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TRIANGULAR AF ALGEBRAS AND NEST SUBALGEBRAS OF UHF ALGEBRAS

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If $\mathfrak{D}$ is a masa in a unital C^*-algebra $\mathfrak{A}$, then a nest $\mathcal{M} \subseteq \mathfrak{D}$ is a linearly ordered set of projections containing 0 and 1. Associated with $\mathcal{M}$ is $\mathrm{Alg}\,\mathcal{M} = \{x \in \mathfrak{A} : p^{\perp}xp = 0, p \in \mathcal{M}\}$, the nest algebra of $\mathcal{M}$. Nest algebras form a class of nonselfadjoint operator algebras; in this paper we will primarily be concerned with nest sublagebras of UHF algebras.

The study of nonselfadjoint subalgebras of UHF, and more generally, of AF algebras was begun in [12], [2], and [9]; these papers deal mostly with triangular subalgebras of AF algebras, called TAF algebras. These are nonselfadjoint subalgebras $\mathcal{T}$ of an AF algebra $\mathfrak{A}$ such that $\mathcal{T} \cap \mathcal{T}^*$ is a certain kind of masa. The present work overlaps with these papers in as much as an important class of nest algebras, namely the nest algebras of multiplicity-free nests, are TAF; furthermore, many of our results are formulated for multiplicity-free nests. Just as in [12], [13], and [9], all masas under consideration are assumed to be of the type studied by Strătilă and Voiculescu in [15], called canonical masas by Power and here denoted simply as masas. Such a masa $\mathfrak{D}$ in an AF algebra $\mathfrak{A}$ is constructed by taking an increasing sequence $\{\mathfrak{A}_n\}_{n=1}^{\infty}$ of finite dimensional subalgebras of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$, choosing masas $\mathfrak{D}_n \subseteq \mathfrak{A}_n$ such that $\mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}$, $n = 1, 2, \ldots$, and setting $\mathfrak{D} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{D}_n}$. The spectrum of $\mathfrak{D}$ is zero dimensional. Recently Blackadar has shown how to construct a masa in the 2^{∞} UHF algebra whose spectrum is not zero dimensional ([4]; also see [14]). A theory of triangular algebras and nest algebras based on such "noncanonical" masas would no doubt be quite different from what is done here. But even aside from the question of the masas, there are problems with nest subalgebras of UHF algebras that appear to have no parallel with nest algebras in $\mathfrak{B}(\mathcal{H}) = \{$bounded operators on a Hilbert space}: two nests may be apparently indistinguishable, yet their nest algebras may not be isomorphic. In [9, Example 4.4], there are two nests which admit a trace-preserving bijection from one to the other, and this bijection even extends uniquely to a C^*-isomorphism of the masas generated by the nests; the nest algebras are nevertheless not isometrically isomorphic. The obstruction here is that the ambient UHF algebras, though C^*-isomorphic, are differently represented; i.e., representing the UHF algebras as inductive limits, the sequences of embeddings are different. This example illustrates the fact that C^*-isomorphism is too coarse a notion in this context. On the other hand, if two nest algebras are isometrically isomorphic, then their ambient UHF algebras are C^*-isomorphic (Theorem 2.11).

The normalized trace on a UHF algebra provides a useful invariant: if two maximal nest have isomorphic nest algebras, then they have the same trace (Proposition 2.16). Given a UHF algebra $\mathfrak{A}$ and masa $\mathfrak{D}$, and $0 < \alpha \leqslant 1$, there is maximal multiplicity-free nest $\mathcal{M}$ satisfying $\sup\{\mathrm{tr}(p) : p \in \mathcal{M},\ p < 1\} = \alpha$ (Corollary 2.21). Thus the nest algebras corresponding to distinct α's are not ismetrically isomorphic. This contradicts [12, Proposition 1.6]; unfortunately there is an error in [12, Lemma 1.5] which nullifies this proposition.

The trace is a relatively crude invariant; a stronger invariant is the diagonal ordering introduced in [9] (see Section 0). For instance, there are maximal, multiplicity-free nests $\mathcal{M}$ and $\mathcal{N}$ in a masa $\mathfrak{D}$ in the 2^∞ UHF algebra such that $\mathrm{tr}(\mathcal{M}) = \mathrm{tr}(\mathcal{N})$ is the set of all dyadic rationals, yet $(\mathfrak{D}, \prec_{\mathrm{Alg}\,\mathcal{M}})$ is not order-isomorphic with $(\mathfrak{D}, \prec_{\mathrm{Alg}\,\mathcal{N}})$ (Example 3.9), and a fortiori, $\mathrm{Alg}\,\mathcal{M}$ and $\mathrm{Alg}\,\mathcal{N}$ are not isometrically isomorphic. It was observed in the previous paragraph that the trace of a nest can have gaps; e.g., $(\mathrm{tr}(\mathcal{M})) \cap (\alpha, 1) = \emptyset$, even if $\mathcal{M}$ is maximal and multiplicity-free. If $\mathcal{M}$ satisfies the stronger condition that $C^*(\mathcal{M})$ is a masa, it can still happen that $\mathrm{tr}(\mathcal{M})$ is a proper subset of the range of the trace (Example 3.17), but we can show $\overline{\mathrm{tr}(\mathcal{M})} = [0, 1]$ (Proposition 3.18). An important fact is that among nests $\mathcal{M}$ for which $C^*(\mathcal{M})$ is a masa, the trace is just as good an invariant as the diagonal ordering (Corollary 3.14).

Section 1 is concerned with isomorphisms of TAF algebras. The first results is that isometric module isomorphisms of algebras are in fact algebra isomorphisms (Theorem 1.1); this is a consequence of work of Arazy and Solel on Jordan isomorphisms. The main theorem in this section, called the Diagonal Extension Theorem, addresses the question of extending C^*-isomorphisms of masas to isomorphisms of their ambient AF algebras (Theorem 1.10); it was motivated by the fundamental relation of [13, §6.2] on the spectrum of the masa. This theorem implies [13, Theorem 6.15].

Sections 2 and 3 are devoted to the theory of nest subalgebras of UHF algebras. In Section 2 we develop a number of isomorphism invariants, and show that there are many nonisomorphic nest algebras, generated by maximal nests, in any given UHF algebra. The primary topic of Section 3 is the relationship between trace-preserving bijections of nests and order isomorphisms of the masas induced by the nest algebras. The Order Isomorphism Theorem for Nest Algebras (Theorem 3.13) gives certain sufficient conditions for extending such a bijection to an order isomorphism. At the end of Section 3, we examine the necessary conditions give in [5, Theorem 4.3] for order isomorphism and algebra isomorphism of TAF algebras, and show they are not sufficient (Example 3.20).

Section 4 deals with the K_0 and K_1-groups of both TAF algebras and nest algebras. In the first case, the K-theory reduces to that of the diagonal masa, and in the case of nest algebras the K-theory coincides with that of the commutant of the nest. Pitts [11] has proved the analogous result for K_0 of nest subalgebras of $\mathfrak{B}(\mathcal{H})$, and Peters [10] has done the same for K_0 and K_1 of semicrossed products. Section 4 also has an interesting structure theorem for nest algebras (Theorem 4.4). Two other structure results are given in Corollaries 2.17 and 3.16.

This paper is a first step toward a theory of nests and nest subalgebras of UHF algebras. While the work on nests in Hilbert space serves as a model for the kinds of questions to be asked, there seems to be very little similarity, either in the techniques or in the results. The fact that UHF algebras are antiliminal means that the rôle played by the compact operators in our theory is nil. One result that is valid in both contexts is that maximal nests are reflexive (Corollary 2.4). Much work remains to be done. In particular, we have not formulated the correct notion of general multiplicity, nor have we resolved the question of the validity of the Arveson distance formula in this setting.

0. PRELIMINARIES

A C^*-algebra $\mathfrak{A}$ is *almost finite dimensional* (AF) if $\mathfrak{A}$ contains a sequence $\{\mathfrak{A}_n : 1 \leqslant n < \infty\}$ of finite dimensional subalgebras such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots$ and $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$. All AF algebras in this paper will be unital, and in this case we require that $\mathfrak{A}_1$ contains the unit 1 of $\mathfrak{A}$. In the special case that each $\mathfrak{A}_n$ is a full matrix algebra, then $\mathfrak{A}$ is called a *UHF algebra*. If $\mathcal{S} \subseteq \mathfrak{A}$, then $\mathcal{S}^c$ will denote $\{x \in \mathfrak{A} : xs = sx$ for all $s \in \mathcal{S}\}$, the commutant of $\mathcal{S}$ in $\mathfrak{A}$.

The term *masa* will be used in the sense of [15], as described in the introduction. If $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$ is an AF algebra with masa $\mathfrak{D}$, where $\mathfrak{A}_n$ is a direct sum of factors

$\mathfrak{A}_{(nk)}$, then a system of matrix units $\{e_{ij}^{(nk)}\}$ can always be chosen for $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ so that $\mathfrak{D}$ is the closed linear span of $\{e_{ii}^{(nk)}\}$ (see [9, §1] for more details). Whenever we use matrix units in $\mathfrak{A}$, we will always assume that they are chose in this manner. Thus, we can think of $\mathfrak{D}_n$ as the usual diagonal of $\mathfrak{A}_n$. We will often write $e_{ii}^{(nk)}$ as $e_i^{(nk)}$. In addition, if $\mathcal{P}(\mathfrak{D})$ (resp. $\mathcal{P}(\mathfrak{D}_n)$) is the set of (selfadjoint) projections in $\mathfrak{D}$ (resp. $\mathfrak{D}_n$), then $\mathcal{P}(\mathfrak{D}) = \bigcup_{n=1}^{\infty} \mathcal{P}(\mathfrak{D}_n)$. It follows that the spectrum of $\mathfrak{D}$ is zero dimensional. In the terminology of [14], a masa is a Cartan subalgebra with locally finite ample semigroup. Thus, given two masas $\mathfrak{D}$ and $\mathfrak{C}$ in an AF algebra $\mathfrak{A}$, Corollary III.1.16 of [14] implies that there is an automorphism α of $\mathfrak{A}$ such that $\mathfrak{C} = \alpha(\mathfrak{D})$.

Given an AF algebra $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ with masa $\mathfrak{D}$, the term $\mathfrak{D}$-*module* will always mean a norm-closed unital $\mathfrak{D}$-bimodule. Subalgebras of $\mathfrak{A}$ will be assumed to also be $\mathfrak{D}$-modules (i.e., norm-closed and containing $\mathfrak{D}$) for some masa $\mathfrak{D}$, usually clear from the context. By Lemma 3.3 and Proposition 3.8 of [9], every $\mathfrak{D}$-module is the closed linear span of the matrix units it contains. Another important property, proved in [9, Theorem 2.2], is that every $\mathfrak{D}$-module $\mathcal{S}$ in $\mathfrak{A}$ is *inductive*. This means that $\mathcal{S} = \overline{\bigcup_{n=1}^{\infty} (\mathcal{S} \cap \mathfrak{A}_n)}$. A $\mathfrak{D}$-module $\mathcal{S}$ is *triangular* if $\mathcal{S} \cap \mathcal{S}^* = \mathfrak{D}$, and $\mathfrak{D}$ is the *diagonal* of $\mathcal{S}$. $\mathcal{S} \subseteq \mathfrak{A}$ is a *triangular AF (TAF) algebra*, if $\mathcal{S}$ is a triangular algebra in $\mathfrak{A}$, and $\mathcal{S}$ is a *triangular UHF (TUHF) algebra* if in addition $\mathfrak{A}$ is UHF. A TAF algebra $\mathcal{S}$ in $\mathfrak{A}$ is *maximal triangular* if it is not contained in any larger TAF subalgebra of $\mathfrak{A}$. A maximal triangular algebra $\mathcal{S}$ is *strongly maximal triangular* if there is some sequence $\{\mathfrak{A}_n\}$ such that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathcal{S} \cap \mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ for all $n >$ some N. This is equivalent to the condition $\overline{\mathcal{S} + \mathcal{S}^*} = \mathfrak{A}$ [V, Theorem 2.1].

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are AF algebras with masas $\mathfrak{D}$ and $\mathfrak{C}$, respectively. Let $\mathcal{S}$ be a $\mathfrak{D}$-module in $\mathfrak{A}$ and $\mathcal{T}$ an $\mathfrak{C}$-module in $\mathfrak{B}$. We say that $\varphi : \mathcal{S} \to \mathcal{T}$ is a *module isomorphism* if φ is an isometric bijection which preserves the module structure (i.e., $\varphi(dsd') = \varphi(d)\varphi(s)\varphi(d')$ for all $d, d' \in \mathfrak{D}$, $s \in \mathcal{S}$) and in addition $\varphi(\mathfrak{D}) = \mathfrak{C}$. Note that $\varphi \mid \mathfrak{D}$ is an isomorphism of abelian C^*-algebras, so it is a C^*-isomorphism. Likewise, we will always assume that an algebra isomorphism φ of two algebras $\mathcal{S}$ and $\mathcal{T}$ is isometric. However, in this case we do *not* also assume that $\varphi(\mathfrak{D}) = \mathfrak{C}$ (although it is true that $\varphi(\mathfrak{D}) = \mathfrak{C}$ if $\mathcal{S}$ and $\mathcal{T}$ are TAF, by [9, Proposition 3.20]).

An important tool in the study of $\mathfrak{D}$-modules is the set $\mathcal{W}_{\mathfrak{D}} = \{$partial isometries $w \in \mathfrak{A} : w^*\mathfrak{D}w \subseteq \mathfrak{D}$ and $w\mathfrak{D}w^* \subseteq \mathfrak{D}\}$. Note that the initial and final projections of $w \in \mathcal{W}_{\mathfrak{D}}$ lie in $\mathfrak{D}$. Also, every matrix unit of $\mathfrak{A}$ is an element of $\mathcal{W}_{\mathfrak{D}}$ [9, Lemma 3.3]. Two partial isometries $v, w \in \mathcal{W}_{\mathfrak{D}}$ are *orthogonal* if their initial projections are orthogonal (i.e., $v^*vw^*w = 0$) and their final projections are also orthogonal. If $\mathcal{S}$ is a $\mathfrak{D}$-module, then we define $\mathcal{W}_{\mathfrak{D}}(\mathcal{S}) = \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$. $\mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ induces a relation $\prec_{\mathcal{S}}$ on $\mathcal{P}(\mathfrak{D})$ by $e \prec_{\mathcal{S}} f$ iff there is some $w \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ with initial projection f and final projection e (we will use the notation $e \leqslant f$ and $e < f$ for the usual ordering on projection). $\prec_{\mathcal{S}}$ is the *diagonal ordering induced by* $\mathcal{S}$. If $\mathcal{S}$ is a TAF algebra then $\prec_{\mathcal{S}}$ is in fact a partial ordering [9, Theorem 3.13]. If $\mathcal{S}$ is a $\mathfrak{D}$-module in $\mathfrak{A}$ and $\mathcal{T}$ is an $\mathfrak{C}$-module in $\mathfrak{B}$, then a C^*-isomorphism $\varphi : \mathfrak{D} \to \mathfrak{C}$ is an *order isomorphism* if it has the property that $e \prec_{\mathcal{S}} f$ iff $\varphi(e) \prec_{\mathcal{T}} \varphi(f)$. We denote such a map by $\varphi : (\mathfrak{D}, \prec_{\mathcal{S}}) \to (\mathfrak{C}, \prec_{\mathcal{T}})$. For more details on $\mathcal{W}_{\mathfrak{D}}$, $\mathcal{W}_{\mathfrak{D}}(\mathcal{S})$, and order isomorphisms, see [9, §3].

Suppose $\mathcal{S}$ is a subset of $\mathfrak{A}$ with $\mathfrak{D} \subseteq \mathcal{S}$. A projection $e \in \mathfrak{A}$ is *invariant* for $\mathcal{S}$ if $e^{\perp}se = 0$ for all $s \in \mathcal{S}$, where $e^{\perp} = 1 - e$. Then e is also invariant for $\mathfrak{D}$, so $e \in \mathfrak{D}^c$ since $\mathfrak{D}$ is selfadjoint. But $\mathfrak{D}$ is a masa, so $\mathfrak{D}^c = \mathfrak{D}$. Thus, the set of invariant projections of $\mathcal{S}$ lies in $\mathfrak{D}$, and therefore is commutative. This set is also a lattice with $e \vee f = e + f - ef$ and $e \wedge f = ef$, and is denoted $\operatorname{Lat} \mathcal{S}$. Conversely, if $\mathcal{P}$ is a set of projections in $\mathfrak{A}$, then we define $\operatorname{Alg} \mathcal{P} = \{a \in \mathfrak{A} : p^{\perp}ap = 0\}$. $\operatorname{Alg} \mathcal{P}$ is a norm-closed algebra, but it contains $\mathfrak{D}$ if and only if $\mathcal{P} \subseteq \mathfrak{D}^c = \mathfrak{D}$, since $\mathfrak{D}$ is selfadjoint. The main focus of this paper, in Sections 2 and 3, will be to consider the case in which $\mathcal{P}$ is a linearly ordered set of projections in $\mathfrak{D}$, i.e., $\mathcal{P}$ is a *nest*, and $\mathfrak{A}$ is a UHF algebra.

One of the main discoveries in [9] was that the isomorphism class of a TAF algebra depends on the way in which each finite dimensional C^*-algebra $\mathfrak{A}_n$ is embedded into the next C^*-algebra $\mathfrak{A}_{n+1}$, even though the isomorphism class of $\mathfrak{A}$ is independent of these embeddings [7, 5]. This use of different embeddings will also be very important in this paper. The *nest embeddings* for UHF algebras will play a particularly important role, and the *standard embeddings* will be used. We will define this below, and define other embeddings as we need them. But first, we note that $\mathbf{M}_n$ will be used throughout this paper to denote a fixed representation of the n^2-dimensional factor as $n \times n$ matrices. Second, we remark that there are two ways to define AF algebras: either as $\overline{\bigcup_n \mathfrak{A}_n}$, as we have done above, or as a Banach algebra inductive limit $\varinjlim(\mathfrak{A}_n, j_n)$, where j_n is a C^*-embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$. In the latter case, $\mathfrak{A}_n$ is isomorphic to a C^*-subalgebra $\tilde{\mathfrak{A}}_n$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_n \tilde{\mathfrak{A}}_n}$. Because of the importance of the embeddings j_n in this paper, we will often define an AF algebra using the inductive limit notation, but then we will identify $\mathfrak{A}_n$ and $\tilde{\mathfrak{A}}_n$. The same comments apply to $\mathfrak{D}$-modules in $\mathfrak{A}$.

Suppose $\{p_n\}$ is a sequence of positive integers such that p_n divides p_{n+1}, and let $q_n = p_{n+1}/p_n$. Let $e_{ij}^{(n)}$ be the usual matrix units for $\mathbf{M}_{p_n}$. The embedding

$\nu_n : \mathbf{M}_{p_n} \hookrightarrow \mathbf{M}_{p_{n+1}}$, denoted by $\nu_n(x) = x \otimes I_{q_n}$, is defined by

$$\nu\left(e_{ij}^{(n)}\right) = \sum_{t=1}^{q_n} e_{(i-1)q_n+t,(j-1)q_n+t}^{(n+1)}$$

and is called the *nest embedding* of $\mathbf{M}_{p_n}$ into $\mathbf{M}_{p_{n+1}}$. Note that $\varinjlim(\mathbf{M}_{p_n}, \nu_n)$ is the UHF algebra of type $(p_1, p_2, \ldots)$, and $\nu_n(\mathfrak{D}_n) \subseteq \mathfrak{D}_{n+1}$ where $\mathfrak{D}_n$ is the diagonal of $\mathbf{M}_{p_n}$.

Alternatively, let p_n, q_n and $e_{ij}^{(n)}$ be the same as above and define $\sigma_n : \mathbf{M}_{p_n} \hookrightarrow$ $\hookrightarrow \mathbf{M}_{p_{n+1}}$ by

$$\sigma_n\left(e_{ij}^{(n)}\right) = \sum_{t=0}^{q_n-1} e_{i+tp_n,j+tp_n}^{(n+1)}.$$

This embedding is denoted by $\sigma_n(x) = I_{q_n} \otimes x$, and is called the *standard emebedding*. $\varinjlim(\mathbf{M}_{p_n}, \sigma_n)$ is once again the UHF algebra of type $(p_1, p_2, \ldots)$, since the isomorphism class of the inductive limit is determined by the dimensions of the finite dimensional factors, and not the form of the embedding. Again, note that σ_n embeds the diagonal of $\mathbf{M}_{p_n}$ into the diagonal of $\mathbf{M}_{p_{n+1}}$.

Let $\mathcal{U}_n$ be the set of upper triangular matrices in $\mathbf{M}_{p_n}$. Then both ν_n and σ_n map $\mathcal{U}_n$ into $\mathcal{U}_{n+1}$, so we can define the Banach algebra inductive limits $\mathcal{T} = \varinjlim(\mathcal{U}_n, \nu_n)$ and $\mathcal{S} = \varinjlim(\mathcal{U}_n, \sigma_n)$. It follows from [9, Theorem 2.6] that these are TUHF algebras. The argument given in Example 1.1 of [9] show that $\operatorname{Lat}\mathcal{S} = \{0, 1\}$, but $\operatorname{Lat}\mathcal{T}$ is a nest $\mathcal{L}$. In fact, $\mathcal{L} = \left\{\sum_{i=1}^{j} e_{ii}^{(n)} : 1 \leqslant j \leqslant p_n; 1 \leqslant n < \infty\right\} \cup \{0\}$.

DEFINITION 0.1. The nest $\mathcal{L}$ defined above is called the *canonical nest* in the UHF algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$, and we will use $\mathcal{L}$ throughout to denote this nest.

1. THE DIAGONAL EXTENSION THEOREM

It was shows in [9, Theorem 3.19 and Proposition 3.20] that an isomorphism of two TAF algebras, or simply two modules, induces an order isomorphism of their respective diagonals. However, the converse does not hold: diagonals may be order isomorphic even though the corresponding algebras are not [9, Example 4.4]. The main result in this section is Theorem 1.10, which gives a sufficient condition for extending diagonal order isomorphisms to module, algebra, and C^*-algebra isomorphisms. First, however, we discuss another question which was raised in [9, Example 4.4]: if two TAF algebras are module isomorphic, must they be algebraically isomorphic? We begin by answering this question in the affirmative.

THEOREM 1.1. *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* AF *algebras with masas* $\mathfrak{D}$ *and* $\mathfrak{C}$, *respectively, and suppose* $\mathcal{S}$ *and* $\mathcal{T}$ *algebras with* $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$ *and* $\mathfrak{C} \subseteq \mathcal{T} \subseteq \mathfrak{B}$. *If* $\varphi : \mathcal{S} \to \mathcal{T}$ *is an isometric module isomorphism, then* φ *is also an algebra isomorphism.*

Proof. Let $\{e_{ij}^{(np)}\}$ be a set of matrix units for $\mathfrak{A}$. Since $\mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ generates $\mathcal{S}$, it is enough to show that $\varphi(e_{ij}^{(np)}e_{k\ell}^{(nq)}) = \varphi(e_{ij}^{(np)})\varphi(e_{k\ell}^{(nq)})$ for all appropriate i, j, k, ℓ, n, p, and q. First suppose $p \neq q$ and let P and Q be orthogonal central projections in $\mathfrak{A}_n$ such that $e_{ij}^{(np)} \leqslant P$ and $e_{k\ell}^{(nq)} \leqslant Q$. Then $0 = \varphi(PQ) = \varphi(P)\varphi(Q)$, so $\varphi(e_{ij}^{(np)})\varphi(e_{k\ell}^{(nq)}) = \varphi(e_{ij}^{(np)}P)\varphi(Qe_{k\ell}^{(nq)}) = \varphi(e_{ij}^{(np)})\varphi(P)\varphi(Q)\varphi(e_{k\ell}^{(nq)}) = 0 = = \varphi(e_{ij}^{(np)}e_{k\ell}^{(nq)})$. Thus we may assume $p = q$, and we can write e_{ij} instead of $e_{ij}^{(np)}$ for convenience.

If $j \neq k$, then

$$\varphi(e_{ij})\varphi(e_{k\ell}) = \varphi(e_{ij}e_j)\varphi(e_k e_{k\ell}) = \varphi(e_{ij})\varphi(e_j)\varphi(e_k)\varphi(e_{k\ell}) =$$

$$= \varphi(e_{ij})\varphi(e_j e_k)\varphi(e_{k\ell}) = 0 = \varphi(e_{ij}e_{k\ell}).$$

It remains to show that $\varphi(e_{ij}e_{j\ell}) = \varphi(e_{ij})\varphi(e_{j\ell})$. Note that we can assume $j \neq \ell$. Now by [1, Corollary 2.10], $\varphi(abc+cba) = \varphi(a)\varphi(b)\varphi(c)+\varphi(c)\varphi(b)\varphi(a)$ for all $a, c \in \mathcal{S}$ and $b \in \mathcal{S} \cap \mathcal{S}^*$. Then

$$\varphi(e_{ij}e_{j\ell}) = \varphi(e_{ij}e_j e_{j\ell}) = \varphi(e_{ij}e_j e_{j\ell} + e_{j\ell}e_j e_{ij}) =$$

$$= \varphi(e_{ij})\varphi(e_j)\varphi(e_{j\ell}) + \varphi(e_{j\ell})\varphi(e_j)\varphi(e_{ij}) =$$

$$= \varphi(e_{ij}e_j)\varphi(e_{j\ell}) + \varphi(e_{j\ell}e_j)\varphi(e_{ij}) =$$

$$= \varphi(e_{ij})\varphi(e_{j\ell}),$$

completing the proof. ■

COROLLARY 1.2. *An isometric module isomorphism of two* TAF *algebras is in fact an algebra isomorphism.*

REMARKS 1.3. (a) Note that φ does not have to be a module isomorphism as we have defined it because it need not map $\mathfrak{D}$ onto $\mathfrak{C}$. The only necessary property is that $\varphi(dsd') = \varphi(d)\varphi(s)\varphi(d')$ for all $d, d' \in \mathfrak{D}$ and $s \in \mathcal{S}$.

(b) Theorem 1.1 shows that the two algebras in [9, Example 4.4] are not even module isomorphic.

We will now consider the problem of extending diagonal mappings to module isomorphisms. We begin with some lemmas.

LEMMA 1.4. *Let* $\mathfrak{A}$ *be an* AF-*algebra with masa* $\mathfrak{D}$. *Suppose* $v, w \in \mathcal{W}_{\mathfrak{D}}$ *satisfy* $vev^* = wew^*$ *for all* $e \in \mathcal{P}(\mathfrak{D})$. *Then there is a unique partial isometry* $u \in \mathfrak{D}$

satisfying $vu = w$ *and* $v^*v = u^*u$ *and a unique partial isometry* $u' \in \mathfrak{D}$ *satisfying* $u'v = w$ *and* $vv^* = u'u'^*$..

Proof. Taking $e = 1$ above, we have $vv^* = ww^*$. Now $ev^*w = ev^*vv^*w = v^*vev^*w = v^*wew^*w = v^*ww^*we = v^*we$ for all $e \in \mathcal{P}(\mathfrak{D})$, so $v^*w \in \mathfrak{D}^c = \mathfrak{D}$. Let $u = v^*w$. Then $u^*u = uu^* = (v^*w)(v^*w)^* = v^*ww^*v = v^*vv^*v = v^*v$ and $vu = vv^*w = ww^*w = w$.

To prove the uniqueness, suppose u is a partial isometry in $\mathfrak{D}$ satisfying $vu = w$ and $vv^* = u^*u$. Then $u = uu^*u = u^*uu = v^*vu = v^*w$.

The second statement is proved similarly. ■

Let $\mathfrak{A}$ and $\mathfrak{B}$ be AF-algebras with masas $\mathfrak{D}$ and $\mathfrak{C}$, respectively. Suppose $\mathcal{S} \subseteq \mathfrak{A}$ is a $\mathfrak{D}$-module and $\mathcal{T} \subseteq \mathfrak{B}$ is an $\mathfrak{C}$-module, and suppose $\varphi : \mathcal{S} \to \mathcal{T}$ is a module isomorphism. From [9, Lemma 3.18 and Theorem 3.19], if $v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$, then $\varphi(v) \in \mathcal{W}_{\mathfrak{C}} \cap \mathcal{T}$, $\varphi(v^*v) = \varphi(v)^*\varphi(v)$, $\varphi(vv^*) = \varphi(v)\varphi(v)^*$, and $\varphi : (\mathfrak{D}, \prec_{\mathcal{S}}) \to (\mathfrak{C}, \prec_{\mathcal{T}})$ is an order isomorphism. More generally, it follows that $\varphi(vev^*) = \varphi\big((ve)(ve)^*\big) = \varphi(ve)\varphi(ve)^* = \varphi(v)\varphi(e)\varphi(v)^*$ for all $e \in \mathcal{P}(\mathfrak{D})$. It turns out that this is the extra condition needed to do the converse: extend a diagonal order isomorphism to a module isomorphism and even further to a C^*-isomorphism. More precisely, suppose $\mathcal{S} \subseteq \mathfrak{A}$ is a $\mathfrak{D}$-module and $\mathcal{T} \subseteq \mathfrak{B}$ is an $\mathfrak{C}$-module, and assume a C^*-isomorphism $\varphi : \mathfrak{D} \to \mathfrak{C}$ is given with the property that for any $v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$ there is a $w \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{T}$ for which

$$(*) \qquad \varphi(vdv^*) = w\varphi(d)w^*$$

for all $d \in \mathfrak{D}$. Let a system of matrix units for $\mathfrak{A}$ with respect to $\mathfrak{D}$ be fixed, as well as a system of matrix units for $\mathfrak{B}$ with respect to $\mathfrak{C}$. Let $\mathcal{W}^0_{\mathfrak{D}}$ denote those elements of $\mathcal{W}_{\mathfrak{D}}$ which are matrix units or sums of orthogonal matrix units. Define $\mathcal{W}^0_{\mathfrak{D}}$ similarly. Note that $\mathcal{W}^0_{\mathfrak{D}}$ contains $\mathcal{P}(\mathfrak{D})$, is multiplicative, and is closed under orthogonal sums and adjoints.

LEMMA 1.5. *If* φ *satisfies* $(*)$, *then for each* $v \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$ *there is a unique* $w \in \mathcal{W}^0_{\mathfrak{C}} \cap \mathcal{T}$ *such that* $\varphi(vdv^*) = w\varphi(d)w^*$ *for all* $d \in \mathfrak{D}$.

Proof. Given v, suppose $w' \in \mathcal{W}_{\mathfrak{C}} \cap \mathcal{T}$ satisfies $(*)$. By [9, Corollary 3.7], there is a $\tilde{w} \in \mathcal{W}_{\mathfrak{C}}\mathcal{T} \cap \mathfrak{B}_n$ for some n and a unitary $u \in \mathfrak{C}$ such that $\tilde{w} = w'u$. By [9, Lemmas 3.1 and 3.4], $\tilde{w} = \sum_{i=1}^{m} \lambda_i x_i$, where the x_i's are orthogonal matrix units of $\mathfrak{B}_n$ and each λ_i is a complex number with $|\lambda_i| = 1$. Let f_i be the initial projection of x_i, let $y = (1 - \tilde{w}^*\tilde{w}) + \sum_{i=1}^{m} \lambda_i^{-1} f_i$, and then define $w = \tilde{w}y$. Then $w = \sum_{i=1}^{m} x_i$ and w satisfies $(*)$ since y and u are unitaries in $\mathfrak{C}$. Uniqueness follows from Lemma 1.4 and [9, Lemma 3.4]. ■

On the basis of Lemma 1.5, we can define a map $\tilde{\varphi} : \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S} \to \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{T}$ by $\tilde{\varphi}(v) = w$ if v, w satisfy $(*)$ with $v \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$, $w \in \mathcal{W}^0_{\mathfrak{C}} \cap \mathcal{T}$. Notice that $\tilde{\varphi}$ extends $\varphi \mid \mathcal{P}(\mathfrak{D})$, for if $e \in \mathcal{P}(\mathfrak{D}) \subseteq \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$, we have $\varphi(ede^*) = \varphi(e)\varphi(d)\varphi(e)^*$ for all $d \in \mathfrak{D}$, and as $\varphi(e) \in \mathcal{P}(\mathfrak{C}) \subseteq \mathcal{W}^0_{\mathfrak{C}} \cap \mathcal{T}$, it follows that $\tilde{\varphi}(e) = \varphi(e)$.

LEMMA 1.6. *If v_1, v_2, $v_1v_2 \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$, then $\tilde{\varphi}(v_1)\tilde{\varphi}(v_2) \in \mathcal{W}^0_{\mathfrak{C}} \cap \mathcal{T}$ and $\tilde{\varphi}(v_1)\cdot \cdot\tilde{\varphi}(v_2) = \tilde{\varphi}(v_1v_2)$. In particular, $\tilde{\varphi}(ve) = \tilde{\varphi}(v)\varphi(e)$ and $\tilde{\varphi}(ev) = \varphi(e)\tilde{\varphi}(v)$ for all $e \in \mathcal{P}(\mathfrak{D})$, $v \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$.*

Proof. Let $w_i = \tilde{\varphi}(v_i)$, $i = 1, 2$. First, $\tilde{\varphi}(v_1v_2dv_2^*v_1^*) = \tilde{\varphi}(v_1v_2)\varphi(d)\tilde{\varphi}(v_1v_2)^*$, $d \in \mathfrak{D}$. On the other hand, we also have $\tilde{\varphi}(v_1(v_2dv_2^*)v_1^*) = w_1\varphi(v_2dv_2^*)w_1^* = w_1(w_2\varphi(d)w_2^*)w_1^* = (w_1w_2)\varphi(d)(w_1w_2)^*$. By uniqueness, $\tilde{\varphi}(v_1v_2) = w_1w_2 = \tilde{\varphi}(v_1)\cdot \cdot\tilde{\varphi}(v_2)$. The second statement follows from this and the fact that $\tilde{\varphi}(e) = \varphi(e)$ for $e \in \mathcal{P}(\mathfrak{D})$. ∎

LEMMA 1.7. *Let φ satisfy $(*)$. For every $v \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$,*

(i) $\varphi(vv^*) = \tilde{\varphi}(v)\tilde{\varphi}(v)^*$, and,

(ii) $\varphi(v^*v) = \tilde{\varphi}(v)^*\tilde{\varphi}(v)$.

Proof. (i) follows from $\varphi(vdv^*) = \tilde{\varphi}(v)\varphi(d)\tilde{\varphi}(v)^*$, setting $d = 1$. Taking $e = v^*v$ in Lemma 1.6, we obtain $\tilde{\varphi}(v) = \tilde{\varphi}(v(v^*v)) = \tilde{\varphi}(v)\varphi(v^*v)$, and it follows that $\tilde{\varphi}(v)^*\tilde{\varphi}(v) = \tilde{\varphi}(v)^*\tilde{\varphi}(v)\varphi(v^*v)$. Thus $\varphi(v^*v)$ dominates $\tilde{\varphi}(v)^*\tilde{\varphi}(v)$. Let $f \in \mathcal{P}(\mathfrak{C})$ be any projection orthogonal to $\tilde{\varphi}(v)^*\tilde{\varphi}(v) \in \mathcal{P}(\mathfrak{C})$. Set $f = \varphi(g)$, $g \in \mathcal{P}(\mathfrak{C})$. Then $0 = \left(\tilde{\varphi}(v)f\right)^*\left(\tilde{\varphi}(v)f\right)$ implies $\tilde{\varphi}(v)f = \tilde{\varphi}(vg) = 0$, so $vg = 0$, and hence $v^*vg = 0$, $\varphi(v^*vg) = 0$, and $\varphi(v^*v)f = 0$, so f is orthogonal to $\varphi(v^*v)$. Thus (ii) follows. ∎

COROLLARY 1.8. *Let φ satisfy $(*)$. Then φ is an order-preserving map from $(\mathfrak{D}, \prec_{\mathcal{S}})$ to $(\mathfrak{C}, \prec_{\mathcal{T}})$.*

Proof. Let $e, f \in \mathcal{P}(\mathfrak{D})$ satisfy $e \prec_{\mathcal{S}} f$, i.e., there is a $v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$ with $v^*v = f$ and $vv^* = e$, and in fact we may take v in $\mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}$ by [9, Corollary 3.7]. By Lemma 1.7, $\tilde{\varphi}(v)$ implements the relation $\varphi(e) \prec_{\mathcal{T}} \varphi(f)$. ∎

LEMMA 1.9. *Let φ satisfy $(*)$, and extend $\tilde{\varphi}$ to $\mathcal{W}^0_{\mathfrak{D}} \cap (\mathcal{S}^*)$ by $\tilde{\varphi}(v^*) = \tilde{\varphi}(v)^*$. Then $\tilde{\varphi}$ has a unique extension from the subsemigroup of $\mathcal{W}^0_{\mathfrak{D}}$ generated by $\mathcal{W}^0_{\mathfrak{D}} \cap (\mathcal{S} \cup \mathcal{S}^*)$ into the semigroup of $\mathcal{W}^0_{\mathfrak{C}}$ generated by $\mathcal{W}^0_{\mathfrak{C}} \cap (\mathcal{T} \cup \mathcal{T}^*)$.*

Proof. Let $v \in \mathcal{W}^0_{\mathfrak{D}} \cap \mathcal{S}, e \in \mathcal{P}(\mathfrak{D})$. Then $\varphi(v^*ev) = \varphi\left((ev)^*(ev)\right) = \tilde{\varphi}(ev)^*\cdot \cdot\tilde{\varphi}(ev) = \tilde{\varphi}(v)^*\varphi(e)\tilde{\varphi}(v)$, so condition $(*)$ is satisfied for $v \in \mathcal{W}^0_{\mathfrak{D}} \cap (\mathcal{S} \cup \mathcal{S}^*)$. Let $v_1, \ldots, v_n \in \mathcal{W}^0_{\mathfrak{D}}(\mathcal{S} \cup \mathcal{S}^*)$ and assume inductively that $\tilde{\varphi}(v_1 \cdots v_n) = \tilde{\varphi}(v_1) \cdots \tilde{\varphi}(v_n)$; in other words, $\varphi\left((v_1 \cdots v_n)d(v_1 \cdots v_n)^*\right) = w\varphi(d)w^*$ for all $d \in \mathfrak{D}$, with $w = \tilde{\varphi}(v_1) \cdots \tilde{\varphi}(v_n)$. Set $v = v_1 \cdots v_n$ and $w_i = \varphi(v_i)$, $i = 1, \ldots, n$. Let $v_{n+1} \in$

$\mathcal{W}^0_{\mathfrak{D}} \cap (\mathcal{S} \cup \mathcal{S}^*)$, $w_{n+1} = \tilde{\varphi}(v_{n+1})$, $v' = vv_{n+1}$, $w' = ww_{n+1}$. Then for all $d \in \mathfrak{D}$,

$$\varphi(v'dv'^*) = \varphi\big(v(v_{n+1}dv^*_{n+1})v^*\big) =$$
$$= w\varphi(v_{n+1}dv^*_{n+1})w^* =$$
$$= ww_{n+1}\varphi(d)w^*_{n+1}w^* = w'\varphi(d)w'^*.$$

Since $w' \in \mathcal{W}^0_{\mathfrak{C}}$, we must have $\tilde{\varphi}(v') = w'$ by the uniqueness assertion of Lemma 1.5, and the induction is complete. ■

Next we extend $\tilde{\varphi}$ linearly to linear combinations of elements of the semigroup generated by $\mathcal{W}^0_{\mathfrak{D}}(\mathcal{S} \cup \mathcal{S}^*)$. Since $\tilde{\varphi}(x^*x) = \tilde{\varphi}(x)^*\tilde{\varphi}(x)$ for any x in this set (denoting the extension again by $\tilde{\varphi}$), $\tilde{\varphi}$ is isometric, and hence $\tilde{\varphi}$ has a unique extension to $C^*(\mathcal{S})$, in which linear combinations of elements of the semigroup generated by $\mathcal{W}^0_{\mathfrak{D}}(\mathcal{S} \cup \mathcal{S}^*)$ are dense. We summarize these results in the following theorem.

THEOREM 1.10 (Diagonal Extension Theorem). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be AF algebras having masas $\mathfrak{D}$ and $\mathfrak{C}$, respectively. Let $\mathcal{S}$ be a norm-closed $\mathfrak{D}$-module in $\mathfrak{A}$ and $\mathcal{T}$ a norm-closed $\mathfrak{C}$-module in $\mathfrak{B}$. Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be a C^*-isomorphism with the property that given $v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$ there is a $w \in \mathcal{W}_{\mathfrak{C}} \cap \mathcal{T}$ satisfying $\varphi(vdv^*) = w\varphi(d)w^*$ for all $d \in \mathfrak{D}$. Then there is a C^*-isomorphism $\tilde{\varphi}$ of $C^*(\mathcal{S})$ into $C^*(\mathcal{T})$ which extends φ. The restriction of $\tilde{\varphi}$ to $\mathcal{S}$ is an isometric module isomorphism from $\mathcal{S}$ into $\mathcal{T}$. If $\mathcal{S}$, $\mathcal{T}$ are TAF algebras, then $\tilde{\varphi}$ is an algebra isomorphism.*

REMARKS 1.11. (a) If in addition to the hypotheses of the theorem we assume that for every $w \in \mathcal{W}^0_{\mathfrak{C}} \cap \mathcal{T}$ there is a $v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{S}$ satisfying $\varphi(vdv^*) = w\varphi(d)w^*$ for all $d \in \mathfrak{D}$, then the C^*-isomorphism $\varphi : \mathfrak{D} \to \mathfrak{C}$ is an order isomorphism. In this case, the map $\tilde{\varphi}$ of the theorem is a C^*-isomorphism from $C^*(\mathcal{S})$ onto $C^*(\mathcal{T})$, and its restriction to $\mathcal{S}$ is a module isomorphism onto $\mathcal{T}$.

(b) If $C^*(\mathrm{Lat}\,\mathcal{S}) = \mathfrak{D}$ and $C^*(\mathrm{Lat}\,\mathcal{T}) = \mathfrak{C}$, then it can be shown that there is only one possible order isomorphism of $(\mathfrak{D}, \prec_{\mathcal{S}})$ onto $(\mathfrak{C}, \prec_{\mathcal{T}})$ (see Section 3). In view of (a), if one is given two specific modules with these properties, then it is often rather easy to prove that they are not isomorphic by showing that condition (*) does not hold.

(c) If $\mathcal{S}$, $\mathcal{T}$ are TAF algebras, the fact that the isometric module isomorphism $\tilde{\varphi}$ is in fact an algebra isomorphism is a consequence of Theorem 1.1, though this does not seem to yield a simplification of the proof of Theorem 1.10.

COROLLARY 1.12. *If $\mathcal{S} \subseteq \mathfrak{A}$ is a $\mathfrak{D}$-module and $\mathcal{T} \subseteq \mathfrak{B}$ is an $\mathfrak{C}$-module with $\mathcal{S} \simeq \mathcal{T}$, then $C^*(\mathcal{S}) \simeq C^*(\mathcal{T})$ via an isomorphism $\tilde{\varphi}$ which also satisfied $\tilde{\varphi}(\mathcal{S}) = \mathcal{T}$. Moreover, for each m, there is some n such that $\tilde{\varphi}(\mathcal{S} \cap \mathfrak{A}_m) \subseteq \mathcal{T} \cap \mathfrak{B}_n$ and $\tilde{\varphi}(C^*(\mathcal{S}) \cap \mathfrak{A}_m) \subseteq C^*(\mathcal{T}) \cap \mathfrak{B}_n$.*

Proof. Suppose $\varphi : \mathcal{S} \to \mathcal{T}$ is an isometric isomorphism. By [9, 3.18–3.20], $\varphi \mid \mathfrak{D}$ is a C^*-isomorphism onto $\mathfrak{C}$, $\varphi(\mathcal{W}_{\mathfrak{D}}(\mathcal{S})) = \mathcal{W}_{\mathfrak{C}}(\mathcal{T})$, and $\varphi(vev^*) = \varphi(v)\varphi(e)\varphi(v)^*$ for all $v \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$. The result then follows from Theorem 1.10, Remark 1.11(a), and the proof of Theorem 1.10. ■

REMARK 1.13. This last corollary generalizes Theorem 6.14 of [13], in which it is required in addition that $\mathfrak{A}$ and $\mathfrak{B}$ be UHF algebras, $C^*(\mathcal{S}) = \mathfrak{A}$, and $C^*(\mathcal{T}) = \mathfrak{B}$. In general, we cannot conclude that $\varphi : \mathcal{S} \to \mathcal{T}$ extends to a C^*-isomorphism of $C^*(\mathcal{S})$ onto $C^*(\mathcal{T})$. $\tilde{\varphi}(x) = \varphi(x)$ for $x \in \mathfrak{D}$, but on $\mathcal{S}$ they are only equal modulo a unitary in $\mathfrak{D}$ (which depends on x). However, if $\mathcal{S}$ and $\mathcal{T}$ are strongly maximal triangular algebras, then it was shown in [9, Theorem 3.26] that φ does in fact extend to a C^*-isomorphism.

Finally, this last corollary allows us to prove a generalization of Theorem 2.7 of [5].

COROLLARY 1.14. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be a $\mathfrak{D}$-module and $\mathcal{T} \subseteq \mathfrak{B}$ be an $\mathfrak{C}$-module, and define $\mathcal{S}_n = \mathcal{S} \cap \mathfrak{A}_n$ and $\mathcal{T}_n = \mathcal{T} \cap \mathfrak{A}_n$. Then $\mathcal{S}$ is isomorphic to $\mathcal{T}$ if and only if $\{\mathcal{T}_n\}$ contains a subsequence $\{\mathcal{T}_{n_k}\}$ and each $\mathcal{T}_{n_k}$ contains an $\mathfrak{C}_{n_k}$-module $\mathcal{R}_k$ such that*

(i) *$\{\mathcal{R}_k\}$ is an increasing sequence and there is an isometric $\left(\bigcup_n \mathfrak{D}_n\right)$-bimodule isomorphism $\tilde{\varphi}$ of $\bigcup_n \mathcal{S}_n$ onto $\bigcup_n \mathcal{R}_n$ such that $\tilde{\varphi}(\mathcal{S}_n) = \mathcal{R}_n$ and $\tilde{\varphi}(\mathfrak{D}_n) \subseteq \mathfrak{C}_n$, and*

(ii) *for every m there is some n such that $\mathcal{T}_m \subseteq \mathcal{R}_n$.*

Proof. Sufficiency follows from the facts that $\bigcup_n \mathcal{T}_n = \bigcup_n \mathcal{R}_n$ (by (ii)) and that $\tilde{\varphi}$ is isometric, and therefore $\tilde{\varphi}$ extends to an isomorphism of $\mathcal{S}$ onto $\mathcal{T}$. Necessity follows from Corollary 1.12 by defining $\mathcal{R}_n = \tilde{\varphi}(\mathcal{S}_n)$ ((ii) follows by considering $\tilde{\varphi}^{-1}$). ■

Let $\mathfrak{A}$ be an AF algebra with masa $\mathfrak{D}$, and suppose $\mathcal{S} \subset \mathfrak{A}$ is an $\mathfrak{D}$-module. We define an ordering $\ll_{\mathcal{S}}$ on X, the spectrum of $\mathfrak{D}$, as follows: for $x, y \in X$, $x \ll_{\mathcal{S}} y$ if there is some $v \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ such that $\langle x, vdv^* \rangle = \langle y, d \rangle$ for all $d \in \mathfrak{D}$. We call this the *spectrum ordering induced by $\mathcal{S}$*. This is the same as the fundamental relation defined in [13], except that Power uses the normalizer $\mathcal{N}_{\mathfrak{D}}(\mathcal{S}) = \{$partial isometries $v \in \mathcal{S} : v\mathfrak{D}v^* \subset \mathfrak{D}\}$ instead of $\mathcal{W}_{\mathfrak{D}}(\mathcal{S})$. It can be shown that $\mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ is strictly smaller than $\mathcal{N}_{\mathfrak{D}}(\mathcal{S})$ for certains $\mathcal{S}$. It was the study of this ordering which motivated condition $(*)$ in the hypotheses of Theorem 1.10.

It is not hard to show, using [9, Lemma 3.18], that the spectrum ordering is a module isomorphism invariant. In other words, suppose $\varphi : \mathcal{S} \to \mathcal{T}$ is a module isomorphism, where $\mathcal{S}$ is a $\mathfrak{D}$-module and $\mathcal{T}$ is an $\mathfrak{C}$-module, and let X and Y be the spectra of $\mathfrak{D}$ and $\mathfrak{C}$, respectively. Now if $\hat{\varphi}: Y \to X$ is the dual homeomorphism of $\varphi \mid \mathfrak{D}$, then $y \ll_{\mathcal{T}} y'$ if and only if $\hat{\varphi} \ll_{\mathcal{S}} \hat{\varphi}(y')$. We say that $\hat{\varphi}$ is a *spectrum order isomorphism*. This raises the following questions. If $\hat{\varphi}: Y \to X$ is a spectrum order isomorphism, is $\varphi \mid \mathfrak{D}$ a diagonal order isomorphism? And conversely, if $\varphi \mid \mathfrak{D}$ is an order isomorphism, is $\hat{\varphi}$ one? We have been unable to resolve the first question, but the next example shows that the second question has a negative answer.

We must first discuss the spectrum and the relation $\ll_{\mathcal{S}}$ in greater detail. To make the notation simpler, assume that $\mathfrak{A}$ is UHF. For $e \in \mathcal{P}(\mathfrak{D})$, let $\hat{e} \in C(X)$ be the image of e under the Gelfand map. Let $\{e_{ij}^{(n)}\}$ be a set of matrix units for $\mathfrak{A}$ with respect to $\mathfrak{D}$, and define $e_i^{(n)} = e_{ii}^{(n)}$. Given $x \in X$, there is a unique sequence $(e_{i_1}^{(1)}, e_{i_2}^{(2)}, e_{i_3}^{(3)}, \ldots)$ such that $\hat{e}_{i_n}^{(n)}(x) = 1$ for all n. Conversely, each such sequence with $e_{i_1}^{(1)} > e_{i_2}^{(2)} > \ldots$ corresponds to a unique $x \in X$. Thus, the spectrum X can be identified with the set of such sequences. Now suppose $x = (e_{i_1}^{(1)}, e_{i_2}^{(2)}, \ldots) \ll_{\mathcal{S}}$ $\ll_{\mathcal{S}} x' = (e_{j_1}^{(1)}, e_{j_2}^{(2)}, \ldots)$ via $v \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$. By [9, Corrolary 3.7], we can assume v lies in some $\mathfrak{A}_m$. It then follows that there is some $N \geqslant m$ such that $v e_{j_n}^{(n)} v^* = e_{i_n}^{(n)}$ and $v^* e_{i_n}^{(n)} v = e_{j_n}^{(n)}$ for all $n \geqslant N$, and by replacing v with $v e_{j_N}^{(N)}$, we have $vv^* = e_{j_N}^{(N)}$ and $v^* v = e_{j_N}^{(N)}$. Thus, $e_{i_n}^{(n)} \prec_{\mathcal{S}} e_{j_n}^{(n)}$ for all $n \geqslant N$.

EXAMPLE 1.15. We use the same algebras defined in [9, Example 4.4]. Specifically, let $\mathfrak{A}_n = M_{2^n}$ with diagonal $\mathfrak{D}_n$, and define $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$, where ν_n is the nest embedding defined in Section 0. Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $\mathfrak{B} = \varinjlim(\mathfrak{A}_n, \operatorname{Ad} P_n \circ \nu_n)$, where $P_n = I \oplus \cdots \oplus I \oplus J$. Finally, let $\mathcal{S}_n$ and $\mathcal{T}_n$ both represent the set of upper triangular matrices in $\mathfrak{A}_n$ and define TAF algebras $\mathcal{S} = \varinjlim(\mathcal{S}_n, \nu_n)$ and $\mathcal{T} = \varinjlim(\mathcal{T}_n, \operatorname{Ad} P_n \circ \nu_n)$. Then, as shown in [9, Example 4.4], $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic but $(\mathfrak{D}, \prec_{\mathcal{S}})$ and $(\mathfrak{C}, \prec_{\mathcal{T}})$ are order isomorphic.

We will show that the diagonal order isomorphism between $\mathfrak{D}$ and $\mathfrak{C}$ does not induce an order isomorphism of their spectra X and Y. In fact, we will prove more: the spectra of $\mathfrak{D}$ and $\mathfrak{C}$ are not order isomorphic at all. Let $\{e_{ij}^{(n)}\}$ and $\{f_{ij}^{(n)}\}$ be sets of matrix units for $\mathfrak{A}$ and $\mathfrak{B}$ such that $e_i^{(n)} \prec_{\mathcal{S}} e_j^{(n)}$ and $f_i^{(n)} \prec_{\mathcal{T}} f_j^{(n)}$ if $i \leqslant j$. Now X contains a unique minimal element x_0 for the ordering $\ll_{\mathcal{S}}$, namely $x_0 =$ $= (e_0^{(1)}, e_0^{(2)}, e_0^{(3)}, \ldots)$. Also, X has a unique maximal element $x_1 = (e_1^{(1)}, e_1^{(2)}, e_1^{(3)}, \ldots)$. Similarly, Y has a unique minimal element $y_0 = (f_0^{(1)}, f_0^{(2)}, f_0^{(3)}, \ldots)$ and maximal element $y_1 = (f_1^{(1)}, f_1^{(2)}, f_1^{(3)}, \ldots)$ for the ordering $\ll_{\mathcal{T}}$. It follows that if $\alpha: X \to Y$ is a spectrum order isomorphism, then $\alpha(x_0) = y_0$ and $\alpha(x_1) = y_1$. This is impossible, however, as $y_0 \ll_{\mathcal{T}} y_1$ via $v = f_{12}^{(1)}$, while $x_0 \not\ll_{\mathcal{S}} x_1$. In fact, it is not hard to show that $(e_{i_1}^{(1)}, e_{i_2}^{(2)}, e_{i_3}^{(3)}, \ldots) \ll_{\mathcal{S}} (e_{j_1}^{(1)}, e_{j_2}^{(2)}, e_{j_3}^{(3)}, \ldots)$ if and only if $i_k \leqslant j_k$ for all $k \geqslant$ some N.

The spectrum ordering is an interesting invariant which merits further study. We close this section with a result related to the nest algebras studied in Sections 2 and 3, but which is true in greater generality. Let $\mathfrak{A}$ be an AF algebra with masa $\mathfrak{D} \subseteq \mathfrak{A}$. If $\mathcal{S}$ is a $\mathfrak{D}$-module in $\mathfrak{A}$, then $\mathcal{S}$ induces the diagonal ordering $\prec_{\mathcal{S}}$ on $\mathcal{P}(\mathfrak{D})$. In Theorem 3.11 of [9], we showed that among all norm-closed $\mathfrak{D}$-modules which induce $\prec_{\mathcal{S}}$ on $\mathcal{P}(\mathfrak{D})$, there is a unique maximal $\mathfrak{D}$-module, denoted $\mathcal{M}_{\max(\mathcal{S})}$. Let $\mathcal{M} \subseteq \mathfrak{D}$ be a set of projections and consider $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$. As $\mathcal{S}$ is defined in terms of $\mathcal{M}$, and since by [9, Lemma 3.22] the ordering $\prec_{\mathcal{S}}$ determines $\mathrm{Lat}\,\mathcal{S}$, and also $\mathrm{Alg}(\mathrm{Lat}\,\mathcal{S}) = \mathcal{S}$, it is natural to expect that $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$ is the maximal $\mathfrak{D}$-module consistent with $\prec_{\mathcal{S}}$. This is in fact the case. This next proposition is the analogue of the result for maximal triangular algebras proved in [9, Corollary 3.14].

PROPOSITION 1.16. $\mathcal{M}_{\max}(\mathrm{Alg}\,\mathcal{M}) = \mathrm{Alg}\,\mathcal{M}$.

Proof. Let $v \in W_{\mathfrak{D}} \cap \mathcal{M}_{\max}(\mathrm{Alg}\,\mathcal{M}) = W_{\max}(\mathrm{Alg}\,\mathcal{M})$. By definition [9, Theorem 3.11], $vev^* \prec_{\mathrm{Alg}\,\mathcal{M}} ev^*ve = v^*ve$ for all $e \in \mathcal{P}(\mathfrak{D})$. By [9, Lemma 3.22], if $p \in \mathcal{M}$, then $vev^* \leqslant p$ whenever $v^*ve \leqslant p$. Taking $e = p$, we have $vpv^* \leqslant p$ for all $p \in \mathcal{M}$. Multiplying both sides by vv^*, we obtain $vpv^* = vpv^*vv^* \leqslant pvv^*$. Thus $p^{\perp}vpv^* \leqslant p^{\perp}(pvv^*) = 0$. But then $0 = p^{\perp}vpv^*p^{\perp} = (p^{\perp}vp)(p^{\perp}vp)^*$, so $p^*vp = 0$ and $v \in W_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{L})$. This shows that $W_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{M}) \supseteq W_{\max}(\mathrm{Alg}\,\mathcal{M})$. The other inclusion is obvious. Now use the fact that every closed $\mathfrak{D}$-module is the closed span of the elements of $W_{\mathfrak{D}}$ it contains. ∎

2. NEST SUBALGEBRAS OF UHF ALGEBRAS

We now turn to the study of a different type of nonselfadjoint subalgebra of an AF algebra. Suppose $\mathfrak{A}$ is an AF algebra with masa $\mathfrak{D}$, and let $\mathcal{M}$ be a commuting set of projections in $\mathfrak{A}$ which contains 0 and 1 and which is linearly ordered by the usual ordering of positive elements in $\mathfrak{A}$. If $\mathrm{Alg}\,\mathcal{M}$ is a $\mathfrak{D}$-module, then $\mathcal{M} \subseteq \mathrm{Lat}(\mathrm{Alg}\,\mathcal{M}) \subseteq \mathrm{Lat}\,\mathfrak{D} \subseteq \mathfrak{D}^c = \mathfrak{D}$. Conversely, if $\mathcal{M} \subseteq \mathfrak{D}$, then $\mathrm{Alg}\,\mathcal{M} \supseteq \mathrm{Alg}\,\mathcal{P}(\mathfrak{D}) = \mathfrak{D}^c = \mathfrak{D}$, so $\mathrm{Alg}\,\mathcal{M}$ is a $\mathfrak{D}$-module. Thus, to use the techniques and results of [9] for $\mathfrak{D}$-modules, we must always assume that $\mathcal{M} \subseteq \mathfrak{D}$. We say that $\mathcal{M}$ is a *nest* and $\mathrm{Alg}\,\mathcal{M}$ is a *nest algebra*. Now $\mathcal{M}$ is always contained in $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{M})$, but if $\mathfrak{A}$ is not UHF, then in general $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{M})$ will be larger than $\mathcal{M}$ and will not be a nest. In the UHF case, however, $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{M})$ is always a nest (Theorem 2.3) and $\mathcal{M} = \mathrm{Lat}(\mathrm{Alg}\,\mathcal{M})$ if $\mathcal{M}$ is maximal in $\mathfrak{D}$ (Corollary 2.4). In addition, a UHF algebra has a unique normalized trace (denoted tr) which, as we will see, is very useful for differentiating between different nests and nest algebras. For these two reasons, we will restrict ourselves to the case in which the AF algebra is UHF. Some of the results are still valid in the AF setting, often with $\mathcal{M}$ replaced by Lat (Alg $\mathcal{M}$). Another direction for generalization is to let $\mathcal{M}$ be a more general lattice than a nest. Some of our results are also valid in this setting, but, as in Hilbert space, the general lattice situation is much more complex.

As mentionated in the preliminaries, given any two masas in $\mathfrak{A}$, there is an automorphism of $\mathfrak{A}$ which maps one onto the other. This implies that the properties of a nest algebra Alg $\mathcal{M}$ do not depend on the particular ambient masa containing $\mathcal{M}$. Thus, if one masa is conveninent for studying $\mathcal{M}$ or Alg $\mathcal{M}$, there is no loss in generality in using that masa.

In this section, $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_n}$ and $\mathfrak{B} = \overline{\bigcup \mathfrak{B}_n}$ will be used to represent UHF algebras with masas $\mathfrak{D}$ and $\mathfrak{C}$, respectively. In view of the remarks in the last paragraph, we will always assume that $\mathfrak{A}_n$ and $\mathfrak{B}_n$ are factors, unless otherwise indicated. We say that $\mathcal{M} \subseteq \mathfrak{D}$ is a *maximal nest* if it is a maximal nest in its masa $\mathfrak{D}$. It is easy to see from Definition 0.1 that the canonical nest is maximal. As in the theory of nests in Hilbert space, we say that $\mathcal{M}$ is *multiplicity-free* if $\mathcal{M}^c$ is a masa, i.e., $\mathcal{M}^c = \mathfrak{D}$. Equivalently, $\mathcal{M}^{cc} = (\mathcal{M}^c)^c = \mathfrak{D}$. Note that $(\mathrm{Alg}\,\mathcal{M}) \cap (\mathrm{Alg}\,\mathcal{M})^* = \mathcal{M}^c$, so Alg $(\mathcal{M})$ is triangular iff $\mathcal{M}$ is multiplicity-free. The canonical nest $\mathcal{L}$ is multiplicity-free since $\mathfrak{D} = C^*(\mathcal{L}) \subseteq \mathcal{L}^{cc} \subseteq \mathfrak{D}$. The following two results illustrate the relationship between nest algebras and maximal triangular algebras in a UHF algebra.

PROPOSITION 2.1. *If $\mathcal{S} \subseteq \mathfrak{A}$ is a maximal triangular algebra with respect to $\mathfrak{D}$, then* Lat $\mathcal{S}$ *is a nest. If in addition* $(\mathrm{Lat}\,\mathcal{S})^{cc} = \mathfrak{D}$, *then* $\mathcal{S} = \mathrm{Alg}(\mathrm{Lat}\,\mathcal{S})$, i.e., *$\mathcal{S}$ is a nest algebra.*

Proof. With minor variations, this is the content of Proposition 2.8 and 2.9 of [9]. ■

PROPOSITION 2.2. *Every triangular nest algebra is maximal triangular.*

Proof. Suppose $\mathcal{M}$ is a nest, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$, such that $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$ is triangular, and suppose $\mathcal{T} \not\supseteq \mathcal{S}$ is maximal triangular. Then there is some $T \in \mathcal{T}$ and $P \in \mathcal{M}$ such that $P^\perp TP \neq 0$. Replace T by $P^\perp TP$, so $T \in \mathcal{T} \backslash \mathcal{S}$ and $T = P^\perp TP \neq 0$. But then $T^* = PT^*P^\perp \in \mathrm{Alg}\,\mathcal{M} \subseteq \mathcal{T}$, so $T \in \mathcal{T} \cap \mathcal{T}^* = \mathfrak{D} \subseteq \mathcal{S}$, a contradiction. ■

More generally, a similar proof shows that a nest algebra $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$ is maximal with respect to property that $\mathcal{S} \cap \mathcal{S}^* = \mathcal{M}^c$.

THEOREM 2.3. *If $\mathcal{M}$ is a nest in $\mathfrak{D} \subseteq \mathfrak{A}$, then* Lat (Alg $\mathcal{M}$) *is a nest in $\mathfrak{D}$ also.*

Proof. $\mathcal{M} \subseteq \mathfrak{D}$ implies that $\mathfrak{D} \subseteq \mathrm{Alg}\,\mathcal{M}$, so $\mathrm{Lat}(\mathrm{Alg}\,\mathcal{M}) \subseteq \mathrm{Lat}\,\mathfrak{D} \subseteq \mathfrak{D}^c = \mathfrak{D}$. Now suppose $M_1, M_2 \in \mathrm{Lat}(\mathrm{Alg}\,\mathcal{M})$, $M_1 \neq M_2$. We must show that $M_1 < M_2$ or $M_2 < M_1$.

We first claim that $\mathcal{M} \cup \{M_1\}$ is a nest. If not, then there is some $P \in \mathcal{M}$ such that $PM_1^{\perp} \neq 0$ and $P^{\perp}M_1 \neq 0$, since P and M_1 commute. $PM_1^{\perp}, P^{\perp}M_1 \in \mathcal{P}(\mathfrak{D}_n)$ for some n, so there are diagonal matrix units $e_i^{(n)} \leqslant PM_1^{\perp}$ and $e_j^{(n)} \leqslant P^{\perp}M_1$, $i \neq j$. Then $e_{ij}^{(n)} \in \operatorname{Alg}\mathcal{M}$ since $P\mathfrak{A}P^{\perp} \subseteq \operatorname{Alg}\mathcal{M}$. But $M_1^{\perp}e_{ij}^{(n)}M_1 = e_{ij}^{(n)} \neq 0$, so $M_1 \notin \operatorname{Lat}(\operatorname{Alg}\mathcal{M})$, a contradiction.

Now $\operatorname{Alg}\mathcal{M} = \operatorname{Alg}(\mathcal{M} \cup \{M_1\})$, so $M_2 \in \operatorname{Lat}\big(\operatorname{Alg}(\mathcal{M} \cup \{M_1\})\big)$, and we can use the same argument to conclude that $\mathcal{M} \cup \{M_1\} \cup \{M_2\}$ is a nest. It follows that $M_1 < M_2$ or $M_2 < M_1$. ∎

COROLLARY 2.4. *If $\mathcal{M}$ is a maximal nest in $\mathfrak{D} \subseteq \mathfrak{A}$, then $\mathcal{M}$ is reflexive,* i.e., $\operatorname{Lat}(\operatorname{Alg}\mathcal{M}) = \mathcal{M}$.

COROLLARY 2.5. *If $\mathcal{M}$ is multiplicity-free nest, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$, then* $\operatorname{Lat}(\operatorname{Alg}\mathcal{M})$ *is a maximal multiplicity-free nest in $\mathfrak{D}$.*

Proof. $\tilde{\mathcal{M}} = \operatorname{Lat}(\operatorname{Alg}\mathcal{M})$ is a nest by Theorem 2.3. Note that $\operatorname{Lat}(\operatorname{Alg}\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$. Now $\tilde{\mathcal{M}} \supseteq \mathcal{M}$, so $\tilde{\mathcal{M}}$ is multiplicity-free since $\mathfrak{D} \subseteq \tilde{\mathcal{M}}^c \subseteq \mathcal{M}^c = \mathfrak{D}$. This same argument shows that if $\mathcal{N}$ is a nest with $\mathcal{N} \supseteq \tilde{\mathcal{M}}$, then $\mathcal{N}$ is multiplicty-free. Therefore $\operatorname{Alg}\mathcal{N}$ and $\operatorname{Alg}\tilde{\mathcal{M}}$ are both triangular, in fact maximal triangular by Proposition 2.2. Since $\operatorname{Alg}\mathcal{N} \subseteq \operatorname{Alg}\tilde{\mathcal{M}}$, it follows that $\operatorname{Alg}\mathcal{N} = \operatorname{Alg}\tilde{\mathcal{M}}$. Thus, $\mathcal{N} \subseteq \operatorname{Lat}(\operatorname{Alg}\mathcal{N}) = \operatorname{Lat}(\operatorname{Alg}\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$, which proves that $\tilde{\mathcal{M}}$ is maximal. ∎

EXAMPLE 2.6. By Corollary 2.4, if two maximal nests are different, then so are their associated nest algebras. Here we will give an example of two different nests, one nonmaximal, which have the same nest algebra. Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ with diagonal $\mathfrak{D}_n$, and let $\{e_{ij}^{(n)}\}$ be a set of matrix units for $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$, where ν_n is the nest embedding defined in the preliminaries. Let $\mathcal{L}$ be the canonical nest (Definition 0.1), and define $\mathcal{N}$ to be the nest $\mathcal{L}\backslash\{e_{11}^{(1)}\}$. Then $\mathcal{N}$ is clearly not maximal, and $\operatorname{Alg}\mathcal{L} \subseteq \operatorname{Alg}\mathcal{N}$. To see that these nest algebras are equal, it is enough to show that every matrix unit in $\operatorname{Alg}\mathcal{N}$ also lies in $\operatorname{Alg}\mathcal{L}$, since $\operatorname{Alg}\mathcal{N}$ is generated by its matrix units. So suppose $v = e_{ij}^{(n)} \in \operatorname{Alg}\mathcal{N}$. Then $v \in \operatorname{Alg}(\mathfrak{D}_n \cap \mathcal{N})$, so either $i \leqslant j$ or else $i - 1 = 2^{n-1} = j$. If $i \leqslant j$, then $v \in \operatorname{Alg}\mathcal{L}$. On the other hand, if $i - 1 = 2^{n-1} = j$, then $\nu_n(v) = e_{2^n+1,2^n-1}^{(n+1)} + e_{2^n+2,2^n}^{(n+1)} \notin \operatorname{Alg}(\mathfrak{D}_{n+1} \cap \mathcal{N}) \supseteq \operatorname{Alg}\mathcal{N}$, a contradiction. Therefore, $v \in \operatorname{Alg}\mathcal{L}$ and the nest algebras are equal.

LEMMA 2.7. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and $\mathcal{S}$ and $\mathcal{T}$ are Banach subalgebras of $\mathfrak{A}$ and $\mathfrak{B}$, respectivley. If $\varphi : \mathcal{S} \to \mathcal{T}$ is an isometric algebra isomorphism, then $\varphi(\mathcal{S} \cap \mathcal{S}^*) = \mathcal{T} \cap \mathcal{T}^*$* and $\varphi \mid \mathcal{S} \cap \mathcal{S}^*$ *is a C^*-isomorphism.*

Proof. The proof of [9, Theorem 3.20] also works in this more general setting. ∎

PROPOSITION 2.8. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are nests, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$ and $\mathcal{N} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$,* *and suppose* $\varphi : \mathrm{Alg}\,\mathcal{M} \to \mathrm{Alg}\,\mathcal{N}$ *is an isometric isomorphism. Then*

(i) $\varphi(\mathcal{M}^c) = \mathcal{N}^c$;

(ii) $\varphi(\mathcal{M}^{cc}) = \mathcal{N}^{cc}$;

(iii) $\varphi \mid \mathcal{M}^c$ *and* $\varphi \mid \mathcal{M}^c$ *are* C^**-isomorphisms;*

(iv) $\varphi(\mathrm{Lat}\,(\mathrm{Alg}\,\mathcal{M})) = \mathrm{Lat}\,(\mathrm{Alg}\,\mathcal{N})$, *and* $\varphi(\mathcal{M}) = \mathcal{N}$ *if* $\mathcal{M}$ *and* $\mathcal{N}$ *are maximal nests.*

Proof. (i) and (ii) follow from Lemma 2.7 since $\mathrm{Alg}\,\mathcal{M} \cap (\mathrm{Alg}\,\mathcal{M})^* = \mathcal{M}^c$ and $\mathcal{M}^{cc} \subseteq \mathcal{M}^c$. Now (ii) and (iv) follow from the fact that φ is an algebraic isomorphism, along with Corrolary 2.4. ■

Note that if $\mathrm{Alg}\,\mathcal{M}$ and $\mathrm{Alg}\,\mathcal{N}$ are not triangular, then we may not have $\varphi(\mathfrak{D}) = \mathfrak{C}$. However, $\varphi(\mathfrak{D})$ still has useful properties, as the proof of the next lemma shows.

LEMMA 2.9. *Suppose* $\mathcal{M}$ *and* $\mathcal{N}$ *are nests,* $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$ *and* $\mathcal{N} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, *and suppose* $\varphi : \mathrm{Alg}\,\mathcal{M} \to \mathrm{Alg}\,\mathcal{N}$ *is an isometric isomorphism. If* $v \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{L})$, *then* $\varphi(v)^*\varphi(v) = \varphi(v^*v)$ *and* $\varphi(v)\varphi(v)^* = \varphi(vv^*)$.

Proof. This same result was proved for module isomorphisms in [9, Lemma 3.18]. In that case $\varphi(\mathfrak{D}) = \mathfrak{C}$, and the proof used the facts that $\varphi \mid \mathfrak{D}$ was a C^*-isomorphism, $\mathfrak{C}$ was maximal abelian in $\mathfrak{B}$, and $\mathrm{sp}(\mathfrak{C})$ was zero-dimensional. The same proof will work in this case if we can show that $\varphi(\mathfrak{D})$ has the latter two properties, since we already know $\varphi \mid \mathfrak{D}$ is a C^*-isomorphism by Proposition 2.8(iii) (because $\mathfrak{D} \subseteq \mathcal{M}^c$).

If $b \in \varphi(\mathfrak{D})^c$, then $b \in \varphi(\mathcal{M}^{cc})^c = (\mathcal{N}^{cc})^c = \mathcal{N}^c = \varphi(\mathcal{M})^c$ by Proposition 2.8, so $b = \varphi(a)$ for some $a \in \mathcal{M}^c$. But then $\varphi(a) \in \varphi(\mathfrak{D})^c$ implies that $a \in \mathfrak{D}^c = \mathfrak{D}$. Therefore $b \in \varphi(\mathfrak{D})$, and it follows that $\varphi(\mathfrak{D})$ is maximal abelian. Now $\mathrm{sp}(\mathfrak{D})$ is zero--dimensional and $\varphi \mid \mathfrak{D} : \mathfrak{D} \to \varphi(\mathfrak{D})$ is a C^*-isomorphism, so $\mathrm{sp}(\mathfrak{D})$ and $\mathrm{sp}(\varphi(\mathfrak{D}))$ are homeomorphic. It follows that $\mathrm{sp}(\varphi(\mathfrak{D}))$ is also zero-dimensional. As noted above, the remainder of the proof is the same as in [9, Lemma 3.18], by replacing $\mathfrak{C}$ in that argument with $\varphi(\mathfrak{D})$. ■

The following lemma is a slight variation of [8, Lemma 6.6.4]. The proof is the same.

LEMMA 2.10. *Let* $\mathfrak{A}$ *be a unital* C^**-algebra,* $\{E_i : 1 \leqslant i \leqslant n\}$ *a family of orthogonal projections in* $\mathfrak{A}$ *such that* $\sum E_i = 1$, $\{E_{i1} : 1 \leqslant i \leqslant n\}$ *a family of partial isometries in* $\mathfrak{A}$ *such that* $E_{i1}E_{i1}^* = E_i$, $E_{i1}^*E_{i1} = E_1$, *and* $E_{11} = E_1$. *Then* $\{F_{ij} = E_{i1}E_{j1}^* : i, j \leqslant n\}$ *is a complete set of matrix units for* $\mathfrak{A}$, *with* $F_{ii} = E_i$, *and* $F_{i1} = E_{i1}$.

THEOREM 2.11. *Suppose that* $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$ *and* $\mathcal{T} = \mathrm{Alg}\,\mathcal{N}$ *are nest subalgebras,* $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$ *and* $\mathfrak{C} \subseteq \mathcal{T} \subseteq \mathfrak{B}$, *and* $\varphi : \mathcal{S} \to \mathcal{T}$ *is an isometric isomorphism. Then* $\mathfrak{A} \simeq \mathfrak{B}$.

Proof. If $\mathcal{M}$ is trivial, then $\mathcal{S} = \mathfrak{A}$ and $\operatorname{Lat}\mathcal{S} = \mathcal{M}$, so $\operatorname{Lat}\mathcal{T}$ is also trivial by Proposition 2.8(iv), and therefore $\mathcal{T} = \mathfrak{B}$. Thus, we may assume $\mathcal{M}$ is nontrivial, and therefore $\mathcal{M} \cap \mathfrak{D}_n$ is nontrivial for all $n \geqslant$ some N. Choose $n \geqslant N$ and let $P \in \mathcal{M} \cap \mathfrak{D}_n$, $P \neq 0, 1$. Let $\{e_{ij} : 1 \leqslant i, j \leqslant m\}$ be a set of matrix units for $\mathfrak{A}_n$ so that $\mathfrak{D}_n = \operatorname{span}\{e_i = e_{ii} : 1 \leqslant i \leqslant m\}$, $e_1 \leqslant P^{\perp}$, and $e_2 \leqslant P$. $P\mathfrak{A}_n P^{\perp} \subseteq \mathcal{S} \cap \mathfrak{A}_n$, so $\varphi(e_{i1})$ is defined for all i such that $e_i \leqslant P$ and $\varphi(e_{2j})$ is defined for all j such that $e_j \leqslant P^{\perp}$.

If $e_i \leqslant P$, define $E_{i1} = \varphi(e_{i1})$. If $e_i \leqslant P^{\perp}$, define $E_{i1} = \varphi(e_{2i})^*\varphi(e_{21})$. Define $E_i = \varphi(e_{ii})$ for all i. $\{E_i\}$ is a family of orthogonal projections in $\mathfrak{B}$ such that $\sum E_i = 1$ by Proposition 2.8(iii), and $E_{11} = \varphi(e_{21})^*\varphi(e_{21}) = \varphi(e_{21}^* e_{21}) = \varphi(e_{11}) = E_1$ by Lemma 2.9. We will show that $\{E_{i1}\}$ satisfies the conditions of Lemma 2.10.

If $e_i \leqslant P$, then $E_{i1}E_{i1}^* = \varphi(e_{i1})\varphi(e_{i1})^* = \varphi(e_{i1}e_{i1}^*)$ by Lemma 2.9, and $\varphi(e_{i1}e_{i1}^*) = \varphi(e_{ii}) = E_i$. Also, $E_{i1}^*E_{i1} = \varphi(e_{i1})^*\varphi(e_{i1}) = \varphi(e_{i1}^*e_{i1}) = \varphi(e_{i1}) = E_1$. On the other hand, if $e_i \leqslant P^{\perp}$, then

$$
\begin{aligned}
E_{i1}E_{i1}^* &= \varphi(e_{2i})^*\varphi(e_{21})\varphi(e_{21})^*\varphi(e_{2i}) = \\
&= \varphi(e_{2i})^*\varphi(e_{21}e_{21}^*)\varphi(e_{2i}) = \varphi(e_{2i})^*\varphi(e_{22})\varphi(e_{2i}) = \\
&= \varphi(e_{2i})^*\varphi(e_{22}e_{2i}) = \varphi(e_{2i})^*\varphi(e_{2i}) = \\
&= \varphi(e_{2i}^*e_{2i}) = \varphi(e_{ii}) = E_i,
\end{aligned}
$$

again by Lemma 2.9. Similarly, $E_{i1}^*E_{i1} = E_1$. Thus, Lemma 2.10 applies to $\{E_i\}$ and $\{E_{i1}\}$, so $\{F_{ij} = E_{i1}E_{j1}^* : 1 \leqslant i, j \leqslant m\}$ is a complete set of matrix units. Therefore, if $\mathfrak{A}_n \simeq \mathbf{M}_{k_n}$, then $\mathfrak{B}$ contains a factor of type I_{k_n}. Since this is true for all $n \geqslant N$, it follows that if $p_1^{m_1}p_2^{m_2}\cdots$ and $q_1^{n_1}q_2^{n_2}\cdots$ are the supernatural numbers of $\mathfrak{A}$ and $\mathfrak{B}$, respectively, then for each $i, p_i =$ some q_j and $m_i \leqslant n_j$ (i.e., the supernatural number of $\mathfrak{A}$ is less than or equal to the supernatural number of $\mathfrak{B}$). But now we can apply the same argument to φ^{-1} and conclude that $\mathfrak{A}$ and $\mathfrak{B}$ have the same supernatural numbers. Therefore, $\mathfrak{A} \simeq \mathfrak{B}$. ∎

REMARK 2.12. With the addition of Corollary 2.17 below, this theorem would be a consequence of Theorem 1.10 and the remarks following it (or of [13, Theorem 6.14]) if we knew that $\varphi(\mathfrak{D})$ was a Strătilă-Voiculescu masa in $\mathfrak{A}$. Of course this holds if $\mathcal{N}$ is a multiplicity-free nest by Proposition 2.8, but we doubt that it holds in general.

DEFINITION 2.13. Suppose $\mathcal{S}$ is a $\mathfrak{D}$-module, $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$. Then $\mathcal{S}$ is *irreducible* if there is no nontrivial projection $p \in \mathcal{S}^c$. $\mathcal{S}$ is *strongly irreducible* if there is a subsequence $\{n_k : 1 \leqslant k < \infty\}$ such that $\mathcal{S} \cap \mathfrak{A}_{n_k}$ is an irreducible $\mathfrak{D}_{n_k}$-module in $\mathfrak{A}_{n_k}$ for each k, i.e., there is no nontrivial projection in $(\mathcal{S} \cap \mathfrak{A}_{n_k})^c \cap \mathfrak{A}_{n_k}$. Equivalently, $\mathcal{S}$ is strongly irreducible if and only if $C^*(\mathcal{S} \cap \mathfrak{A}_{n_k}) = \mathfrak{A}_{n_k}$ for some subsequence

$\{n_k\}$. Since every projection in $\mathcal{S}^c$ must lie in $\mathcal{P}(\mathfrak{D}) = \bigcup_n \mathcal{P}(\mathfrak{D}_n)$, it follows that strongly irreducible modules are irreducible. However, the converse is false, as shown in Example 2.14 below. Note that if $\mathcal{S} \cap \mathfrak{A}_n$ is maximal triangular for each n, then $\mathcal{S}$ is strongly irreducible.

EXAMPLE 2.14. We will give an example of an irreducible TAF algebra which is not strongly irreducible. Let $\mathfrak{A}_n = \mathrm{M}_{2^n}$ with diagonal $\mathfrak{D}_n$, let $\{e_{ij}^{(n)}\}$ be a set of matrix units for $\mathfrak{A}_n$, and let $\sigma_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ be the standard embedding, defined in Section 0. Define $\mathcal{V}_n$ to be the set of matrices (a_{ij}) in $\mathfrak{A}_n$ such that $a_{ij} = 0$ for $i > j$ and $a_{1j} = 0$ for $j > 1$. Note that each $\mathcal{V}_n$ is a triangular subalgebra of $\mathfrak{A}_n$ and $\sigma_n(\mathcal{V}_n) \subseteq \mathcal{V}_{n+1}$, so $\mathcal{V} = \varinjlim(\mathcal{V}_n, \sigma_n)$ is a TAF algebra in the UHF algebra $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \sigma_n)$ by [9, Theorem 2.6]. Also $\mathcal{V}_n = \mathcal{V}_{n+1} \cap \mathfrak{A}_n$, so $\mathcal{V}_n = \mathcal{V} \cap \mathfrak{A}_n$ by [9, Proposition 2.5].

Now the projection $e_{11}^{(n)}$ commutes with $\mathcal{V}_n$, so each $\mathcal{V}_n$ is reducible. However, $\mathcal{V}$ is irreducible. To see this, suppose p is a projection which commutes with $\mathcal{V}$. Then $p \in \mathfrak{D}^c = \mathfrak{D}$ and therefore lies in some $\mathcal{P}(\mathfrak{D}_n)$. Also, p commutes with $\mathcal{V}_n$, so $p = 0, e, e^\perp$, or 1, where $e = e_{11}^{(n)}$. But $\sigma_n(e)$ and $\sigma_n(e^\perp)$ do not commute with $\mathcal{V}_{n+1}$, so $p = 0$ or 1.

PROPOSITION 2.15. *If $\mathcal{S}$ is a nest algebra, $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$, then $\mathcal{S}$ is strongly irreducible. In fact, $\mathcal{S} \cap \mathfrak{A}_n$ is irreducible for all $n >$ some N.*

Proof. Let $\mathcal{M} = \operatorname{Lat} \mathcal{S}$. If $\mathcal{M} = \{0, 1\}$, then $\mathcal{S} = \mathfrak{A}$ and the result is trivial, so assume $\mathcal{M}$ is nontrivial. Since $\mathcal{M} = \bigcup_n (\mathcal{M} \cap \mathfrak{D}_n)$, it follows that $\mathcal{M} \cap \mathfrak{D}_n$ is nontrivial for all $n >$ some N.

Now suppose $n > N$ and $\mathcal{S} \cap \mathfrak{A}_n$ is reducible, i.e., there is some projection $p \neq 0, 1$ in $\mathfrak{A}_n$ such that $ps = sp$ for all s in $\mathcal{S} \cap \mathfrak{A}_n$. $p \in \mathfrak{D}_n$ since $\mathfrak{D}_n \subseteq \mathcal{S} \cap \mathfrak{A}_n$ and $\mathfrak{D}_n$ is a masa in $\mathfrak{A}_n$. We first show that there is no $q \in \mathcal{M} \cap \mathfrak{D}_n$ such that $p \geqslant q$. $q\mathfrak{A}_n q^\perp \subseteq \operatorname{Alg} \mathcal{M} = \mathcal{S}$ for any such q, so in particular there is some matrix unit $e_{ij}^{(n)} \in \mathcal{S} \cap \mathfrak{A}_n$ such that $q e_{ij}^{(n)} p^\perp = e_{ij}^{(n)}$. But then $q e_{ij}^{(n)} p^\perp = p\left(q e_{ij}^{(n)} p^\perp\right)p = 0$, a contradiction. Similarly, there is no $q \in \mathcal{M} \cap \mathfrak{D}_n$ such that $p \leqslant q$. Thus, since $\mathcal{M} \cap \mathfrak{D}_n$ is nontrivial, there is some $q \in \mathcal{M} \cap \mathfrak{D}_n$, $q \neq 0, 1$, such that $p^\perp q \neq 0$ and $pq^\perp \neq 0$. Let $e_{k\ell}^{(n)}$ be a matrix unit such that $(p^\perp q) e_{k\ell}^{(n)} (pq^\perp) = e_{k\ell}^{(n)}$. Then $e_{k\ell}^{(n)} \in \mathcal{S} \cap \mathfrak{A}_n$ since $q\mathfrak{A}_n q^\perp \subseteq \operatorname{Alg} \mathcal{M} = \mathcal{S}$, but $p e_{k\ell}^{(n)} = 0$ and $e_{k\ell}^{(n)} p = e_{k\ell}^{(n)} \neq 0$, a contradiction. Therefore, $\mathcal{S} \cap \mathfrak{A}_n$ is irreducible for all $n > N$. ■

Strong irreducibility is the key to showing that certain diagonal maps preserve trace. If $\mathfrak{D}$ is a masa in $\mathfrak{A}$, then there are C^*-isomorphisms of $\mathfrak{D}$ which do not preserve trace on $\mathfrak{D}$ (see Example 3.6). However, it turns out that diagonal order isomorphisms do preserve trace in some cases. This will be discussed in Section 3. Unfortunately, as noted earlier, an isomorphism of two nest algebras does not necessarily induce an order isomorphism of their diagonal masas. However, their nests are isomorphic by Proposition 2.8(iv), and we can use Lemma 2.9 to show that the trace is preserved.

Proposition 2.16. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are nests, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$, $\mathcal{N} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, and $\varphi : \operatorname{Alg}\mathcal{M} \to \operatorname{Alg}\mathcal{N}$ is an isometric isomorphism. Then $\varphi \mid \mathfrak{D}$ preserves trace. In particular, φ preserves trace on $\operatorname{Lat}(\operatorname{Alg}\mathcal{M})$.*

Proof. Let $\mathcal{S} = \operatorname{Alg}\mathcal{M}$. By Proposition 2.15, $\mathcal{S} \cap \mathfrak{A}_n$ is irreducible for all $n >$ $>$ some N. If we let $e_1, \dots, e_m$ be the minimal projections in $\mathfrak{D}_n$, $n > N$, it is enough to show that $\operatorname{tr}(\varphi(e_i)) = \operatorname{tr}(e_i)$ for all i. Write $e_i \sim e_j$ if $e_i \prec_{\mathcal{S}} e_j$ or $e_j \prec_{\mathcal{S}} e_i$, and write $e_i \approx e_j$ if there are $k_1, k_2, \dots, k_\ell$ such that $e_i \sim e_{k_1} \sim \cdots \sim e_{k_\ell} \sim e_j$. Then $\approx$ is an equivalence relation, and irreducibility of $\mathcal{S} \cap \mathfrak{A}_n$ implies that $[e_1]$, the equivalence class of e_1, consists of the set $\{e_1, \dots, e_m\}$, i.e., there is just one equivalence class $\left(\text{otherwise } p = \sum_{e_i \in [e_1]} e_i \text{ commutes with } \mathcal{S} \cap \mathfrak{A}_n\right)$.

If $e_i \prec_{\mathcal{S}} e_j$, then there is some $v \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ such that $vv^* = e_i$ and $v^*v = e_j$. Lemma 2.9 then implies that

$$\operatorname{tr}(\varphi(e_i)) = \operatorname{tr}(\varphi(vv^*)) = \operatorname{tr}(\varphi(v)\varphi(v)^*) = \operatorname{tr}(\varphi(v)^*\varphi(v)) =$$

$$= \operatorname{tr}(\varphi(v^*v)) = \operatorname{tr}((\varphi(e_j)).$$

It follows that $\operatorname{tr}(\varphi(e_i)) = \operatorname{tr}(\varphi(e_j))$ if $e_i \approx e_j$, and therefore $\operatorname{tr}(\varphi(e_j)) = \operatorname{tr}(\varphi(e_1))$ for all j. Finally,

$$1 = \operatorname{tr}(\varphi(1)) = \sum_{j=1}^{m} \operatorname{tr}(\varphi(e_j)) = \sum_{j=1}^{m} \operatorname{tr}(\varphi(e_1)),$$

so

$$\operatorname{tr}(\varphi(e_j)) = \operatorname{tr}(\varphi(e_1)) = \frac{1}{m} = \operatorname{tr}(e_j) \quad \text{for all } j.$$

■

It is interesting to note that the same result holds for strongly maximal TUHF algebras. For in this case, $\varphi : \mathcal{S} \to \mathcal{T}$ extends to a C^*-isomorphism $\tilde{\varphi}$ of $\mathfrak{A}$ onto $\mathfrak{B}$ by [9, Theorem 3.26]. $\tilde{\varphi}$ can then be used to show directly that $\operatorname{tr}(\varphi(e_i)) = \operatorname{tr}(\varphi(e_j))$ for each pair of minimal projections e_i and e_j in $\mathfrak{D}_n$.

If $\mathcal{S} = \operatorname{Alg}\mathcal{M}$ is a nest algebra, and $e_1, \dots, e_m$ are the minimal projections in $\mathfrak{D}_n$, then as we saw in the last proof, $e_1 \approx e_j$ for all j if n is large enough. Actually, the relation $\approx$ can be implemented in at most two steps. To see this, let p be a projection in $\mathfrak{D}_n \cap \mathrm{Lat}\, \mathcal{S}$. If $e_i^{(n)} \leqslant p$ and $e_j^{(n)} \leqslant p^\perp$, then $e_{ij}^{(n)} \in \mathcal{S}$ and $e_i^{(n)} \prec_{\mathcal{S}} e_j^{(n)}$ via $e_{ij}^{(n)}$. Suppose $e_i^{(n)}, e_j^{(n)} \leqslant p$. Let $e_k^{(n)} \leqslant p^\perp$. Then $e_i^{(n)} \prec_{\mathcal{S}} e_k^{(n)}$ and $e_j^{(n)} \prec_{\mathcal{S}} e_k^{(n)}$, so $e_i^{(n)} \sim e_k^{(n)} \sim e_j^{(n)}$. Note also that $e_{ij}^{(n)} = e_{ik}^{(n)} e_{kj}^{(n)} \in \mathcal{S}\mathcal{S}^*$. Similarly, if $e_i^{(n)}, e_j^{(n)} \leqslant p^\perp$, then $e_i^{(n)} \sim e_\ell^{(n)} \sim e_j^{(n)}$ for some $e_\ell^{(n)} \leqslant p$ and $e_{ij}^{(n)} \in \mathcal{S}^*\mathcal{S}$. Therefore, we have

CORROLARY 2.17. *Suppose $\mathcal{S} = \mathrm{Alg}\,\mathcal{M}$ is a nest algebra, and let $\mathcal{S}_n = \mathfrak{A}_n \cap \cap \mathrm{Alg}\,\mathcal{M}$. Then $\mathfrak{A}_n = \mathrm{span}\{\mathcal{S}_n\mathcal{S}_n^*, \mathcal{S}_n^*\mathcal{S}\}$ for all $n >$ some N, $\mathfrak{A} = \overline{\mathrm{span}\{\mathcal{S}\mathcal{S}^*, \mathcal{S}^*\mathcal{S}\}}$, $\mathfrak{A} = C^*(\mathcal{S})$, and $\mathcal{S}^c = \mathbf{C}I$.*

If in addition $C^*(\mathcal{M}) = \mathfrak{D}$, then a stronger result is true (see Corollary 3.16). We are now in position to use the trace and Proposition 2.16 to show that there are many nonisomorphic nest algebras in any given UHF algebra.

THEOREM 2.18. *Suppose $\mathfrak{A}$ is a UHF algebra with masa $\mathfrak{D}$ and $0 < \alpha < 1$. Then there is a maximal nest $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$ such that $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}, P < 1\} = \alpha$.*

Proof. We will carry out the construction for the 2^∞ UHF algebra; the construction is similar in general case. Thus, let $\{e_{ij}^{(n)}\}$ be the usual matrix units for M_{2^n}, and let $\mathfrak{A} = \varinjlim(\mathrm{M}_{2^n}, \nu_n)$, where ν_n is the nest embedding. Let $e_i^{(n)} = e_{ii}^{(n)}$. Considering the infinite dyadic expansion $.d_1d_2d_3\ldots$ of α, let $\{d_{n_j}\}$ be the subsequence of 1's in the sequence $\{d_n\}$, and let $\alpha_j = .d_1d_2\cdots d_{n_j}$ for all j. Thus, $\alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha$ and $\alpha_j \to \alpha$. Finally, let $\beta_j = \alpha_j - \alpha_{j-1}$.

We will define a sequence $\{E_j : 1 \leqslant j < \infty\}$ of mutually orthogonal projections in $\mathcal{P}(\mathfrak{D})$ with $\mathrm{tr}(E_j) = \beta_j$ and such that if $E \in \mathcal{P}(\mathfrak{D})$, then $EE_j \neq 0$ for some j. Then define $P_j = \sum_{i=1}^{j} E_i$. It follows that $\sup\{\mathrm{tr}(P_j) : 1 \leqslant j < \infty\} = \alpha$ and that there is no $Q \in \mathcal{P}(\mathfrak{D})$, $Q \neq 1$, such that $Q \geqslant P_j$ for all j. The proof can then be completed by choosing any maximal nest $\mathcal{M}$ in $\mathfrak{D}$ containing the sequence $\{P_j\}$.

First find k_1 such that $\mathrm{tr}\big(e_1^{(k_1)}\big) = \beta_1$. Let $E_1 = e_1^{(k_1)}$. Now find k_2 such that $\mathrm{tr}\big(e_1^{(k_2)}\big) = \beta_2$ and let $E_2 = e_{2^{k_2-1}+1}^{(k_2)}$ and $m_1 = 2$. Then $e_1^{(1)}E_1 \neq 0$ and $e_2^{(1)}E_2 \neq 0$. Let $f_1^{(2)}, \ldots, f_{m_2}^{(2)}$ be the minimal projections in $\mathfrak{D}_2$ such that $E_j f_i^{(2)} = 0$ for all i, j. Actually, $m_2 = 1$ if $\alpha > 1/2$ and $m_2 = 2$ if $\alpha \leqslant 1/2$. For each $i, 1 \leqslant i \leqslant m_2$, find a new k_i such that $\mathrm{tr}\big(e_1^{(k_i)}\big) = \beta_{m_1+i}$ and let $E_{m_1+i} = e_{n_i}^{(k_i)}$, where n_i is chosen so that $E_{m_1+i} \leqslant f_i^{(2)}$. Now let $f_1^{(3)}, \ldots, f_{m_3}^{(3)}$ be the minimal projections in $\mathfrak{D}_3$ such that $E_j f_i^{(3)} = 0$ for all i, j. For each i with $1 \leqslant i \leqslant m_3$, find a new k_i so that $\mathrm{tr}\big(e_1^{(k_i)}\big) = \beta_{m_2+i}$ and let $E_{m_2+i} = e_{n_i}^{(k_i)}$, where a new n_i is chosen so that $E_{m_2+i} \leqslant \leqslant f_1^{(3)}$. Continue in this manner to get the sequence $\{E_j\}$. ∎

COROLLARY 2.19. *There are uncountably many nonisomorphic nest subalgebras in any given UHF algebra.*

Proof. For each $\alpha, 0 < \alpha \leqslant 1$, construct a nest $\mathcal{M}_\alpha$ by Theorem 2.18 so that $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}_\alpha, P < 1\} = \alpha$. Then the nest algebras $\{\mathrm{Alg}\,\mathcal{M}_\alpha\}$ are pairwise nonisomorphic by Proposition 2.16. ■

Before proving the next lemma, we make the following observations. If $\mathfrak{A}$ is a UHF algebra with masa $\mathfrak{D}$, and $E \in \mathcal{P}(\mathfrak{D})$, then $E\mathfrak{A}E$ is a UHF algebra with masa $\mathfrak{D}E = E\mathfrak{D}E$. Also, if $\mathcal{M}$ is a maximal nest in $\mathfrak{D}$, then $\mathcal{M}E = \{PE : P \in \mathcal{M}\} = EME = E\mathcal{M}$ is a maximal nest in $\mathfrak{D}E$, and if $\mathcal{M}$ is multiplicity-free, then so is $\mathcal{M}E$ (i.e., $(\mathcal{M}E)^c \cap E\mathfrak{A}E = \mathfrak{D}E$). The proofs of these facts are straightforward and will be omitted.

Furthermore, given any $\mathfrak{A}$ and $\mathfrak{D}$, a nest $\mathcal{K}$ can always be constructed with the property that $\mathcal{K} \cap \mathfrak{D}_n$ is a maximal nest in $\mathfrak{D}_n$. Simply let $\mathcal{K}_1$ be a maximal nest in $\mathfrak{D}_1$ and inductively define $\mathcal{K}_{n+1}$ to be any maximal nest in $\mathfrak{D}_{n+1}$ containing $j_n(\mathcal{K}_n)$, where j_n is the embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$. Then let $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_n$. Note that $\mathcal{K}$ is maximal and $C^*(\mathcal{K}) = \mathfrak{D}$, so $\mathcal{K}$ is multiplicity-free.

Let $\mathcal{N}$ be a countable nest in $\mathfrak{D}$. Suppose $\mathcal{N}$ admits a finite partition $\mathcal{N} = \bigcup_{i=0}^{N} A_i$ with $A_i < A_{i+1}$ (i.e., $p < q$ for all $p \in A_i$, $q \in A_{i+1}$), $0 \leqslant i < N$, such that

(i) If $p, q \in A_i$ and $p < q$, then there is a finite chain $p = p_0 < p_1 < \cdots < p_m = q$ such that p_{j+1} is the immediate succesor of p_j in $\mathcal{N}$ (and, necessarily, $p_1, \ldots, p_{m-1} \in A_i$).

(ii) For each i, there is no $R \in \mathcal{P}(\mathfrak{D})$ such that $p < R < q$ for all $p \in A_i$, $q \in A_{i+1}$.

(iii) For each i, there is no $e \in \mathcal{P}(\mathfrak{D})$ with $e \leqslant q - p$ for all $p \in A_i$, $q \in A_{i+1}$.

LEMMA 2.20. *Let $\mathcal{N}$ be as above, and let $\mathcal{K}$ be a nest in $\mathfrak{D}$ such that $\mathcal{K} \cap \mathfrak{D}_n$ is a maximal nest in $\mathfrak{D}_n$ for all n. Define $\tilde{\mathcal{N}} = \bigcup_{p \in \mathcal{N}} (p + (p' - p)\mathcal{K})$, where p' is the immediate succesor of p in $\mathcal{N}$ if p has such, and $p' = p$ otherwise. Then $\tilde{\mathcal{N}}$ is a maximal, multiplicity-free nest containing $\mathcal{N}$.*

Proof. To show $\tilde{\mathcal{N}}$ is maximal, let $r \in \mathcal{P}(\mathfrak{D})$ be comparable with $\tilde{\mathcal{N}}$, $r \notin \tilde{\mathcal{N}}$. Set $r_- = \{p \in \mathcal{N} : p < r\}$ and $r_+ = \{p \in \mathcal{N} : r < p\}$. Then $\mathcal{N} = r_- \cup r_+$. Consider two cases:

Case (a). There exists an $i_0 \in \{0, \ldots, N\}$ such that $r_- \cap A_{i_0} \neq \emptyset$ and $r_+ \cap A_{i_0} \neq \emptyset$. Let $p \in r_- \cap A_{i_0}$, $q \in r_+ \cap A_{i_0}$. By (i), there is a chain $p = p_0 < p_1 < \cdots < p_m = q$ in A_{i_0}, where p_{j+1} the immediate succesor of p_j in $\mathcal{N}$. Thus, $p_j \not< r < p_{j+1}$ for some j. Change notation and set $p = p_j$, $q = p_{j+1}$. Then r is comparable with $p + (q-p)\mathcal{K} \subseteq \tilde{\mathcal{N}}$; equivalently, $r - p$ is comparable with $(q-p)\mathcal{K}$. But $r - p \in (q-p)\mathfrak{D}$, and $(q-p)\mathcal{K}$ is a maximal, multiplicity free nest in $(q-p)\mathfrak{D}$ by the remarks preceding this lemma. It follows that $r - p \in (q-p)\mathcal{K}$, so $r \in p + (q-p)\mathcal{K}$, a contradiction.

Case (b). There exists an i_0 such that $A_{i_0} \subseteq r_-$, $A_{i_0+1} \subseteq r_+$. But then r violates hypothesis (ii).

To show that $\tilde{\mathcal{N}}$ is multiplicity free, we first establish a claim: if $e \in \mathfrak{D}$ is any nonzero projection, there exists $p \in \mathcal{N}$ having an immediate succesor $p' \in \mathcal{N}$ such that $(p'-p)e \neq 0$. Suppose, on the contrary, that $(p'-p)e = 0$ for all $p \in \mathcal{N}$ having an immediate succesor. Then if $q, p \in A_i$, it follows that $pe = qe$. Indeed, if $p < q$, write $p = p_0 < p_1 < \cdots < p_m = q$ by (i), and then $qe = pe + (p_1 - p_0)e + \cdots + (p_m - p_{m-1})e = pe$. This shows that if we set $e_i = pe$, where p is any element of A_i, then e_i is well-defined. Since the function $\mathcal{N} - \mathcal{P}(\mathfrak{D})$ defined by $p \to pe$ increases from 0 to e in $\mathcal{P}(\mathfrak{D})$ as p increases from 0 to 1, i.e., $0 = e_0 \leqslant e_1 \leqslant \cdots \leqslant e_N \leqslant e$, it must be true that $e_i \neq e_{i+1}$ for some i. Let $f = e_{i+1} - e_i$. For all $p \in A_i$ and $q \in A_{i+1}$, $f = qe - pe = (q-p)e \leqslant p - q$. But this violates hypothesis (iii). We conclude that $(p'-p) \neq 0$ for some $p \in \mathcal{N}$, establishing the claim.

Now $\tilde{\mathcal{N}}^c$ is a $\mathfrak{D}$-module, so it is generated by the matrix units it contains. Suppose $e_{ij}^{(n)} \in \tilde{\mathcal{N}}^c$, $i \neq j$. By the claim, there is some $p \in \mathcal{N}$ with succesor $p' \in \mathcal{N}$ such that $(p'-p)e_i^{(n)} \neq 0$. Choose $m \geqslant n$ so that $p, p' \in \mathfrak{D}_m$. Then $0 \neq (p'-p)e_{ij}^{(n)} = e_{ij}^{(n)}(p'-p)$ is a sum matrix units in $\tilde{\mathcal{N}}^c$ of the form $e_{i',j'}^{(m)}$, $i' \neq j'$, so $e_{i',j'}^{(m)} \in (p'-p)\mathcal{K}^c = (p'-p)\mathfrak{D}$ for some i', j'. As $i' \neq j'$, this is a contradiction. It follows that $\tilde{\mathcal{N}}^c = \mathfrak{D}$. ■

COROLLARY 2.21. *Let $\mathfrak{A}$ be a UHF algebra with masa $\mathfrak{D}$, and let $0 < \alpha \leqslant 1$. Then there is a maximal multiplicity-free nest $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$ such that* $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}, P < 1\} = \alpha$.

Proof. As in the proof of Theorem 2.18, there is an increasing sequence $\{P_j\}$ of projections in $\mathfrak{D}$ such that $\sup\{\mathrm{tr}(P_j)\} = \alpha$ and such that there is no $Q \in \mathcal{P}(\mathfrak{D})$, $Q \neq 1$, with $Q \geqslant P_j$ for all j. Now apply Lemma 2.20 with $N = 1$, $A_0 = \{P_j\}$, and $A_1 = \{1\}$ to get $\mathcal{M}$. ■

COROLLARY 2.22. *There are uncountably many nonisomorphic triangular nest subalgebras in any given* UHF *algebra.*

Proof. For each $\alpha, 0 < \alpha \leqslant 1$, find a nest $\mathcal{M}_\alpha$ using Corollary 2.21. The nest algebras $\{\mathrm{Alg}\,\mathcal{M}_\alpha\}$ are nonisomorphic by Proposition 2.16 and triangular since each $\mathcal{M}_\alpha$ is multiplicity-free. ■

We can do even more:

COROLLARY 2.23. *For each α, $0 < \alpha \leqslant 1$, there are uncountably many nonisomorphic triangular nest algebras $\mathcal{S}_\beta$ in a given* UHF *algebra such that* $\mathrm{Lat}\,\mathcal{S}_\beta$ *is a maximal nest with* $\sup\{\mathrm{tr}(P) : P \in \mathrm{Lat}\,\mathcal{S}_\beta, P < 1\} = \alpha$.

Proof. First find a sequence of orthogonal projections $\{E_j\} \subseteq \mathfrak{D}$ as in the proof of Theorem 2.18. Let $\delta = \mathrm{tr}(E_1)$, let $P_j := \sum_{i=1}^{j} E_i$, and let $A_1 = \{P_j\}$. Now fix β, $0 < \beta \leqslant \delta$, and find a sequence of projections $\{F_j\} \subseteq \mathfrak{D}E_1$ as in the proof of Theorem 2.18 such that $\sup\{\mathrm{tr}(F_j)\} = \beta$ and such that there is no projection $Q \in \mathcal{P}(\mathfrak{D})$ with $F_j \leqslant Q < E_1$ for all j. Let $A_0 = \{0\} \cup \{F_j\}$, $A_2 = \{1\}$, and let $\mathcal{N} = A_0 \cup A_1 \cup A_2$. Now use Lemma 2.20 to obtain a maximal, multiplicity-free nest $\mathcal{M}_\beta \supseteq \mathcal{N}$. Then $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}_\beta, P < 1\} = \alpha$ and $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}, P < E_1\} = \beta$. Now, if $\mathcal{S}_\beta = \mathrm{Alg}\,\mathcal{M}_\beta$, then the $\mathcal{S}_\beta$'s are pairwise non-isomorphic by Propositon 2.16. ■

By Proposition 2.2, the nest algebras in the last two corollaries are maximal triangular, but they are not necessarily strongly maximal triangular. Indeed, a triangular nest algebra need not in general be strongly maximal triangular (Example 2.26). However, if we choose the projections $\{E_j\}$ in the proof of Theorem 2.18 in a certain way, then we can in fact obtain a strongly maximal triangular nest algebra. It is not easy to see this using arbitrary embeddings, however. Instead, we will use special embeddings to create the strongly maximal algebra, and then show that it is a nest algebra. By rewriting the construction given in the following theorem in terms of the nest embeddings, for example, one can see that it is the same as in Theorem 2.18 with particular choices for the E_j's. This illustrates the value of both techniques. If one wants to create a maximal triangular algebra with certain properties, it is generally easiest to use certain embeddings, as was done in [9]. If one wants to create nest algebras with certain properties, it is generally easiest to first construct an appropiate nest (and if possible use the nest embeddings), as we have done in the last few results and in various examples in this paper.

Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$, let $\{e_{ij}^{(n)} : 1 \leqslant i,j \leqslant 2^n\}$ be matrix units for $\mathfrak{A}_n$, and define $e_i^{(n)} = e_{ii}^{(n)}$. For every positive integer N, let $Q(N)$ be a permutation matrix in $\mathbf{M}_{2^N}$ such that

$$Q(N)\mathrm{diag}\left(a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_1^{(N)}, a_2^{(N)}\right) Q(N)^{\mathrm{T}} =$$
$$= \mathrm{diag}\left(a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(N)}, a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(N)}\right).$$

Here $\mathrm{diag}(b_1, \ldots, b_\ell)$ denotes the diagonal matrix in $\mathbf{M}_\ell$ with diagonal entries $b_1, \ldots \ldots, b_\ell$. For each n and $1 \leqslant m < 2^n$, let $R(n,m) = I_{2m} \oplus Q(2^n - m)$. Observe, as in [9, Theorem 4.5], that $\mathrm{Ad}\,R(n,m) \circ \nu_n$ maps the upper triangular matrices in $\mathfrak{A}_n$ to the upper triangular matrices in $\mathfrak{A}_{n+1}$.

For $\alpha \in (0,1]$, let $\alpha = \sum_{n=1}^{\infty} \frac{k_n}{2^n}$ be the nonterminating binary expansion. Set $M_n = \sum_{i=1}^{n} 2^{n-i} k_i$ and $\alpha_n = \sum_{i=1}^{n} \frac{k_i}{2^i} = \frac{M_n}{2^n}$. Finally, let $j_n = \mathrm{Ad}\,R(n, M_n) \circ \nu_n$,

$\mathfrak{A} = \varinjlim(\mathfrak{A}_n, j_n)$ and $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, j_n)$, where $\mathfrak{D}_n$ is the diagonal of $\mathfrak{A}_n$.

THEOREM 2.24. *There is a maximal nest $\mathcal{M}$ in $\mathfrak{D}$ such that* $\operatorname{Alg}\mathcal{M}$ *is a strongly maximal* TAF *nest algebra and* $\sup\{\operatorname{tr}(P) : P \in \mathcal{M}, P < 1\} = \alpha$.

Proof. We will prove the result for the 2^∞ UHF algebra; the general case is similar. Let $M_n, \alpha_n, j_n, \mathfrak{A}$, and $\mathfrak{D}$ be as defined above. Set $P_j^{(n)} = \sum_{i=1}^{j} e_i^{(n)}$, $j = 1, \ldots, M_n$, and $\mathcal{M} = \{P_j^{(n)} : j = 1, \ldots, M_n, n = 1, 2, \ldots\} \cup \{0, 1\}$. Then $\sup\{\operatorname{tr}(P) : P \in \mathcal{M}, P < 1\} = \sup\{\alpha_n\} = \alpha$. For $m > n$, let $j_{n,m} = j_{m-1} \circ \cdots \circ j_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_m$. Set $\mathcal{T} = \operatorname{Alg}\mathcal{M}$ and $\mathcal{T}_n = (\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n$. Observe that if $\ell < n$, then $\{P_j^{(\ell)} : j = 1, \ldots, M_\ell\} \subseteq \{P_j^{(n)} : j = 1, \ldots, M_n\}$, so $\mathcal{M}$ is a nest. Now if $1 \leqslant i \leqslant j \leqslant 2^n$, then $e_{ij}^{(n)} \in \mathcal{T}_n$. This is clear if $i = j$, so let $i < j$ and $p \in \mathcal{M}$, say $p = P_k^{(m)}$. If $m \leqslant n$, then by the observation above $p = P_{k'}^{(n)}$ for some k', and therefore $p^\perp e_{ij}^{(n)} p = 0$. Suppose $m > n$. As $j_{n,m}(e_{ij}^{(n)})$ is a sum of upper triangular matrix units of the form $e_{i'j'}^{(m)}$, $i' < j'$, we have $P_k^{(m)\perp} e_{i'j'}^{(m)} P_k^{(m)} = 0$ for all i', j', and hence $p^\perp j_{n,m}(e_{ij}^{(n)}) p = 0$.

Next we show that if $i > j$, then $e_{ij}^{(n)} \notin \mathcal{T}_n$. If $j \leqslant M_n$, then $P_j^{(n)\perp} e_{ij}^{(n)} P_j^{(n)} = e_{ij}^{(n)} \neq 0$. Now suppose $j > M_n$, and write $j = M_n + j_0$.

Claim. There is an $m > n$ such that

$$2^{m-n} M_n + j_0 \leqslant M_m.$$

To see this, write

$$\alpha_n + \frac{j_0}{2^m} = \frac{M_n}{2^n} + \frac{j_0}{2^m} = \frac{2^{m-n} M_n + j_0}{2^m}.$$

Since the binary expansion of α is nonterminating, there exists $m_1 > n$ such that $\alpha_n < \alpha_{m_1}$. If m is sufficiently large, then $\frac{j_0}{2^m} \leqslant \alpha_{m_1} - \alpha_n$. As the sequence $\{\alpha_k\}$ is nondecreasing, replacing m_1 by $m \geqslant m_1$ only increases the right side, so $\frac{j_0}{2^m} \leqslant \alpha_m - \alpha_n$, or $\alpha_n + \frac{j_0}{2^m} \leqslant \alpha_m$. Hence

$$\frac{2^{m-n} M_n + j_0}{2^m} = \alpha_n + \frac{j_0}{2^m} \leqslant \alpha_m = \frac{M_m}{2^m},$$

from which the claim follows.

Let m be the least integer $t, t > n$, satisfying the inequality $2^{t-n} M_n + j_0 > M_k$. Note that $j_n(e_j^{(n)}) = e_{2M_n + j_0}^{(n+1)} + e_{M_n + j_0 + 2^n}^{(n+1)}$. If $m > n + 1$ (i.e., $2M_n + j_0 > M_{n+1}$), then $j_{n+1} \circ j_n(e_j^{(n)})$ contains the summand $e_{4M_n + j_0}^{(n+2)}$. Continuing inductively, we obtain that $j_{n,m}(e_j^{(n)})$ contains the summand $e_k^{(m)}$, $k = 2^{m-n} M_n + j_0$. It follows that $j_{n,m}(e_{ij}^{(n)}) = j_{n,m}(e_{ij}^{(n)}) j_{n,m}(e_j^{(n)})$ contains a summand of the form $e_{\ell k}^{(m)}$ for some $\ell > k$. This is because $j_{n,m}$ maps the lower triangular matrix unit $e_{ij}^{(n)}$ $(i > j)$ in

$\mathfrak{A}_n$ to a sum of lower triangular matrix units in $\mathfrak{A}_m$. As $k \leqslant M_m$, $P_k^{(m)} \in \mathcal{M}$, and $P_k^{(m)\perp} e_{\ell k}^{(m)} P_k^{(m)} = e_{\ell k}^{(m)} \neq 0$. Hence $P_k^{(m)\perp} j_{n,m}(e_{ij}^{(n)}) P_k^{(m)} \neq 0$, so that $e_{ij}^{(n)} \notin \mathcal{T}_n = (\mathrm{Alg}\,\mathcal{M}) \cap \mathfrak{A}_n$.

We have shown that $\mathcal{T}_n$ consists precisely of all upper triangular matrices in $\mathfrak{A}_n$, and thus $\mathcal{T}$ is a strongly maximal triangular nest algebra. In fact, $\mathcal{T}$ is just the algebra $\mathcal{T}_{(\alpha)}$ of Theorem 4.5 of [9] (assuming we choose the nonterminating expansion for α; see remark (b) below). It is shown in the proof of that theorem that $\mathrm{Lat}\,\mathcal{T} = \mathcal{M}$, so it follows from Corollary 2.5 that $\mathcal{M}$ is maximal. ■

REMARK 2.25. (a) $\mathrm{tr}\mathcal{M} = \left\{ \frac{k}{2^n} : k, n \in \mathbb{Z}^+, \frac{k}{2^n} < \alpha \right\}$.

(b) In case $\alpha \in (0,1)$ is a dyadic rational, we specified in the proof of 4.5 of [9] a terminating binary expresion for α. However, the proof works equally well if we use the nonterminating binary expansion. If we use the terminating binary expansion, the resulting $\mathcal{T}_{(\alpha)}$ of [9, Theorem 4.5] is not a nest algebra; rather, it is isomorphic to a direct sum of a nest algebra and a TAF algebra with no notrivial invariant projections.

EXAMPLE 2.26. We give an example of a nest subalgebra of a UHF algebra which is triangular but not strongly maximal. Let $\mathfrak{A}_n = \mathrm{M}_{2^n}$ with matrix units $\{e_{ij}^{(n)}\}$, $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and $\nu_n : \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ the nest embedding. Let $\mathrm{diag}\,(b_1, \ldots, b_m)$ represent the diagonal matrix in M_m with diagonal entries $b_1, \ldots, b_m$, and define U_n to be the permutation matrix in $\mathfrak{A}_n$ which satisfies

$$U_n \mathrm{diag}\,(a_1, \ldots, a_{2^n}) U_n^* = \mathrm{diag}\,(a_1, \ldots, a_{2^n-4}, a_{2^n-3}, a_{2^n-1}, a_{2^n}, a_{2^n-2}).$$

Let $j_n = \mathrm{Ad}\, U_{n+1} \circ \nu_n$, and let $j_{n,m} = j_{m-1} \circ \cdots \circ j_n$. Define

$$\mathcal{M}_n = \left\{ 0, 1, \sum_{i=1}^{k} e_i^{(n)} : 1 \leqslant k \leqslant 2^n - 2 \right\}.$$

Note that $j_n(\mathcal{M}_n) \subseteq \mathcal{M}_{n+1}$, so $\mathcal{M} = \overline{\bigcup_{n=1}^{\infty} \mathcal{M}_n}$ is a nest.

Claim. $(\mathrm{Alg}\,\mathcal{M}) \cap \mathfrak{A}_n = \mathrm{span}\Big(\{e_{ij}^{(n)} : i \leqslant j\} \setminus \{e_{2^n-1,2^n}^{(n)}\}\Big)$.

It follows from the claim and [9, Theorem 2.6] that $\mathrm{Alg}\,\mathcal{M}$ is a TAF algebra, and therefore a maximal TAF algebra by Proposition 2.2. First let $w = e_{ij}^{(n)}$, $i > j$, $j \leqslant 2^n - 2$. Let $p = \sum_{\ell=1}^{j} e_\ell^{(n)}$. Note that n must be greater than 1, so p is a nonzero projection in $\mathcal{M}$. But $p^\perp w p \neq 0$, and therefore $w \notin \mathrm{Alg}\,\mathcal{M}$. Next let $w = e_{ij}^{(n)}$, $i < j$, $j \neq 2^n$. Let $p = \sum_{\ell=1}^{i} e_\ell^{(n)}$. $p \in \mathcal{M}$ since $n > 1$ and $j \neq 2^n$. Then $pwp^\perp = w$, and it follows that $w \in \mathrm{Alg}\,\mathcal{M}$ since $q\mathfrak{A}q^\perp \subseteq \mathrm{Alg}\,\mathcal{M}$ for every $q \in \mathcal{M}$.

Now let $w = e^{(n)}_{2^n,2^n-1}$ and $p = \sum_{\ell=1}^{2^{n+1}-3} e^{(n+1)}_\ell$. Then $j_n(w) = e^{(n+1)}_{2^{n+1}-2,2^{n+1}-3} + e^{(n+1)}_{2^{n+1}-1,2^{n+1}}$, so $p^\perp j_n(w)p \neq 0$, and therefore $w \notin \operatorname{Alg}\mathcal{M}$. Finally, let $w = e^{(n)}_{2^n-1,2^n}$. Then $j_n(w) = e^{(n+1)}_{2^{n+1}-3,2^{n+1}-2} + e^{(n+1)}_{2^{n+1},2^{n+1}-1}$. As we just saw above, the first term on the right is in $\operatorname{Alg}\mathcal{M}$, but the second term is not. Thus, $w \notin \operatorname{Alg}\mathcal{M}$. This completes the proof of the claim.

Claim. $\operatorname{Alg}\mathcal{M}$ is not strongly maximal.

Let $\mathcal{T} = \operatorname{Alg}\mathcal{M}$ and $\mathcal{T}_n = (\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n$. It suffices to show that $\overline{\mathcal{T}+\mathcal{T}^*} \neq \mathfrak{A}$. Indeed, $e^{(1)}_{1,2} \notin \overline{\mathcal{T}+\mathcal{T}^*}$. For if it were, then there would be a sequence $\{t_k\} \subseteq \bigcup_{n=1}^{\infty} (\mathcal{T}_n + \mathcal{T}_n^*)$ such that $t_k \to e^{(1)}_{1,2}$. By passing to a subsequence of $\{\mathcal{T}_n + \mathcal{T}_n^*\}$, we can assume that $t_k \in \mathcal{T}_k + \mathcal{T}_k^*$. Choose k_0 large enough so that $\left\| t_{k_0} - j_{1,m}\left(e^{(1)}_{1,2}\right)\right\| < 1$, where m is the integer satisfying $\dim \mathfrak{A}_{k_0} = (2^m)^2$. Multiplying on the left and right by $e^{(m)}_{2^m-1}$ and $e^{(m)}_{2^m}$, we have $\left\| e^{(m)}_{2^m-1}\left(t_{k_0} - j_{1,m}\left(e^{(1)}_{1,2}\right)\right)e^{(m)}_{2^m}\right\| < 1$ and $\left\| e^{(m)}_{2^m}\left(t_{j_0} - j_{1,m}\left(e^{(1)}_{1,2}\right)\right)e^{(m)}_{2^m-1}\right\| < 1$. But $e^{(m)}_{2^m-1} t_{k_0} e^{(m)}_{2^m} = 0 = e^{(m)}_{2^m} t_{k_0} e^{(m)}_{2^m-1}$ by the previous claim, and it follows from the definition of $j_{1,m}$ that either $e^{(m)}_{2^m-1} j_{1,m}\left(e^{(1)}_{1,2}\right)e^{(m)}_{2^m} = e^{(m)}_{2^m-1,2^m}$ (m odd) or $e^{(m)}_{2^m} j_{1,m}\left(e^{(1)}_{1,2}\right)e^{(m)}_{2^m-1} = e^{(m)}_{2^m,2^m-1}$ (m even). This contradiction completes the proof.

This example is especially interesting in light of Corollary 3.16, which gives a simple condition for a nest algebra to be strongly maximal triangular.

DEFINITION 2.27. Let $\mathfrak{A}$ be a UHF algebra with masa $\mathfrak{D}$, and let m be a positive integer. We say that a nest $\mathcal{M} \subseteq \mathfrak{D}$ has *uniform multiplicity* m if there is a set of partial isometries $\{W_{ij} : 1 \leqslant i,j \leqslant m\} \subseteq \mathcal{W}_{\mathfrak{D}}(\mathcal{M}^c)$ such that $W_{ij}W_{k\ell} = \delta_{jk}W_{i\ell}$ for all i,j,k,ℓ, $\sum_{i=1}^{m} W_{ii} = 1$, and the nest $\mathcal{M}_0 = \mathcal{M}W_{11} \subseteq W_{11}\mathfrak{A}W_{11}$ is multiplicity-free. It follows that $\mathcal{M}$ is unitarily equivalent to $\mathcal{M}_0^{(m)} = \{p \oplus \cdots \oplus p \ (m \text{ times}) : p \in \mathcal{M}_0\}$ via the unitary $U = \bigoplus_{i=1}^{m} W_{i1}$. U also carries $\mathcal{M}^c$ onto $(\mathcal{M}_0^{(m)})^c = \mathcal{M}_0^c \otimes \mathbf{M}_m$, and the argument given in [6, Lemma 7.12] then shows that uniform multiplicity is well-defined. If $\mathcal{M}$ has uniform multiplicity m, then so does $\operatorname{Lat}(\operatorname{Alg}\mathcal{M})$. It follows from Proposition 2.8 that if $\mathcal{M}$ and $\mathcal{N}$ are nests of uniform multiplicity m and n, respectively, and $\operatorname{Alg}\mathcal{M} \simeq \operatorname{Alg}\mathcal{N}$, then $m = n$. Note that unlike the Hilbert space situation, a nest cannot have uniform infinite multiplicity, since then $\mathfrak{D}$ would have an infinite number of projections with the same trace. Also, uniform multiplicity one is just another way of saying multiplicity-free.

It is perhaps surprising that there are maximal nests with uniform multiplicity greater than one. The following proposition shows exactly which multiplicities are possible.

PROPOSITION 2.28. *Suppose $\mathfrak{A}$ is a* UHF *algebra of type $(p_1^{m_1}p_2^{m_2}p_3^{m_3}\cdots)$ with masa $\mathfrak{D}$. Let m be a positive integer with prime factorization $(q_1^{n_1}q_2^{n_2}\cdots q_s^{n_s})$. Then $\mathfrak{D}$ contains a maximal nest $\mathcal{M}$ of uniform multiplicity m if and only if for each i, $q_i = p_j$ for some j and $n_i \leqslant m_j$.*

Proof. Let j_k be the embedding of $\mathfrak{A}_k$ into $\mathfrak{A}_{k+1}$, and let $[k] = \sqrt{\dim \mathfrak{A}_k}$. Necessity is now clear, for $\{W_{ii} : 1 \leqslant i \leqslant m\}$ is a set of projections, with identical trace, which lies in some $\mathfrak{D}_n$. Therefore, $m \mid [n]$, and the result follows.

To prove sufficiency, we will construct the nest. Let $\{e_{ij}^{(n)}\}$ be a set of matrix units for $\mathfrak{A}$, and denote $e_{ii}^{(n)}$ by $e_i^{(n)}$. From the hypothesis, there is some n such that $m \mid [n]$. Let $d = [n]/m$ and define $\mathcal{M}_{n,0} = \left\{\sum_{i=1}^{k} e_i^{(n)} : 1 \leqslant k \leqslant d\right\} \cup \{0\}$. For $1 \leqslant i,j \leqslant m$, define $W_{ij} = \sum_{k=1}^{d} e^{(n)}_{(i-1)d+k,(j-1)d+k}$. Now for $p \geqslant n$, inductively define $\mathcal{M}_{p+1,0}$ to be a maximal nest in $\mathfrak{D}_{p+1}W_{11}$ containing $j_p(\mathcal{M}_{p,0})$, and let $\mathcal{M}_0 = \bigcup_p \mathcal{M}_{p,0}$. $\mathcal{M}_0$ is multiplicity-free since $C^*(\mathcal{M}_0) = \mathfrak{D}W_{11}$. Finally, define $\mathcal{M}$ to be the nest $\left\{\sum_{i=1}^{m} W_{i1}^* e W_{i1} : e \in \mathcal{M}_0\right\}$. Maximality follows from the construction and the fact that each $\mathcal{M}_{p,0}$ is maximal in $\mathfrak{D}_p W_{11}$. ∎

EXAMPLE 2.29. Let $\mathfrak{A}_n = \mathrm{M}_{2^n}$ and define $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$, where ν_n is the nest embedding defined in Section 0. Then the construction for $m = 2$ in the preceding proof yields the nest $\mathcal{M} = \left\{\sum_{i=1}^{k}(e_i^{(n)} + e_{i+2^{n-1}}^{(n)}) : 1 \leqslant k \leqslant 2^{n-1}, 1 \leqslant n < \infty\right\} \cup \{0\}$.

We have not formulated a general theory of multiplicity for a nest. One possibility would be to embed the UHF algebra in a II_1 factor and then consider the usual multiplicity of the strong closure of the nest. In view of the fact that all such factors are isomorphic, we feel that this is probably not the correct formulation.

It is not hard to prove analogues of Corollaries 2.21, 2.22, and 2.23 for nests of uniform multiplicity m, where m is any allowable multiplicity given by Proposition 2.28. For example, to show that there is maximal multiplicity m nest $\mathcal{M}$ with $\sup\{\mathrm{tr}(P) : P \in \mathcal{M}, P < 1\} = \alpha$, first define $\{W_{ij} : 1 \leqslant i,j \leqslant m\}$ as in the proof of Proposition 2.28. Now use Corollary 2.21 to obtain a maximal multiplicity-free nest $\mathcal{M}_0 \subseteq W_{11}\mathfrak{D}$ such that $\sup\{\mathrm{tr}(P) : p \in \mathcal{M}_0, P < W_{11}\} = \alpha/m$. Then just define $\mathcal{M} = \left\{\sum_{i=1}^{m} W_{i1}^* e W_{i1} : e \in \mathcal{M}_0\right\}$. It follows that there are an uncountable number of nonisomorphic nest algebras in $\mathfrak{A}$ whose nests have uniform multiplicity m. By using Corollary 2.23 instead of 2.21, we can obtain an uncountable number of such nest algebras $\mathcal{S}$ which are nonisomorphic and have the additional property that $\sup\{\mathrm{tr}(P) : : P \in \mathrm{Lat}\,\mathcal{S}, P < 1\} = \alpha$ for any given α.

3. NEST ALGEBRAS: TRACE AND ORDER ISOMORPHISM

While in the previous section the primary focus was on nest algebras, in this section we turn our attention to the relation between masas and nests. In [9, Corollary 3.23], it was shown that an order isomorphism of masas preserves nests, and in Proposition 3.7 below we show that it must also preserve trace. Conversely, if there is a trace-preserving bijection between two nests, can it be extended to an order isomorphism of masas? In general the answer is no (Example 3.9), but in the special case that the nests generate the masas, such an extension exists (Theorem 3.13). This is the principal result of this section. We begin with a lemma that concerns projections generated by a nest, and which is used repeatedly.

LEMMA 3.1. *Let $\mathfrak{D}$ be a masa in a* UHF *algebra $\mathfrak{A}$, and let $\mathcal{M} \subseteq \mathfrak{D}$ be a nest. If $p \in C^*(\mathcal{M})$ is a projection, then there exist projections $f_1 > f_1' > f_2 > \cdots > f_n > f_n'$, $f_i, f_i' \in \mathcal{M}$, $1 \leqslant i \leqslant n$, such that $p = (f_1 - f_1') + (f_2 - f_2') + \cdots + (f_n - f_n')$. The projections f_i, f_i', $1 \leqslant i \leqslant n$, and the integer n are uniquely determined by p.*

Proof. Initially we prove uniqueness. Suppose $p = \sum_{i=1}^{n}(f_i - f_i') = \sum_{j=1}^{m}(g_j - g_j')$, where $g_1 > g_1' > g_2 > \cdots > g_m > g_m'$ are projections in the nest. First note that $f_1 = g_1$. If $f_1 > g_1$, then $0 \neq f_1 - g_1 \vee f_1' \leqslant p$, but $(f_1 - g_1 \vee f_1') \perp \sum_{i=1}^{m}(g_j - g_j')$. Reversing the roles of f_1 and g_1, we likewise obtain a contradiction by assuming $g_1 > f_1$. Next, we have $f_1' = g_1'$. For if $f_1' > g_1'$, then $0 \neq f_1' - f_2 \vee g_1' \leqslant g_1 - g_1' \leqslant p$, whereas $f_1' - f_2 \vee g_1' \leqslant (f_1' - f_2) \perp \sum_{i=1}^{n}(f_i - f_i')$. Similarly, assuming $g_1' > f_1'$ yields a contradiction. Now suppose $m \geqslant n$, and apply the first paragraph to $p - (f_1 - f_1')$, $p - (f_1 - f_1') - (f_2 - f_2')$, etc. to conclude that $f_i = g_i$, $f_i' = g_i'$, $1 \leqslant i \leqslant n$. If $m > n$, then we are left with $0 = \sum_{j=n+1}^{m} (g_j - g_j')$, which is absurd, as the right side is a sum of orthogonal projections.

Let $\mathfrak{C}_0 = \Bigg\{ a \in \mathfrak{D} : a = \sum_{i=1}^{n} \lambda_i (f_i - f_i')$ for some $n \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$ $(1 \leqslant i \leqslant n)$, and $f_1 > f_1' > f_2 > f_2' > \cdots > f_n > f_n' \in \mathcal{M} \Big\}$. It is elementary to verify that $\mathfrak{C}_0$ is an algebra over $\mathbb{C}$, and the closure of $\mathfrak{C}_0$ in $\mathfrak{D}$ is evidently the samllest C^*-subalgebra of $\mathfrak{D}$ containing $\mathcal{M}$, i.e., $\overline{\mathfrak{C}_0} = C^*(\mathcal{M})$. View $C^*(\mathcal{M})$ as $C(X)$ via the Gelfand representation and let $a \in \mathfrak{C}_0$ satisfy $\|a - p\| < 1/2$. Express $a = \sum_{i=1}^{n} \lambda_i(f_i - f_i')$. We claim that $\bigvee_{i=1}^{n}(f_i - f_i') = \sum_{i=1}^{n}(f_i - f_i') \geqslant p$. For if $\sum_{i=1}^{n}(f_i - f_i') < p$, then there is an $x_0 \in X$ with $p(x_0) = 1$ and $(f_i - f_i')(x_0) = 0$, $1 \leqslant i \leqslant n$. But then $1 = |p(x_0) - a(x_0)| \leqslant \|p - a\| < 1/2$. There is no loss in assuming $(f_i - f_i')p \neq 0$ for all i, for if $(f_i - f_i')p = 0$ for some i, then $b = \sum_{j \neq i} \lambda_j(f_j - f_j')$ also satisfies $\|p - b\| < 1/2$. Now suppose $\bigvee_{i=1}^{n}(f_i - f_i') > p$. Then there is an $i_0 \in \{1, \ldots, n\}$ and an $x_1 \in X$ for which $(f_{i_0} - f_{i_0}')(x_1) = 1$ and $p(x_1) = 0$. As $(f_{i_0} - f_{i_0}')p \neq 0$, there is an $x_2 \in X$ for which $(f_{i_0} - f_{i_0}')(x_2) = 1$ and $p(x_2) = 1$. Thus we have $|\lambda_{i_0}| = |p(x_1) - a(x_1)| \leqslant \|p - a\| < 1/2$, and $|1 - \lambda_{i_0}| = |p(x_2) - a(x_2)| \leqslant \|p - a\| < 1/2$. From this we conclude that $p = \bigvee_{i=1}^{n}(f_i - f_i')$. ∎

COROLLARY 3.2. *Let $\mathcal{M} \subseteq \mathcal{N}$ be nests in $\mathfrak{D}$. If $C^*(\mathcal{M}) = C^*(\mathcal{N})$, then $\mathcal{M} = \mathcal{N}$.*

Proof. Suppose $F \in \mathcal{N} \setminus \mathcal{M}$. Since f is a projection in $C^*(\mathcal{M})$, we can express $f = \sum_{i=1}^{n}(f_i - f_i')$, where $f_1 > f_1' > \cdots > f_n > f_n'$ are unique projections in $\mathcal{M}$. But $f_i, f_i' \in \mathcal{N}$, $1 \leqslant i \leqslant n$, and the unique decomposition of f in $C^*(\mathcal{N})$ is $f - 0$, so it follows that $n = 1$, $f_1' = 0$, and in particular $f_1 = f$. ∎

COROLLARY 3.3. *If $\mathcal{M}$ is a nest in $\mathfrak{D}$ such that $C^*(\mathcal{M}) = \mathfrak{D}$, then $\mathcal{M}$ is maximal.*

Proof. Suppose $\mathcal{N}$ is a nest in $\mathfrak{D}$ with $\mathcal{N} \supseteq \mathcal{M}$. Then $\mathfrak{D} \supseteq C^*(\mathcal{N}) \supseteq C^*(\mathcal{M}) = \mathfrak{D}$, so $C^*(\mathcal{N}) = C^*(\mathcal{M})$, and therefore $\mathcal{N} = \mathcal{M}$ by the previous corollary. ∎

REMARKS 3.4. (a) Corollary 3.2 is false without the hypothesis $\mathcal{M} \subseteq \mathcal{N}$. Example 3.17 gives a nest $\mathcal{M}$ with $C^*(\mathcal{M}) = \mathfrak{D}$; the canonical nest $\mathcal{L}$ also has this property. But 1/2 is not the trace of any projection in $\mathcal{M}$, whereas $\mathrm{tr}(\mathcal{L})$ includes all dyadic rationals, so $\mathcal{L} \neq \mathcal{M}$.

(b) If $\mathcal{M}$ is a nest in $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$, then $C^*(\mathcal{M} \cap \mathfrak{D}_n)$ is in general properly contained in $C^*(\mathcal{M}) \cap \mathfrak{D}_n$. For example, let $\dim \mathfrak{D}_n = 2^n$ with basis $\{ e_i^{(n)} : 1 \leqslant i \leqslant 2^n \}$, and let $\nu_n : \mathfrak{D}_n \hookrightarrow \mathfrak{D}_{n+1}$ be the nest embedding. Define $\mathcal{M} = \{0, p_1, p_2, 1\}$, where $p_1 = e_1^{(2)}$ and $p_2 = e_1^{(2)} + e_2^{(1)}$. Then $e_2^{(1)} \in C^*(\mathcal{M}) \cap \mathfrak{D}_1$, but $C^*(\mathcal{M} \cap \mathfrak{D}_1) = \mathbb{C}1$. If $\mathcal{N}$ is any nest (in particular, a maximal nest) containing $\mathcal{M}$, it will still be the case that $C^*(\mathcal{N}\cap\mathfrak{D}_1) \neq C^*(\mathcal{N})\cap\mathfrak{D}_1$. For $e_2^{(1)} = p_2 - p_1$ is the unique expression of $e_2^{(1)}$ by Lemma 3.1, and since $p_1 \notin \mathfrak{D}_1$, it follows that $e_2^{(1)} \notin C^*(\mathcal{N}\cap\mathfrak{D}_1)$.

COROLLARY 3.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be nests, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$, $\mathcal{N} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$. If $\operatorname{tr}(\mathcal{M}) = \operatorname{tr}(\mathcal{N})$, then there is a unique C^*-isomorphism $\varphi : C^*(\mathcal{M}) \to C^*(\mathcal{N})$ which preserves trace on $\mathcal{M}$.*

Proof. If $p \in \mathcal{M}$ with $\operatorname{tr}(p) = t$, let $\varphi(p)$ be the unique element $q \in \mathcal{N}$ with $\operatorname{tr}(q) = t$. As in the proof of Lemma 3.1, each element a in the $*$-algebra generated by $\mathcal{M}$ has a unique representation $a = \sum_{i=1}^{n} \lambda_i(f_i - f_i')$, with $\lambda_i \in \mathbb{C}$, $f_i, f_i' \in \mathcal{M}$, $f_1 > f_1' > \cdots > f_n > f_n'$. Set $\varphi(a) = \sum_{i=1}^{n} \lambda_i\big(\varphi(f_i) - \varphi(f_i')\big)$. Then φ is trace-preserving, and its image is the $*$-subalgebra of $\mathfrak{C}$ generated by $\mathcal{N}$. As $\|a\| = \max_{1\leqslant i\leqslant n} |\lambda_i| = \|\varphi(a)\|$, φ extends to an isomorphism of $C^*(\mathcal{M})$ onto $C^*(\mathcal{N})$. ∎

Suppose $\varphi : \mathfrak{D} \to \mathfrak{C}$ is an isomorphism of masas. We saw in Proposition 2.16 that if φ is the restriction of an isometric isomorphism of nest algebras, then φ is trace-preserving. More generally, it turns out that if φ is an order isomorphism, and the ordering on $\mathcal{P}(\mathfrak{D})$ is given by a nest algebra, then φ preserves the trace. But first we observe that in the absence of additional hypotheses, φ need not preserve the trace.

EXAMPLE 3.6. A C^*-isomorphism of masas need not preserve trace. Let $\mathfrak{A}_n = M_{2^n}$ for $n \geqslant 2$, and let $\{e_{ij}^{(n)} : 1 \leqslant i,j \leqslant 2^n\}$ be a set of matrix units for $\mathfrak{A}_n$. Write $[n] = 2^n$. Let $\mathfrak{D}_n$ be the span of the diagonal matrix units $\{e_i^{(n)} = e_{ii}^{(n)} : 1 \leqslant i \leqslant [n]\}$. Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$ and $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \nu_n)$, where ν_n is the nest embedding. We construct an automorphism φ of $\mathfrak{D}$ as follows:

$$\varphi(e_i^{(n)}) = \begin{cases} e_{4i-3}^{(n+1)} + e_{4i-2}^{(n+1)} + e_{4i-1}^{(n+1)} + e_{4i}^{(n+1)}, & i = 1, \ldots, [n-2] \\ e_{[n-1]+2i-1}^{(n+1)} + e_{[n-1]+2i}^{(n+1)}, & i = [n-2]+1, \ldots, [n-1] \\ e_{[n]+i}^{(n+1)}, & i = [n-1]+1, \ldots, [n]. \end{cases}$$

Note that φ is well-defined since $\varphi(e_i^{(n)}) = \varphi(e_{2i-1}^{(n+1)}) + \varphi(e_{2i}^{(n+1)})$. The linear extension of φ is clearly a unital embedding of $\mathfrak{D}$ into $\mathfrak{D}$. To see that φ is onto, ovserve that

$$e_i^{(n)} = \begin{cases} \varphi(e_i^{(n+1)}), & i = 1, \ldots, [n-1] \\ \varphi(e_{i-[n-2]}^{(n)}), & i = [n-1]+1, \ldots, [n-1]+[n-2] \\ \varphi(e_{i-[n-1]}^{(n-1)}), & i = [n-1]+[n-2]+1, \ldots, [n]. \end{cases}$$

PROPOSITION 3.7. *Suppose $\mathcal{S}$ is a strongly irreducible $\mathfrak{D}$-module, $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$, and $\mathcal{T}$ is any $\mathfrak{C}$-module, $\mathfrak{C} \subseteq \mathcal{T} \subseteq \mathfrak{B}$. Let $\varphi : (\mathfrak{D}, \prec_{\mathcal{S}}) \to (\mathfrak{C}, \prec_{\mathcal{T}})$ be an order* *isomorphism. Then φ preserves trace. In particular, the result holds when $\mathcal{S}$ is a nest algebra.*

Proof. The proof is almost the same as the proof of Proposition 2.16. The only differences are that we must assume strong irreducibility, and then use the order isomorphism property instead of Lemma 2.9. Proposition 2.15 then implies the result for nest algebras. ∎

REMARK 3.8. If $\mathcal{S}$ is a strongly maximal TUHF algebra, $\mathfrak{D} \subseteq \mathcal{S} \subseteq \mathfrak{A}$, $\mathcal{T}$ is any $\mathfrak{C}$-module, $\mathfrak{C} \subseteq \mathcal{T} \subseteq \mathfrak{B}$ with $\mathfrak{B}$ UHF, and $\varphi : (\mathfrak{D}, \prec_{\mathcal{S}}) \to (\mathfrak{C}, \prec_{\mathcal{T}})$ is an order isomorphism, then φ also preserves trace. The proof is a slight generalization of the argument in Proposition 2.16. First, it follows from [16, Lemma 2.4] that $\mathfrak{A}_n \subseteq$ $\subseteq (C^*(\mathfrak{A}_n, \mathfrak{D}_m) \cap \mathcal{S}) + (C^*(\mathfrak{A}_n, \mathfrak{D}_m) \cap \mathcal{S}^*)$ for some $m \geqslant n$. Choose a particular minimal projection e_i in $\mathfrak{D}_n$, $i \neq 1$, and let e_{1i} be the partial isometry with initial projection e_i and final projection e_1. Then there is some $p \in \mathcal{P}(\mathfrak{D}_m)$ such that $e_{1i} = pe_{1i} + p^{\perp}e_{1i}$ with $pe_{1i} \in \mathcal{S}$ and $p^{\perp}e_{1i} \in \mathcal{S}^*$. Letting q and q' be the initial projections of pe_{1i} and $p^{\perp}e_{1i}$, respectively, it follows that $pe_1 \prec_{\mathcal{S}} q$, $p^{\perp}e_1 \prec_{\mathcal{S}^*} q'$, and $q + q' = e_i$. It follows that $\operatorname{tr}(\varphi(pe_1)) = \operatorname{tr}(\varphi(q))$ and $\operatorname{tr}(\varphi(p^{\perp}e_1)) = \operatorname{tr}(\varphi(q'))$ since φ is an order isomorphism, and therefore $\operatorname{tr}(\varphi(e_1)) = \operatorname{tr}(\varphi(e_i))$. Now just apply the last line of the proof of Proposition 2.16.

It is now easy to see that if $\mathcal{M} \subseteq \mathfrak{D}$ and $\mathcal{N} \subseteq \mathfrak{C}$ are maximal nests such that $C^*(\mathcal{M}) = \mathfrak{D}$ and $C^*(\mathcal{N}) = \mathfrak{C}$, then there is only one possible order isomorphism of $(\mathfrak{D}, \prec_{\operatorname{Alg}\mathcal{M}})$ onto $(\mathfrak{C}, \prec_{\operatorname{Alg}\mathcal{N}})$. For suppose φ is an order isomorphism. Then $\varphi : \operatorname{Lat}(\operatorname{Alg}\mathcal{M}) \to \operatorname{Lat}(\operatorname{Alg}\mathcal{M})$ is a trace-preserving bijection by [9, Corollary 3.23] and Proposition 3.7. $\mathcal{M} = \operatorname{Lat}(\operatorname{Alg}\mathcal{M})$ and $\mathcal{N} = \operatorname{Lat}(\operatorname{Alg}\mathcal{N})$ since $\mathcal{M}$ and $\mathcal{N}$ are maximal, and the result now follows from Corollary 3.5. In Theorem 3.13, we will prove the existence of an order isomorphism in this case, given a trace-preserving bijection of the nests. The next example shows that an order isomorphism does not exist in general.

EXAMPLE 3.9. Lattice isomorphism does not imply order isomorphism, even among maximal, multiplicity free nests.

Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ with a given set of matrix units $\{e_{ij}^{(n)} : 1 \leqslant i, j \leqslant [n] = 2^n\}$. Set $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$ and $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \nu_n)$, where $\mathfrak{D}_n$ is the diagonal of $\mathfrak{A}_n$ spanned by $\{e_i^{(n)} = e_{ii}^{(n)}, 1 \leqslant i \leqslant [n]\}$. Let $\mathcal{L}$ be the canonical nest (Definition 0.1). Next, let R be the 8×8 permutation matrix $Q(4)$, defined just prior to Theorem 2.24, which converts the nest embedding to the standard embedding; i.e., $\operatorname{Ad} R \circ \nu_2 = \sigma_2 : \mathbf{M}_4 \to \mathbf{M}_8$. Define a sequence of permutation matrices $P_n = I_{[n+1]-8} \oplus R$, $n \geqslant 2$, and define $\gamma_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ by $\gamma_n = \operatorname{Ad} P_n \circ \nu_n$.

Let $\mathfrak{B} = \varinjlim(\mathfrak{A}_n, \gamma_n)$ and $\mathfrak{C} = \varinjlim(\mathfrak{D}_n, \gamma_n)$. Now define $\mathcal{N} = \bigcup_{n=3}^{\infty} \mathcal{N}_n \subseteq \mathfrak{C}$, where $\mathcal{N}_n = \left\{ q_k^{(n)} : q_k^{(n)} = \sum_{i=1}^{k} e_i^{(n)}, 1 \leqslant k \leqslant [n] - 4 \right\} \cup \{0, 1\} \subseteq \mathfrak{D}_n$. To show that $\mathcal{N}$ is a nest, it is enough to show $\mathcal{N}_n \subseteq \mathcal{N}_{n+1}$. Let $q \in \mathcal{N}_n$, $q = \sum_{i=1}^{k} e_i^{(n)} = q_k^{(n)}$ for some k, $1 \leqslant k \leqslant [n] - 4$. Then

$$\gamma_n(q) = \operatorname{Ad} P_n \left(\sum_{i=1}^{2k} e_i^{(n+1)} \right) = \sum_{i=1}^{2k} e_i^{(n+1)} = q_{2k}^{(n+1)} \text{ since } 2k \leqslant [n+1] - 8.$$

Maximality now follows since $\operatorname{tr}(\mathcal{N}) = \{ k/2^n : 0 \leqslant k \leqslant [n]-4, 3 \leqslant n < \infty \} \cup \{1\}$ is the set of all dyadic rationals. In addition, since $\operatorname{tr}(\mathcal{L}) = \operatorname{tr}(\mathcal{N})$, there is a trace-preserving bijection of $\mathcal{L}$ onto $\mathcal{N}$.

To see that $\mathcal{N}$ is multiplicity free, note that because $\mathcal{N}^c$ is a norm-closed $\mathfrak{C}$--module, it is the closed linear span of the matrix units $e_{ij}^{(n)}$ it contains. Thus to show $\mathcal{N}^c = \mathfrak{D}$, it is enough to show that $e_{ij}^{(n)} \in \mathcal{N}^c$ implies $i = j$. Now if $1 \leqslant i \leqslant [n] - 4$, $i < j$, we have $q_i^{(n)} e_{ij}^{(n)} = e_{ij}^{(n)}$, whereas $e_{ij}^{(n)} q_i^{(n)} = 0$. If $1 \leqslant j \leqslant [n] - 4$, $j < i$, then $q_j^{(n)} e_{ij}^{(n)} = 0$, but $e_{ij}^{(n)} q_j^{(n)} = e_{ij}^{(n)}$. So suppose $[n] - 4 < i, j \leqslant [n]$. Then $\gamma_n(e_{ij}^{(n)}) = e_{i_1 j_1}^{(n+1)} + e_{i_2 j_2}^{(n+1)}$, where $[n+1] - 8 < i_1,\ j_1 \leqslant [n+1] - 4 < i_2, j_2 \leqslant [n+1]$, and $i_1 \neq j_1$, $i_2 \neq j_2$. Thus if $i_1 < j_1$, then

$$q_{i_1}^{(n+1)} \gamma_n(e_{ij}^{(n)}) = q_{i_1}^{(n+1)} e_{i_1 j_1}^{(n+1)} = e_{i_1 j_1}^{(n+1)} \text{ and } \gamma_n(e_{ij}^{(n)}) q_{i_1}^{(n+1)} = 0.$$

On the other hand, if $j_1 < i_1$, then

$$q_{j_1}^{(n+1)} \gamma_n(e_{ij}^{(n)}) = 0 \text{ and } \gamma_n(e_{ij}^{(n)}) q_{j_1}^{(n+1)} = e_{i_1 j_1}^{(n+1)}.$$

Thus $e_{ij}^{(n)}$ does not commute with $\mathcal{N}_{n+1}$.

Suppose now that $\varphi : \mathfrak{D} \to \mathfrak{C}$ is an order-preserving C^*-isomorphism. By [9, Corollary 3.23], φ maps $\operatorname{Lat}(\operatorname{Alg}\mathcal{L})$ onto $\operatorname{Lat}(\operatorname{Alg}\mathcal{N})$. By Corollary 2.4, $\varphi(\mathcal{L}) = \mathcal{N}$, and therefore $\varphi(C^*(\mathcal{L})) = C^*(\mathcal{N})$. As $C^*(\mathcal{L}) = \mathfrak{D}$, we will arrive at a contradiction by showing that $C^*(\mathcal{N}) \neq \mathfrak{C}$.

We claim that $e_4^{(2)} \notin C^*(\mathcal{N})$. If on the contrary $e_4^{(2)} \in C^*(\mathcal{N})$, then by Lemma 3.1 there is a unique representation $e_4^{(2)} = (q_1 - q_1') + (q_2 - q_2') + \cdots + (q_m - q_m')$, with $q_1 > q_1' > \cdots > q_m > q_m'$ projections in $\mathcal{N}$. Choose n sufficiently large so that $q_i, q_i' \in \mathcal{N}_n$, $1 \leqslant i \leqslant m$. Expressing $e = \gamma_{n-1} \circ \cdots \circ \gamma_2(e_4^{(2)})$ as a sum of minimal projections in $\mathfrak{D}_n$, observe that $e_{[n]}^{(n)} \leqslant e$ and $e_i^{(n)} \perp e$ for $[n] - 4 < i < [n]$. Note that for any $q \in \mathcal{N}_n$, $q < 1$, we have $q \perp e_{[n]}^{(n)}$. Thus q_1 must be 1. But also $q < 1$,

$q \in \mathcal{N}_n$, implies $q \perp e_i^{(n)}$, $[n] - 4 < i \leqslant [n]$. Thus $e \geqslant q_1 - q_1' \geqslant \sum_{i=[n]-3}^{[n]} e_i^{(n)}$, which is a contradiction.

We conclude this example by remarking that the diagonals $\mathfrak{D}$ and $\mathfrak{C}$ are isomorphic; in fact, there is an isomorphism $\psi : \mathfrak{A} \to \mathfrak{B}$ with $\psi(\mathfrak{D}) = \mathfrak{C}$. Define $\psi_n : \mathfrak{A}_n \to \mathfrak{A}_n$ so that the diagram

$$\begin{array}{ccc} \mathfrak{A}_n & \xrightarrow{\nu_n} & \mathfrak{A}_{n+1} \\ \Big\downarrow \psi_n & & \Big\downarrow \psi_{n+1} \\ \mathfrak{A}_n & \xrightarrow[\gamma_n = \mathrm{Ad}\, P_n \circ \nu_n]{} & \mathfrak{A}_{n+1} \end{array}$$

commutes. One checks inductively that this is accomplished by taking $\psi_n = \mathrm{Ad}\, Q_n$, $n \geqslant 2$, where $Q_2 = 1$, $Q_3 = R = P_2$, $Q_4 = P_3 \nu_3(P_2), \ldots, Q_{n+1} = P_n \nu_n(Q_n)$. Note that $\psi(\mathfrak{D}_n) = \mathfrak{D}_n$. Define ψ on $\bigcup_n (\mathfrak{A}_n, \nu_n)$ by $\psi(x) = \psi_n(x)$ if $x \in \mathfrak{A}_n$. ψ is consistently defined, and is an isometric $*$-isomorphism onto $\bigcup_n (\mathfrak{A}_n, \gamma_n)$. Hence ψ extends uniquely to a C^*-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. As $\psi(\mathfrak{D}_n) = \mathfrak{D}_n$, it follows that $\psi(\mathfrak{D}) = \mathfrak{C}$. (Note: This argument works for any embeddings, not just ν_n and γ_n).

REMARK 3.10. If $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic UHF algebras with masas $\mathfrak{D}$ and $\mathfrak{C}$, respectively, and if $\mathcal{M} \subseteq \mathfrak{D}$ and $\mathcal{N} \subseteq \mathfrak{C}$ are nests with $\mathrm{tr}(\mathcal{M}) = \mathrm{tr}(\mathcal{N})$, then the bijection $\varphi : \mathcal{M} \to \mathcal{N}$ given by the trace extends uniquely to a C^*-isomorphism $C^*(\mathcal{M}) \to$ $\to C^*(\mathcal{N})$ by Corollary 3.5. A necessary condition for this mapping to have an extension to an order isomorphism $\mathfrak{D} \to \mathfrak{C}$ is that $\mathrm{codim}\, C^*(\mathcal{M})$ in $\mathfrak{D}$ equals $\mathrm{codim}\, C^*(\mathcal{N})$ in $\mathfrak{C}$. In the previous example, $\mathcal{M} = \mathcal{L}$ generates $\mathfrak{D}$, whereas $\mathrm{codim}\, C^*(\mathcal{N}) = 3$. Theorem 3.13 says that if the codimensions are zero, then the map is an order isomorphism. However, it seems unlikely that the conditions $\mathrm{tr}(\mathcal{M}) = \mathrm{tr}(\mathcal{N})$ and $\mathrm{codim}\, C^*(\mathcal{M}) = \mathrm{codim}\, C^*(\mathcal{N})$ would in general imply $\mathfrak{D}$ and $\mathfrak{C}$ are order isomorphic.

PROPOSITION 3.11. *Let $\mathcal{M}$ be a nest in $\mathfrak{D} \subseteq \mathfrak{A}$. Suppose $w \in \mathcal{W}_{\mathfrak{D}}$ with $w^*w \perp ww^*$ and either $w^*w \in C^*(\mathcal{M})$ or $ww^* \in C^*(\mathcal{M})$. Then there are projections $P_1, \ldots, P_{m_1}$ and $R_1, \ldots, R_{m_2}$ in $\mathcal{M}$ and orthogonal partial isometries $w_1, \ldots, w_{m_1}, \tilde{w}_{m_1}, \ldots, \tilde{w}_{m_2}$ in $\mathcal{W}_{\mathfrak{D}}$ such that $P_i w_i P_i^{\perp} = w_i$ and $R_i^{\perp} \tilde{w}_i R_i = \tilde{w}_i$ for all i and $w = \sum w_i + \sum \tilde{w}_i$. (Note that each $w_i \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\, \mathcal{M})$ since $P \mathfrak{A} P_i^{\perp} \subseteq \mathrm{Alg}\, \mathcal{M}$, and similarly each $\tilde{w}_i \in$ $\in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\, \mathcal{M})^*$).*

Proof. Suppose $ww^* \in C^*(\mathcal{M})$. Then by Lemma 3.1, we can write ww^* uniquely as $ww^* = \sum_{k=1}^{m} (Q_{2k} - Q_{2k-1})$, where each $Q_i \in \mathcal{M}$ and $Q_1 < Q_2 < \cdots < Q_{2m}$. Since

$w^*w \perp ww^*$, it follows that $w^*w = \sum_{k=0}^{m} F_k$ with $F_0 = Q_1 w^*w$ and $F_k = (Q_{2k+1} - Q_{2k})w^*w$ (letting $Q_{2m+1} = 1$). Let $R_1 = Q_1$, $\tilde{w}_1 = wR_1 = R_1^\perp w R_1$, and $v_1 = w - \tilde{w}_1$. Note that $v_1^*v_1 \perp v_1v_1^*$, $v_1^*v_1 = \sum_{k=1}^{m} F_k$, and $v_1v_1^* = \sum_{k=1}^{m} E_k$ with $E_k = (Q_{2k} - Q_{2k-1})v_1v_1^*$.

Now let $P_1 = Q_2$, $w_1 = P_1 v_1$, and $v_2 = v_1 - w_1$. Then $v_2^*v_2 \perp v_2v_2^*$, $v_2v_2^* = \sum_{k=2}^{m} E_k$, and $v_2^*v_2 = \sum_{k=1}^{m} F_k^{(2)}$ with $F_k^{(2)} = (Q_{2k+1} - Q_{2k})v_2^*v_2$. Next, let $R_2 = Q_3$, $\tilde{w}_2 = v_2 R_2$, and $v_3 = v_2 - \tilde{w}_2$. Then $v_3^*v_3 \perp v_3v_3^*$, $v_3^*v_3 = \sum_{k=2}^{m} F_k^{(2)}$, and $v_3v_3^* = \sum_{k=2}^{m} E_k^{(3)}$ with $E_k^{(3)} = (Q_{2k} - Q_{2k-1})v_3v_3^*$. Continue in this manner. Eventually, $P_i = Q_{2i}$ and $R_i = Q_{2i-1}$ for all i, $1 \leqslant i \leqslant m$, and $R_{m+1} = Q_{2m+1}$. However, we may have $w_i = 0$ or $\tilde{w}_j = 0$ for some i's and j's. In this case, delete these w_i's and $\tilde{w}_j$'s and corresponding P_i's and R_j's from the lists and relabel. Note that the w_i's and $\tilde{w}_j$'s are orthogonal from the construction.

If instead $w^*w \in C^*(\mathcal{M})$, then apply the same argument to w^* and $\mathcal{M}^\perp = \{1 - P : P \in \mathcal{M}\}$, obtaining projections $P_1', \ldots, P_{m_1}', R_1', \ldots, R_{m_2}' \in \mathcal{M}^\perp$ and orthogonal partial isometries $w_1', \ldots, w_{m_1}', \tilde{w}_1, \ldots, \tilde{w}_{m_2}'$ such that $P_i' w_i' P_i'^\perp = w_i'$ and $R_i'^\perp \tilde{w}_i' R_i' = \tilde{w}_i'$ for all i and $w = \sum w_i' + \sum \tilde{w}_i'$. Then just let $P_i = P_i'^\perp$, $R_i = R_i'^\perp$, $w_i = w_i'^*$, and $\tilde{w}_i = \tilde{w}_i'^*$. ■

COROLLARY 3.12. *Suppose $\mathcal{M}$ is a nest in $\mathfrak{D} \subseteq \mathfrak{A}$ and $w \in \mathcal{W}_{\mathfrak{D}}$ sucht that $w^*w \perp ww^*$ and either $w^*w \in C^*(\mathcal{M})$ or $ww^* \in C^*(\mathcal{M})$. Then $w \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{M})$ if and only if there are projections $P_1, \ldots, P_m \in \mathcal{M}$ and orhogonal partial isometries $w_1, \ldots, w_m \in \mathcal{W}_{\mathfrak{D}}$ such that $P_i w_i P_i^\perp = w_i$ for all i and $w = \sum_{i=1}^{m} w_i$. (Note that each $w_i \in \mathrm{Alg}\,\mathcal{M}$).*

Proof. The "if" direction is clear. Now if $w \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{M})$, use Proposition 3.11 to write $w = \sum w_i + \sum \tilde{w}_i$ with $w_i \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{M})$ and $\tilde{w}_i \in \mathcal{W}_{\mathfrak{D}}(\mathrm{Alg}\,\mathcal{M})^*$. But then if e and f are the initial and final projections of $\tilde{w}_i$, then we have $\tilde{w}_i = ewf \in \mathrm{Alg}\,\mathcal{M}$, which implies that each $\tilde{w}_i = 0$. ■

THEOREM 3.13 (Order Isomorphism Theorem for Nest Algebras). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are nests, $\mathcal{M} \subseteq \mathfrak{D} \subseteq \mathfrak{A}$ and $\mathcal{N} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, such that $C^*(\mathcal{M}) = \mathfrak{D}$ and $C^*(\mathcal{N}) = \mathfrak{C}$. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a trace-preserving bijection. Then φ extends uniquely to an order isomorphism of $(\mathfrak{D}, \prec_{\mathrm{Alg}\,\mathcal{M}})$ and $(\mathfrak{C}, \prec_{\mathrm{Alg}\,\mathcal{N}})$.*

Proof. By Corollary 3.5, φ extends uniquely to a C^*-isomorphism of $\mathfrak{D}$ onto $\mathfrak{C}$.

Since φ preserves trace on $\mathcal{M}$, it must also preserve trace on $\mathcal{P}(\mathfrak{D}) = \mathcal{P}(C^*(\mathcal{M}))$ because of the unique decomposition of projections in $C^*(\mathcal{M})$ and $C^*(\mathcal{N})$ (Lemma 3.1).

Let $\mathcal{S} = \operatorname{Alg}\mathcal{M}$ and $\mathcal{T} = \operatorname{Alg}\mathcal{N}$, and suppose $p \prec_{\mathcal{S}} q$ via $w \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$, i.e., $ww^* = p$ and $w^*w = q$. By [9, Corollary 3.7], there is some $\tilde{w} \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S}) \cap \mathfrak{A}_n$ and unitary u in $\mathfrak{D}$ such that $w = \tilde{w}u$. Then $\tilde{w}\tilde{w}^* = p$ and $\tilde{w}^*\tilde{w} = q$, so by replacing w with $\tilde{w}$, we can assume without loss of generality that $w \in \mathfrak{A}_n$. Let $\{e_{ij} : 1 \leqslant i,j \leqslant k_n\}$ be the matrix units of $\mathfrak{A}_n$ and denote e_{ii} by e_i. By [9, Lemma 3.4], we can write $w = \sum_{i\in I_1} \lambda_i e_{i,j(i)} + \sum_{i\in I_2} \lambda_i e_i$, where $\lambda_i \in \mathbb{C}$, $|\lambda_i| = 1$, $I_1, I_2 \subseteq \{1,\ldots,k_n\}$, $(I_1 \cup j(I_2)) \cap$ $\cap I_2 = \emptyset$, and $i \neq j(i)$ for all $i \in I_1$. Note that each $\lambda_i e_{i,j(i)} \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ [9, Lemma 3.1] and has orthogonal initial and final projections. Also, $p = \sum_{i\in I_1} e_i + \sum_{i\in I_2} e_i$ and $q = \sum_{i\in I_1} e_{j(i)} + \sum_{i\in I_2} e_i$. Thus, it is enough to find for each $i \in I_1$ some $v_i \in \mathcal{W}_{\mathfrak{C}}(\mathcal{T})$ such that $v_i v_i^* = \varphi(e_i)$ and $v_i^* v_i = \varphi(e_{j(i)})$, for then $\varphi(p) \prec_{\mathcal{T}} \varphi(q)$ via $v = \sum_{i\in I_1} v_i + \sum_{i\in I_2} \varphi(e_i)$.

Fix $i \in I_1$. By Corollary 3.12, there are projections $P_1,\ldots,P_m \in \mathcal{M}$ and orthogonal partial isometries $w_1,\ldots,w_m \in \mathcal{W}_{\mathfrak{D}}(\mathcal{S})$ such that $P_k w_k P_k^\perp = w_k$ and $\lambda_i e_{i,j(i)} = \sum_{k=1}^{m} w_k$. Now $\operatorname{tr}(w_k^* w_k) = \operatorname{tr}(w_k w_k^*)$, so $\operatorname{tr}(\varphi(w_k^* w_k)) = \operatorname{tr}(\varphi(w_k w_k^*))$. Also, $\varphi(w_k w_k^*) \leqslant \varphi(P_k)$, $\varphi(w_k^* w_k) \leqslant \varphi(P_k^\perp) = \varphi(P_k)^\perp$, and $\varphi(P_k) \in \mathcal{N}$. Since $\varphi(P_k)\cdot$ $\cdot\mathfrak{B}\varphi(P_k)^\perp \subseteq \mathcal{T}$, there is a partial isometry $x_k \in \mathcal{W}_{\mathfrak{C}}(\mathcal{T})$ with $x_k x_k^* = \varphi(w_k w_k^*)$ and $x_k^* x_k = \varphi(w_k^* x_k)$. Let $v_i = \sum_{k=1}^{m} x_k$. Orthogonality is preserved by φ, so the x_k's are orthogonal, and therefore $v_i \in \mathcal{W}_{\mathfrak{C}}(\mathcal{T})$. ∎

Corollary 3.14. *If $C^*(\mathcal{M}) = \mathfrak{D}$ and $C^*(\mathcal{N}) = \mathfrak{C}$, then $(\mathfrak{D}, \operatorname{Alg}\mathcal{M})$ and $(\mathfrak{C}, \operatorname{Alg}\mathcal{N})$ are order isomorphic if and only if there is a trace-preserving bijection $\varphi : \mathcal{M} \to \mathcal{N}$.*

The techniques of Proposition 3.11 can be used to give additional information, especially when $C^*(\mathcal{M}) = \mathfrak{D}$.

Corollary 3.15. *If $C^*(\mathcal{M}) = \mathfrak{D}$ and $w \in \mathcal{W}_{\mathfrak{D}}$, then there are projections $P_1,\ldots,P_{m_1}$ and $R_1,\ldots,R_{m_2}$ in $\mathcal{M}$ and orthogonal partial isometries $w_1,\ldots,w_{m_1}$, $\tilde{w}_1,\ldots,\tilde{w}_{m_2}$ in $\mathcal{W}_{\mathfrak{D}}$ such that $P_i w_i P_i^\perp = w_i$ and $R_i^\perp \tilde{w}_i R_i = \tilde{w}_i$ for all i, and $w-$ $-\sum w_i - \sum \tilde{w}_i \in \mathfrak{D}$. $w \in \operatorname{Alg}\mathcal{M}$ if and only if $w - \sum w_i \in \mathfrak{D}$. Also, if $w^*w \perp ww^*$, then $w = \sum w_i + \sum \tilde{w}_i$.*

Proof. First, write $w = \tilde{w}u$ with $\tilde{w} \in \mathcal{W}_{\mathfrak{D}} \cap \mathfrak{A}_n$ and $u \in \mathcal{U}(\mathfrak{D})$, by [9, Theorem 3.6].

As in the proof of Theorem 3.13, $\tilde{w} = \sum_{i \in I_1} e_{i,j(i)} + \sum_{i \in I_2} e_i$ with $(I_1 \cup j(I_1)) \cap I_2 = \emptyset$ and $i \neq j(i)$ for all $i \in I_1$. Apply Proposition 3.11 to each $e_{i,j(i)}u$ to get $e_{i,j(i)}u = \sum_{k=1}^{s_i} w_{ik} + \sum_{\ell=1}^{t_i} \tilde{w}_{i\ell}$ and projections $P_{i1}, \ldots, P_{is_i}, R_{i1}, \ldots, R_{it_i} \in \mathcal{M}$ such that $P_{ik} w_{ik} P_{ik}^{\perp} = w_{ik}$ and $R_{i\ell}^{\perp} \tilde{w}_{i\ell} R_{i\ell} = \tilde{w}_{i\ell}$. Then $w - \sum_{i \in I_1} \left(\sum_{k=1}^{s_i} w_{ik} + \sum_{k=1}^{t_1} \tilde{w}_{ik} \right) = \sum_{I_2} e_i u \in \mathfrak{D}$. Now just relabel $\{P_{ik} : i \in I_1, 1 \leqslant k \leqslant s_i\}$ as $P_1, \ldots, P_{m_1}$, and similarly for $\{R_{i\ell}\}$, $\{w_{ik}\}$, and $\{\tilde{w}_{i\ell}\}$. ■

COROLLARY 3.16. *If* $C^*(\mathcal{M}) = \mathfrak{D}$, *then* $\overline{\operatorname{Alg}\mathcal{M} + (\operatorname{Alg}\mathcal{M})^*} = \mathfrak{A}$. *In fact, the stronger statement* $\bigcup_n ((\operatorname{Alg}\mathcal{M} + (\operatorname{Alg}\mathcal{M})^*) \cap \mathfrak{A}_n) = \bigcup_n \mathfrak{A}_n$ *holds. Therefore,* $\operatorname{Alg}\mathcal{M}$ *is strongly maximal.*

Proof. $w = e_{ij}^{(n)}$, $i \neq j$, satisfies the hypothesis of Proposition 3.11, so $w = \sum w_i + \sum \tilde{w}_i$ with $w_i \in \operatorname{Alg}\mathcal{M}$ and $\tilde{w}_i \in (\operatorname{Alg}\mathcal{M})^*$. A closer examination of the proof of Proposition 3.11 shows that in fact each w_i and $\tilde{w}_i$ lies in some $\mathfrak{A}_k$, $k \geqslant n$. By [16, Theorem 2.1], $\operatorname{Alg}\mathcal{M}$ is strongly maximal. ■

Note, however, that for a particular sequence $\{\mathfrak{A}_n\}$, we can still have $((\operatorname{Alg}\mathcal{M}) + (\operatorname{Alg}\mathcal{M})^*) \cap \mathfrak{A}_n \not\subseteq \mathfrak{A}_n$ for all n.

We already have several examples of maximal, multiplicity free nests for which the trace has large gaps; e.g., $(\operatorname{tr}(\mathcal{M})) \cap (\alpha, 1) = \emptyset$ if $\mathcal{M}$ is the nest of Theorem 2.18 with $0 < \alpha < 1$. In view of Theorem 3.13, one would suspect that the trace of a maximal nest $\mathcal{M}$ satisfying $C^*(\mathcal{M}) = \mathfrak{D}$ ought to be more constrained. Indeed, Proposition 3.18 below shows that $\operatorname{tr}(\mathcal{M})$ intersects every nonempty interval in this case. However, there can still be "point gaps", as the following example shows.

EXAMPLE 3.17. Let $\mathfrak{A}_n = M_{2^n}$ with diagonal $\mathfrak{D}_n$, given the matrix units $\{e_{ij}^{(n)}\}$ for $\mathfrak{A}_n$. Denote $e_{ii}^{(n)}$ by $e_i^{(n)}$. Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$ and $\mathfrak{D} = \varinjlim(\mathfrak{D}_n, \nu_n)$, where ν_n is the nest embedding. We will give an example of a nest $\mathcal{M} \subseteq \mathfrak{D}$ with $C^*(\mathcal{M}) = \mathfrak{D}$, but such that no projection in $\mathcal{M}$ has trace 1/2.

Construct an increasing sequence $\{p_n : 0 \leqslant n < \infty\}$ of projections in $\mathfrak{D}$ as follows: $p_0 = 0$, $p_1 = e_1^{(2)}$, $p_2 = p_1 + e_5^{(3)}, \ldots, p_n = p_{n-1} + e_{2^{n+1}-3}^{(n+1)}, \ldots$. Define a decreasing sequence $\{q_n : 0 \leqslant n < \infty\}$ of projections in $\mathfrak{D}$: $q_0 = 1$, $q_1 = q_0 - e_2^{(3)}$, $q_2 = q_1 - e_6^{(3)}, \ldots, q_n = q_{n-1} - e_{2^{n+1}-2}^{(n+1)}, \ldots$. Observe that $p_n \leqslant q_m$ for all n, m, and that $\operatorname{tr}(p_n) < 1/2 < \operatorname{tr}(q_m)$ for all n, m.

Let $A_0 = \{p_n : 0 \leqslant n < \infty\}$ and $A_1 = \{q_n : 0 \leqslant n < \infty\}$. Let $\mathcal{L}$ be the canonical nest, and set $\mathcal{L}_i = p_{i-1} + (p_i - p_{i-1})\mathcal{L}$ and $\mathcal{L}^i = q_i + (q_{i-1} - q_i)\mathcal{L}$. Using $\mathcal{L}$ for

$\mathcal{K}$ in Lemma 2.20, it follows that there is a maximal multiplicity-free nest $\mathcal{M} \subseteq \mathfrak{D}$ containing $A_0 \cup A_1$. Obseve that $1/2 \neq \mathrm{tr}(p)$ for all $p \in \mathcal{M}$. For if $p \in \mathcal{L}_i$, then $p \leqslant p_i$, so $\mathrm{tr}(p) \leqslant \mathrm{tr}(p_i) < 1/2$. If $p \in \mathcal{L}^j$, then $p \geqslant q_j$, so $\mathrm{tr}(p) \geqslant \mathrm{tr}(q_j) > 1/2$.

We claim that $C^*(\mathcal{M}) \cap \mathfrak{D}_n \supseteq \mathfrak{D}_{n-1}$. Let $\mathfrak{C}_n = C^*(\mathcal{M}) \cap \mathfrak{D}_n$. Now $p_1 - p_0 = = p_1 = e_1^{(2)} = e_1^{(n)} + \cdots + e_{2^{n-2}}^{(n)} \in \mathfrak{C}_n$, so $\{e_1^{(n)}, \ldots, e_{2^{n-2}}^{(n)}\} \subseteq \mathfrak{C}_n$. Also, $q_0 - q_1 = = 1 - q_1 = e_2^{(2)} = e_{2^{n-2}+1}^{(n)} + \cdots + e_{2^{n-1}}^{(n)} \in \mathfrak{C}_n$, so $\{e_{2^{n-2}+1}^{(n)}, \ldots, e_{2^{n-1}}^{(n)}\} \subseteq \mathfrak{C}_n$. Next, $p_2 - -p_1 = e_5^{(3)} = e_{2^{n-1}+1}^{(n)} + \cdots + e_{5\cdot 2^{n-3}}^{(n)} \in \mathfrak{C}_n$ implies that $\{e_{2^{n-1}+1}^{(n)}, \ldots, e_{5\cdot 2^{n-3}}^{(n)}\} \subseteq \mathfrak{C}_n$, and $q_1 - q_2 = e_6^{(3)} = e_{5\cdot 2^{n-3}+1}^{(n)} + \cdots + e_{6\cdot 2^{n-3}}^{(n)} \in \mathfrak{C}_n$ implies that $\{e_{5\cdot 2^{n-3}+1}^{(n)}, \ldots, e_{6\cdot 2^{n-3}}^{(n)}\} \subseteq \subseteq \mathfrak{C}_n$. Continue this process $n-1$ times. For the last step, $p_{n-1} - p_{n-2} = e_{2^n-3}^{(n)} \in \in \mathfrak{C}_n$ and $q_{n-2} - q_{n-1} = e_{2^n-2}^{(n)} \in \mathfrak{C}_n$. Thus, $\mathfrak{C}_n \supseteq \{e_1^{(n)}, \ldots, e_{2^n-2}^{(n)}\}$. As $1 \in \mathfrak{C}_n$, $1 - \sum_{j=1}^{2^n-2} e_j^{(n)} = e_{2^n-1}^{(n)} + e_{2^n}^{(n)} \in \mathfrak{C}_n$. We conclude that $\mathfrak{C}_n$ contains the projections $e_1^{(n)} + e_2^{(n)}, e_3^{(n)} + e_4^{(n)}, \ldots, e_{2^n-1}^{(n)} + e_{2^n}^{(n)}$, which are exactly the minimal projections of $\mathfrak{D}_{n-1}$. This finishes the claim, and it follows immediately that $C^*(\mathcal{M}) = \mathfrak{D}$.

Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ be a UHF algebra, and set $[n]^2 = \dim \mathfrak{A}_n$. Let $\{e_{ij}^{(n)} : 1 \leqslant i, j \leqslant \leqslant [n]\}$ be a set of matrix units for $\mathfrak{A}_n$, and let $\mathfrak{D}_n$ be the diagonal of $\mathfrak{A}_n$ with respect to this set of matrix units. Set $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$.

PROPOSITION 3.18. *If $\mathcal{M} \subseteq \mathfrak{D}$ is a nest such that $C^*(\mathcal{M}) = \mathfrak{D}$, then*

$$\overline{\{\mathrm{tr}(p) : p \in \mathcal{M}\}} = [0,1].$$

Proof. If the conclusion fails, there exist $0 \leqslant a < b \leqslant 1$ with $(a,b) \subset [0,1] \setminus \{\mathrm{tr}(p) : : p \in \mathcal{M}\}$. Let $\mathcal{M}_a = \{f \in \mathcal{M} : \mathrm{tr}(f) \leqslant a\}$, $\mathcal{M}_b = \{f \in \mathcal{M} : \mathrm{tr}(f) \geqslant b\}$. Then $\mathcal{M} = \mathcal{M}_a \cup \mathcal{M}_b$, and $\mathrm{tr}(g - f) \geqslant b - a$ for all $g \in \mathcal{M}_b$, $f \in \mathcal{M}_a$.

Choose $f_a \in \mathcal{M}_a$, $f_b \in \mathcal{M}_b$ (e.g., $f_a = 0$, $f_b = 1$) and n sufficiently large so that $f_a, f_b \in \mathfrak{D}_n$ and $[n]^{-1} < b - a$. Express $f_b - f_a$ as a sum of minimal projections in $\mathfrak{D}_n$; i.e., there is a set $I \subseteq \{1, \ldots, [n]\}$ of indices for which $f_b - f_a = \sum_{i \in I} e_i^{(n)}$. By Lemma 3.1, each $e_i^{(n)}$ has a unique expression as $e_i^{(n)} = (f_1(e_i^{(n)}) - f_1'(e_i^{(n)})) + \cdots + (f_{n_i}(e_i^{(n)}) - -f_{n_i}'(e_i^{(n)}))$, with $f_1(e_i^{(n)}) > f_1'(e_i^{(n)}) > f_2(e_i^{(n)}) > \cdots > f_{n_i}'(e_i^{(n)})$ projections in $\mathcal{M}$. Let $f_1(e_{i_1}^{(n)}) = \max\{f_1(e_i^{(n)}) : i \in I\}$; then $f_1'(e_{i_1}^{(n)}) \geqslant \max\{ f_1(e_i^{(n)}) : i \in \in I \setminus \{i_1\} \}$. Otherwise, the projections $f_1(e_{i_1}^{(n)}) - f_1'(e_{i_1}^{(n)})$ and $f_1(e_i^{(n)}) - f_1'(e_i^{(n)})$ are not orthogonal for some $i \in I \setminus \{i_1\}$. But they are subprojections of $e_{i_1}^{(n)}$ and $e_i^{(n)}$, respectively, which are orthogonal.

Let $g_1 > g_2 > \cdots > g_m$ be the projections $f_j(e_i^{(n)})$ written in decreasing order, and let $g_1' > g_2' > \cdots > g_m'$ be the projections $f_j'(e_i^{(n)})$ written in decreasing order. We just showed that $g_1' \geqslant g_2$. Continue inductively to show that $g_k' \geqslant g_{k+1}$, $k =$

$= 1,\dots,m-1$. As $\{g_k - g'_k : 1 \leqslant k \leqslant m\} = \{f_j(e_i^{(n)}) - f'_j(e_i^{(n)}) : j = 1,\dots,n_i, i \in I\}$ we can write

$$f_b - f_a = \sum_{i\in I} e_i^{(n)} = \sum_{i\in I}\sum_{j=1}\left(f_j(e_i^{(n)}) - f'_j(e_i^{(n)})\right) =$$

$$= \sum_{k=1}^{m}(g_k - g'_k).$$

By the uniqueness result of Lemma 3.1, we must have $f_b = g_1$, $f_a = g'_m$, and $g'_1 = g_2$, $g'_2 = g_3,\dots,g'_{m-1} = g_m$. Since $\mathcal{M} = \mathcal{M}_a \cup \mathcal{M}_b$, there is a largest g_k in $\mathcal{M}_a$, say g_{k_0}, such that $g_{k_0}, g_{k_0+1},\dots,g_m \in \mathcal{M}_a$ and $g_1, g_2,\dots,g_{k_0-1} \in \mathcal{M}_b$. Thus $g_{k_0-1} - g'_{k_0-1} = g_{k_0-1} - g_{k_0}$ is a difference of projections in $\mathcal{M}_b$ and $\mathcal{M}_a$, and hence $\operatorname{tr}(g_{k_0-1} - g_{k_0}) \geqslant b - a$. However, $g_{k_0-1} - g'_{k_0-1}$ is a subprojection of some $e_j^{(n)}$, with trace at most $[n]^{-1} < b - a$. This contradiction establishes the result. ∎

At this point it is worthwhile to reexamine Theorem 4.3 and Example 4.4 of [9] in light of the results obtained in this paper. Let $\mathcal{L}$ be the canonical nest in $\mathfrak{D} \subseteq \mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$ and let $\mathcal{S} = \operatorname{Alg}\mathcal{L}$. If $\mathcal{T}$ is a strongly maximal triangular algebra with diagonal $\mathfrak{C}$, contained in a UHF algebra $\mathfrak{B}$ isomorphic to $\mathfrak{A}$, then Theorem 4.3 gives necessary conditions for order isomorphism of $(\mathfrak{D}, \prec_{\mathcal{S}})$ and $(\mathfrak{C}, \prec_{\mathcal{T}})$, as well as conditions implying isomorphism of $\mathcal{S}$ and $\mathcal{T}$. Now if $\varphi : (\mathfrak{D}, \prec_{\mathcal{S}}) \to (\mathfrak{C}, \prec_{\mathcal{T}})$ is an order isomorphism, then [9, Corollary 3.23] implies that $\varphi(\mathcal{L}) = \operatorname{Lat}(\mathcal{T})$, so $\mathcal{M} = \operatorname{Lat}(\mathcal{T})$ is a nest and $\varphi(C^*(\mathcal{L})) = C^*(\mathcal{M})$. Since $C^*(\mathcal{L}) = \mathfrak{D}$, it follows that $C^*(\mathcal{M}) = \mathfrak{C}$. Thus $\mathcal{M}$ is maximal by Corollary 3.3 and $\mathcal{T}$ is a nest algebra by Proposition 2.1. Therefore, Theorem 4.3 can be reinterpreted as a result about nest algebras.

To briefly review of Theorem 4.3, one can first assume that $\mathfrak{B} = \varinjlim(\mathfrak{A}_n, \operatorname{Ad} P_n \circ \nu_n)$, where P_n is a permutation matrix, and $\mathcal{T} \cap \mathfrak{A}_n$ is the set of upper triangular matrices in $\mathfrak{A}_n$. Define $[n] = \sqrt{\dim \mathfrak{A}_n}$. Now if $(\mathfrak{D}, \prec_{\mathcal{S}})$ and $(\mathfrak{C}, \prec_{\mathcal{T}})$ are order isomorphic, then for each k there is some $\ell(k)$ such that for all $\ell > \ell(k)$, $P_\ell = \bigoplus_{i=1}^{[k]} P(\ell, i)$ and $P(\ell, i)$'s are permutation matrices of size $[\ell+1] / [k]$. If $\mathcal{S}$ and $\mathcal{T}$ are isomorphic, then in addition $P(n,i) = P(n,j)$ for all i and j. To obtain Example 4.4 of [9], let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ and define each P_n to be of the form $I \oplus \cdots \oplus I \oplus J$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic by the theorem. However, the diagonals are order isomorphic. This can be seen directly, or alternatively it can be easily shown that $\mathcal{M} = \operatorname{Lat}\mathcal{T} = \left\{\sum_{i=1}^{k} f_{ii}^{(n)} : 1 \leqslant k \leqslant 2^n, 1 \leqslant n < \infty\right\} \cup \{0\}$, where

$\{f_{ij}^{(n)}\}$ is a set of matrix units for $\mathfrak{B}$. It then follows that $C^*(\mathcal{M}) = \mathfrak{C}$ and there is a trace-preserving bijection of $\mathcal{L}$ onto $\mathcal{M}$, and therefore $(\mathfrak{D}, \prec_S)$ and $(\mathfrak{C}, \prec_T)$ are order isomorphic by Theorem 3.13. In addition, this order isomorphism is unique. As a consequence, one can also see that $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic by showing that condition $(*)$ of Section 1 cannot hold. We leave this rather interesting exercise for the reader. Still another proof was given in Example 1.15.

Example 4.4 is somewhat special, however. We will show below that the conditions of Theorem 4.3 are not sufficient to ensure either order isomorphism or algebra isomorphism. Indeed, Lat $\mathcal{T}$ may not even be multiplicity-free. First we need the following lemma, which was used implicity in [9, Example 3.27].

LEMMA 3.19. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be finite dimensional factors, and let $\mathcal{S} \subseteq \mathfrak{A}$ and $\mathcal{T} \subseteq \mathfrak{B}$ be maximal triangular algebras with diagonals $\mathfrak{D} = \mathcal{S} \cap \mathcal{S}^*$ and $\mathfrak{C} = \mathcal{T} \cap \mathcal{T}^*$. Let $e_1 \prec_S e_2 \prec_S \dots \prec_S e_M$ and $f_1 \prec_T f_2 \prec_T \dots \prec_T f_N$ be the unique orderings of the minimal projections of $\mathfrak{D}$ and $\mathfrak{C}$, respectively, induced by $\mathcal{S}$ and $\mathcal{T}$. Let $\varphi : \mathfrak{D} \to \mathfrak{C}$ be an injective unital C^*-homomorphism which is order-preserving ($\varphi(e) \prec_T \varphi(f)$ if $e \prec_S f$). Then for $1 \leqslant j \leqslant M$,*

$$(*) \qquad f_1 + \dots + f_j \leqslant \varphi(e_1 + \dots + e_j).$$

Proof. Suppose $(*)$ holds for all $j < j_0$ and fails for $j = j_0$. Since

$$\sum_{j<j_0} f_i \leqslant \varphi\left(\sum_{j<j_0} e_j\right), \quad f_{j_0} \not\leqslant \varphi\left(\sum_{j\leqslant j_0} e_j\right),$$

and f_{j_0} is a minimal projection, there is an m, $j_0 < m \leqslant M$, for which $f_{j_0} \leqslant \varphi(e_m)$. $\sum_{j\leqslant j_0} f_i \leqslant \varphi\left(\sum_{j<j_0} e_i\right) + \varphi(e_m)$, so $\varphi(e_{j_0}) \perp \sum_{j\leqslant j_0} f_i$. Write $\varphi(e_j) = \sum_{i\in I_j} f_i$. Then $\varphi(e_{j_0}) \prec_T \varphi(e_m)$ since $e_{j_0} \prec_S e_m$. Let $v \in \mathcal{W}_{\mathfrak{C}} \cap \mathcal{T}$ implement this relation, i.e., $v^*v = \varphi(e_m)$ and $vv^* = \varphi(e_{j_0})$. Express $v = \sum_{i\in I_m} v_i$, where $v_i = vf_i$. Then $v_i^*v_i = f_i$ and $v_iv_i^* = f_{\psi(i)}$ for some $\psi(i) \in I_{j_0}$. Thus, this defines a bijection $\psi : I_m \to I_{j_0}$ with $\psi(i) < i$ for all $i \in I_m$. Now $j_0 \in I_m$, so $\psi(j_0) < j_0$. On the other hand, $i > j_0$ for all $i \in I_{j_0}$, which implies that $\psi(j_0) > j_0$. This contradiction completes the proof. ∎

EXAMPLE 3.20. We show that the converses of Theorem 4.3 (i) and (ii) of [9] fail. Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ and $\{e_{ij}^{(n)} : 1 \leqslant i,j \leqslant 2^n\}$ be a given set of matrix units for $\mathfrak{A}_n$. Denote the diagonal matrix in $\mathbf{M}_4$ with diagonal entries a, b, c, and d by $\mathrm{diag}(a,b,c,d)$. Let $\mathcal{S}_n$ and $\mathcal{T}_n$ each denote the upper triangular matrices in $\mathfrak{A}_n$, and let $\nu_n : \mathcal{S}_n \to \mathcal{S}_{n+1}$ be the nest embedding. Define $j_n : \mathcal{T}_n \to \mathcal{T}_{n+1}$ by $j_n = \mathrm{Ad}(R^{(2^{n-1})}) \circ \nu_n$, where R is the

4×4 permutation matrix satisfying $\operatorname{Ad} R(\operatorname{diag}(x_1,x_2,x_3,x_4)) = \operatorname{diag}(x_1,x_3,x_2,x_4)$ and $R^{(k)} = R\oplus\cdots\oplus R$ (k factors). Set $\mathcal{S} = \varinjlim(\mathcal{S}_n,\nu_n)$ and $\mathcal{T} = \varinjlim(\mathcal{T}_n, j_n)$. Let $\mathfrak{D}_n$ and $\mathfrak{C}_n$ each denote the diagonal matrices in $\mathfrak{A}_n$, and set $\mathfrak{D} = \varinjlim(\mathfrak{D}_n,\nu_n)$ and $\mathfrak{C} = \varinjlim(\mathfrak{C}_n, j_n)$, so $\mathfrak{D}$ and $\mathfrak{C}$ are masas in $\mathfrak{A} = \varinjlim(\mathfrak{A}_n,\nu_n)$ and $\mathfrak{B} = \varinjlim(\mathfrak{A}_n, j_n)$, respectively. $\mathfrak{D} = \mathcal{S}\cap\mathcal{S}^*$, $\mathfrak{C} = \mathcal{T}\cap\mathcal{T}^*$, and $\mathcal{P}(\mathfrak{D})$ and $\mathcal{P}(\mathfrak{C})$ have diagonal orderings $\prec_{\mathcal{S}}$ and $\prec_{\mathcal{T}}$ induced by $\mathcal{S}$ and $\mathcal{T}$.

We will give two proofs that $(\mathfrak{D},\prec_{\mathcal{S}})$ and $(\mathfrak{C},\prec_{\mathcal{T}})$ are not order isomorphic. The first proof is direct and uses the preceding lemma. Suppose $\varphi : (\mathfrak{D},\prec_{\mathcal{S}}) \to (\mathfrak{C},\prec_{\mathcal{T}})$ is an order isomorphism. Then there exist integers $1 \leqslant k < k+1 \leqslant \ell$ so that the diagram

$$\begin{array}{ccc}
\mathfrak{C}_1 & \xrightarrow{\;j_{\ell-1}\circ j_{\ell-2}\circ\cdots\circ j_1\;} & \mathfrak{C}_\ell \\
{\scriptstyle\varphi^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\
\mathfrak{D}_k & \xrightarrow[\nu_k]{\qquad} & \mathfrak{D}_{k+1}
\end{array}$$

commutes. Let $j = j_{\ell-1}\circ j_{\ell-2}\circ\cdots\circ j_1$. Then $j(e_1^{(1)}) = \sum\limits_{k=1}^{2^{\ell-1}} e_{2k-1}^{(\ell)} \in \mathfrak{C}_\ell$. By Lemma 3.19, $\nu_k(\varphi^{-1}(e_1^{(1)})) \geqslant \nu_k(e_1^{(k)}) = e_1^{(k+1)} + e_2^{(k+1)}$. Again by the lemma, $\varphi(e_1^{(k+1)} + e_2^{(k+1)}) \geqslant e_1^{(\ell)} + e_2^{(\ell)}$. As $j(e_1^{(1)}) \perp e_2^{(\ell)}$, the diagram in fact does not commute. Thus, $(\mathfrak{D},\prec_{\mathcal{S}})$ and $(\mathfrak{C},\prec_{\mathcal{T}})$ are not order isomorphic, and it also follows that $\mathcal{S}$ and $\mathcal{T}$ are not isomorphic by [9, Proposition 3.20].

The second proof uses information about $\operatorname{Lat}\mathcal{T}$, specifically that $C^*(\operatorname{Lat}\mathcal{T}) \neq \mathfrak{C}$. As mentioned above, this precludes an order isomorphism. We first show that $\operatorname{Lat}\mathcal{T}$ is a nest and that $\operatorname{tr}(\operatorname{Lat}\mathcal{T})$ is the set of dyadic rationals. $\mathcal{T}$ is an $\mathfrak{C}$-module, so as noted in the preliminaries, every projection in $\operatorname{Lat}\mathcal{T}$ must lie in some $\mathfrak{C}_n$. Now any projection $p\in\mathfrak{C}_n\cap\operatorname{Lat}\mathcal{T}$ must be invariant for $\mathcal{T}_n$, so p has the form $p = \sum\limits_{j=1}^{k} e_j^{(n)}$. If $p = \sum\limits_{j=1}^{2\ell} e_j^{(n)}$, $1\leqslant \ell \leqslant 2^{n-1}$, then p is invariant for $\mathcal{T}_n$, and one easily checks that for any $m>n$, $j_{m-1}\circ\cdots\circ j_n(p)$ is invariant for $\mathcal{T}_m$, so $p\in\operatorname{Lat}\left(\bigcup\mathcal{T}_n\right) = \operatorname{Lat}\left(\overline{\bigcup\mathcal{T}_n}\right) = \operatorname{Lat}(\mathcal{T})$. On the other hand, if $p = \sum\limits_{j=1}^{2\ell-1} e_j^{(n)}$, then $j_n(p) = \sum\limits_{j=1}^{4\ell-3} e_j^{(n+1)} + e_{4\ell-1}^{(n+1)} \notin \operatorname{Lat}(\mathcal{T}_n+1)$. We conclude that all $p\in\operatorname{Lat}(\mathcal{T})$ have the form $p = \sum\limits_{j=1}^{2\ell} e_j^{(n)}$ for some n, $1\leqslant\ell\leqslant 2^{n-1}$, or $p=0$. It follows that $\operatorname{Lat}\mathcal{T}$ is a nest and that $\operatorname{tr}(\operatorname{Lat}\mathcal{T})$ is the set of all dyadic rationals. Thus, $\operatorname{Lat}\mathcal{T}$ is a maximal nest in $\mathfrak{C}$ and, since $\operatorname{Lat}\mathcal{S}$ is the canonical nest, there is a trace-preserving bijection of $\operatorname{Lat}\mathcal{S}$ onto $\operatorname{Lat}\mathcal{T}$.

Now $(\operatorname{Lat}\mathcal{T})^c \neq \mathfrak{C}$ because $e_{21}^{(1)} \in (\operatorname{Lat}\mathcal{T})^c$. To see this, let $p \in \operatorname{Lat}\mathcal{T}$, $p = \sum_{j=1}^{2\ell} e_j^{(n)}$. Write $j_{n-1} \circ \cdots \circ j_1(e_{21}^{(1)}) = \sum_{k=1}^{2^{n-1}} e_{2k,2k-1}^{(n)}$, and compute

$$p^{\perp} j_{n-1} \circ \cdots \circ j_1(e_{21}^{(1)}) p = p^{\perp} \left(\sum_{k=1}^{\ell} e_{2k,2k-1}^{(n)} \right) = 0.$$

Similarly, $p j_{n-1} \circ \cdots \circ j_1(e_{21}^{(1)}) p^{\perp} = 0$. It follows that $C^*(\operatorname{Lat}\mathcal{T}) \neq \mathfrak{C}$. This also shows, by Proposition 2.8(i), that $\mathcal{S} = \operatorname{Alg}(\mathcal{L})$ and $\operatorname{Alg}(\operatorname{Lat}\mathcal{T})$ are not isomorphic.

REMARKS 3.21. This example shows more than was claimed. It shows that the necessary condition (ii) of [9, Theorem 4.3] is not even sufficient for the weaker conclusion (i) that the diagonals be order isomorphic.

Example 4.4 of [9] illustrates some of the problems that remain. Here we have two strongly maximal triangular nest algebras with order isomorphic diagonals and isomorphic nests. In addition, the two nests $\operatorname{Lat}\mathcal{S}$ and $\operatorname{Lat}\mathcal{T}$ are as "nice" as possible in that $(\operatorname{Lat}\mathcal{S}) \cap \mathfrak{A}_n$ is maximal in $\mathfrak{D}_n$ and likewise for $\operatorname{Lat}\mathcal{T}$. However, the algebras are still not isomorphic. Again, the different embeddings are the problem. In other words, the nests look the same, but they are different in the way they lie in their respective UHF algebras. This phenomenon is still not well understood in terms of trying to classify such nest algebras, and remains an object of further study.

4. K-THEORY FOR TAF ALGEBRAS AND NEST ALGEBRAS

Let $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$ be an AF algebra and let $\mathfrak{D} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{D}_n}$ be a masa in $\mathfrak{A}$. Let $\mathcal{S} \subseteq \mathfrak{A}$ be a TAF algebra with diagonal $\mathfrak{D}$. Set $\mathcal{S}^0 = \bigcup_{n=1}^{\infty} \mathcal{S} \cap \mathfrak{A}_n$ and $\mathfrak{D}^0 = \bigcup_{n=1}^{\infty} \mathfrak{D}_n$; then $\mathcal{S}^0$ and $\mathfrak{D}^0$ are local Banach algebras whose completions are $\mathcal{S}$ and $\mathfrak{D}$, respectively. In the notation of [3, 5.1.2], $\mathcal{V}(\mathcal{S}^0) \simeq \mathcal{V}(\mathcal{S})$ and $\mathcal{V}(\mathfrak{D}^0) \simeq \mathcal{V}(\mathfrak{D})$, where $\mathcal{V}(\mathcal{S})$, for example, represents the homotopy classes of idempotents in $\mathrm{M}_\infty(\mathcal{S}) = \bigcup_{m=1}^{\infty} \mathrm{M}_m(\mathcal{S})$. Now $\mathcal{S}^0$ is the algebraic inductive limit $\varinjlim(\mathcal{S}_n, \varphi_n)$, where $\mathcal{S}_n = \mathcal{S} \cap \mathfrak{A}_n$, and φ_n is the restriction of the embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$; similarly, $\mathfrak{D}^0 = \varinjlim(\mathfrak{D}_n, \varphi_n)$. By [3, 5.2.4], $\mathcal{V}(\mathcal{S}^0) = \varinjlim(\mathcal{V}(\mathcal{S}_n), (\varphi_n)_*)$ and $\mathcal{V}(\mathfrak{D}^0) = \varinjlim(\mathcal{V}(\mathfrak{D}_n), (\varphi_n)_*)$.

We claim that $\mathcal{V}(\mathcal{S}_n) = \mathcal{V}(\mathfrak{D}_n)$. Fix n and let $e \in \mathrm{M}_m(\mathcal{S}_n)$ be an idempotent. Let $\{e_{ij}^{(nr)}\}$ be a set of matrix units for $\mathfrak{A}_n$, chosen so that all matrices in $\mathcal{S}_n$ are in upper triangular form. View e as a matrix $e = (a_{ij}^{(r)})$ with $a_{ij}^{(r)} = e_{ij}^{(nr)} \otimes A_{ij}^{(r)}$,

$A_{ij}^{(r)} \in M_m(\mathbb{C})$ and $A_{ij}^{(r)} = 0$ if $e_{ij}^{(nr)} \notin \mathcal{S}_n$. If $e_t = (t^{j-i}a_{ij}^{(r)})$, $0 \leqslant t \leqslant 1$, then one can check directly that e_t is an idempotent in $M_m(\mathcal{S}_n)$, $e_1 = e$, and $e_0 \in M_m(\mathfrak{D}_n)$.

Next, let $e, f \in M_m(\mathfrak{D}_n)$ be idempotents. If e and f are homotopic as elements of $M_\infty(\mathcal{S}_n)$, then they are homotopic as elements of $M_\infty(\mathfrak{D}_n)$. For if $h : [0,1] \to M_\infty(\mathcal{S}_n)$ is a homotopy with $h(0) = e$ and $h(1) = f$, then by defining h_t as above (i.e., $h_t(s) = (h(s))_t$), we see that h_t is a homotopy joining e and f for every $t, 0 \leqslant t \leqslant 1$. In particular, h_0 is such a homotopy; but $h_0(s) \in M_\infty(\mathfrak{D}_n)$ for all s, $0 \leqslant s \leqslant 1$.

PROPOSITION 4.1. *Let $\mathcal{S}$ be* TAF *with diagonal $\mathfrak{D}$. Then*

(i) $K_0(\mathcal{S}) \simeq K_0(\mathfrak{D})$,

(ii) $K_1(\mathcal{S}) = \{0\}$.

Proof. (i) By the above, we have $\mathcal{V}(\mathfrak{D}) = \mathcal{V}(\mathcal{S})$, and the conclusion follows from [3, 5.3.1 and 5.5.5]. (ii) By a similar argument with invertibles in place of idempotents, we obtain $K_1(\mathcal{S}) \simeq K_1(\mathfrak{D})$. But $\mathfrak{D}$ is an AF algebra, so $K_1(\mathfrak{D})$ is trivial [3, 8.1.2(a)]. ∎

We now turn to an analysis of the structure of $(\mathrm{Alg}\,\mathcal{M}) \cap \mathfrak{A}_n$ for a nest $\mathcal{M}$ in $\mathfrak{D}$, where $\mathfrak{D}$ is a masa in UHF algebra $\mathfrak{A} = \bigcup_n \mathfrak{A}_n$. This will allow us to determine the K-theory of nest algebras, and is also of independent interest. $(\mathrm{Alg}\,\mathcal{M}) \cap \mathfrak{A}_n$ is of course a subalgebra of the matrix algebra $\mathfrak{A}_n$, and we will show that it can be put into "upper block triangular form". Additional information will be given concerning the component "blocks".

DEFINITION 4.2. A *partition* $\mathcal{P} = (E_1, E_2, \ldots, E_m)$ of $\mathfrak{D}_n$ is an ordered set of orthogonal projections $\{E_1, \ldots, E_m\}$ in $\mathfrak{D}_n$ such that $\sum_{i=1}^{m} E_i = 1$. The number m of projections in the partition is the *length* of the partition and is denoted length($\mathcal{P}$). If $\mathcal{S} \subseteq \mathfrak{A}$, then $E_i\mathcal{S}E_j$ defines a *block of* $\mathcal{S}$ *with respect to* $\mathcal{P}$. We say that the block $E_i\mathcal{S}E_j$ is *full in* $\mathfrak{A}$ if $E_i\mathcal{S}E_j = E_i\mathfrak{A}E_j$, and $E_i\mathcal{S}E_j$ is *full in* $\mathfrak{A}_n$ if $E_i(\mathcal{S} \cap \mathfrak{A}_n)E_j = E_i\mathfrak{A}_nE_j$. On the other hand, $E_i\mathcal{S}E_j$ is *zero in* $\mathfrak{A}_n$ if $E_i(\mathcal{S} \cap \mathfrak{A}_n)E_j = 0$. If $\mathcal{S} \subseteq \mathfrak{A}_n$, then $\mathcal{S}$ is *block triangular* with respect to partition $\mathcal{P} = (E_1, \ldots, E_m)$ if there exist no i and j with $i \neq j$ such that both $E_i\mathcal{S}E_j$ and $E_j\mathcal{S}E_i$ are nonzero. $\mathcal{S}$ is *upper block triangular* with respect to $\mathcal{P}$ if $E_i\mathcal{S}E_j = 0$ for $i > j$.

The next lemma is undoubtably known, but we have been unable to find a reference.

LEMMA 4.3. *Let $\mathfrak{A}$ be a matrix algebra with masa $\mathfrak{D}$. Suppose $\mathcal{S}$ is a block triangular algebra in $\mathfrak{A}$ with respect to a partition $\mathcal{P} = (E_1, \ldots, E_m)$ of $\mathfrak{D}$. Then* *there is a permutation $\mathcal{Q}$ of $\mathcal{P}$ such that $\mathcal{S}$ is upper block triangular with respect to $\mathcal{Q}$.*

Proof. Use induction on the length of the partition. The result is trivial if length$(\mathcal{P}) = 1$. Let $Q_k = \sum_{i=1}^{k} E_i$. Now $Q_k\mathfrak{D}Q_k$ is a masa in the matrix algebra $Q_k\mathfrak{A}Q_k$, and $Q_k\mathcal{S}Q_k$ is a block triangular algebra with respect to the partition $(E_1, \ldots, E_k)$, so we can for simplicity suppose the result is true for any block triangular algebra with respect to a partition of length less than m, and then prove the result for $\mathcal{S}$. Let $Q = \sum_{i=1}^{m-1} E_i$. Then by the induction hypothesis there is a permutation of the partition $\mathcal{P}_Q = (E_1, \ldots, E_{m-1})$ so that $Q\mathcal{S}Q$ is upper block triangular in $Q\mathfrak{A}Q$. Let $\mathcal{P}' = (F_1, \ldots, F_m)$ be the new partition of $\mathfrak{D}$ (so $F_m = E_m$ and $(F_1, \ldots, F_{m-1})$ is a permutation of $(E_1, \ldots, E_{m-1})$). Define the blocks $F_{ij} = F_i\mathcal{S}F_j$, and note that $F_{m-1,j} = 0$ for all $j < m - 1$.

If $F_{mj} = 0$ for all $j < m$, then we are done. So suppose $F_{mk} \neq 0$ for some $k < m$. If $k = m - 1$, then $F_{m-1,m} = 0$ since $\mathcal{S}$ is block triangular. If $k < m - 1$, and $F_{m-1,m} \neq 0$, then $F_{m-1,k} \neq 0$ since $\mathcal{S}$ is an algebra, a contradiction. Thus in either case $F_{m-1,m} = 0$. Now interchange F_{m-1} and F_m, relabel the partition as $\mathcal{P}'' = (G_1, \ldots, G_m)$, and define the blocks $G_{ij} = G_i\mathcal{S}G_j$. Then $G_{mj} = 0$ for all $j < m$. Finally, let $Q' = \sum_{i=1}^{m-1} G_i$ and permute $(G_1, \ldots, G_{m-1})$ so that $Q'\mathcal{S}Q'$ is upper block triangular form. Let $\mathcal{Q} = (H_1, \ldots, H_m)$ be the new partition and define the blocks $H_{ij} = H_i\mathcal{S}H_j$. $H_m = F_m$, so the bottom block row will not be affected by this permutation, i.e., $H_{mj} = 0$ for all $j < m$, and this completes the proof. ■

THEOREM 4.4. *Suppose $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is a UHF algebra with masa $\mathfrak{D}$. If $\mathcal{M}$ is a nest in $\mathfrak{D}$, $\mathcal{S} = \operatorname{Alg}\mathcal{M}$, and $\mathcal{S}_n = (\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n$, then there is a partition $\mathcal{P} = (E_1, \ldots, E_m)$ of $\mathfrak{D}_n$ such that*

(i) *$\mathcal{S}_n$ is upper block triangular with respect to $\mathcal{P}$,*

(ii) *for $i < j$, $E_i\mathcal{S}E_j$ is either full in $\mathfrak{A}_n$ or zero in $\mathfrak{A}_n$,*

(iii) *$E_i\mathcal{S}E_i$ is full in $\mathfrak{A}_n$ for all i,*

(iv) *$\mathcal{M}^c \cap \mathfrak{A}_n = \bigoplus_{i=1}^{m} E_i\mathcal{S}_nE_i$,*

(v) *if $\mathcal{M} \cap \mathfrak{D}_n = \{P_i : 0 \leqslant i \leqslant k\}$, $0 = P_0 < P_1 < \cdots < P_k = 1$, and $F_i = P_i - P_{i-1}$,*

then there are integers $0 = m_0 < m_1 < \cdots < m_k = m$ such that $F_i = \sum_{i=m_{i-1}+1}^{m_i} E_i$, and $F_i\mathcal{S}F_j$ is full in $\mathfrak{A}$ for $i < j$ and $F_i\mathcal{S}F_j = 0$ for $i > j$.

Proof. $P_i\mathfrak{A}P_i^\perp \subseteq \operatorname{Alg}\mathcal{M} = \mathcal{S}$ for all i, so $F_i\mathfrak{A}F_j = F_iP_i\mathfrak{A}P_i^\perp F_j \subseteq F_i\mathcal{S}F_j$ if $i < j$. Also, $P_j^\perp \mathcal{S} P_j = 0$ for all j, so $F_i\mathcal{S}F_j = F_iP_j^\perp \mathcal{S}P_jF_j = 0$ if $i > j$. It follows that $\mathcal{S}_n$ is upper block triangular with respect to $\{F_1, \dots, F_k\}$. We will now partition each F_i and then combine these partitions to finish the proof.

Fix ℓ, $1 \leqslant \ell \leqslant k$. Define $\mathfrak{B}_n$ to be the matrix algebra $F_\ell\mathfrak{A}_nF_\ell$. Then $\mathcal{M}^c\cap\mathfrak{B}_n$ is a C^*-subalgebra of $\mathfrak{B}_n$, so there are minimal orthogonal central projections $G_1, \dots, G_{k_\ell}$ in $\mathcal{M}^c \cap \mathfrak{B}_n$ with $\sum G_j = F_\ell$ so that $\mathcal{M}^c \cap \mathfrak{B}_n = \sum_{j=1}^{k_\ell} G_j\mathfrak{B}_nG_j$. $F_\ell\mathfrak{D}_nF_\ell$ is a masa in $\mathcal{M}^c\cap\mathfrak{B}_n$, so $G_j \in F_\ell\mathfrak{D}_nF_\ell$ for all j. Also, $\mathcal{M}^c\cap\mathfrak{B}_n \subseteq F_\ell\mathcal{S}_nF_\ell$ so $G_j\mathcal{S}G_j = G_jF_\ell\mathcal{S}F_\ell G_j$ is full in $\mathfrak{A}_n$.

Now let $\{e_{ij}\}$ be a system of matrix units for $\mathfrak{B}_n$ so that $G_1 = e_1 + \cdots + e_{t_1}$, $G_2 = e_{t_1+1} + \cdots + e_{t_2}$, etc., where $e_i = e_{ii}$. Suppose $e_{ij} \in \mathcal{S}_n$ such that $G_pe_{ij}G_q = e_{ij}$, $p \neq q$, i.e., e_{ij} is in an "off-diagonal" block of $\mathcal{S}_n$. Then $e_{ji} \notin \mathcal{S}_n$ since otherwise $e_{ij} \in \mathcal{M}^c$, a contradiction of the choice of the G_j's. Suppose e_{rs} is in the same block (i.e., $G_pe_{rs}G_q = e_{rs}$). Then $t_{p-1} + 1 \leqslant i$, $r \leqslant t_p$ so $e_{ri} \in G_p\mathfrak{A}_nG_p \subseteq \mathcal{M}^c$. Similarly, $e_{js} \in \mathcal{M}^c$, so $e_{rs} = e_{ri}e_{ij}e_{js} \in \mathcal{S}_n$ also. Therefore, $G_p\mathfrak{A}_nG_q \subseteq \mathcal{S}_n$, so $G_p\mathcal{S}G_q$ is full in $\mathfrak{A}_n$ and $G_q\mathcal{S}G_p$ is zero in $\mathfrak{A}_n$. $F_\ell\mathcal{S}_nF_\ell$ is thus a block triangular algebra in $\mathfrak{B}_n$ with respect to $\{G_1, \dots, G_{k_\ell}\}$, so the G_j's can be rearranged by Lemma 4.3 so that $F_\ell\mathcal{S}_nF_\ell$ is upper block triangular. Denote these permuted projections by $\{E_{m_{\ell-1}}, \dots, E_{m_\ell}\}$ where $m_0 = 0$ and $m_\ell - m_{\ell-1} = k_\ell$, and the result now follows. ∎

Now suppose $\mathcal{M}$ is a nest in $\mathfrak{D} \subseteq \mathfrak{A}$, and let $\mathcal{P} = (E_1, \dots, E_p)$ be the partition of $\mathfrak{D}_n$ given by Theorem 4.4. Let $\{e_{ij}^{(n)}\}$ be a system of matrix units for $\mathfrak{A}_n$ such that $E_1 = \sum_{i=1}^{k_1} e_{ii}^{(n)}$, $E_2 = \sum_{i=k_1+1}^{k_2} e_{ii}^{(n)}$, etc., and suppose $e \in \mathrm{M}_m((\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n)$ is an idempotent. Then e can viewed as a matrix (a_{ij}) with $a_{ij} = e_{ij}^{(n)} \oplus A_{ij}$, $A_{ij} \in \mathrm{M}_m(\mathbf{C})$ and $A_{ij} = 0$ if $e_{ij}^{(n)} \notin (\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n$. For each i and j, there are unique integers k and ℓ such that $E_ke_{ij}^{(n)}E_\ell = e_{ij}^{(n)}$. Define $e_t = (t^{\ell-k}a_{ij})$, $0 \leqslant t \leqslant 1$. Then each e_t is also an idempotent (by direct calculation), and lies in $\mathrm{M}_m((\operatorname{Alg}\mathcal{M}) \cap \mathfrak{A}_n)$ by Theorem 4.4. $\{e_t : 0 \leqslant t \leqslant 1\}$ is a homotopy joining $e = e_1$ to $e_0 \in \mathrm{M}_m(\mathcal{M}^c \cap \mathfrak{A}_n)$, so the same proof as Proposition 4.1, using blocks instead of matrix units, shows that $\mathrm{K}_0(\operatorname{Alg}\mathcal{M}) \simeq \mathrm{K}_0(\mathcal{M}^c)$. Similarly, if e is an invertible element in $\mathrm{M}_m((\operatorname{Alg}\mathcal{M})\cap\mathfrak{A}_n)$, then so is each e_t, and we obtain the same result for K_1. Thus, we have proved the following proposition:

PROPOSITION 4.5. *Let $\mathcal{M}$ be a nest in $\mathfrak{D} \subseteq \mathfrak{A}$. Then*

(i) $\mathrm{K}_0(\operatorname{Alg}\mathcal{M}) \simeq \mathrm{K}_0(\mathcal{M}^c)$,

(ii) $\mathrm{K}_1(\operatorname{Alg}\mathcal{M}) \simeq \mathrm{K}_1(\mathcal{M}^c) = \{0\}$.

Proposition 4.1 and 4.5 are analogues of Pitt's result [11] for K_0 of nest subalgebras of $\mathfrak{B}(\mathcal{H})$ and Peters' results [10] for K_0 and K_1 of semicrossed products.

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