$ (where $\gamma'\in H$) is the modular function of the group H. We shall also set $\delta(\gamma)=\delta(G(r(\gamma)),\gamma)$. It is a 1-cocycle on G. The following lemma provides the disintegration with respect to the quotient map of G onto R of a Haar system for G. It can be found in [28] (see also [1]) in the transitive case. Recall that a Haar system β for the isotropy bundle G' has been fixed. We set $\beta^x_y=\gamma\beta^y$ where $\gamma\in xGy$.

LEMMA 1.7. *Let G be a second countable* l.c.c. *groupoid and $\beta=(\beta^x_y)$ be as above. Then there is a 1–1 correspondence between Borel Haar systems λ for G and Borel Haar systems α for R given by* $\lambda^x=\int\beta^x_y\,\mathrm{d}\alpha^x(y)$.

Proof. If α is a Borel Haar system for R, the above formula defines a Borel Haar system for G. Conversely, suppose that a Borel Haar system for G is given. For a non--negative Borel function f on G, define $\lambda(f)(x)=\int f\,\mathrm{d}\lambda^x$ and $\beta(f)(x,y)=\int f\,\mathrm{d}\beta^x_y$. A straightforward change of variable using the relations $\gamma\beta^{s(\gamma)}_{s(\gamma)}\gamma^{-1}=\delta(\gamma)\beta^{r(\gamma)}_{r(\gamma)}$ and $\beta^x=\delta_{G(x)}\beta_x$ gives the symmetry property $\lambda^x(f\beta(g))=\lambda^x(\beta(f)g)$ for every non--negative Borel functions f and g. Since xG is locally compact and second countable and β is bounded on compact sets, we can find a non-negative Borel function F on G such that $\beta(F)(x,y)=1$ for every $(x,y)\in xR$. Since β has the conditional expectation property with respect to the quotient map from G onto R, every non--negative Borel function on xR is of the form $\beta(f)$ and $\lambda^x(f)$ depends only on $\beta(f)$, thus we can define the measure α^x on xR by $\alpha^x(\beta(f))=\lambda^x(f)$. Since $\lambda=(\lambda^x)$ is left invariant, so is $\alpha=(\alpha^x)$. We now have to show that, for every non-negative function h on R, the function $x\mapsto\alpha^x(h)$ is measurable. It suffices to show that its restriction to every compact subset L of $G^{(0)}$ is measurable. But this is clear since we can find a non-negative Borel function f on G such that $\beta(f)=h$ on LR and then $\alpha^x(h)=\lambda^x(f)$. ■

COROLLARY 1.8 (cf. Theorem 4.4. of [13]). *Let G be a* l.c.c. *groupoid as above and Λ be a transverse measure for G. Then:*

i) *There exist a transverse measure $\underline{\Lambda}$ for the equivalence relation R such that* $\Lambda=\underline{\Lambda}\circ\beta_G$.

ii) *The Radon-Nikodym derivative* Δ *of* Λ *and* $\underline{\Delta}$ *of* $\underline{\Lambda}$ *are related by* $\Delta(\gamma) = \delta(\gamma)\underline{\Delta}(\dot\gamma)$ a.e.
iii) *The Radon-Nikodym derivative* Δ *of* Λ *can be chosen such that it coincides with* δ *on* G'.

Proof. We fix the Haar systems λ on G, β on G' and α on R as in the previous proposition. Then the transverse measure Λ is given by a measure μ on $G^{(0)}$ such that $\mu\circ\lambda$ and $\mu\circ\lambda^{-1}$ are equivalent to Δ as a version of the Radon-Nikodym derivative $\mu\circ\lambda/\mu\circ\lambda^{-1}$. Since they are pseudo-images of equivalent measures, $\mu\circ\alpha$ and $\mu\circ\alpha^{-1}$ are also equivalent and thus define a transverse measure $\underline{\Lambda}$ on R which depends only on Λ and β. Let $\underline{\Delta}$ be the Radon-Nikodym derivative of $\underline{\Lambda}$. Using the disintegration of λ^x, the relation $\mu\circ\alpha = \underline{\Delta}\mu\circ\alpha^{-1}$ and the quasi-invariance of β with respect to the action of G by conjugation, one obtains that

$$\int f(\gamma)\,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\,\mu(x) = \int f(\gamma^{-1})\delta(\gamma^{-1})\underline{\Delta}(\dot\gamma^{-1})\,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\mu(x)$$

for every non-negative Borel function f on G and therefore that

$$\Delta(\gamma) = \delta(\gamma)\underline{\Delta}(\dot\gamma) \quad \text{for } \mu\circ\lambda \text{ a.e. } \gamma.$$

Since both sides are homomorphisms, the equality holds on the reduction of G to a saturated conull subset of $G^{(0)}$. We obtain (iii) by replacing Δ by $\delta\underline{\Delta}\circ(r,s)$. ∎

We turn now to the second point, which is to get some control of the quotient space $G^{(0)}/G$. Given a subset K of the topological groupoid G, we introduce the relation on $G^{(0)}$ $x \overset{K}{\sim} y$ iff $(x,y)\in R|K = (r,s)(K)$. We shall see that for K symmetric and conditionally compact and when suitably restricted, it becomes an equivalence relation. A similar equivalence relation has already been used by P. Forrest in [9] to construct countable sections for group actions. It is also related to the partition of a distinguished open subset of a foliated manifold into plaques.

LEMMA 1.9. *Let* K *be a conditionally compact subset of* G. *Then*
i) $R|K$ *is a closed subset of* $G^{(0)}\times G^{(0)}$,
ii) *the quotient and the product topologies coincide on* $R|K$.

Proof. Let (γ_i) be a net in K such that $(r(\gamma_i), s(\gamma_i))$ converges to (x,y) in $G^{(0)}\times G^{(0)}$. Because K is conditionally compact, there exists a subnet (γ_i) converging to $\gamma\in K$. By continuity of r,s and uniqueness of the limit in $G^{(0)}$, $(x,y) = (r(\gamma), s(\gamma))\in R|K$. Moreover $(r(\gamma_i), s(\gamma_i))$ converges to (x,y) in the quotient topology and this still holds for the whole net. ∎

LEMMA 1.10. *Let* K *be a conditionally compact subset of* G, x *a point in* $G^{(0)}$ *and* N *be a neighborhood of* $G(x)$ *in* G. *Then there exists a neighborhood* V *of* x *in* $G^{(0)}$ *such that* VKV *is contained in* N.

Proof. Suppose no such V exists. Then there exists a net (γ_i) in $K\cap N^c$ such that $(r(\gamma_i))$ and $(s(\gamma_i))$ converge to x. Let (γ_i) be a subnet converging to γ. Then $\gamma\in G(x)$, which is impossible. ∎

PROPOSITION 1.11 (cf. Theorem 1.4 of [11]). *Let* x *be a point of continuity of the isotropy,* K *be a symmetric conditionally compact neighborhood of* x *in* G *and*

M be a neighborhood of the diagonal in R. Then there exists a neighborhood V of x in $G^{(0)}$ such that the relation $y\overset{K}{\sim}z$ becomes on V an open equivalence relation with a closed graph contained in M.

Proof. We know from Lemma 1.1.(iii) that $G'K$ is a neighborhood of $G(x)$ in G. Therefore there exists an open neighborhood N of $G(x)$ in G such that $N^2 \subset \subset G'K \cap (r,s)^{-1}(M)$. By Lemma 1.10 there exists an open neighborhood V of x in $G^{(0)}$ such that $VKV \subset N$ and $V \subset N$. Then for $y, z \in V$,

$$y\overset{K}{\sim}z \Rightarrow y\overset{N}{\sim}z \Rightarrow y\overset{N^2}{\sim}z \Rightarrow y\overset{K}{\sim}z.$$

Therefore, the relation $y\overset{K}{\sim}z$ restricted to V is:

reflexive, because $V \subset N$;

symmetric, because K is symmetric;

transitive, because the saturation of an open set U contained in V is just $r(NU) \cap V$, which is open.

Its graph, which is $R|K \cap V \times V$ is closed by Lemma 1.9. It is contained in M by construction.

This proposition will be used to solve the equation:

$$e(x) \int_{R|K} e(y)\, \mathrm{d}\alpha^x(y) = b(x)$$

where e is the unknown function. In order to apply it, we shall reduce G to a subset C of $G^{(0)}$ such that the restriction to it of the isotropy is continuous. The next lemma ensures that the restrictions of the orbital measures α^x are non-zero.

LEMMA 1.12. *Let R be a topological principal groupoid such that for each x in the unit space X, xR is a second countable locally compact space, $\alpha = (\alpha^x)$ a Borel Haar system and C a compact subset of X. Then,*

i) *For each $x \in X$, let C^x denote the support of the restriction of α^x to RC. Then the union of the C^x's is of the form RC' where C' is a Borel subset of C which we call the support of $\alpha|C$.*

ii) *$C \setminus C'$ is a null-set for every measure μ on X quasi-invariant with respect to α.*

Proof. The first assertion results from the equivariance of the subsets C^x: for $(x,y) \in R$, $C^x = (x,y)C^y$. The set C' can be defined as $C' = \{x \in C: (x,x) \in C^x\}$. Let μ be a measure on X quasi-invariant with respect to α. By definition, $R(C \setminus C')$ is a null-set for every measure α', hence a null-set for $\mu \circ \alpha$. By quasi-invariance, $(C \setminus C')R$ is also a null-set for $\mu \circ \alpha$ and this implies that $(C \setminus C')$ is a null-set for μ. ∎

We are going to apply the preceding results to crossed product C^*-algebras. Let G be a locally compact groupoid and A a G-bundle of C^*-algebras, that is, a bundle of C^*-algebras over $G^{(0)}$ on which G acts continuously — on the left — by automorphisms. Then G also acts on the primitive ideal space $\operatorname{Prim} A$ of the

C^*-algebra $C_0(G^{(0)}, A)$ of continuous sections of A vanishing at infinity. We shall write it as a right action: for $\gamma \in G$ and $x \in \operatorname{Prim} A_{r(y)}$, we write $x\gamma = \gamma^{-1}(x)$. The projection map of $\operatorname{Prim} A$ onto $G^{(0)}$ will be denoted by p. Since the Jacobson topology on $\operatorname{Prim} A$ is not necessarily Hausdorff, we use instead, as in [11], the regularized topology. We recall that it is the coarsest topology making continuous the functions $x \mapsto \|a + x\|$, for each $a \in C_0(G^{(0)}, A)$. It is Hausdorff, and Polish when A is separable and $G^{(0)}$ second countable. Just as in [11, Proposition 1.6] we have:

PROPOSITION 1.14. *Suppose that G acts continuously on the C^*-algebra bundle A, then it acts continuously on* Prim *A endowed with the regularized topology.*

Proof. Since the regularized topology is finer than the Jacobson topology, the projection map p of $\operatorname{Prim} A$ onto $G^{(0)}$ is continuous — but it may fail to be open —. Let (x_i, y_i) be a net converging to (x, y) in the space $\operatorname{Prim} A \rtimes G$ of composable pairs in the product $\operatorname{Prim} A \times G$. For each a in $C_0(G^{(0)}, A)$, we have

$$\Big| \|a + x_i\gamma_i\| - \|a + x\gamma\| \Big| = \Big| \|\gamma_i a \circ s(\gamma_i) + x_i\| - \|\gamma a \circ s(\gamma) + x\| \Big|.$$

We can find $b \in C_0(G^{(0)}, A)$ such that $b \circ r(\gamma) = \gamma a \circ s(\gamma)$ and majorize the above by

$$\Big| \|\gamma_i a \circ s(\gamma_i) + x_i\| - \|b \circ r(\gamma_i) + x_i\| \Big| + \Big| \|b \circ r(\gamma_i) + x_i\| - \|b \circ r(\gamma) + x\| \Big| \leqslant$$

$$\leqslant \|\gamma_i a \circ s(\gamma_i) - b \circ r(\gamma_i)\| + \Big| \|b + x_i\| - \|b + x\| \Big|.$$

Both terms tend to zero and (x_i, γ_i) converges to $x\gamma$ in $\operatorname{Prim} A$. ∎

The semi-direct product $\operatorname{Prim} A \rtimes G = \{(x, y) \in \operatorname{Prim} A \times G : p(x) = r(\gamma)\}$, where the groupoid structure is defined by

$$r(x, \gamma) = x, \quad s(x, \gamma) = x\gamma \quad \text{and } (x, \gamma)(x\gamma, \gamma') = (x, \gamma\gamma')$$

and the topology is induced from $\operatorname{Prim} A \times G$, is a l.c.c. groupoid in the sense defined at the beginning of the section. We shall assume from now on that G is second countable and has a continuous Haar system λ and that A is separable. Then $(\lambda^x = \varepsilon_x \times \lambda^{p(x)})$, where ε_x is the point mass at x, is a continuous Haar system for the semi-direct product. Moreover, all the separability assumptions which were needed in this section are satisfied.

2. HOMOGENOUS DISINTEGRATION OF REPRESENTATIONS

Recall from [18] the definition of a twisted C^*-dynamical system, or dynamical system in short. It is a triplet (G, Σ, A) consisting of

a) a locally compact groupoid G admitting a Haar system λ,

b) an extension Σ of G by a G-bundle S of locally compact abelian groups admitting a G-invariant Haar system dt,

c) a Σ-bundle A of C^*-algebras such that S is unitarily implemented. That is, there exists a homomorphism x of S into the unitary elements of the multiplier algebra bundle $M(A)$ with the following properties:

i) the map $(s,a) \mapsto \chi(s)a$ is continuous from $S*A$ into A,
ii) for every $(s,a) \in S*A$, $sa = \chi(s)a\chi(s)^{-1}$,
iii) for every $(\sigma, s) \in \Sigma * S$, $\chi(\sigma s \sigma^{-1}) = \sigma\chi(s)$.

When G is a group, this is N. Dang Ngoc [3] and P. Green [12]'s definition of a twisted covariant system in the particular case when the normal subgroup is abelian. We are mostly interested in the case when S is the trivial G-bundle $G^{(0)} \times S$ (the circle group) (see [14]).

Given a dynamical system (G, Σ, A) and a Haar system λ for G, fixed throughout this paper, one can construct in the usual way (see [18]) the crossed-product C^*-algebra $C^*(G, \Sigma, A)$. The main result of [18] is the disintegration of an arbitrary representation of $C^*(G, \Sigma, A)$ over the unit space $G^{(0)}$. We shall need a finer result, namely the disintegration over the primitive ideal space $\operatorname{Prim} A$, which uses the homogenous disintegration theorem of E. Effros [4]. The techniques are the same as in [18]. A direct proof would be possible but we find more convenient to start from the disintegration theorem of [18].

We first recall that the homogeneous disintegration theorem of Effros [4]. A representation M of a C^*-algebra A in a Hilbert space $\mathcal{H}$ is called homogeneous if, for every non-zero projection $e \in M(A)'$, the representations M and M_e have the same kernel. Let M be an arbitrary representation of A in $\mathcal{H}$. Each (closed two-sided) ideal I of A defines a unique projection $e(I)$ in the centre of $M(A)''$ namely the central cover of $M(I)$. This extends to a representation of the $*$-algebra $B(\operatorname{Prim} A)$ of bounded Borel functions on $\operatorname{Prim} A$. When A is separable, it defines a measure class ν on $\operatorname{Prim} A$ and a measurable field of Hilbert spaces $x \mapsto H_x$ such that $\mathcal{H} = \int^{\oplus} H_x \, d\nu(x)$ and $M = \int^{\oplus} M_x \, d\nu(x)$. Then for ν-almost every x, M_x is homogeneous with kernel x. We are now looking for an equivariant version of the disintegration. Suppose that (G, Σ, A) is a dynamical system. As we have seen before, Σ acts continuously on $\operatorname{Prim} A$. Since S acts trivially, G also acts continuously on $\operatorname{Prim} A$ and we can form the semi-direct product $\underline{G} = \operatorname{Prim} A \rtimes G$. We shall denote by $\underline{A} = p^* A$ the pull back of A to $\operatorname{Prim} A$ and set $\underline{\Sigma} = \operatorname{Prim} A \rtimes \Sigma$.

DEFINITION 2.1 (cf. 3.4 of [18]). A representation over $\operatorname{Prim} A$ of the dynamical system (G, Σ, A) is a pair (ν, H) consisting of a transverse measure class ν for $\underline{G}$ and a measurable Hilbert bundle H defined on a Borel subset U of $\operatorname{Prim} A$, of ν-conegligible saturation, endowed with measurable actions L and M of $\underline{\Sigma}_U$ and $\underline{A}_U$ such that

i) $L(x,\sigma)M_{x\sigma}(a)L(x,\sigma)^{-1} = M_x(\sigma a)$ for $(x,\sigma,a) \in \underline{\Sigma}_U * A$,
ii) $L(x,s) = M_x(\chi(s))$ for $(x,s) \in \underline{S}_U$,
iii) For every $x \in U$, the representation M_x of $\underline{A}_x$ on H_x is homogeneous with kernel x.

Two representations (ν, H) and (ν', H') are said to be equivalent if $\nu = \nu'$ and there exists a Borel subset U of $\operatorname{Prim} A$ of ν-conegligible saturation and an isomorphism V of H_U onto H'_U intertwining the actions.

A representation (ν, H) over $\operatorname{Prim} A$ of (G, Σ, A) can be integrated to yield a non-degenerate representation L of $C^*(G, \Sigma, A)$ in the Hilbert space $\mathcal{H} = \int^{\oplus} H_x \, d\nu(x)$

according to the formula

$$\langle \xi, L(f)\eta\rangle = \int \langle \xi(x), M_x(f(\sigma))L(x,\sigma)\eta(x\sigma)\rangle_x \Delta^{1/2}(x,\dot\sigma)\,\mathrm{d}\lambda^x(\dot\sigma)\,\mathrm{d}\nu(x)$$

for $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, \Sigma, A)$. In the formula, we have written λ^x instead of $\lambda^{p(x)}$ and we have identified the transverse measure class ν with the quasi-invariant measure it provides in presence of the Haar system λ. The Radon-Nikodym derivative $\frac{\nu\circ\lambda^{-1}}{\nu\circ\lambda}$ has been denoted by Δ. From the definition of $C_c(G, \Sigma, A)$, the function under the integrand depends only on $(x, \dot\sigma)$, where $\dot\sigma$ is the image of σ in G. One can check as in [18] that equivalent representations of $C_c(G, \Sigma, A)$ yield equivalent representations of $C^*(G, \Sigma, A)$. Under the usual separability assumptions, every non-degenerate representation of $C_c(G, \Sigma, A)$ is so obtained:

THEOREM 2.2 (cf. §4 of [18]). *Let (G, Σ, A) be a dynamical system with Σ second countable and A separable and λ be a Haar system for G. Then every non--degenerate representation of $C^*(G, \Sigma, A)$ in a separable Hilbert space is equivalent to a representation obtained by integration.*

Proof. Let L be a non-degenerate representation of $C_c(G, \Sigma, A)$ in a separable Hilbert space $\mathcal{H}$. From [18, Lemma 4.6] there exist unique non-degenerate representation of $C^*(\Sigma)$ and $C_0(G^{(0)}, A)$, denoted respectively by L and M, such that for every $f \in C_c(G, \Sigma, A)$,

$$L(\varphi * f) = L(\varphi)L(f) \quad \text{for } \varphi \in C_c(\Sigma)$$

$$L(hf) = M(h)L(f) \quad \text{for } h \in C_0(G^{(0)}, A).$$

They can be disintegrated to yield a representation of (G, Σ, A) in the sense of [18, Definition 3.4]: a pair (μ, K) consisting of a quasi-invariant measure μ for G and a Hilbert bundle K defined over a Borel subset V of $G^{(0)}$, which can be chosen μ--negligible, endowed with measurable actions L and M of Σ_V and A_V such that

i) $L(\sigma)M_{s(\sigma)}(a)L(\sigma)^{-1} = M_{r(\sigma)}(\sigma a)$ for $(\sigma, a) \in \Sigma_V * A$

ii) $L(s) = M_{r(s)}(\chi(x))$ for $s \in S_V$.

Identifying $\mathcal{H}$ with $\int^{\oplus} K_u\,\mathrm{d}\mu(u)$, M decomposes as $\int^{\oplus} M_u\,\mathrm{d}\mu(u)$. As we have seen before, M defines a representation M'' of $B(\operatorname{Prim} A)$ and a measure class μ on $\operatorname{Prim} A$. Similarly, for every $u \in V$, M_u defines a representation M'' of $B(\operatorname{Prim} A_u)$ and a measure class ν^u on $\operatorname{Prim} A_u$. Since M'' decomposes as $\int^{\oplus} M''_u\,\mathrm{d}\mu(u)$, ν decomposes as $\int^{\oplus} \nu_u\,\mathrm{d}\mu(u)$. The covariance relation (i) holds also for M'': we have

i′) $L(\sigma)M''_{s(\sigma)}(h)L(\sigma)^{-1} = M''_{r(\sigma)}(\sigma h)$ for $\sigma \in \Sigma_V$ and $h \in B(\operatorname{Prim} A)$, where $\sigma h(x) = h(x\sigma)$.

This shows that $\nu^{s(\gamma)}\gamma^{-1} \sim \nu^{r(\gamma)}$ for $\gamma \in G_V$. Together with the quasi-invariance of μ, this implies the quasi-invariance of ν. We choose Radon-Nikodym derivatives

$$\delta = \frac{\mu\circ\lambda}{\mu\circ\lambda^{-1}}, \quad \Delta = \frac{\nu\circ\lambda}{\nu\circ\lambda^{-1}} \text{ and } \rho(\cdot,\gamma) = \frac{\nu^{s(\gamma)}\gamma^{-1}}{\nu^{r(\gamma)}}.$$

They are related by $\rho(x,y)\Delta(x,y) = \delta(\gamma)$ for almost every (x,y).

The next task is the construction of the Hilbert bundle $x \mapsto H_x$ over $\operatorname{Prim} A$ and the operators $L(x,\sigma)$ and $M_x(a)$. This can be done by using the groupoid $\underline{\Sigma} = \operatorname{Prim} A \times \Sigma$. Let $B(\underline{\Sigma})$ denote the space of bounded Borel functions on $\underline{\Sigma}$ which vanish outside a rectangle $\operatorname{Prim} A \times K$, where K is compact in Σ. For $f \in B(\underline{\Sigma})$ and $\sigma \in \Sigma$, $f(\sigma)$ will denote the function $f(\cdot,\sigma)$ defined on $\operatorname{Prim} A_{r(\sigma)}$. The space $B(\underline{\Sigma})$ is made into a $*$-algebra with the usual convolution and involution. The representation L of $C^*(G,\Sigma,A)$ defines a non-degenerate representation, still denoted by L, of $B(\underline{\Sigma})$ through the formula

$$\langle \xi, L(f)\eta\rangle = \int \langle \xi(u), M''_u[f(\sigma)]\, L(\sigma)\eta \circ s(\sigma)\rangle_u \delta^{-1/2}(\dot\sigma)\, d\sigma\, d\mu(u)$$

where $f \in B(\underline{\Sigma})$, ξ and $\eta \in \mathcal{H} = \int^{\oplus} K_u\, d\mu(x)$ and $d\sigma$ is the Haar system $dt\, d\lambda$ on Σ. The construction of the G-Hilbert bundle H given in the proof of Proposition 4.2 of [18] uses only the standard Borel structure of G, hence applies to $\underline{\Sigma}$. Therefore there exist a representation (ν, H) of $\underline{\Sigma}$ and an isometry V of $\mathcal{H}$ onto $\tilde{\mathcal{H}} = \int^{\oplus} H_x\, d\nu(x)$ which intertwines L and the integrated representation $\tilde L$ given by

$$\langle \xi, \tilde L(f)\eta\rangle = \int \langle \xi(x), f(x,\sigma)L(x,\sigma)\eta(x\sigma)\rangle_x \Delta^{-1/2}(x,\dot\sigma)\, d\sigma\, d\nu(x)$$

where $f \in \underline{\Sigma}$, ξ and $\eta \in \tilde{\mathcal{H}}$. Since for $h \in B(\operatorname{Prim} A)$ and $f \in B(\underline{\Sigma})$, $M''(h)L(h) = L(hf)$, V intertwines $M''(h)$ and the operator of multiplication by h. In particular, if we decompose $\mathcal{H}$ as $\int^{\oplus} K_u\, d\mu(u)$ and $\tilde{\mathcal{H}}$ as $\int^{\oplus} \tilde K_u\, d\mu(u)$ with $\tilde K_u = \int^{\oplus} H_x\, d\nu^u(u)$, then V decomposes as $\int^{\oplus} V_u\, d\mu(u)$. Let us define $\tilde L(\sigma) = V_{r(\sigma)} \cdot L(\sigma) V^{-1}_{s(\sigma)}$ for $\sigma \in \Sigma_V$ and $M_u(a) = V_u M_u(a) V_u^{-1}$ for $(u,a) \in V * A$. Comparing both formulas for the coefficients of $\tilde L$, we see that, up to a null-set in $\underline{\Sigma}$,

$$\tilde L(\sigma)\xi(x) = \rho^{1/2}(x,\dot\sigma) L(x,\sigma)\xi(x\sigma) \quad \text{for every } \xi \in \tilde{\mathcal{H}}.$$

Because $B(\operatorname{Prim} A_u)$ acts by multiplication operators, $\tilde M_u$ decomposes as $\int^{\oplus} M_x \cdot d\nu^u(x)$. The covariance relation (i), which also holds for the representations $\tilde L$ and $\tilde M$ of Σ_V and A_V on $\tilde K$, and a standard argument on null-sets of $\underline{\Sigma}$ give the existence of a conull-set U in $\operatorname{Prim} A$ such that

$$L(x,\sigma)M_{x\sigma}(a) = M_x(\sigma a)L(x,\sigma) \quad \text{for } (x,\sigma,a) \in \underline{\Sigma}_U * A.$$

Because of the relation (ii), we can choose U to have also

$$L(x,s) = M_x(\chi(s)) \quad \text{for } (x,s) \in \underline{S}_U.$$

Since from its construction, for almost every u in $G^{(0)}$ and ν^u almost every x, M_x is a homogeneous representation of A_u with kernel x, we can choose U to have also M_x homogeneous with kernel x for $x \in U$.

We have thus obtained a representation (ν, H) over $\operatorname{Prim} A$ of the dynamical system (G, Σ, A). It remains to check that V intertwines L and the representation $\tilde{L}$ obtained by integration. This is straightforward: for $f \in C_c(G, \Sigma, A)$ and $\xi,\ \eta \in \mathcal{H}$,

$$\langle \xi, L(f)\eta\rangle = \int \langle \xi(u), M_u\,[f(\sigma)]\,L(\sigma)\eta \circ s(\sigma)\rangle_u \delta^{-1/2}(\dot{\sigma})\,\mathrm{d}\lambda^u(\dot{\sigma})\,\mathrm{d}\mu(u) =$$

$$= \int \langle \tilde{\xi}(x), M_x\,[f(\sigma)]\,L(x,\sigma)\tilde{\eta}(x,\sigma)\rangle_x \cdot$$

$$\cdot \rho^{1/2}(x,\dot{\sigma})\delta^{-1/2}(\dot{\sigma})\,\mathrm{d}\nu^u(x)\,\mathrm{d}\lambda^u(\dot{\sigma})\,\mathrm{d}\mu(u) =$$

$$= \int \langle \tilde{\xi}(x), M_x\,[f(\sigma)]\,L(x,\sigma)\tilde{\eta}(x,\sigma)\rangle_x \Delta^{-1/2}(x,\dot{\sigma})\,\mathrm{d}\lambda(\dot{\sigma})\,\mathrm{d}\nu(x) =$$

$$= \langle \tilde{\xi}, \tilde{L}(f)\tilde{\eta}\rangle, \quad \text{where } \tilde{\xi} = V\xi \text{ and } \tilde{\eta} = V\eta.$$

∎

The rest of this section will deal with restricting and inducing representations. Let (G, Σ, A) be a dynamical system. As before, we form $\underline{G} = \operatorname{Prim} A \rtimes G$, $\underline{\Sigma} = \operatorname{Prim} A \rtimes \Sigma$ and $\underline{A} = \operatorname{Prim} A * A = p^*A$. To a closed subgroupoid G' of $\underline{G}$ one can associate the dynamical system (G', Σ', A') where Σ' is the pull-back of $\underline{\Sigma}$ to G' and A' is the reduction of A to $G'^{(0)}$. Definition 2.1 of a representation of a dynamical system applies as well to (G', Σ', A'). From now on we shall assume that G' is a closed subbundle of a isotropy bundle of $\underline{G}$, equipped with the Borel Haar system β inherited from the space of closed subgroups of G and that G is equipped with a fixed Haar system λ.

Then a representation $L = (\nu, H)$ of (G, Σ, A) defines by restriction a representation $L' = (\nu', H')$ of (G', Σ', A'). Here we view ν as a measure on $\operatorname{Prim} A$ and ν' is its restriction to $G'^{(0)}$. The Hilbert bundle is the reduction of H to $G'^{(0)}$ endowed with the actions of Σ'_U and A'_U. We shall sometimes write $L' = \operatorname{Res}^G_{G'} L$.

Conversely, suppose that we are given a representation $L' = (\nu', H')$ of (G', Σ', A'). To define the induced representation $L = \operatorname{Ind}^G_{G'} L'$ of (G, Σ, A), one can proceed in two steps. First $\underline{\Sigma}$, viewed as a principal left Σ'-space, establishes an equivalence (see [18, Definition 5.3]) between (G', Σ', A') and $(\underline{\Sigma} * \underline{\Sigma}/S' \ltimes \underline{\Sigma}', \underline{\Sigma} * \Sigma/\Sigma', \underline{\Sigma} * A'/\Sigma')$ and therefore carries L' into a representation $L'' = (\nu'', H'')$ of $(\underline{\Sigma} * \underline{\Sigma}/S' \ltimes \Sigma', \underline{\Sigma} * \underline{\Sigma}/\Sigma'$, $\underline{\Sigma} * A'/\Sigma')$. Explicitly the measure ν'' on $\underline{\Sigma}/\Sigma' = \underline{G}/G'$ can be defined as in 1.10 by

$$\int f(x,\gamma)\delta^{-1}(x,\gamma)\,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\nu'(x) = \int f(x,\gamma'\gamma)\,\mathrm{d}\beta_x(\gamma')\,\mathrm{d}\nu''\overbrace{(x,\gamma)}.$$

The Hilbert bundle H'' is $\underline{\Sigma} * H'/\Sigma'$, where $(\underline{\sigma}, \xi') \sim (\sigma'\underline{\sigma}, L'(\sigma')\xi')$ for $\sigma' \in \Sigma'$. The action of $\underline{\Sigma} * \underline{\Sigma}/\underline{\Sigma}'$ is given by $L''\overbrace{(\underline{\sigma},\underline{\tau})}\overbrace{(\underline{\tau},\xi')} = \overbrace{(\underline{\sigma},\xi')}$ and that of $\underline{\Sigma} * A'/\Sigma'$ by $M''\overbrace{(\underline{\sigma},a)}(\underline{\sigma},\xi') = \overbrace{(\underline{\sigma}, M'_{r(\sigma)}(a)\xi')}$.

In the second step, we write $Y = \underline{\Sigma}/\Sigma' = \underline{G}/G'$ and we identify $(\underline{\Sigma} * \underline{\Sigma}/S' \ltimes \underline{\Sigma}$, $\underline{\Sigma} * \Sigma/\Sigma', \underline{\Sigma} * A'/\Sigma')$ with $(Y \rtimes \underline{G}, Y \rtimes \underline{\Sigma}, Y * \underline{A})$. The identification maps are $\overbrace{(\underline{\sigma},\underline{\tau})} \in$

$\in \underline{\Sigma} * \underline{\Sigma}/\underline{\Sigma}' \mapsto (\dot{\underline{\sigma}}, \underline{\sigma}^{-1}\underline{\tau}) \in Y \times \underline{\Sigma}$ and $\widehat{(\underline{\sigma}, a)} \in \underline{\Sigma}*A'/\Sigma' \mapsto (\dot{\underline{\sigma}}, \sigma^{-1}a) \in Y* *\underline{A}$. Let us review how a representation $L'' = (\nu'', H'')$ of the semi-direct product $(Y \rtimes \underline{G}, Y \rtimes \underline{\Sigma}, Y*\underline{A})$ induces a representation $L = (\nu, H)$ of $(\underline{G}, \underline{\Sigma}, \underline{A})$. We choose for ν a pseudo-image of ν'' by the map $s: Y \mapsto \operatorname{Prim} A$ and disintegrate ν'' with respect to s: $\nu'' = \int \tau_x \, d\nu(x)$. Since ν'' is quasi-invariant, so is ν and the measure system $\{\tau_x\}$ quasi-invariant under G. Let Δ and δ'' denote respectively the modules of ν and ν'' and set $\rho(y, \underline{\gamma}) = \dfrac{\Delta(\gamma)}{\delta''(y, \underline{\gamma})}$. Then we have $\dfrac{\tau_{x\gamma} \cdot \gamma^{-1}}{\tau_x}(y) = \rho(y, \underline{\gamma})$ for a.e. $(y, \underline{\gamma})$, where $\underline{\gamma} = (x, \gamma)$.

The Hilbert bundle H is defined by its fibres $H_x = L^2(Y, \tau_2, H'')$ and the measurable sections given by measurable sections of H''. The action of $\underline{\Sigma}$ on H is given by

$$[L(\sigma)\xi](y) = \rho(y, \dot{\underline{\sigma}})^{1/2} L''(y, \underline{\sigma})\xi(y, \underline{\sigma})$$

and that of $\underline{A}$ by

$$[M_x(a)\xi](y) = M''(y, a)\xi(y) \quad \text{for } \xi \in H_x.$$

One can check that this is indeed a representation of (G, Σ, A) in the sense of 2.1 and that its equivalence class is independent of the choises we made. We call L the representation induced by L'' — or by L' in our case — and write $L = \operatorname{Ind}_{G'}^{G} L'$. Let us just quote the theorem on induction by stages: for $G' \subset G'' \subset G$, we have $\operatorname{Ind}_{G'}^{G} L' = \operatorname{Ind}_{G''}^{G} \circ \operatorname{Ind}_{G'}^{G''} L'$. We recall that the regular representations of (G, Σ, A) are the representations induced by $G^{(0)}$ — or more exactly from $\operatorname{Prim} A$ with our notation —.

We conclude this section by writing down the coefficients of an induced representation. Let $L = (\nu, H)$ be the representation of (G, Σ, A) induced from a representation $L'' = (\nu'', H'')$ of $(Y \rtimes \underline{G}, Y \rtimes \underline{\Sigma}, Y*\underline{A})$. The Hilbert space of the integrated representation is $\mathcal{H} = \int^{\oplus} H''_y \, d\nu''(y)$. For $\xi, \eta \in \mathcal{H}$ and $f \in C_c(G, \Sigma, A)$,

$$\langle \xi, L(f)\eta \rangle = \int \langle \xi(y), M''(y, f(\sigma))L''(y, \underline{\sigma})\eta(y\underline{\sigma})\rangle \delta''^{-1/2}(y, \dot{\underline{\sigma}}) \, d\lambda^y(\dot{\sigma}) \, d\nu''(y)$$

where, as before, $\underline{\sigma} = (s(y), \sigma)$. Moreover, if the representation L'' arises from a representation $L' = (\nu', H')$ as above, the Hilbert space $\mathcal{H}$ may be viewed as the space of measurable sections $\xi: \underline{\Sigma} \mapsto H'$ such that $\xi(\underline{\sigma}) \in H'_{r(\underline{\sigma})}$, $\xi(\sigma'\underline{\sigma}) = L'(\sigma')\xi(\underline{\sigma})$ for $\sigma' \in \Sigma'$ and $\|\xi\|^2 = \int \|\xi(\underline{\sigma})\|^2 \, d\nu''(\dot{\sigma}) < \infty$. The coefficients of L are given by

$$\langle \xi, L(f)\eta \rangle = \int \langle \xi(x, \tau), M'_x[\tau f(\sigma)] \, \eta(x, \tau\sigma)\rangle_x \delta^{-1/2}(x\tau, \dot{\sigma}) \, d\lambda^{x\tau}(\dot{\sigma}) \, d\nu'' \widehat{(x, \tau)}.$$

As it is well known, the construction of the induced representation can be viewed as a change-of-ring operation. This is summarized in the next lemma.

LEMMA 2.3. *Let $L' = (\nu', H')$ be a representation of the subsystem (G', Σ', A') as above and L the induced representation of the Hilbert space $\mathcal{H}$.*

i) *There exists a unique map of* $C_c(G,\Sigma,A)\otimes L^2(G'^{(0)},\nu',H')$ *into* $\mathcal{H}$ *sending* $f\otimes\xi$ *onto* $f\hat{\otimes}\xi(x,\sigma) = \int M'_x\left[\sigma f(\sigma^{-1}\sigma')\right]\delta^{1/2}(x,\sigma'^{-1}\sigma)L'(x,\sigma')\xi(x)\,\mathrm{d}\beta^x(\dot{\sigma}')$ *and it has dense range.*

ii) *For* $f_i\in C_c(G,\Sigma,A)$ *and* $\xi_i\in L^2(G'^{(0)},\nu',H')$, $i=1,2$

$$\langle f_1\hat{\otimes}\xi_1, f_2\hat{\otimes}\xi_2\rangle = \int \langle \xi_1(x), M'_x\left[f_1^* * f_2(\sigma')\right]L'(x,\sigma')\xi_2(x)\rangle\cdot$$

$$\cdot\delta^{-1/2}(x,\sigma')\,\mathrm{d}\beta^x(\dot{\sigma}')\,\mathrm{d}\nu'(x).$$

iii) *For* $f,g\in C_c(G,\Sigma,A)$ *and* $\xi\in L^2(G'^{(0)},\nu',H')$,

$$L(f)g\hat{\otimes}\xi = f*g\hat{\otimes}\xi.$$

Proof. i) The function $f\hat{\otimes}\xi$ of $\underline{\Sigma}$ into H' is measurable and satisfies $f\hat{\otimes}\xi(x,\sigma)\in H'_x$ and $f\hat{\otimes}\xi(x,\sigma'\sigma) = L'(\sigma')f\hat{\otimes}\xi(x,\sigma)$. Moreover, since

$$\left\|f\hat{\otimes}\xi(x,\sigma)\right\| \leqslant \int \left\|f(\sigma^{-1}\sigma')\right\|\,\|\xi(x)\|\,\delta^{1/2}(x,\sigma'^{-1}\sigma)\,\mathrm{d}\beta^x(\dot{\sigma}')$$

the integral $\int \|f\hat{\otimes}\xi\,\overset{\cdot}{\overbrace{(x,\sigma)}}\|^2\,\mathrm{d}\nu''\overset{\cdot}{\overbrace{(x,\sigma)}}$ is less than

$$\int \left\|f(\sigma^{-1}\sigma')\right\|\,\left\|f(\sigma^{-1}\sigma'')\right\|\,\|\xi(x)\|^2\,\delta^{1/2}(x,\sigma'^{-1}\sigma)\cdot$$

$$\cdot\delta^{1/2}(x,\sigma''^{-1}\sigma)\,\mathrm{d}\beta^x(\sigma')\,\mathrm{d}\beta^x(\sigma'')\,\mathrm{d}\nu''\overset{\cdot}{\overbrace{(x,\sigma)}} =$$

$$= \int \left\|f(\sigma^{-1}\sigma')\right\|\,\left\|f(\sigma^{-1}\sigma'\sigma'')\right\|\,\|\xi(x)\|^2\,\delta^{1/2}(x,\sigma'^{-1}\sigma)\cdot$$

$$\cdot\delta^{1/2}(x,\sigma''^{-1}\sigma'^{-1}\sigma)\,\mathrm{d}\beta^x(\sigma'')\,\mathrm{d}\beta^x(\sigma')\,\mathrm{d}\nu''\overset{\cdot}{\overbrace{(x,\sigma)}} =$$

(by the change of variable $\sigma''\mapsto\sigma'\sigma''$)

$$= \int \|f^*(\sigma'\sigma)\|\,\left\|f(\sigma'\sigma^{-1}\sigma'')\right\|\,\|\xi(x)\|^2\,\delta(x,\sigma'\sigma)\cdot$$

$$\cdot\delta^{-1/2}(x,\sigma'')\,\mathrm{d}\beta^x(\sigma'')\,\mathrm{d}\beta_x(\sigma')\,\mathrm{d}\nu''\overset{\cdot}{\overbrace{(x,\sigma)}} =$$

(by definition of β_x)

$$= \int \|f^*(\sigma)\|\,\left\|f(\sigma^{-1}\sigma'')\right\|\,\|\xi(x)\|^2\,\delta^{-1/2}(x,\sigma'')\,\mathrm{d}\beta^x(\sigma'')\,\mathrm{d}\lambda^x(\sigma)\,\mathrm{d}\nu'(x) =$$

(by definition of ν'')

$$= \int (\|f^*\| * \|f\|)(\sigma'')\,\|\xi(x)\|^2\,\delta^{-1/2}(x,\sigma'')\,\mathrm{d}\beta^x(\sigma'')\,\mathrm{d}\nu'(x) \leqslant$$

$$\leqslant \sup_x \left(\int (\|f^*\| * \|f\|)(\sigma'') \, d\beta^x(\sigma'') \right) \|\xi\|^2 .$$

This shows that $f\hat{\otimes}\xi$ belongs to $\mathcal{H}$. Since $f\hat{\otimes}\xi$ is linear in each of the variables, there is a unique linear map of $C_c(G, \Sigma, A) \otimes L^2(G^{(0)}, \nu', H')$ into $\mathcal{H}$ sending $f \otimes \xi$ onto $f\hat{\otimes}\xi$. Suppose next that $\eta \in \mathcal{H}$ is orthogonal to every $f\hat{\otimes}\xi$, where $f \in C_c(G, \Sigma, A)$ and $\xi \in L^2(G^{(0)}, \nu', H')$. Then

$$\langle f\hat{\otimes}\xi, \eta \rangle = \int \langle f\hat{\otimes}\xi(x,\sigma), \eta(x,\sigma)\rangle \, d\nu'' \overbrace{(x,\sigma)}^{\cdot} =$$

$$= \int \langle M'_x \left[\sigma f(\sigma^{-1}\sigma')\right] L'(x,\sigma')\xi(x), \eta(x,\sigma)\rangle \cdot$$

$$\cdot \delta^{1/2}(x, \sigma'^{-1}\sigma) \, d\beta^x(\sigma') \, d\nu'' \overbrace{(x,\sigma)}^{\cdot} =$$

$$= \int \langle M'_x \left[\sigma'\sigma f(\sigma^{-1}\sigma'^{-1})\right] \xi(x), \eta(x, \sigma'\sigma)\rangle \delta^{1/2}(x, \sigma'\sigma) \, d\beta_x(\sigma') \, d\nu'' \overbrace{(x,\sigma)}^{\cdot} =$$

$$= \int \langle M'_x \left[\sigma f(\sigma^{-1})\right] \xi(x), \eta(x,\sigma)\rangle \delta^{-1/2}(x,\sigma) \, d\lambda^x(\dot\sigma) \, d\nu'(x) = 0.$$

This implies that for every $f \in C_c(G, \Sigma, A)$ and $\xi \in L^2(G^{(0)}, \nu', H')$

$$\langle M'_x \left[f(\sigma)\right] \xi(x), \mu(x,\sigma)\rangle = 0 \quad \nu' \circ \lambda\text{-almost everywhere.}$$

The separability assumptions and the nondegeneracy of the representation M_x then imply that $\eta(x,\sigma) = 0$ $\nu' \circ \lambda$-almost everywhere and therefore that $\eta = 0$ in $\mathcal{H}$.

ii) The computation of $\langle f_1\hat{\otimes}\xi_1, f_2\hat{\otimes}\xi_2\rangle$ is similar to — and justified by — the computations done in i).

iii) This is a straightforward computation. On one hand, we have seen in (i) that

$$\langle \eta (f{*}g)\hat{\otimes}\xi = \int \langle \eta(x,\sigma), M'_x \left[\sigma(f{*}g)(\sigma^{-1})\right] \xi(x)\rangle \delta^{-1/2}(x,\dot\sigma) \, d\lambda^x(\dot\sigma) \, d\nu'(x)$$

for every $\eta \in \mathcal{H}$. On the other hand, by definition of the induced representation L, the coefficient $\langle \eta, L(f) g\hat{\otimes}\xi\rangle$ is equal to

$$\int \langle \eta(x,\sigma), M'_x \left[\sigma f(\tau)\right] g\hat{\otimes}\xi(x, \sigma\tau)\rangle \delta^{-1/2}(x,\sigma,\dot\tau) \, d\lambda^{s(\sigma)}(\dot\tau) \, d\nu' \overbrace{(x,\sigma)}^{\cdot} =$$

$$= \int \langle \eta(x,\sigma), M'_x \left[\sigma f(\tau)\right] M'_x \left[\sigma\tau g(\tau^{-1}\sigma^{-1}\sigma')\right] L'(x,\sigma')\xi(x)\rangle \cdot$$

$$\cdot \delta^{1/2}(x, \sigma'^{-1}\tau) \delta^{-1/2}(x\sigma, \tau) \, d\beta^x(\dot\sigma') \, d\lambda^{s(\sigma)}(\dot\tau) \, d\nu'' \overbrace{(x,\sigma)}^{\cdot} =$$

$$= \int \langle \eta(x, \sigma'\sigma), M'_x \left[\sigma'\sigma f(\tau)\tau g(\tau^{-1}\sigma^{-1}\sigma'^{-1})\right] \xi(x)\rangle \cdot$$

$$\cdot\delta^{-1/2}(x\sigma,\sigma^{-1}\sigma'^{-1})\,\mathrm{d}\lambda^{s(\sigma)}(\dot\tau)\,\mathrm{d}\beta_x(\dot\sigma')\,\mathrm{d}\nu''\widehat{(x,\sigma)} =$$

$$= \int \langle \eta(x,\sigma), M'_x\left[\sigma f(\tau)\tau g(\tau^{-1}\sigma^{-1})\right]\,\xi(x)\rangle\cdot$$

$$\cdot\delta^{-1/2}(x\sigma,\sigma^{-1})\delta^{-1}(x,\sigma)\,\mathrm{d}\lambda^{s(\sigma)}(\dot\tau)\,\mathrm{d}\lambda^{x}(\dot\sigma)\mathrm{d}\nu'(x) =$$

$$= \int \langle \eta(x,\sigma), M'_x\left[\sigma(f{*}g)(\sigma^{-1})\right]\,\xi(x)\rangle\delta^{-1/2}(x,\dot\sigma)\,\mathrm{d}\lambda^{x}(\dot\sigma)\,\mathrm{d}\nu'(x).$$

■

The next lemma relates the representation of the twisted crossed product $C_c(G,\Sigma,A)$ and those of the ordinary crossed product $C^*(\Sigma,A)$. As in [18], one defines

$$\chi(F)(\sigma) = \int F(t\sigma)\chi(t)\,\mathrm{d}t \quad \text{for } F\in C_c(\Sigma,\Sigma{*}A).$$

LEMMA 2.4. *Let (G,Σ,A) be a dynamical system and G' be a closed subbundle of the isotropy bundle of* Prim A.

i) *The map χ extends to a $*$-homorphism from $C^*(\Sigma,A)$ onto $C^*(G,\Sigma,A)$.*

ii) *If L is the representation of $C^*(G,\Sigma,A)$ obtained by integrating a representation of (G,Σ,A), then $L\circ\chi$ is the representation of $C^*(\Sigma,A)$ obtained by integrating the same representation, viewed as a representation of (Σ,A).*

(iii) *If L is the representation of $C^*(G,\Sigma,A)$ induced from a representation of (G',Σ',A'), then $L\circ\chi$ is the representation of $C^*(\Sigma,A)$ induced from the same representation, viewed as a representation of (Σ',A').*

Proof. i) This results from [18, lemme 3.3].
ii) and iii) These result from the definitions and straightforward calculations. ■

3. PROOF OF THE MAIN THEOREM

Let (G,Σ,A) be a dynamical system in the sense of the previous section and let L be a (non-degenerate) representation of $C^*(G,\Sigma,A)$ in a separable Hilbert space $\mathcal{H}$. As we have seen, it can be disintegrated over Prim A. Identifying $\mathcal{H}$ with $L^2(\mathrm{Prim}\,A,\nu,H)$, the coefficients of L are given by

$$\langle\xi,L(f)\zeta\rangle = \int \langle\xi(x),M_x(f(\sigma))L(x,\sigma)\zeta(x,\sigma)\rangle_x\Delta^{-1/2}(x,\dot\sigma)\,\mathrm{d}\lambda^x(\dot\sigma)\,\mathrm{d}\nu(x)$$

where $f\in C_c(G,\Sigma,A)$ and $\xi,\zeta\in\mathcal{H}$. Replacing $M_x(f(\sigma))$ by $M_x(f(x,\sigma))$ in this formula, one can define a bounded operator $L(f)$ when f is in $\mathcal{L}^1(G,\Sigma,A)$, the space of Borel functions of Σ into $\underline{A}$ such that $f(x,\sigma)\in A_{p(x)}$, $f(x,s\sigma)\chi(s^{-1})$ and

$$\|f\|_{1,\lambda} = \max\left(\sup\int\|f(x,\sigma)\|\,\mathrm{d}\lambda^x(\dot\sigma),\ \sup\int\left\|f(x\sigma,\sigma^{-1})\right\|\,\mathrm{d}\lambda^x(\dot\sigma)\right)$$

is finite. Then the norm of $L(f)$ is bounded by $\|f\|_{1,\lambda}$. The space $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ is made into a normed $*$-algebra with the following operations

$$f*_{\lambda}g(x,\sigma) = \int f(x,\tau)\left[\tau g(x\tau, \tau^{-1}\sigma)\right] \, \mathrm{d}\lambda^x(\dot{\tau})$$

$$f^*(x,\sigma) = \sigma\left[f(x\sigma, \sigma^{-1})\right]^* .$$

Let G' be the isotropy group bundle of $\underline{G} = \mathrm{Prim}\, A \times G$ equipped with its Borel Haar system β as in Section 1. We define in a similar fashion the normed $*$-algebra $\mathcal{L}^1(G', \Sigma', A')$ by using β. There are two pieces of structure relating $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ and $\mathcal{L}^1(G', \Sigma', A')$. First $\mathcal{L}^1(\underline{G'}, \underline{\Sigma'}, \underline{A'})$ acts on $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ by left and right multipliers according to

$$g*_{\beta}f(x,\sigma) = \int g(x,\tau)\left[\dot{\tau} f(x\tau, \tau^{-1}\sigma)\right] \, \mathrm{d}\beta^x(\dot{\tau}) =$$

$$= \int g(x,\sigma\tau)\left[\sigma\tau f(x\sigma\tau, \tau^{-1})\right] \, \mathrm{d}\beta_x^{x\sigma}(\dot{\tau})$$

$$f*_{\beta}g(x,\sigma) = \int f(x,\sigma\tau)\left[\sigma\tau g(x\sigma\tau, \tau^{-1})\right] \, \mathrm{d}\beta^{x\sigma}(\dot{\tau}) =$$

$$= \int f(x,\tau)\left[\tau g(x\tau, \tau^{-1}\sigma)\right] \, \mathrm{d}\beta^x_{x\sigma}(\dot{\tau})$$

where $g \in \mathcal{L}^1(G', \Sigma', A')$ and $f \in \mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$. Second the restriction map P of $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ into $\mathcal{L}^1(G', \Sigma', A')$ is a (not everywhere defined) conditional expectation in the sense that

$$P(g*_{\beta}f) = g*_{\beta}P(f) \quad \text{and} \quad P(f*_{\beta}g) = f*_{\beta}P(g)$$

whenever these expressions are defined. Note that $C_c(\underline{G}, \underline{\Sigma}, \underline{A})$ is the domain of P. Finally we define the restriction L' of L to $\mathcal{L}^1(G', \Sigma', A')$ by its coefficients:

$$\langle \xi, L'(g)\zeta \rangle = \int \langle \xi(x), M_x(g(x,\sigma)) L(x,\sigma)\zeta(x)\rangle_x \delta^{-1/2}(x,\dot{\sigma}) \, \mathrm{d}\beta^x(\dot{\sigma}) \, \mathrm{d}\nu(x)$$

where $g \in \mathcal{L}^1(G', \Sigma', A')$, ξ and $\zeta \in \mathcal{H}$ and δ is the modular function of G'. The norm of $L'(g)$ is bounded by $\|g\|_{1,\beta}$.

LEMMA 3.1. i) *The $*$-algebra $\mathcal{L}^1(G', \Sigma', A')$ acts on $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ by multipliers.*
ii) *The representation L and its restriction L' are related by*

$$L(g*_{\beta}f) = L'(g)L(f) \quad \text{for } g \in \mathcal{L}^1(G', \Sigma', A') \text{ and } f \text{ in } \mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A}).$$

Proof. i) We have to check the relations

$$(g*_{\beta}f)^* = f^* *_{\beta} g^* \quad \text{where } g \in \mathcal{L}^1(G', \Sigma', A') \text{ and } f \in \mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$$

$$f_1 *_{\lambda}(g*_{\beta}f_2) = (f_1*_{\beta}g)*_{\lambda}f_2 \quad \text{where } g \in \mathcal{L}^1(G', \Sigma', A') \text{ and } f_1, f_2 \in \mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A}).$$

The first relation presents no difficulty. To obtain the second relation, we use the disintegration 1.7 of λ:

$$f_1 *_\lambda (g *_\beta f_2)(x,\sigma) = \int f_1(x,\tau)\left[\tau(g *_\beta f_2)(x\tau,\tau^{-1}\sigma)\right] \mathrm{d}\lambda^x(\dot\tau) =$$

$$= \int f_1(x,\tau) \int \tau\left[g(x\tau,\tau^{-1}\sigma\rho)\right] \tau^{-1}\sigma\rho\left[f_2(x\sigma\rho,\rho^{-1})\right] \mathrm{d}\beta^{x\sigma}_{x\tau}(\dot\rho)\,\mathrm{d}\lambda^x(\dot\tau) =$$

$$= \int f_1(x,\tau)\tau\left[g(x\tau,\tau^{-1}\sigma\rho)\right] \sigma\rho\left[f_2(x\sigma\rho,\rho^{-1})\right] \mathrm{d}\beta^{x\sigma}_{x\tau}(\dot\rho)\,\mathrm{d}\beta^x_y(\dot\tau)\,\mathrm{d}\alpha^x(y) =$$

$$= \int f_1(x,\tau)\tau\left[g(x\tau,\tau^{-1}\sigma\rho)\right] \sigma\rho\left[f_2(x\sigma\rho,\rho^{-1})\right] \mathrm{d}\beta^x_{x\sigma\rho}(\dot\tau)\,\mathrm{d}\beta^{x\sigma}_{x\tau}(\dot\rho)\,\mathrm{d}\alpha^x(y) =$$

$$= \int \left\{ \int f_1(x,\tau)\tau\left[g(x\tau,\tau^{-1}\sigma\rho)\right] \mathrm{d}\beta^x_{x\sigma\rho}(\dot\tau)\right\} \cdot$$

$$\cdot \sigma\rho\left[f_2(x\sigma\rho,\rho^{-1})\right] \mathrm{d}\beta^{x\sigma}_y(\dot\rho)\,\mathrm{d}\alpha^x(y) =$$

$$= \int f_1 *_\beta g(x,\sigma\rho)\sigma\rho\left[f_2(x\sigma\rho,\rho^{-1})\right] \mathrm{d}\lambda^{x\sigma}(\dot\rho) =$$

$$= (f_1 *_\beta g) *_\lambda f_2(x,\sigma).$$

ii) It suffices to check the equality of the coefficients

$$\langle \xi, L(g *_\beta f)\zeta\rangle = \langle \xi, L'(g)L(f)\zeta\rangle.$$

By Corollary 1.8, we can choose the module Δ of ν so that it coincides with δ on G'. The computation then consists of a change of order of integration. ∎

We construct a family of (not everywhere defined) linear maps Q of $\mathcal{L}^1(\underline{G}, \underline{\Sigma}, \underline{A})$ into itself in the following fashion. The family is indexed by (K, M, C)'s where K is a symmetric c.c. neighborhood of $G^{(0)}$ in G, M is a neighborhood of the diagonal in the graph R of the equivalence relation with the quotient topology and C is a compact subset of $\operatorname{Prim} A$ such that the restriction to C of the isotropy is continuous. Let (K, M, C) be such an index. By Proposition 1.11, there exists a finite cover $V_1, \dots, V_n$ of C such that the relation $x \overset{K}{\sim} y$ becomes on each V_i an open equivalence relation with a closed graph contained in M. Let $b_1, \dots, b_n$ be a continuous partition of unity subordinate to it. We define the functions $e_1, \dots, e_n$ on $\operatorname{Prim} A$ by

$$e_i(x) = \begin{cases} b_i(x) \Big/ \left[\displaystyle\int_{R|K} b_i(y)\,\mathrm{d}\alpha^x(y) \right]^{1/2} & \text{if } x \in C' \text{ and } b_i(x) > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where C' is the support of $\alpha|C$ as defined in Lemma 1.12. Then it holds that

$$e_i(x) \int_{R|K} e_i(y)\,\mathrm{d}\alpha^x(y) = b_i(x)$$

for every $x \in C'$. We define the linear map $Q = Q(K, M, C)$ by

$$Q(f)(x,\sigma) = \sum_{i=1}^{n} e_i(x) f(x,\sigma) e_i(x\sigma).$$

It depends in fact on the choice of the cover and the partition of unity but this does not matter for our purpose. Note also that $C_c(G, \Sigma, A)$ is in its domain.

LEMMA 3.2. *With above notation:*

i) *for $f \in C_c(G, \Sigma, A)$ with support modulo S contained in K,*

$$\|Q(f)\|_{1,\lambda} \leqslant \sup\left\{\int \|f(\sigma)\| \, d\beta_y^x(\sigma) \, : \, (x,y) \in M\right\}$$

ii) *let L be a non-degenerate representation of $C_c(G, \Sigma, A)$ in a separable Hilbert space $\mathcal{H}$ and L' be its restriction to $\mathcal{L}(G', \Sigma', A')$. Then for each $f \in C_c(G, \Sigma, A)$, $L' \circ P(f)$ is in the weak closure of the $L \circ Q(f)$'s.*

Proof. i) This is a straightforward estimate:

$$\int \|Q(f)(x,\sigma)\| \, d\lambda^x(\sigma) \leqslant \sum_{i=1}^{n} \int e_i(x) e_i(x\sigma) \, \|f(\sigma)\| \, d\lambda^x(\dot\sigma) \leqslant$$

$$\leqslant \sum_{i=1}^{n} \int e_i(x) e_i(y) \, \|f(\sigma)\| \, d\beta_y^x(\dot\sigma) \, d\alpha^x(y) \leqslant$$

$$\leqslant \sup \int \|f(\sigma)\| \, d\beta_y^x(\dot\sigma) \sum_{i=1}^{n} \int_{R|K} e_i(x) e_i(y) \, d\alpha^x(y) \leqslant$$

(where (x, y) runs over M)

$$\leqslant \sup\left\{\int \|f(\sigma)\| \, d\beta_y^x(\dot\sigma) \, : \, (x,y) \in M\right\}.$$

ii) Let f be given in $C_c(G, \Sigma, A)$. We fix a c.c. neighborhood K of $G^{(0)}$ in G containing supp$f$. Given $\xi_1, \zeta_1, \ldots, \xi_m, \zeta_m$ in $\mathcal{H}$ and $\varepsilon > 0$, we have to find Q such that

$$|\langle \xi_j, L \circ Q(f)\zeta_j\rangle - \langle \xi_j, L' \circ P(f)\zeta_j\rangle| \leqslant \varepsilon \quad \text{for } j = 1, \ldots, m.$$

Since L is non-degenerate and the family $(L \circ Q(f))$ is bounded, we may assume that ζ_j is of the form $L(g_j)\zeta_j$, where g_j is in $C_c(G, \Sigma, A)$ for $j = 1, \ldots, m$. By Lusin's theorem, we can find a compact subset C of Prim A such that the restriction to C of the isotropy is continuous and

$$\int_{\text{Prim } A \setminus C} \|\xi_j(x)\|^2 \, d\nu(x) \leqslant \varepsilon^2 \quad \text{for } j = 1, \ldots, m.$$

Because of the continuity of the isotropy, the function h_j, defined by

$$h_j(x,y,\sigma) = \int f(\sigma')\sigma'[g_j(\sigma'^{-1}\sigma)]\, \mathrm{d}\beta_y^x(\dot\sigma')$$

on the set of (x,y,σ)'s in $R \times \Sigma$ such that $x \in C$ and $r(\sigma) = p(\sigma)$, is continuous. Therefore, there exists a neighborhood M of the diagonal in R such that for $j = 1, \ldots, m$ the conditions $x \in C, \sigma \in \operatorname{supp} f \circ \operatorname{supp} g_j$ and $(x,y) \in M$ imply that $\|h_j(x,y,\sigma) - h_j(x,x,\sigma)\| \leqslant \varepsilon$. Let Q be $Q(K,M,C)$. Then

$$\langle \xi_j, L \circ Q(f)L(g_j)\zeta_j\rangle - \langle \xi_j, L' \circ P(f)L(g_j)\zeta_j\rangle = \langle \xi_j, L\,[Q(f)*_\lambda g_j - P(f)*_\beta g_j]\,\zeta_j\rangle$$

is the sum of an integral over C and an integral over its complement. The integral over the complement is in absolute value less than

$$\|L\,[Q(f)*_\lambda g_j - P(f)*_\beta g_j]\|\,\|\zeta_j\| \left[\int_{\operatorname{Prim} A\setminus C} \|\xi_j(x)\|^2\, \mathrm{d}\nu(x)\right]^{1/2}.$$

This is less than a constant times ε, where the constant depends only on the data f, g_j, ζ_j. The integral over C can be replaced, because of Lemma 1.12, by an integral over the support C' of $\alpha|C$ and is in absolute value less than

$$\|\operatorname{char}(C')\,[Q(f)*_\lambda g_j - P(f)*_\beta g_j]\|_{1,\lambda}\, \|\xi_j\|\,\|\zeta_j\|,$$

where $\operatorname{char}(C')$ is the characteristic function of C'. Since the supports modulo S of $Q(f)*_\lambda g_j$ and $P(f)*_\beta g_j$ are contained in a fix compact set $\operatorname{supp} f \circ \operatorname{supp} g_j$, the $\|\cdot\|_{1,\lambda}$-norm is less than, up to a multiplicative constant, the sup-norm. Now

$$Q(f)*_\lambda g_j(x,\sigma) = \sum_{i=1}^{n} \int_{R|K} e_i(x)e_i(y)h_j(x,y,\sigma)\, \mathrm{d}\alpha^x(y)$$

and

$$P(f)*_\beta g_j(x,\sigma) = h_j(x,x,\sigma).$$

Hence if x is in C', the norm of $Q(f)*_\lambda g_j(x,\sigma) - P(f)*_\beta g_j(x,\sigma)$ is less than

$$\sum_{i=1}^{n} \int_{R|K} e_i(x)e_i(y)\,\|h_j(x,y,\sigma) - h_j(x,x,\sigma)\|\, \mathrm{d}\alpha^x(y).$$

Since $(x,y) \in V_i(R|K)V_i$ implies that $(x,y) \in M$, this is less than ε. ∎

THEOREM 3.3. *Let L be a non-degenerate representation of $C^*(G,\Sigma,A)$ in a separable Hilbert space $\mathcal{H}$ and L' its restriction to the isotropy G' on* Prim A. *Then the representation $\hat{L}$ of $C^*(G,\Sigma,A)$ induced from L' is weakly contained in L.*

Proof. By Lemma 2.3, the space $\hat{\mathcal{H}}$ of the induced representation $\hat{L}$ is the Hilbert space completion of the algebraic tensor product $C_c(G, \Sigma, A) \otimes \mathcal{H}$ with respect to the scalar product defined on elementary tensors by

$$\langle f_1 \hat{\otimes} \zeta_1, f_2 \hat{\otimes} \zeta_2 \rangle = \langle \xi_1, L' \circ P(f_1^* *_\lambda f_2) \xi_2 \rangle$$

where L' is viewed as a representation of $B(G', \Sigma', A')$ and $f_1^* *_\lambda f_2$ as an element of $B(G, \Sigma, A)$. The induced representation acts on elementary tensors by

$$\hat{L}(f) g \hat{\otimes} \xi = (f * g) \hat{\otimes} \xi.$$

For given $f_1, f_2, f \in C_c(G, \Sigma, A)$ and $\xi_1, \xi_2 \in \mathcal{H}$, the coefficient

$$\langle f_1 \hat{\otimes} \xi_1, \hat{L}(f) f_2 \hat{\otimes} \xi_2 \rangle = \langle \xi_1, L' \circ P(f_1^* * f * f_2) \xi_2 \rangle$$

is a limit, by Lemma 3.2, of terms of the form

$$\langle \xi_1, L \circ Q(f_1^* * f * f_2) \xi_2 \rangle = \sum_{i=1}^{n} \langle \xi_1^i, L(f) \xi_2^i \rangle$$

where $Q(f) = \sum_{i=1}^{n} e_i f e_i$ as above and $\xi_j^i = L(f_j) M(e_i) \xi_j$ for $j = 1, 2$. Since $\sum_{i=1}^{n} \|\xi_j^i\|^2 = \langle \xi_j, L \circ Q(f_j^* * f_j) \xi_j \rangle$ is bounded, by Lemma 3.2.(i), the convergence still holds when f is replaced by an arbitrary element a of $C_c(G, \Sigma, A)$. Thus the coefficients of $\hat{L}$ are limits of linear combinations of coefficients of L. ∎

The reverse inclusion holds under an amenability assumption. This is a straightforward generalization of [17, Proposition 3.2, page 87]. Let us first recall a definition of [17, page 86].

DEFINITION 3.4. Let G be a Borel groupoid endowed with a Borel Haar system λ. A quasi-invariant measure μ on $G^{(0)}$ is called *amenable* if there exists a net (f_i) of complex-valued functions on G such that

i) The functions $x \mapsto \int |f_i(\gamma)|^2 \, d\lambda^x(\gamma)$ are essentially bounded and this family is bounded in $L^\infty(G^{(0)}, \mu)$.

ii) The functions $\gamma \mapsto f_i * f_i^*(\gamma) = \int f_i(\gamma') \overline{f_i}(\gamma^{-1}\gamma') \, d\lambda^{r(\gamma)}(\gamma')$ converge to 1 in the weak $*$-topology of $L^\infty(G, \mu \circ \lambda)$.

REMARK 3.5. This is in fact a property of the transverse measure class defined by μ and λ.

THEOREM 3.6. *Let L be a representation of $C^*(G, \Sigma, A)$ and L' be its restriction to the isotropy G' of the action of G on* Prim *A. Assume that the quasi-invariant measure ν on* Prim *A provided by the disintegration of L is amenable with respect to the equivalence relation R. Then the representation $\hat{L}$ of $C^*(G, \Sigma, A)$ induced from L' is weakly equivalent to L.*

Proof. It suffices to show that $\hat{L}$ weakly contains L. Realize the space $\hat{\mathcal{H}}$ of the induced representation $\hat{L}$ as the space of measurable sections $\xi : \operatorname{Prim} A \times \Sigma \mapsto H$ such that $\xi(x,\sigma) \in H_x$, $\xi(\sigma'\underline{\sigma}) = L(\sigma')\xi(\underline{\sigma})$ for $\sigma' \in \Sigma'$ and $\underline{\sigma} = (x,\sigma)$, and $\|\xi\|^2 = = \int \|\xi(x,\sigma)\|^2 \, \mathrm{d}\alpha^x(x\sigma)\, \mathrm{d}\nu(x) < \infty$. The coefficient $\langle \xi, \hat{L}(f)\mu\rangle$ is given by

$$\int \langle \xi(x,\tau), M_x[\tau f(\sigma)]\, \eta(x,\tau\sigma)\rangle_x \delta^{-1/2}(x\tau,\dot{\sigma})\, \mathrm{d}\lambda^{x\tau}(\dot{\sigma})\, \mathrm{d}\alpha^x(x\tau)\, \mathrm{d}\nu(x).$$

We want to approximate the vector state L defined by $\xi \in L^2(\operatorname{Prim} A, \nu, H)$. Let (f_i) be a net provided by the amenability of ν with respect to R and the Haar system α. Define then

$$\xi_i(x,\sigma) = \underline{\Delta}^{-1/2}(x,x\sigma) f_i(x\sigma,x) L(x,\sigma)\xi(x\sigma)$$

where $\underline{\Delta}$ is the module of ν as a quasi-invariant measure with respect to R. Recall from Corollary 1.11 that Δ and $\underline{\Delta}$ are related by

$$\Delta(x,\gamma) = \delta(x,\gamma)\underline{\Delta}(x,x\gamma) \quad \text{a.e.}$$

We first check that $\xi_i \in \hat{\mathcal{H}}$. The first two conditions are immediate. Moreover

$$\|\xi_i\|^2 = \int \|\xi_i(x,\sigma)\|^2 \, \mathrm{d}\alpha^x(x\sigma)\, \mathrm{d}\nu(x) = \int \|\xi(x)\|^2 \left(\int |f_i(x,y)|^2 \, \mathrm{d}\alpha^x(y) \right) \mathrm{d}\nu(x)$$

is uniformly bounded. A routine computation gives that for $f \in C_c(G,\Sigma,A)$,

$$\langle \xi_i, \hat{L}(f)\xi_i\rangle = \int \left(\int^{-} f_i(x,y) f_i(x\sigma,y)\, \mathrm{d}\alpha^x(y) \right) \langle \xi(x), M_x[f(\sigma)]\, L(x,\sigma)\xi(x\sigma)\rangle \cdot$$

$$\cdot \Delta^{-1/2}(x,\sigma)\, \mathrm{d}\lambda^x(\sigma)\, \mathrm{d}\nu(x).$$

By condition (ii) of 3.4, this tends to $\langle \xi, L(f)\xi\rangle$. Because of the boundedness of (ξ_i), the convergence still holds for $f \in C^*(G,\Sigma,A)$. ∎

REMARK 3.7. The amenability of R, that is, the existence of a net (f_i) of Borel complex-valued functions on R such that

(i) $\int |f_i(x,y)|^2 \, \mathrm{d}\alpha^x(y)$ is bounded uniformly in x and in i,

(ii) the net of functions $f_i *_\alpha f_i^*$, where $f_i *_\alpha f_i^*(x,y) = \int f_i(x,z)\overline{f}_i(y,z)\, \mathrm{d}\alpha^x(z)$, converges to the function 1 against every finite measure on R,

will ensure the weak equivalence of L and $\hat{L}$ for every representation L. A stronger condition is the amenability of $\operatorname{Prim} A \times G$. In fact, this condition is equivalent to the amenability of R and of G'. A still stronger condition is the amenability of G.

4. SIMPLICITY OF SOME REDUCED CROSSED PRODUCT C^*-ALGEBRAS

In the last section, we have compared an arbitrary representation of the crossed product C^*-algebra $C^*(G,\Sigma,A)$ with a representation induced from the isotropy.

We want now to compare arbitrary representations and regular representations, in the case when the isotropy is small, in a sense that we shall make precise. The idea is that induction preserves weak containment and that it suffices therefore to compare arbitrary representations of the isotropy group bundle with certain regular representations of this group bundle.

DEFINITION 4.1. Let G be a topological groupoid with a non necessarily Hausdorff unit space and $\underline{x}$ be a unit. The isotropy will be said to be *discretely trivial* at $\underline{x}$ if for each compact set K in G, there exists a neighborhood V of $\underline{x}$ in $G^{(0)}$ such that for each x in V, $G(x) \cap K$ contains at most x.

REMARK 4.2. Let us say that the isotropy is discrete at $\underline{x}$ if there exists a neighborhood V of $\underline{x}$ in $G^{(0)}$ and a neighborhood U of V in G such that for each x in V, $G(x) \cap U$ is reduced to $\{x\}$. Assume that G is locally conditionally compact. If the isotropy is discrete trivial at $\underline{x}$, then it is discrete at $\underline{x}$. Suppose conversely that the isotropy is discrete at $\underline{x}$ and that $G(\underline{x}) = \{x\}$, then the isotropy is discretely trivial at $\underline{x}$ provided that for each $x \neq \underline{x}$, there is a neighborhood of $\underline{x}$ which does not contain x. In particular, for a groupoid which is r-discrete in the sense of [17] or the graph of a foliation, the isotropy is discretely trivial at $\underline{x}$ iff $G(\underline{x}) = \{\underline{x}\}$.

As before, (G, Σ, A) denotes a groupoid dynamical system. In this section, Prim A is equipped with the null-kernel topology.

THEOREM 4.3. *Let (G, Σ, A) be a dynamical system, L a representation of $C^*(G, \Sigma, A)$ and x_0 a point of* Prim A *where the isotropy is discretely trivial. Then the regular representations induced from x_0 are weakly contained in L if and only if x_0 belongs to the support of the restriction of L to $C_0(G^{(0)}, A)$.*

Proof. By Lemma 2.4 it is sufficient to consider the untwisted crossed product $C^*(G, A)$. The only if part is clear, because restriction to $C_0(G^{(0)}, A)$ preserves weak containment. Let us suppose that x_0 belongs to the support of the restriction M of L to $C_0(G^{(0)}, A)$. Because of Theorem 3.3, we may assume that L is the representation induced from the representation $L' = (\nu, H)$ of the isotropy dynamical system (G', A'). The Hilbert space $\mathcal{H}$ of the representation L consists of measurable sections ξ of $\underline{G} = \operatorname{Prim} A \rtimes G$ into H such that $\xi(x,\gamma) \in H_x$, $\xi(x,\gamma'\gamma) = L'(x,\gamma')\xi(x,\gamma)$ for $\gamma' \in G(x)$ and $\|\xi\|^2 = \int \|\xi(x,\gamma)\|^2 \, d\alpha^x(x\gamma)\, d\nu(x)$ is finite. The coefficients of L are given by

$$\langle \xi, L(f)\xi\rangle = \int \langle \xi(x,\gamma), M_x[\gamma f(\gamma_1)]\,\xi(x,\gamma\gamma_1)\rangle\cdot$$
$$\cdot\delta^{-1/2}(x\gamma,\gamma_1)\, d\lambda^{x\gamma}(\gamma_1)\, d\alpha^x(x\sigma)\, d\nu(x).$$

Since the isotropy is discrete in a neighborhood of x_0 and we shall choose ξ supported in the reduction of $\underline{G}$ to this neighborhood, the modulus function δ will take the value 1. Moreover, after a change of variable, we obtain

$$\langle \xi, L(f)\xi\rangle = \int \langle \xi(x,\gamma), M_x\left[\gamma f(\gamma^{-1}\gamma_1)\right]\xi(x,\gamma_1)\rangle\, d\lambda^x(\gamma_1)\, d\alpha^x(x\sigma)\, d\nu(x).$$

On the other hand, the regular representation R induced from the homogeneous representation M_0 of $C_0(G^{(0)}, A)$ with kernel x_0 on the Hilbert space H_0 acts on the Hilbert space $L^2([x_0], \alpha^{x_0}, H_0)$, where $[x_0]$ is the orbit of x_0 on $\operatorname{Prim} A$. The coefficients of R are given by

$$\langle \eta, R(f)\eta \rangle = \int \langle \eta(x_0\gamma), M_0 \left[\gamma f(\gamma^{-1}\gamma_1)\right] \eta(x_0\gamma_1) \rangle \, d\alpha^{x_0}(\gamma_1) \, d\alpha^{x_0}(x_0\gamma).$$

We want to approximate coefficients of R by sums of coefficients of L. It suffices to consider a coefficient $\langle \eta, R(f)\eta \rangle$ where η is of the form $\eta(x) = \varphi(x)\eta_0$ where φ is a continuous function on $\operatorname{Prim} A$ such that $\int |\varphi(x)|^2 \, d\alpha^{x_0}(x) = 1$ and η_0 is a unit vector in H_0. Since M_0 is weakly contained in M, there exist vectors ξ'_{ij} in $L^2(\operatorname{Prim} A, \mu, H)$ such that, for each i, $\sum_j \|\xi'_{ij}\|^2 = 1$ and the net of states $\omega_i(a) = \sum_j \langle \xi'_{ij}, M(a)\xi'_{ij} \rangle$ converges weakly to the vector state $\omega_0(a) = \langle \eta_0, M_0(a)\eta_0 \rangle$. In other words, for each $a \in C_0(G^{(0)}, A)$, we have

$$(*) \qquad \lim_i \sum_j \int \langle \xi'_{ij}(x), M_x(a)\xi'_{ij}(x), \rangle \, d\nu(x) = \langle \eta_0, M_0(a)\eta_0 \rangle.$$

The convergence still holds for a in the multiplier algebra of $C_0(G^{(0)}, A)$. In particular, the probability measure $\sum_j \|\xi'_{ij}(x)\|^2 \, d\nu(x)$ converge vaguely to the point mass at x_0.

We choose an increasing sequence (K_n) of conditionally compact symmetric neighborhoods of $G^{(0)}$ in G such that $U \overset{\circ}{K}_n = G$ and a decreasing fundamental sequence (V_n) of neighborhoods of x_0 in $\operatorname{Prim} A$ such that for each n and each x in V_n, $G(x) \cap K_n^3 = \{x\}$. The existence of such neighborhoods comes from the discrete triviality of the isotropy at x_0. Since for each n, the net $\sum_j \int_{V_n} \|\xi'_{ij}(x)\|^2 \, d\nu(x)$ converges to 1, there exists a subnet (i_n) such that $\sum_j \int_{V_n} \|\xi'_{i_n j}(x)\|^2 \, d\nu(x)$ converges to 1 $(**)$.

We associate to each n the linear functional ω_n on $C^*(G, A)$ defined by

$$\omega_n(f) = \sum_j \langle \xi_{nj}, L(f)\xi_{nj} \rangle$$

where ξ_{nj} is defined by

$$\xi_{nj}(x, \gamma'\gamma) = \begin{cases} L'(x, \gamma')\varphi(x\gamma)\xi'_{i_n j}(x) & \text{if } x \in V_n,\ \gamma' \in G(x) \text{ and } \gamma \in K_n \\ 0 & \text{if not.} \end{cases}$$

This is well defined because, for $x \in V_n$, the elements of $G(x)K_n$ are uniquely written in the form $\gamma'\gamma$ with γ' in $G(x)$ and $\gamma \in K_n$, because $G(x) \cap K_n^2 = \{x\}$. Moreover, it is easily checked that ξ_{nj} is a vector in $\mathcal{H}$ and that

$$\sum_j \|\xi_{nj}\|^2 = \sum_j \int \|\xi_{nj}(x, \gamma)\|^2 \, d\alpha^x(x\gamma) \, d\nu(x) =$$

$$= \sum_j \int_{V_n} \|\xi'_{i_nj}(x)\|^2 \left(\int_{K_n} |\varphi(x\gamma)|^2 \,\mathrm{d}\,\lambda^x(\gamma) \right) \mathrm{d}\nu(x).$$

Let us show that this converges to 1. Because of the continuity of λ, we can find, given $\varepsilon > 0$, n_0 such that for $x \in V_{n_0}$ and every $n \geqslant n_0$,

$$1 - \varepsilon \leqslant \int_{K_n} |\varphi(x\gamma)|^2 \,\mathrm{d}\lambda^x(\gamma) \leqslant 1 + \varepsilon.$$

Because of the choice of (i_n), we can find $n_1 \geqslant n_0$ such that for every $n \geqslant n_1$,

$$1 - 2\varepsilon \leqslant \sum_j \|\xi_{nj}\|^2 \leqslant 1 + 2\varepsilon.$$

We want to show that for each $f \in C^*(G, A)$, the sequence $(\omega_n(f))$ converges to the coefficient $\langle \eta, R(f)\eta \rangle$. Because of the boundedness of (ω_n), it suffices to check the convergence for $f \in C_c(G, A)$. Let us choose n such that K_n contains supp$f$. A straightforward computation taking into account the fact that $G(x) \cap K_n^3 = \{x\}$ gives

$$\omega_n(f) = \sum_j \int_{V_n} \int_{K_n} \int_{K_n} \overline{\varphi(x\gamma)} \varphi(x\gamma_1) \langle \xi'_{i_nj}(x), M_x \left[\gamma f(\gamma^{-1}\gamma_1)\right] \xi'_{i_nj}(x) \rangle \cdot$$

$$\cdot \,\mathrm{d}\lambda^x(\gamma_1)\,\mathrm{d}\lambda^x(\gamma)\,\mathrm{d}\nu(x) =$$

$$= \sum_j \int_{V_n} \langle \xi'_{i_nj}(x), M_x \left[F_n(x)\right] \xi'_{i_nj}(x) \rangle \,\mathrm{d}\nu(x)$$

where we have set

$$F_n(x) = \int_{K_n} \int_{K_n} \overline{\varphi(\gamma)} \varphi(x\gamma_1) \gamma f(\gamma^{-1}\gamma_1)\,\mathrm{d}\lambda^x(\gamma_1)\,\mathrm{d}\lambda^x(\gamma).$$

We do not alter the convergence by replacing $F_n(x)$ by $F(x)$, where

$$F(x) = \int \int \overline{\varphi(x\gamma)} \varphi(x\gamma_1) \gamma f(\gamma^{-1}\gamma_1)\,\mathrm{d}\lambda^x(\gamma_1)\,\mathrm{d}\lambda^x(\gamma)$$

because of the estimate, obtained by Cauchy-Schwarz inequality

$$\|F_n(x) - F(x)\| \leqslant \|f\|_1 \int_{G \setminus K_n} |\varphi(x\gamma)|^2 \,\mathrm{d}\lambda^x(\gamma)$$

where as before

$$\|f\|_1 = \max\left(\sup_u \int \|f(\gamma)\| \,\mathrm{d}\lambda^u(\gamma), \sup_u \int \|f(\gamma^{-1})\| \,\mathrm{d}\lambda^u(\gamma) \right).$$

The proof will be complete after we show:

i) F is a continuous section of the pull-back bundle p^*A over $\operatorname{Prim} A$.

ii) For every continuous section F of p^*A,

$$\lim_n \sum_j \int_{V_n} \langle \xi'_{i_n j}(x), M_x[F(x)]\xi'_{i_n j}(x)\rangle \, \mathrm{d}\nu(x) = \langle \eta_0, M_0[F(x_0)]\,\eta_0\rangle.$$

It suffices to check (ii) for F of the form $F(x) = F_1(x)a \circ p(x)$, where F_1 is scalar valued and $a \in C_0(G^{(0)}, A)$. Then the result follows from $(*)$ and $(**)$. The assertion (i) follows from the following lemma.

LEMMA 4.4. *Let π be a continuous and open map from the locally compact space Y onto the locally compact Hausdorff space Z, λ a continuous π-system and E a Banach bundle over Z. Then for every continuous map p from a topological space X onto Z, every open Hausdorff subset U of Y, every compact subset K of U and every section f of the pull-back of E over the fibered product $X *_Z Y$ such that*

i) for every x in X, the function $f(x,\cdot)$ has its support contained in K,

ii) the restriction of f to $X *_Z U$ is continuous,

the section $x \mapsto \int f(x,y)\,\mathrm{d}\lambda^{p(x)}(y)$ is continuous.

Proof. Let x_0 be a point in X. The section f can be uniformly approximated by linear combinations of sections of the form $f_1(x,y)e \circ p(x)$ where f_1 is scalar and satisfies (i) and (ii) and e is a continuous section of E. This reduces the problem to the scalar case. By normality of U, there exists $g \in C_c(U)$ such that $f(x_0, y) = g(y)$ for every $y \in U \cap \pi^{-1}(p(x_0))$. The difference $\int f(x,y)\,\mathrm{d}\lambda^{p(x)}(y) - \int f(x_0,y)\,\mathrm{d}\lambda^{p(x_0)}(y)$ can be decomposed as the sum of two terms A and B where

$$A = \int (f(x,y) - g(y))\,\mathrm{d}\lambda^{p(x)}(y)$$

$$B = \int g(y)\,\mathrm{d}\lambda^{p(x)}(y) - \int g(y)\,\mathrm{d}\lambda^{p(x_0)}(y).$$

Because of the continuity of λ and p, the term B tends to 0 when x tends to x_0. Let us show that the term A also tends to 0. It suffices to show that for every $\varepsilon > 0$, there exists a neighborhood V of x_0 such that for every x in V and every $y \in U \cap \pi^{-1}(p(x))$, we have $|f(x,y) - g(y)| < \varepsilon$. Suppose this is not true. Then there exists $\varepsilon > 0$ and $(x_n), (y_n)$ such that x_n tends to x_0, $y_n \in U \cap \pi^{-1}(p(x_n))$ and $|f(x_n, y_n) - g(y_n)| \geqslant \varepsilon$. Each y_n belongs to the compact set $K \cup \operatorname{supp} g$. Passing to a subnet converging to y_0, one obtains $|f(x_0, y_0) - g(y_0)| \geqslant \varepsilon$. This is impossible because of the choice of g. ■

REMARK 4.5. Under the sole hypothesis that x_0 is a point with trivial isotropy, the conclusion of the theorem does not hold. Let G be the unit disk $\{z \in \mathbb{C} : |z| \leqslant 1\}$. It is a group-bundle over [0,1] with bundle map $z \mapsto |z|$ and multiplication $(ae^{i\theta}) \cdot (ae^{i\varphi}) = ae^{i(\theta+\varphi)}$. We provide it with the Haar system $\int f\,\mathrm{d}\lambda^a = \frac{1}{2\pi i}\int_{\mathbb{S}^1} f(az)\,\frac{\mathrm{d}z}{z}$.

The regular representation R induced from 0 is given by $R(f) = f(0)$. It is not weakly contained in the representation L on $L^2[0,1]$ given by $L(f)\xi(a)=\left(\frac{1}{2\pi i}\int_{\mathbb{S}^1} f(az)\,dz\right)\cdot$ $\cdot\xi(a)$.

As a corollary to their solution of the generalized Effros-Hahn conjecture, Gootman and Rosenberg obtain the simplicity of the reduced crossed-product $C^*_{red}(G,A)$ when the action of G on $\operatorname{Prim} A$ is free and minimal. By using Theorem 4.3 one can slightly relax the assumption of freeness. Among a number of results on the simplicity of crossed product C^*-algebras, the following corollary was motivated by [17, Chapter II, Section 4] which was used to prove the simplicity of the Cuntz algebras and by [7, théorème 26], where it is shown that the C^*-algebra of a minimal foliation (with Hausdorff holonomy groupoid) is simple. A related result of G. Elliot [6] on the simplicity of a crossed product by a discrete group should also be mentioned here.

COROLLARY 4.6. *Let (G,Σ,A) be a dynamical system such that G is Hausdorff, the action of G on* $\operatorname{Prim} A$ *is minimal and there exists a point in* $\operatorname{Prim} A$ *with discretely trivial isotropy. Then the reduced crossed product $C^*_{red}(G,\Sigma,A)$ is simple.*

Proof. Because G is Hausdorff, the restriction map p from $C_c(G,\Sigma,A)$ onto $C_c(G^{(0)},A)$ is well defined. It is a generalized conditional expectation in the sense of [19, Definition 4.12] and the regular representations of $C^*(G,\Sigma,A)$ are exactly the representations induced via P. Let L be a representation of $C^*(G,\Sigma,A)$. By minimality of the action of G on $\operatorname{Prim} A$, the support of its restriction to $C_0(G^{(0)},A)$ is $\operatorname{Prim} A$. By Theorem 4.3, L weakly contains the regular representations induced from a point with trivial isotropy. Because these points are dense and induction via P preserves weak containment, L weakly contains every regular representations. In particular, if L is a representation of $C^*_{red}(G,\Sigma,A)$, it is faithful.

REMARK 4.7. This result does not necessarily hold when G is not Hausdorff. G. Skandalis gives a counterexample in the appendix.

The next result on the ideal structure of the crossed product when the action of the primitive ideal space is no longer minimal will require two hypothesis. One of them is the amenability of this action as defined in 3.7. More precisely, this mens that each isotropy subgroup is amenable and each quasi-invariant measure on $\operatorname{Prim} A$ is amenable with respect to the equivalence relation R. The other is the essential freeness of this action as defined below.

DEFINITION 4.8. A continuous action of a topological groupoid on a topological space X will be said to be *essentialy free* if for every invariant closed subset F of X, the subset of points of F with discretely trivial isotropy is dense in F.

COROLLARY 4.9. *Let (G,Σ,A) be a dynamical system such that G is Hausdorff and its action on* $\operatorname{Prim} A$ *is amenable and essentially free. Then restriction to $C_0(G^{(0)},A)$ yields an isomorphism between the lattice of ideals of the crossed product $C^*(G,\Sigma,A)$ and the lattice of invariant open subsets of* $\operatorname{Prim} A$.

Proof. Let L be a representation of $C^*(G,\Sigma,A)$, L' its restriction to the isotropy G' of the action of G on $\operatorname{Prim} A$ and L_0 its restriction to $C_0(G^{(0)},A)$. The support of L_0 is a closed invariant subset F of $\operatorname{Prim} A$ which depends only on the kernel of L. By Theorem 4.3, L weakly contains the regular representations induced from points of F which have discretely trivial isotropy. Since these points are dense in F and induction via the generalized conditional expectation P from $C_c(G,\Sigma,A)$ onto $C_c(G^{(0)},A)$ preserves weak containment, L weakly contains the representation $\operatorname{Ind} L_0$ induced from L_0. On the other hand, since the isotropy subgroups are amenable, the regular representation of the isotropy subgroup $G(x)$ weakly contains $L'|G(x)$ for almost every x and therefore $\operatorname{Ind} L_0$ weakly contains the representation $\hat{L}$ induced from L'. Furthermore, because the equivalence relation R is amenable, L and $\hat{L}$ are weakly equivalent by Theorem 3.6. Hence L and $\operatorname{Ind} L_0$ are also weakly equivalent and induction via P provides an inverse to the restriction map. ■

REMARK 4.10. Does this result hold without the assumption of amenability for the reduced crossed product $C^*_{\mathrm{red}}(G,\Sigma,A)$? The answer is likely to be no, due to the bad behaviour of the reduced crossed product with respect to exact sequences. Let us illustrate this by an example which was given to me by G. Skandalis. It is well known (see for example [25, Lemma 1]), and easily shown by using the disintegration theorem for representation that, given a locally compact groupoid G with Haar system, an invariant open subset U of $G^{(0)}$ yields an exact sequence

$$0 \longrightarrow C^*(G_U) \xrightarrow{\ j\ } C^*(G) \xrightarrow{\ p\ } C^*(G_F) \longrightarrow 0,$$

where F is the complement of U in $(G^{(0)})$. Defined on functions, the map j is the extension by 0 while the map p is the restriction map. If G_F is amenable, then the sequence

$$0 \longrightarrow C^*_{\mathrm{red}}(G_U) \longrightarrow C^*_{\mathrm{red}}(G) \longrightarrow C^*_{\mathrm{red}}(G_F) \longrightarrow 0$$

is also exact. This is an immediate consequence of the previous exact sequence and has been noticed and extensively used in [27]. If G_F is not amenable, the sequence is not always exact. Let Γ be a countable discrete group acting on the right by diffeomorphisms on a manifold M. Connes' construction of the tangent groupoid of M can be modified to incorporate Γ. It yields the normal groupoid, in the sense of [26, Remark 3.19] of the injective embedding of $M \rtimes \Gamma$ into $M \times M$. This groupoid G is the disjoint union of groupoids G_t where t runs over [0,1]. For $t \neq 0$, G_t is the groupoid $M \times M$ (the trivial equivalence relation on M) and G_0 is the semi-direct product $\mathrm{T}M \rtimes \Gamma$, where the tangent bundle $\mathrm{T}M$ is viewed as a group bundle and the multiplication is given by $(x,X,\gamma)(x,Y,\gamma_1) = (x, X+Y\gamma^{-1}, \gamma\gamma_1)$ where $Y\gamma^{-1}$ is the image of Y by $\gamma'(x)^{-1}$. These groupoids are glued together according to the following rules: for $t \neq 0$, $(t_n,x_n,y_n) \to (t,x,y)$ if and only if $t_n \to t$, , $x_n \to x$, $y_n \to y$ and $(t_n,x_n,y_n) \to (0,x,X,\gamma)$ if and only if $t_n \to 0$, , $x_n \to x$, $y_n \to x\gamma$ and $\dfrac{y_n\gamma^{-1} - x_n}{t_n} \to$ $\to X$ (this last condition does not depend on the local chart around). This makes G into a locally compact Hausdorff groupoid. A choice of a non-vanishing density ρ on M fixes a Haar system for G: for $f \in C_c(G)$ and $x \in M$, one defines for $t \neq 0$

$$\int f \, d\lambda^{(t,x)} \int f(t,x,y) t^{-1} \rho(y) \, dy$$

and

$$\int f \, d\lambda^{(0,x)} = \sum_{\gamma} \int f(0,x,X,\gamma) \rho(x\gamma) \, |\gamma'(x)| \, dX \quad \text{if } t=0.$$

The continuity of the Haar system results from the change of variable formula. We let U be $]0,1] \times M$ and F be $\{0\} \times M$. Then G_U is isomorphic to $]0,1] \times M \times M$ and $G_F = \mathrm{T}M \rtimes \Gamma$. Consider the family $\{L_t, t \in [0,1]\}$ of representations of $C^*(G)$ on the Hilbert space $L^2(M, \rho)$: for $t \neq 0$

$$\langle \xi, L_t(f)\eta \rangle = \int f(t,x,y)\overline{\xi(x)}\eta(y)t^{-1}\rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\langle \xi, L_0(f)\eta \rangle = \sum_{\gamma} \int f(0,x,X,\gamma)\overline{\xi(x)}\eta(x\gamma)\rho(x\gamma)\rho(x)\,|\gamma'(x)|\,\mathrm{d}X\,\mathrm{d}x.$$

Since $\langle \xi, L_0(f)\eta \rangle$ tends to $\langle \xi, L_0(f)\eta \rangle$ as t tends to 0, L_0 is weakly contained in $\{L_t, t \in]0,1]\}$. But for $t \neq 0$, L_t is weakly equivalent to the regular representation of G_t and factors through $C^*_{\mathrm{red}}(G)$. Moreover L_0 vanishes on $C^*_{\mathrm{red}}(G_U)$. However L_0 is not necessarily a representation of $C^*_{\mathrm{red}}(G_F)$. Suppose indeed that Γ is a non--amenable group preserving a density ρ of finite volume. Then the restriction of L_0 to $C^*(\Gamma)$ has a fixed vector, hence cannot factor through $C^*_{\mathrm{red}}(\Gamma)$. Therefore L_0 itself cannot factor $C^*_{\mathrm{red}}(\mathrm{T}M \rtimes \Gamma)$. Examples are provided by [24], where it is shown that an odd-dimensional sphere $\mathbb{S}^{2n+1} (n \geqslant 1)$ admits a non-commutative free group of isometries which acts freely.

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Added in proof. A result similar to Theorem 4.3 on the primitive ideal space of $C^*(G, A)$, but in a more general setting of an algebraic C^*-bundle A over G, is announced in [29].

APPENDICE

UN EXAMPLE DE FEUILLETAGE MINIMAL DONT LA C^*-ALGEBRE N'EST PAS SIMPLE

GEORGES SKANDALIS

Nous construisons un feuilletage (V,F) minimal pour lequel la C^*-algèbre $C^*\ (V,F)$ de Connes (cf. [1]) n'est pas simple. Cet exemple n'est pas en contradiction avec le théorème 2.1 de [3], comme le graphe du feuilletage considéré n'est pas separé.

Soit M une variété, $\tilde{M}$ son revêtement universel et $\Gamma = \pi_1(M)$ son groupe fondamental. Si Γ agit dans une variété T, $V = \tilde{M} \times_\Gamma T$ est muni d'un feuilletage horizontal F dont les feuilles sont les images de $\tilde{M} \times \{x\}$ dans V $(x \in T)$. Le graphe de ce feuilletage restreint à la transversale fidèle T, noté G_T^T, est le quotient du groupoïde $T \rtimes \Gamma$ par la relation d'équivalence $\mathcal{R}$ oú $T \rtimes \Gamma$ est la variété $T \times \Gamma$ munie de la structure de groupoïde $r(x,g) = x$, $s(x,g) = g^{-1}x$ et $(x,g)\circ(g^{-1}x, g') = (x, gg')$, et $\mathcal{R}$ est défini par $(x,g)\mathcal{R}(x',g')$ ssi $x = x'$ et les germes de g^{-1} et g'^{-1} en x coïncident.

Notons que (V,F) est minimal si et seulement si l'action de Γ dans T est minimale. De plus $C^*(V,F)$ et $C^*_{\text{red}}(G_T^T)$ sont équivalentes au sens de Morita ([5], [2], [4]) où $C^*_{\text{red}}(G_T^T)$ désigne la C^*-algèbre réduite du groupoïde étale G_T^T (on a en fait $C^*(V,F) = C^*_{\text{red}}(G_T^T) \otimes \mathcal{K}$ [4]). Mais $C^*_{\text{red}}(G_T^T)$ est le séparé complété de $C_c(T \rtimes \Gamma)$ pour les représentations naturelles π_x dans $\ell^2(G_x^T)$, $x \in T$ (cf. [2], §6).

Posons $T = \mathbb{S}^1$ et soient g_1 et g_2 deux difféomorphismes de T de points fixes respectivement les segments orientés $[a,b]$ et $[b,a]$ où a et b sont deux points distincts de $T = \mathbb{S}^1$. Soit g_0 une rotation irrationelle. Comme g_1 et g_2 commutent entre eux, les difféomorphismes g_1, g_2 et g_0 déterminent une action du produit libre $(\mathbb{Z}\times\mathbb{Z})*\mathbb{Z} = \Gamma$ dans T. Soit M une variété avec $\pi_1(M) = \Gamma$ et (V,F) le feuilletage associé. Comme $g_0 \in \Gamma$, (V,F) est minimal.

Soit $f \in C_c(T \rtimes \Gamma)$ donnée par

$$f(x,g) = \begin{cases} 1 & \text{si } g = 1 \text{ ou } g_1g_2, \\ -1 & \text{si } g = g_1 \text{ ou } g_2, \\ 0 & \text{sinon.} \end{cases}$$

Soit $x \in T$. Notons $(e_z)_{z\in\ell^2(G_x^T)}$ la base canonique de $\ell^2(G_x^T)$. On a

$$\pi_x(f)(e_{\overline{(gx,g)}}) = e_{\overline{(gx,g)}} + e_{\overline{(g_1g_2gx,g_1g_2g)}} - e_{\overline{(g_1gx,g_1g)}} - e_{\overline{(g_2gx,g_2g)}}$$

où si $(y,h) \in T \rtimes \Gamma$, $\overline{(y,h)}$ désigne sa classe dans G_T^T. Si $gx \in]a,b[$, on a $(g_1gx, g_1g) = (gx,g)$ et $(g_1g_2gx, g_1g_2g)\,\mathcal{R}\,(g_2gx, g_2g)$ (car $g_2gx \in]a,b[$) et donc $\pi_x(f)(e_{\overline{(gx,g)}}) = 0$. De même si $g(x) \in]b,a[$, $\pi_x(f)(e_{\overline{(gx,g)}}) = 0$. Donc $\pi_x(f)$ est différent de zéro si et seulement si x appartient à l'orbite de a ou à celle de b. Comme $\pi_a(f) \neq 0$, l'image de f dans $C^*_{\text{red}}(G_T^T) \neq 0$, et donc π_x n'est pas fidèle si x n'est pas dans l'orbite de a ou de b.

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