J. OPERATOR THEORY
25(1991), 141–162

ON SOME REFLEXIVE LATTICES OF SUBSPACES

EDWARD KISSIN

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{L}$ be a collection of closed subspaces of a Hilbert space H containing $\{0\}$ and H which form a complete subspace lattice under the operation $\wedge$ (intersection) and $\vee$ (closed linear span). We denote by $\operatorname{Alg}\mathcal{L}$ the algebra of all bounded operators which leave all subspaces in $\mathcal{L}$ invariant and by $\operatorname{Lat}\operatorname{Alg}\mathcal{L}$ the lattice of all subspaces in H invariant under $\operatorname{Alg}\mathcal{L}$. A lattice is *reflexive* if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$.

Halmos [3] studied lattices which contain precisely five elements. Out of five isomorphism types of such lattices three (the chain, the pendulum and the pendulum upside down) have the property that every realization as a subspace lattice is reflexive. The realization of the pentagon as a subspace lattice $\mathcal{P} = \left\{\{0\}, K, M, N, H\right\}$ $M \subset N, K \wedge N = \{0\}$ and $K \vee M = H$ can be reflexive or non-reflexive. Halmos showed that if $\dim(N/M) = 1$, it is reflexive. Longstaff and Rosenthal [7] considered an example of the pentagon such that $\dim(N/M) = 2$. They proved that it is non--reflexive. Longstaff [8] (cf. Exemple 2.7) proved that for any realization $\mathcal{P}$ of the pentagon, $\operatorname{Alg}\mathcal{P}$ always contains a rank one operator.

Halmos [3] also considered realizations of the double triangle as a subspace lattice $\mathcal{T} = \left\{\{0\}, K, M, N, H\right\}$, $K\wedge M = K\wedge N = \{0\}$ and $K\vee M = K\vee N = M\vee N = H$. He established that all such realizations in finite-dimensional spaces are non-reflexive. For the infinite dimensional case Longstaff [9] showed that if any of the vector sums $K+L$, $K+M$, $L+M$ is closed, $\mathcal{T}$ is non-reflexive. He also proved that in this case $\operatorname{Alg}\mathcal{T}$ contains an operator of rank two (it was proved earlier in [8] that for any realization $\mathcal{T}$ of the double triangle, $\operatorname{Alg}\mathcal{T}$ does not contain a rank one operator). He also studied the operator double triangles $\left\{\{0\}, G(A), G(B), G(C), H\oplus H\right\}$, where A, B, C are operators on H and for an operator T, $G(T)$ denotes its graph, and proved an interesting sufficient condition that such a double triangle be non-reflexive.

In Section 3 we shall show that $\mathcal{T}$ is non-reflexive if $\operatorname{Alg}\mathcal{T}$ contains a non-zero finite rank operator. However, the question whether all realizations of the double triangle in infinite dimensional spaces are non-reflexive is still open. In Remark 3.4 we shall consider $\mathcal{T}$ such that Alg $\mathcal{T}$ does not contains a non-zero finite rank operator.

In this paper we investigate a special type of subspace lattices which we shall call s-lattices.

DEFINITION. A reflexive lattice $\mathcal{L}$ is an *s-lattice* if $\mathcal{L}$ has a non-trivial subspace K which is not comparable to any other non-trivial subspace in $\mathcal{L}$ and if Alg $\mathcal{L}$ contains a non-zero finite rank operator.

From a result of Longstaff [8] it follows (cf. Example 2.7) that Lat Alg $\mathcal{P}$ is always an s-lattice. If $\dim(N/M) = 1$, then $\mathcal{P}$ is itself an s-lattice. In Section 3 we shall show that for any realization $\mathcal{T}$ of the double triangle as a subspace lattice, such that Alg $\mathcal{T}$ contains a non-zero finite rank operator, Lat Alg $\mathcal{T}$ is also an s-lattice.

In Section 2 we establish that s-lattices, which have at least five subspaces, belong to two classes. Lattices from one class contain the double triangle as a sublattice. Lattices from another class do not contain the double triangle. Instead they contain the pentagon. Theorem 2.6 describes those s-lattices which do not contain the double triangle.

Section 3 investigates the structure of an s-lattice $\mathcal{L}$ in the most interesting case when it contains $\mathcal{T}$. Theorem 3.2 gives a detailed description of $\mathcal{L}\backslash\left\{\{0\}, H\right\}$ which is the union of disjoint segments $\mathcal{K}_t$, $t \in S^2 = \mathbb{C} \cup \infty$. It follows from Theorem 3.2 that any realization $\mathcal{T} = \left\{\{0\}, K, M, N, H\right\}$ of the double triangle such that Alg $\mathcal{T}$ contains a non-zero finite rank operator is non-reflexive.

Section 3 also establishes the structure of the algebras Alg $\mathcal{L}$ when $\mathcal{L}$ is an s--lattice which contains $\mathcal{T}$. It shows that there exist closed linear transformations F and G from $K^{\perp}$ into K such that Alg $\mathcal{L}$ is a subalgebra of the algebra $\mathcal{A}(F,G)$. (The algebras $\mathcal{A}(F,G)$ were considered in [4].) Conversely, if a subalgebra $\mathcal{A}$ of $\mathcal{A}(F,G)$ contains a non-zero finite operator and if the algebras $P\mathcal{A}P$ and $(1-P)\mathcal{A}(1-P)$ are transitive (P is the projection onto K), then Lat $\mathcal{A}$ is an s-lattice which contains $\mathcal{T}$.

The s-lattices $\mathcal{L}$ which contain $\mathcal{T}$ constitute a large class of reflexive lattices, since the structure of the segments $\mathcal{K}_t$ in $\mathcal{L}$ can be very varied. It is not even known when different segments $\mathcal{K}_t$ and $\mathcal{K}_s$, $t \neq s$, are isomorphic. Thus, in [6] an algebra $\mathcal{A}(F,G)$ was considered such that $\mathcal{K}_0$ and $\mathcal{K}_\infty$ in Lat $\mathcal{A}(F,G)$ contain only one subspace while all the other $\mathcal{K}_t$ contain at least two subspaces. In Section 4 we study s-lattices $\mathcal{L}$ which satisfy the following condition: $\mathcal{L}$ has a subspace K such that the metric distance

$$d(K,M) = \|P_K - P_M\| < 1$$

for all non-trivial M in $\mathcal{L}$ (P_K and P_M are the projections onto K and M). Using a result of Davidson and Harrison [1] about close projections, we obtain that all such lattices are isomorphic and that all segments $\mathcal{K}_t$ contain only one subspace.

Let $\mathcal{L}$ be a reflexive lattice. For M and N in $\mathcal{L}$, $M \subset N$, set

$$[M,N] = \{L \in \mathcal{L} : M \subseteq L \subseteq N\}.$$

Then $\mathcal{L}(M,N) = \{K \subseteq N \ominus M : M \oplus K \in \mathcal{L}\}$ is a subspace lattice in $N \ominus M$ isomorphic to $[M,N]$. $\mathcal{A}(M,N) = \{PAP : A \in \operatorname{Alg}\mathcal{L},\ P$ is the projection onto $N \ominus$ $\ominus M\}$ is an operator algebra on $N \ominus M$. By Theorem 4.2 [10], $\mathcal{L}(M,N) = \operatorname{Lat}\mathcal{A}(M, N)$, so that $\mathcal{L}(M,N)$ is reflexive.

Very often arbitrary subspace lattices contain segments isomorphic to s-lattices. For $L \in \mathcal{L}$, write $L_- = \vee\{M \in \mathcal{L} : M \subset L\}$ and $L_+ = \wedge\{M \in \mathcal{L} : L \subset M\}$. Then L_- and L_+ belong to $\mathcal{L}$ and $L_- \subseteq L \subseteq L_+$. If $L_- \neq L \neq L_+$ and if $\operatorname{Alg}\mathcal{L}(L_-, L_+)$ contains a non-zero finite rank operator, $\mathcal{L}(L_-, L_+)$ is an s-lattice.

It is sometimes possible to solve the problem of "synthesis" and to obtain a description of the structure of $\mathcal{L}$ when the structure of the segments $[M,N]$ is known for "sufficiently many" pairs $\{M,N\}$ in $\mathcal{L}$. Section 5 considers a large class of reflexive lattices which we call *chains of s-lattices.* Every such lattice has many segments $[M,N]$ isomorphic to s-lattices. It also has a *nest* $\mathcal{N}$ as a sublattice and all the "ends" M and N of the segments belong to $\mathcal{N}$. Theorems 5.2 and 5.3 describe the structure of $\mathcal{L}$ and of $\operatorname{Alg}\mathcal{L}$. In [4] certain reflexive operator algebras were investigated whose lattices are chains of s-lattices.

I should like to thank the referee for many helpful suggestions and for drawing my attention to papers [8] and [9]. I should also like to thank the referee and Professor W. E. Longstaff for providing me with the examples cited in Remark 3.4.

2. TWO CLASSES OF s-LATTICES

Let K be an non-trivial element of an s-lattice $\mathcal{L}$ which is not comparable to any other non-trivial subspace in $\mathcal{L}$. Set $\mathcal{A} = \operatorname{Alg}\mathcal{L}$. Then $H = K \oplus K^{\perp}$ and every operator in $\mathcal{A}$ has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$. Let P be the projection onto K. From the reflexivity of $\mathcal{L}$ we obtain the following condition:

(C_1) the algebras $\mathcal{A}_1 = P\mathcal{A}P$ and $\mathcal{A}_2 = (1-P)\mathcal{A}(1-P)$ are transitive on K and $K^{\perp}$ respectively.

If $\mathcal{L} = \Big\{\{0\}, K, H\Big\}$, then $\operatorname{Alg}\mathcal{L}$ consists of all operators $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ from $B(H)$.

If $M \neq K$ is a non-trivial subspace in $\mathcal{L}$, then

$$K \wedge M = \{0\} \text{ and } K \vee M = H, \tag{1}$$

since $K \wedge M \subseteq K \subseteq K \vee M$ and $K \wedge M, K \vee M \in \mathcal{L}$. If $z \neq 0$ belongs to M, then $z = x + y$, $x \in K^{\perp}$, $y \in K$ and $x \neq 0$. Define $Fx = y$. It is easy to see that F is a closed linear transformation from $K^{\perp}$ into K such that

$$\text{(2)} \qquad D(F) = (1-P)M \text{ is dense in } K^{\perp} \text{ and } M = \left\{ \begin{pmatrix} Fx \\ x \end{pmatrix} : x \in D(F) \right\},$$

where P is the projection on K. We shall denote such a subspace by M_F.

It easily follows that for every $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in \operatorname{Alg}\mathcal{L}$,

$$\text{(C}_2\text{)} \qquad A_{22}D(F) \subseteq D(F) \text{ and } A_{12} \mid D(F) = (FA_{22} - A11F) \mid D(F).$$

If $\mathcal{L} = \Big\{\{0\}, K, M, H\Big\}$ is the Boolean algebra with four elements then $\mathcal{L}$ is reflexive and $\operatorname{Alg}\mathcal{L}$ consists of all operators $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ in $B(H)$ which satisfy (C_2).

If $\mathcal{L}$ has more than four elements, then there are four following possibilities:

(P_1) $\mathcal{L}$ contains non-trivial subspaces M_i and N_i, $i = 1, 2$, distinct from K such that $M_1 \wedge M_2 = \{0\}$ and $N_1 \vee N_2 = H$;

(P_2) $M_1 \wedge M_2 \neq \{0\}$ for any non-trivial $M_i, i = 1, 2$, in $\mathcal{L}$ and $\mathcal{L}$ contains non-trivial subspaces $N_i, i = 1, 2$, such that $N_1 \vee N_2 = H$ (M_i and N_i are distinct from K);

(P_3) $\mathcal{L}$ contains non-trivial subspaces $M_i, i = 1, 2$, such that $M_1 \wedge M_2 = \{0\}$ and $N_1 \vee N_2 \neq H$ for all non-trivial $N_i, i = 1, 2$, in $\mathcal{L}$ (M_i and N_i are distinct from K);

(P_4) $M_1 \wedge M_2 \neq \{0\}$ and $M_1 \vee M_2 \neq H$ for all non-trivial M_1, M_2 in $\mathcal{L}$ distinct from K.

Before studying all these cases we shall consider a few technical lemmas. For all non-zero $x, y \in H$ we denote by $x \otimes y$ the rank one operator $z \mapsto (z, x)y$. Using standard arguments from linear algebra one can obtain the following lemma.

LEMMA 2.1. *Let D be a dense linear manifold in H.*

(i) *If $\{x_i\}_{i=1}^n$ are orthonormal elements in H, then there exist elements $\{z_i\}_{i=1}^n$ in D such that* $(z_i, x_j) = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$

(ii) *If A is a non-zero finite rank operator such that $AD \subseteq D$, then there exist orthonormal elements $\{z_i\}_{i=1}^n$ in H and linearly independent elements $\{y_i\}_{i=1}^n$ in D such that* $A = \sum_{i=1}^n x_i \otimes y_i$.

The following lemma was proved in [4].

LEMMA 2.2. *Let F be a closed linear transformation from H_1 into H_2 and let A and B be bounded operators on H_1 and H_2 correspondingly. Let $D \subseteq D(F)$ be such that $\mathrm{cl}(F \mid D) = F$. If $AD \subseteq D$ and if $(FA - BF) \mid D$ extends to a bounded operator C from H_1 into H_2, then*

$$B^* D(F^*) \subseteq D(F^*) \text{ and } C^* \mid D(F^*) = (A^* F^* - F^* B^*) \mid D(F^*).$$

We shall now prove a useful lemma about transitive operator algebras.

LEMMA 2.3. *Let $\mathcal{B}$ be a transitive operator algebra on H and let $\mathcal{B}$ contain a non-zero finite rank operator. Then there exists a dense linear manifold D in H which is invariant under $\mathcal{B}$ and is contained in every non-zero linear manifold of H invariant under $\mathcal{B}$.*

Proof. Let A be a non-zero finite rank operator in $\mathcal{B}$ and let $A = \sum_{i=1}^{n} x_i \otimes y_i$ where all x_i are orthonormal. By D we denote the smallest linear manifold in H invariant under $\mathcal{B}$ which contains all $\{y_i\}_{i=1}^n$. Since $\mathcal{B}$ is transitive, D is dense in H. Let D_1 be a non-zero linear manifold in H invariant under $\mathcal{B}$. Then D_1 is dense in H. By Lemma 2.1(i), there are elements $\{z_j\}_{j=1}^n$ in D_1 such that $(z_j, x_i) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$. Then

$$Az_j = \sum_{i=1}^{n} (z_j, x_i) y_i = y_i \in D_1,$$

so that $D \subseteq D_1$. The lemma is proved.

REMARK. Obviously D does not depend on the choice of the finite rank operator in $\mathcal{B}$.

Let $B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ be a non-zero finite rank operator in $\mathrm{Alg}\,\mathcal{L}$. By ($C_2$), $B_{12} \mid D(F) = (FB_{22} - B_{11}F) \mid D(F)$. If $B_{11} = B_{22} = 0$, then $B_{12} = 0$, so that $B = 0$. Therefore

(3) $$\text{either } B_{11} \neq 0 \text{ or } B_{22} \neq 0.$$

The following theorem investigates the case when $B_{22} \neq 0$.

THEOREM 2.4. *Let $\mathcal{L}$ be an s-lattice and let $B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ be a finite rank operator in $\mathcal{A} = \mathrm{Alg}\,\mathcal{L}$ such that $B_{22} \neq 0$. Let $\mathcal{A}_1 = P\mathcal{A}P$ and $\mathcal{A}_2 = (1-P)\mathcal{A}(1-P)$ where P is the projection onto K.*

(i) *Let M_1 and M_2 be non-trivial subspaces in $\mathcal{L}$ distinct from K and such that $M_1 \wedge M_2 = \{0\}$. Let F_i, $i = 1, 2$, be closed linear transformations from $K^\perp$ into K such that* $M_i = \left\{ \begin{pmatrix} F_i x \\ x \end{pmatrix} : x \in D(F_i) \right\}$. *Then*

1) $B_{11} \neq 0$;
2) *$F_2 - F_1$ is densely defined and closable;*
3) $\mathrm{Ker}(G) = \{0\}$ *and* $\mathrm{Im}(G)$ *is dense in K where* $G = \mathrm{cl}(F_2 - F_1)$;
4) *for every* $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in \mathcal{A}$ *the following condition holds:*

$$(\mathrm{C}_3) \qquad A_{22}D(G) \subseteq D(G) \text{ and } A_{11}G \mid D(G) = GA_{22} \mid D(G);$$

5) $M_1 \vee M_2 = H$.

(ii) *If $M_1 \wedge M_2 \neq \{0\}$ for all non-trivial M_1, M_2 in $\mathcal{L}$ distinct from K, then there is a non-trivial subspace $\mathcal{M}$ in $\mathcal{L}$ such that* $\mathcal{L} = \Big\{ \{0\}, K, [\mathcal{M}, H] \Big\}$.

Proof. Let $M_1 \wedge M_2 = \{0\}$. By (2) and by (C_2), $D(F_i)$, $i = 1, 2$, are dense manifolds in $K^\perp$ invariant under $\mathcal{A}_2$. Since B_{22} is a non-zero finite rank operator in the transitive algebra $\mathcal{A}_2$, by Lemma 2.3, there exists a dense linear manifold D invariant under $\mathcal{A}_2$ which is contained in $D(F_1)$ and in $D(F_2)$. Set $R = F_2 - F_1$. Then R is a linear transformation from $K^\perp$ into K, $D(R) = D(F_1) \cap D(F_2)$ is dense in $K^\perp$ and $D \subseteq D(R)$. Since $M_1 \wedge M_2 = \{0\}$, $\mathrm{Ker}(R) = 0$. It follows from (C_2) that for all $A \in \mathcal{A}$

$$(4) \qquad A_{22}D(R) \subseteq D(R) \text{ and } RA_{22} \mid D(R) = A_{11}R \mid D(R).$$

If $B_{11} = 0$, by (4), $RB_{22} \mid D(R) = 0$. Since D is invariant under $\mathcal{A}_2$, D is invariant under B_{22}. We have that $B_{22} = \sum_{i=1}^{n} x_i \otimes y_i$, where x_i and y_i, $i = 1, \ldots, n$, satisfy Lemma 2.1(ii). Choosing $\{z_j\}_{j=1}^{n}$ in D as in Lemma 2.1(i), we obtain that

$$0 = RB_{22}z_j = R\sum_{i=1}^{n}(z_j, x_i)y_i = Ry_j.$$

This contradicts the fact that $\mathrm{Ker}(R) = \{0\}$. Therefore $B_{11} \neq 0$ and 1) is proved.

We shall now prove that R is closable. From (C_2) and from Lemma 2.2 it follows that for every $A_{11} \in \mathcal{A}_1$.

$$(5) \qquad A_{11}^*D(F_1^*) \subseteq D(F_1^*) \text{ and } A_{11}^*D(F_2^*) \subseteq D(F_2^*).$$

Since B_{11} is a non-zero finite rank operator, B_{11}^* is also a non-zero finite rank operator. Since $\mathcal{A}_1$ is transitive, the algebra $\mathcal{A}_1^*$ is also transitive on K and $B_{11}^* \in \mathcal{A}_1^*$. By

Lemma 2.3, there is a dense linear manifold D_* in K invariant under $\mathcal{A}_1^*$ and contained in every linear manifold of K invariant under $\mathcal{A}_1^*$. By (5),

$$D_* \subseteq D(F1^*) \cap D(F_2^*).$$

Therefore the linear transformation $F_2^* - F_1^*$ is densely defined. Since $F_2^* - F_1^* \subseteq R^*$, R^* is densely defined. Therefore the linear transformation R^{**} is closed. Since $R \subseteq \subseteq R^{**}$, R is closable. Part 2) is proved.

Since $G = \mathrm{cl}(R)$, using the standard argument we obtain from (4) that condition (C_3) holds. Part 4) is proved.

If $\mathrm{Ker}(G) \neq \{0\}$, it follows from (C_3) that $\mathrm{Ker}\, G$ is invariant under $\mathcal{A}_2$. Hence $\mathrm{Ker}\, G$ contains D. Therefore $G \mid D = R \mid D = 0$ which contradicts the fact that $\mathrm{Ker}(R) = 0$. It follows from (C_3) that $\mathrm{Im}(G)$ is invariant under $\mathcal{A}_1$. By (C_1), $\mathrm{Im}(G)$ is either $\{0\}$ or dense in K. Since $G \neq 0$, $\mathrm{Im}(G)$ is dense in K. Part 3) is proved.

Let now $M = M_1 \vee M_2$. If $M \neq H$, then, by (1) and by (2), there exists a closed transformation F from $K^\perp$ into K such that $M = M_F$. Since M_1 and M_2 are contained in M, $D(F_i) \subseteq D(F)$ and $F \mid D(F_i) = F_i$, $i = 1, 2$. Since $D(F_1) \cap D(F_2) = = D(R)$ is dense in $K^\perp$, we obtain that

$$F \mid D(R) = F_1 \mid D(R) = F_2 \mid D(R),$$

so that $R = F_2 - F_1 = 0$. This contradiction shows that $M_1 \vee M_2 = H$ which completes the proof of (i).

Let now $M_1 \wedge M_2 \neq \{0\}$ for all non-trivial M_i, $i = 1, 2$, in $\mathcal{L}$ distinct from K. By (2), $M_i = M_{F_i}$. Since $M_1 \wedge M_2 \neq \{0\}$ and $M_1 \wedge M_2 \in \mathcal{L}$, there is a linear manifold L in $D(F_1) \cap D(F_2)$ invariant under $\mathcal{A}_2$ such that

$$F_1 \mid L = F_2 \mid L \text{ and } M_1 \wedge M_2 = \left\{ \begin{pmatrix} F_1 x \\ x \end{pmatrix} : x \in L \right\}.$$

Since $B_{22} \neq 0$, by Lemma 2.3, there exists a dense linear manifold D invariant under $\mathcal{A}_2$ which is contained in every linear manifold of H invariant under $\mathcal{A}_2$. Therefore $D \subseteq L$ and $F_1 \mid D = F_2 \mid D$.

Set $T = \mathrm{cl}(F_1 \mid D)$. Then the subspace $M_T = \left\{ \begin{pmatrix} Tx \\ x \end{pmatrix} : x \in D(T) \right\}$ is contained in every $M \in \mathcal{L} \setminus \Big\{ \{0\}, K, H \Big\} = \hat{\mathcal{L}}$. Therefore

$$\mathcal{M} = \bigwedge_{M \in \hat{\mathcal{L}}} M \supseteq M_T \neq \{0\}$$

and the theorem is proved.

Using Theorem 2.4 we shall prove the following theorem which complements it.

THEOREM 2.5. *Let $\mathcal{L}$ be an s-lattice and let* $B=\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$ *be a finite rank operator in* $\operatorname{Alg}\mathcal{L} = \mathcal{A}$ *such that* $B_{11} \neq 0$.

(i) *Let M_1 and M_2 be non-trivial subspaces in $\mathcal{L}$ distinct from K and such that $M_1 \vee M_2 = H$. Let F_1 and F_2 be closed linear transformations from $K^\perp$ into K such that $M_i = M_{F_i}$. Then $B_{22} \neq 0$, Theorem 2.4(i) holds and $M_1 \wedge M_2 = \{0\}$.*

(ii) *If $M_1 \vee M_2 \neq H$ for all non-trivial M_1, M_2 in $\mathcal{L}$ distinct from K, then there is a non-trivial subspace $\mathcal{N}$ in $\mathcal{L}$ such that $\mathcal{L} = \left\{[\{0\}, \mathcal{N}], K, H\right\}$.*

Proof. $B^* = \begin{pmatrix} B_{11}^* & 0 \\ B_{12}^* & B_{22}^* \end{pmatrix}$ is a non-zero finite rank operator in $\mathcal{A}^*$, the subspace $K^\perp$ belongs to $\mathcal{L}^* = \{L^\perp : \in \mathcal{L}\}$ and $\mathcal{L}^*$ does not have subspaces comparable to $K^\perp$. Therefore $\mathcal{L}^*$ is an s-lattice. The subspaces $M_1^\perp$ and $M_2^\perp$ belong to $\mathcal{L}^*$. If $M_1 \vee M_2 = H$, then $M_1^\perp \wedge M_2^\perp = (M_1 \vee M_2)^\perp = \{0\}$. Since $B_{11}^* \neq 0$, it follows from Theorem 2.4(i) 1) and 5) that $B_{22}^* \neq 0$ and $M_1^\perp \vee M_2^\perp = H$. Therefore

$$B_{22} \neq 0 \quad \text{and} \quad M_1 \wedge M_2 = (M_1^\perp \vee M_2^\perp)^\perp = \{0\}.$$

The rest of (i) follows from Theorem 2.4(i).

If $M_1 \vee M_2 \neq H$ for all non-trivial M_1, M_2 in $\mathcal{L}$ distinct from K, then $M_1^\perp \wedge \wedge M_2^\perp \neq \{0\}$. By Theorem 2.4(ii), there is a non-trivial $\mathcal{M}$ in $\mathcal{L}^*$ such that $\mathcal{L}^* = = \left\{\{0\}, K^\perp, [\mathcal{M}, H]\right\}$. Therefore $\mathcal{L} = \left\{[\{0\}, \mathcal{N}], K, H\right\}$, where $\mathcal{N} = \mathcal{M}^\perp$. The theorem is proved.

Now we shall use the results of Theorems 2.4 and 2.5 in order to caracterize $\mathcal{L}$ and $\mathcal{A} = \operatorname{Alg}\mathcal{L}$ in all four cases (P_1)–(P_4). As in Theorem 2.4, let $\mathcal{A}_1 = P\mathcal{A}P$ and $\mathcal{A}_2 = (1-P)\mathcal{A}(1-P)$ where P is the projection onto K.

THEOREM 2.6. *Let $\mathcal{L}$ be an s-lattice and let $\mathcal{L}$ contain at least five subspaces. Then conditions (C_1) and (C_2) hold and every operator in $\mathcal{A}$ has the form* $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, $A_{11} \in \mathcal{A}_1$, $A_{22} \in \mathcal{A}_2$.

(i) *Let the case (P_1) hold, i.e., there exist non-trivial M_i and N_i, $i = 1, 2$, in $\mathcal{L}$ distinct from K and such that $M_1 \wedge M_2 = \{0\}$ and $N_1 \vee N_2 = H$. Then*

1) *$A_{11} \neq 0$ and $A_{22} \neq 0$ for every non-zero finite rank operator in $\mathcal{A}$.*

2) *$M_1 \vee M_2 = H$ and $N_1 \wedge N_2 = \{0\}$, so that $\mathcal{L}$ contains the double triangle as a sublattice;*

3) *if $M_i = M_{F_i}$, $i = 1, 2$, then $G = \operatorname{cl}(F_2 - F_1)$ is a densely defined linear transformation from $K^\perp$ into K such that $\operatorname{Ker}(G) = \{0\}$ and $\operatorname{Im}(G)$ is dense in K; for all A in $\mathcal{A}$*

$$(\mathrm{C}_3) \qquad A_{22}D(G) \subseteq D(G) \quad \text{and} \quad A_{11}G \mid D(G) = GA_{22} \mid D(G).$$

(ii) *Let the case* (P_2) *hold,* i.e., $M_1 \wedge M_2 \neq \{0\}$ *for all* M_1 *and* M_2 *in* $\mathcal{L}$, *and there exist* N_1 *and* N_2 *in* $\mathcal{L}$ *such that* $N_1 \vee N_2 = H$ (M_i *and* N_i *are non-trivial and distinct from* K). *Then*

1) $A_{11} = 0$ *for every finite rank operator* A *in* $\mathcal{A}$, i.e., $\mathcal{A}_1$ *does not contain non-zero finite rank operators;*

2) *there is a non-trivial* $\mathcal{M}$ *in* $\mathcal{L}$ *such that* $\mathcal{L} = \Big\{\{0\}, K, [\mathcal{M}, H]\Big\}$.

(iii) *let the case* (P_3) *hold,* i.e., $N_1 \vee N_2 \neq H$ *for all* N_1 *and* N_2 *in* $\mathcal{L}$, *and there exist* M_1 *and* M_2 *in* $\mathcal{L}$ *such taht* $M_1 \wedge M_2 = \{0\}$ (M_i *and* N_i *are non-trivial and distinct from* K). *Then*

1) $A_{22} = 0$ *for every finite rank operator* A *in* $\mathcal{A}$, i.e., $\mathcal{A}_2$ *does not contain non-zero finite rank operators;*

2) *there is a non-trivial* $\mathcal{N}$ *in* $\mathcal{L}$ *such that* $\mathcal{L} = \Big\{[\{0\}, \mathcal{N}], K, H\Big\}$.

(iv) *Let the case* (P_4) *hold,* i.e., $M_1 \wedge M_2 \neq \{0\}$ *and* $M_1 \vee M_2 \neq H$ *for all non-trivial* M_1 *and* M_2 *in* $\mathcal{L}$ *distinct from* K.

1) *If* $\mathcal{A}_1$ *has and* $\mathcal{A}_2$ *does have a non-zero finite rank operator, then* (iii) 2) *holds.*

2) *If* $\mathcal{A}_2$ *has and* $\mathcal{A}_1$ *does not have a non-zero finite rank operator, then* (ii) 2) *holds.*

3) *If* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *have non-zero finite rank operators, then there are non-trivial* $\mathcal{M}$ *and* $\mathcal{N}$ *in* $\mathcal{L}$ *such that* $\mathcal{L} = \Big\{\{0\}, K, H[\mathcal{M}, \mathcal{N}]\Big\}$.

Proof. Let A be a non-zero finite rank operator in $\mathcal{A}$. By (3), either $A_{11} \neq 0$ or $A_{22} \neq 0$. Let (P_1) hold. If $A_{11} \neq 0$, part (i) follows from Theorem 2.5(i). If $A_{22} \neq 0$, part (i) follows from Theorem 2.4(i).

Let (P_2) hold. If $A_{11} \neq 0$, by Theorem 2.5(i), $N_1 \wedge N_2 = \{0\}$. Therefore $A_{11} = 0$, so that $A_{22} \neq 0$ and (ii) 2) follows from Theorem 2.4(ii). Similarly one can prove part (iii) and part (iv) 1) and 2).

If (P_4) holds and if $\mathcal{A}_1$ and $\mathcal{A}_2$ have non-zero finite rank operators, then, by Theorems 2.4(ii) and 2.5(ii), there are non-trivial $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{L}$ such that

$$\mathcal{L} = \Big\{\{0\}, K, [\mathcal{M}, H]\Big\} = \Big\{[\{0\}, \mathcal{N}], K, H\Big\}.$$

Therefore every non-trivial $M \neq K$ in $\mathcal{L}$ is contained in $[\mathcal{M}, \mathcal{N}]$ which concludes the proof of the theorem.

Remark. In the case (P_1) $\mathcal{L}$ contains the double triangle $\mathcal{T} = \Big\{\{0\}, K, M, N, H\Big\}$ as a sublattice. In the cases (P_2)–(P_4) $\mathcal{L}$ does not contain the double triangle but contains the pentagon.

Example 2.7. Let $H = K \oplus K^{\perp}$ and let F_1 and F_2 be closed and densely defined linear transformations from $K^{\perp}$ into K such that $F_1 \subset F_2$. Set $M = M_{F_1}$

and $N = M_{F_2}$. Then $M \subset N$ and $\mathcal{P} = \Big\{\{0\}, K, M, N, H\Big\}$ is the pentagon, since $N \wedge K = \{0\}$ and $M \vee K = H$. It is clear that every realization of the pentagon has this form. Set $\mathcal{A} = \operatorname{Alg}\mathcal{P}$. Then $\mathcal{A}$ consists of all $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ in $B(H)$ such that

$$A_{22}D(F_i) \subseteq D(F_i), \ \ i = 1, 2, \ \text{and} \ A_{12} \mid D(F_2) = (F_2A_{22} - A_{11}F_2) \mid D(F_2).$$

The algebra $\mathcal{A}_1 = \{A_{11} : A \in \mathcal{A}\}$ contains all rank one operators $x \otimes y$, $x \in D(F_2^*)$, $y \in K$ and the algebra $\mathcal{A}_2 = \{A_{22} : A \in \mathcal{A}\}$ contains all rank one operators $x \otimes y$, $x \in K^\perp$, $y \in D(F_1)$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are transitive algebras on K and $K^\perp$, so that $\operatorname{Lat}\mathcal{A}$ does not have non-trivial subspaces comparable to K. Hence $\operatorname{Lat}\mathcal{A}$ is an s-lattice. It is easy to show that $\operatorname{Lat}\mathcal{A} = \operatorname{Lat}\operatorname{Alg}\mathcal{P} = \Big\{\{0\}, K, H, \ [M, N]\Big\}$, so that the example corresponds to the case (P_4) 3). If $\dim(D(F_2)/D(F_1)) = 1$, then $[M, N] = \{M, N\}$, so that $\operatorname{Lat}\mathcal{A} = \mathcal{P}$, and $\mathcal{P}$ is reflexive. This case was considered by Halmos [3]. The case when $\dim(D(F_2)/D(F_1)) = 2$ was studied by Longstaff and Rosenthal [7]. They constructed the operators F_1 and F_2 in such a way that $[M, N]$ contains at least three subspaces, so that $\operatorname{Lat}\mathcal{A} \neq \mathcal{P}$ and $\mathcal{P}$ is not reflexive.

REMARK. Longstaff [8] showed earlier that if $\mathcal{P}$ is a realization of the pentagon, $\operatorname{Alg}\mathcal{P}$ always contains a non-zero rank one operator and that $\operatorname{Lat}\operatorname{Alg}\mathcal{P}$ is always an s-lattice.

3. STRUCTURE OF s-LATTICES WHICH CONTAIN THE DOUBLE TRIANGLE

Theorem 2.6(ii), (iii) and (iv) describes the structure of $\mathcal{L}$ in the case when it does not contain a sublattice isomorphic to the double triangle with least element $\{0\}$ and greatest element H. There is not much more we can say about the structure of $\mathcal{L}$ in this case. However, in the most interesting case (P_1) when $\mathcal{L}$ contains a sublattice isomorphic to the double triangle, a detailed description of $\mathcal{L}$ can be obtained. We shall do it in Theorem 3.2 using conditions (C_1), (C_2) and (C_3) which $\operatorname{Alg}\mathcal{L}$ satisfies.

Firstly, we shall prove the following lemma.

LEMMA 3.1. *Let R be a densely defined linear transformation on H and let $\mathcal{B}$ be a transitive algebra of bounded operators on H which contains a non-zero finite rank operator B. If there is a dense manifold X in $D(R)$ such that for all $A \in \mathcal{B}$*

$$AX \subseteq X \ \ \text{and} \ \ RA \mid X = AR \mid X,$$

then there exist a complex t and a dense manifold $Z \subseteq X$ such that

$$AZ \subseteq Z, \ \ \text{for all } A \in \mathcal{B} \ \ \text{and} \ \ R \mid Z = tI \mid Z.$$

Proof. By Lemma 2.1(ii), $B = \sum_{i=1}^{n} x_i \otimes y_i$ where $\{x_i\}_{i=1}^n$ are orthonormal in H and $\{y_i\}_{i=1}^n$ belong to X. Choosing $\{z_i\}_{i=1}^n$ in X as in Lemma 2.1(i), we obtain that

$$RBz_j = R\sum_{i=1}^{n}(z_j, x_i)y_i = Ry_j = BRz_j = \sum_{i=1}^{n}(Rz_j, x_i)y_i.$$

Therefore the n-dimensional subsapce H_n generated by $\{y_i\}_{i=1}^n$ is invariant under R. Hence R has an eigenvector $u \in H_n \subset X$ and $Ru = tu$. Set $Z = \{Au : A \in \mathcal{B}\}$. Then $Z \subseteq X$ and $AZ \subseteq Z$ for all $A \in \mathcal{B}$. Since $\mathcal{B}$ is transitive, Z is dense in H. For every $A \in \mathcal{B}$, $RAu = ARu = tAu$, so that $R \mid Z = tI \mid Z$. The lemma is proved.

Let now $\mathcal{L}$ be an s-lattice which satisfies (P_1) and let non-trivial subspaces M and N in $\mathcal{L}$ be distinct from K and such that $M \wedge N = \{0\}$. By (2), there are closed and densely defined linear transformations F_1 and F_2 from $K^\perp$ into K such that $M = M_{F_1}$ and $N = M_{F_2}$. Then the linear transformation $G = \mathrm{cl}(F_2 - F_1)$ satisfies Theorem 2.6(i) and $M \vee N = H$. Set $F = F_1$ and let $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$ be as in Theorem 2.6. Since

$$D(F_2) \cap D(F) \subseteq D(G) \cap D(F) \subseteq D(G),$$

we have

$$\mathrm{cl}(G \mid D(F) \cap D(G)) = G.$$

By Theorem 2.6(i), the transitive algebra $\mathcal{A}_2$ has a non-zero finite rank operator and $D(F)$ and $D(G)$ are invariant under $\mathcal{A}_2$. It follows from Lemma 2.3 that $D(F) \cap D(G)$ is dense in $K^\perp$ and invariant under $\mathcal{A}_2$. By (C_3), $G(D(F)\cap D(G))$ is invariant under $\mathcal{A}_1$. Since $\mathcal{A}_1$ is transitive and since $\mathrm{Ker}(G) = \{0\}$, $G(D(F)\cap D(G))$ is dense in K. Similarly we obtain that $D(F^*)\cap D(G^*)$ is dense in K and that $G^*(D(F^*)\cap D(G^*))$ is dense in $K^\perp$. In [4] it was proved that in this case the linear transformations $F+tG$ and $F^* + tG^*$ are closable for any complex t. Set

$$S_t = \mathrm{cl}((F + tG) \mid D(F) \cap D(G)) \text{ and } R_t = ((F^* + \bar{t}G^* \mid D(F^*) \cap D(G^*))^*.$$

Then $S_t \subseteq R_t$ and $S_0 \subseteq F \subseteq R_0$.

By Lemma 2.3, there exists the smallest dense linear manifold D in $K^\perp$ invariant under $\mathcal{A}_2$ which is contained in every linear manifold of $K^\perp$ invariant under $\mathcal{A}_2$. Similarly, there exists the smallest dense linear manifold D_* in K invariant under $\mathcal{A}_1^*$ which is contained in every linear manifold of K invariant under $\mathcal{A}_1^*$. Then $D \subseteq D(F) \cap D(G)$ and $D_* \subseteq D(F^*) \cap D(G^*)$.

For every complex t set

$$P_t = \mathrm{cl}(S_t \mid D) \text{ and } Q_t = ((F^* + \bar{t}G^* \mid D_*)^*.$$

Then

(6) $$Q_t^* = \mathrm{cl}((F^* + \bar{t}G^*) \mid D_*) \quad \text{and} \quad P_t \subseteq S_t \subseteq R_t \subseteq Q_t.$$

Set $S^2 = \mathbb{C} \cup \infty$, $\mathcal{K}_\infty = K$ and

$$\mathcal{K}_t = [M_{P_t}, M_{Q_t}] = \{L \in \mathcal{L} : M_{P_t} \subseteq L \subseteq M_{Q_t}\}, \quad t \in \mathbb{C},$$

where $M_T = \left\{ \begin{pmatrix} Tx \\ x \end{pmatrix} : x \in D(T) \right\}$ for any closed linear transformation T from $K^\perp$ into K. The following theorem gives a detailed description of $\mathcal{L}$.

THEOREM 3.2. *Let an s-lattice $\mathcal{L}$ satisfy* (P_1) *and let $M = M_{F_1}$ and $N = M_{F_2}$ be non-trivial subspaces in $\mathcal{L}$ distinct from K and such that $M \wedge N = \{0\}$.*

(i) *The linear transformations $F = F_1$ and $G = \mathrm{cl}(F_2 - F_1)$ from $K^\perp$ into K are densely defined and closed. $D(F) \cap D(G)$ and $G^*(D(F^*) \cap D(G^*))$ are dense in $K^\perp$, $D(F^*) \cap D(G^*)$ and $G(D(F) \cap D(G))$ are dense in K, $\mathrm{Ker}(G) = \{0\}$ and $\mathrm{cl}(G \mid D(F) \cap D(G)) = G$. The algebra $\mathcal{A}$ and the transformations F and G satisfy conditions* (C_1), (C_2) *and* (C_3). *There are the smallest dense linear manifolds D in $K^\perp$ invariant under $\mathcal{A}_2$ and D_* in K invariant under $\mathcal{A}_1^*$.*

(ii) $\mathcal{L} = \left\{ \{0\}, H, \bigcup_{t \in S^2} \mathcal{K}_t \right\}$. *Every $\mathcal{K}_t$, $t \in \mathbb{C}$, consists of all subspaces M_T where T is any closed linear transformation from $K^\perp$ into K such that $P_t \subseteq T \subseteq Q_t$ and that $A_{22}D(T) \subseteq D(T)$ for all $A_{22} \in \mathcal{A}_2$. In particular, T can be P_t, S_t, R_t, Q_t and F if $t = 0$.*

(iii) *All $\mathcal{K}_t$ are disjoint. Any non-trivial comparable subspaces from $\mathcal{L}$ belong to the same segment $\mathcal{K}_t$.*

Proof. Part (i) was already proved above. A subspace M_T belongs to $\mathcal{L}$ if and only if $D(T)$ is dense in $K^\perp$ and for every $A \in \mathcal{A}$

(7) $$A_{22}D(T) \subseteq D(T) \quad \text{and} \quad A_{12} \mid D(T) = (TA_{22} - A_{11}T) \mid D(T).$$

D is invariant under $\mathcal{A}_2$ and, by (C_2) and (C_3),

(8) $$(P_tA_{22} - A_{11}P_t) \mid D = (FA_{22} - A_{11}F) \mid D + t(GA_{22} - A_{11}G) \mid D = A_{12} \mid D.$$

Since $P_t = \mathrm{cl}(S_t \mid D)$, we obtain in the usual way that $D(P_t)$ is invariant under $\mathcal{A}_2$ and that (8) also holds on $D(P_t)$. Therefore $M_{P_t} \in \mathcal{L}$ for all complex t.

Let us show that $M_{Q_t} \in \mathcal{L}$. From (C_2), (C_3) and from Lemma 2.2 it follows that $A_{11}^*D(G^*) \subseteq D(G^*)$, $A_{11}^*D(F^*) \subseteq D(F^*)$ and that

(9) $$G^*A_{11}^* \mid D(G^*) = A_{22}^*G^* \mid D(G^*), \quad A_{12}^* \mid D(F^*) = (A_{22}^*F^* - F^*A_{11}^*) \mid D(F^*).$$

The linear manifold D_* is invariant under $\mathcal{A}_1^*$ and is contained in $D(F^*) \cap D(G^*)$. Therefore, by (6), (8) and (9),

$$(Q_t^* A_{11}^* - A_{22}^* Q_t^*) \mid D_* = (F^* A_{11}^* - A_{22}^* F^*) \mid D_* + \bar{t}(G^* A_{11}^* - A_{22}^* G^*) \mid D_* = -A_{12}^* \mid D_*.$$

Applying Lemma 2.2 and replacing D by D^*, F by Q_t^*, A by A_{11}^*, B by A_{22}^* and C by $-A_{12}^*$, we obtain that

$$A_{22}D(Q_t) \subseteq D(Q_t) \text{ and } A_{12} \mid D(Q) = (Q_t A_{22} - A_{11} Q_t) \mid D(Q_t).$$

Therefore, by (7), $M_{Q_t} \in \mathcal{L}$. In the same way one can prove that the subspaces M_{S_t}, M_{R_t} and M_F belong to $\mathcal{L}$ and that $M_T \in \mathcal{L}$ if T satisfies the conditions in (ii).

Suppose now that $M_T \in \mathcal{L}$, so that (7) holds. Let us show that there exists a complex t such that $P_t \subseteq T \subseteq Q_t$. Since $D \subseteq D(T)$, we obtain from (C_2) and from (7) that

$$A_{12} \mid D = (TA_{22} - A_{11}T) \mid D = (FA_{22} - A_{11}F) \mid D.$$

Therefore $(T - F)A_{22} \mid D = A_{11}(T - F) \mid D$. Since $A_{22}D \subseteq D \subseteq D(G)$, by (C_3),

$$A_{11}(T - F) \mid D = (T - F)A_{22} \mid D = (T - F)G^{-1}GA_{22} \mid D = (T - F)G^{-1}A_{11}G \mid D.$$

It follows from (C_3) that GD is invariant under $\mathcal{A}_1$. Since $\mathcal{A}_1$ is transitive and since $\mathrm{Ker}(G) = \{0\}$, GD is dense in K. Set $R = (T - F)G^{-1}$. Then

$$GD \subseteq D(R) \text{ and } A_{11}R \mid GD = RA_{11} \mid GD.$$

By Theorem 2.6(i) 1), $\mathcal{A}_1$ contains a non-zero finite rank operator. Therefore it follows from Lemma 3.1 that there exist a dense manifold $Z \subseteq GD$ and a complex t such that $R \mid Z = tI \mid Z$ and that $A_{11}Z \subseteq Z$ for all $A_{11} \in \mathcal{A}_1$. Hence

$$Y = G^{-1}Z \subseteq D \text{ and } (T - F) \mid Y = tG \mid Y.$$

By (C_3),

$$Y = G^{-1}Z \supseteq G^{-1}A_{11}Z = G^{-1}A_{11}GY = G^{-1}GA_{22}Y = A_{22}Y,$$

so that $A_{22}Y \subseteq Y$ for all $A_{22} \in \mathcal{A}_2$. Since D is the smallest invariant manifold under $\mathcal{A}_2$, $D \subseteq Y$. Therefore $D = Y$ and

$$(10) \qquad T \mid D = (F + tG) \mid D.$$

Since T is closed, $P_t \subseteq T$.

Now let us prove that $T \subseteq Q_t$. From Lemma 2.2 and from (7) we obtain that

$$A_{11}^* D(T^*) \subseteq D(T^*) \text{ and } A_{12}^* \mid D(T^*) = (A_{22}^* T^* - T^* A_{11}^*) \mid D(T^*).$$

Since $D(T^*)$ is invariant under $\mathcal{A}_1^*$, $D_* \subseteq D(T^*)$. It follows from (9)

$$A_{12}^* \mid D_* = (A_{22}^* T^* - T^* A_{11}^*) \mid D_* = (A_{22}^* F^* - F^* A_{11}^*) \mid D_*,$$

so that $A_{22}^*(T^* - F^*) \mid D_* = (T^* - F^*) A_{11}^* \mid D_*$.

Repeating the same argument as we used in order to prove (10), we obtain that there exists a complex r such that

$$T^* \mid D_* = (F^* + \bar{r} G^*) \mid D_*.$$

By (6), $Q_r^* \subseteq T^*$, so that $T \subseteq Q_r$. Let us show that $r = t$. By (6),

$$T \mid D = Q_r \mid D = P_r \mid D = (F + rG) \mid D.$$

Comparing this to (10) we obtain that $(r - t)G \mid D = 0$. Since $\operatorname{Ker}(G) = \{0\}$, $r = t$. Thus $P_t \subseteq T \subseteq Q_t$ and $M_T \in \mathcal{K}_t$.

If $L \in \mathcal{L}$ and $L \neq K$, then $L \wedge K = \{0\}$ and $L \vee K = H$. By (2), there exists a closed linear transformation T from $K^\perp$ into K such that $L = M_T$. Thus part (ii) is proved.

Finally, let $L, M \in \mathcal{L}$ and let $L \subset M$. By (ii), $L \in \mathcal{K}_t$ and $M \in \mathcal{K}_u$. Then $M_{P_t} \subseteq L \subset M \subseteq M_{Q_u}$. Therefore

$$P_t \mid D = (F + tG) \mid D = Q_u \mid D = (F + uG) \mid D,$$

so that $(u - t)G \mid D = 0$. Since $\operatorname{Ker}(G) = \{0\}$, $t = u$ which completes the proof of the theorem.

REMARK 3.3. It follows from Theorem 3.2 that any realization of the double triangle $\mathcal{T}$ such that $\operatorname{Alg}\mathcal{T}$ has a non-zero finite rank operator is non-reflexive. A particular such realization was considered by Halmos in [3]. Longstaff [9] studied operator double triangles of the form $\mathcal{T} = \left\{ \{0\}, H \oplus \{0\}, G(A), G(B), H \oplus H \right\}$ where A and B are operators on H and $G(A)$ and $G(B)$ are their graphs. He showed that if there exists a complex t such that $R(A) + R(B) \subseteq R(A - tB)$ ($R(T)$ is the range of T) and such that the image of $\operatorname{Ker}(A - tG)$ under A is not dense in H, then $\mathcal{T}$ is non-reflexive.

REMARK 3.4. We shall consider now an example of an operator double triangle $\mathcal{T} = \left\{ \{0\}, H \oplus \{0\}, G(A), G(B), H \oplus H \right\}$ in $H \oplus H$ such that $\operatorname{Alg}\mathcal{T}$ does not contain a non-zero finite rank operator. Let A and B be positive injective operators in $B(H)$ such that $R(A) \cap R(B) = \{0\}$ (see [2]). Then

$$\operatorname{Alg}\mathcal{T} = \left\{ \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} : YA = A(X + ZA) \text{ and } YB = B(X + ZB) \right\}$$

and $YR(A) \subseteq R(A)$ and $YR(B) \subseteq R(B)$. If $\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} \in \operatorname{Alg}\mathcal{T}$ and has a finite rank then Y has finite rank. Therefore $R(Y) = Y\overline{R(A)} \subseteq \overline{YR(A)} = YR(A) \subseteq R(A)$ and similarly $R(Y) \subseteq R(B)$. Thus $R(Y) \subseteq R(A) \cap R(B) = \{0\}$, so that $Y = 0$. Then $A(X+ZA) = B(X+ZB) = 0$ and, since A and B are injective, $X+ZA = X+ZB = 0$. Therefore $Z(A - B) = 0$. Since $R(A) \cap R(B) = \{0\}$, $A - B$ is also injective and selfadjoint, so that $Z = 0$. Therefore $X = 0$ and $\operatorname{Alg}\mathcal{T}$ does not contain a non-zero finite rank operator. Of course, for this $\mathcal{T}$, $\operatorname{Lat}\operatorname{Alg}\mathcal{T}$ is not an s-lattice.

In Section 2 we showed that if an s-lattice $\mathcal{L}$ contains a sublattice isomorphic to the double triangle then $\operatorname{Alg}\mathcal{L}$ satisfies conditions (C_1), (C_2) and (C_3). Using the results of Section 2 and of Theorem 3.2 one can prove the following theorem.

THEOREM 3.5. *Let $H = K \oplus K^{\perp}$ and let F and $G \neq 0$ be closed and densely defined linear transformations from $K^{\perp}$ into K. Let an operator algebra $\mathcal{A}$ have a non-zero finite rank operator, let $K \in \operatorname{Lat}\mathcal{A}$ and let F and G satisfy conditions* (C_1), (C_2) *and* (C_3). *Then* $\operatorname{Lat}\mathcal{A}$ *is an s-lattice which contains a sublattice isomorphic to the double triangle. Theorem 3.2 describes the structure of* $\operatorname{Lat}\mathcal{A}$.

DEFINITION. Let $H = K \oplus K^{\perp}$ and let densely defined and closed linear transformations F and G from $K^{\perp}$ into K satisfy Theorem 3.2(i). By $\mathcal{A}(F,G)$ we denote the algebra of *all* operators $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ in $B(H)$ which satisfy conditions (C_2) and (C_3).

For every $y \in D(F) \cap D(G)$ and every $x \in D(F^*) \cap D(G^*)$ set $A_{11} = x \otimes Gy$, $A_{22} = (G^*x) \otimes y$ and $A(x,y) = \begin{pmatrix} A_{11} & FA_{22} - A_{11}F \\ 0 & A_{11} \end{pmatrix}$. Then $A(x,y)$ are finite rank operators and they belong to $\mathcal{A}(F,G)$. From the properties of F and G it follows that the algebras $P\mathcal{A}(F,G)P$ and $(1-P)\mathcal{A}(F,G)(1-P)$ are transitive on K and $K^{\perp}$ respectively. By Theorem 3.5, $\operatorname{Lat}\mathcal{A}(F,G)$ is an s-lattice which contains a sublattice isomorphic to the double triangle and whose structure is described in Theorem 3.2.

The algebras $\mathcal{A}(F,G)$ were investigated in [4]. It was shown that $D = D(F) \cap \cap D(G)$ and $D_* = D(F^*) \cap D(G^*)$, so that $P_t = S_t$, $Q_t = R_t$ and $\mathcal{K}_t = [M_{S_t}, M_{R_t}]$, $t \in \in \mathbf{C}$. It was also proved that $\mathcal{A}(F,G)$ is reflexive if either

a) $\bigcap_{t \in \mathbf{C}} D(S_t) = D$ and $\operatorname{cl}(G \mid D) = G$, or

b) $\bigcap_{t\in\mathbb{C}} D(R_t^*) = D_*$ and $\mathrm{cl}(G^* \mid D_*) = G^*$.

From Theorems 3.2(i) and 3.5 we obtain easily the following theorem.

THEOREM 3.6. *Let an s-lattice $\mathcal{L}$ contain a sublattice isomorphic to the double triangle. Then there are linear transformations F and G from $K^\perp$ into K which satisfy Theorem 3.2(i) and such that $Alg\mathcal{L}$ is a subalgebra of $\mathcal{A}(F,G)$. Conversely, if a subalgebra of $\mathcal{A}(F,G)$ contains a non-zero finite rank operator and satisfies* (C_1), *then* $\mathrm{Lat}\,\mathcal{A}$ *is an s-lattice which contains a sublattice isomorphic to the double triangle.*

4. TOPOLOGY ON s-LATTICES

In this section we consider a certain class of subspace lattices which have a sublattice isomorphic to the double triangle with least element $\{0\}$ and greatest element H. We define them using the topology on $\mathcal{L}$. Let $\mathcal{L}$ be a reflexive lattice which has a subspace K such that $\mathrm{d}(K,M) < 1$ for any non-trivial M in $\mathcal{L}$ and such that $\mathrm{Alg}\,\mathcal{L}$ has a non-zero finite rank operator. We also assume that $\mathcal{L}$ has at least five subspaces. By $\mathcal{D}$ we denote the class of all such lattices.

We shall show that all lattices $\mathcal{L}$ in $\mathcal{D}$ are isomorphic and describe their structure.

Davidson and Harrison [1] studied some properties of close projections (P and Q are close if $\|P-Q\| < 1$). They proved the following very useful lemma.

LEMMA 4.1. *Let L, M and N be subspaces and let M and N be comparable. If the projection P_L is close to both P_M and P_N, then $M = N$.*

Let $\mathcal{L} \in \mathcal{D}$. Since $d(K,M) < 1$ for all non-trivial M in $\mathcal{L}$, K is not comparable to any other non-trivial subspace in $\mathcal{L}$. Thus $\mathcal{L}$ is an s-lattice. From Lemma 4.1 it follows immediately that if M and N are distinct non-trivial subspaces in $\mathcal{L}$, they are not comparable.

THEOREM 4.2. *Let $\mathcal{L}$ belong to the class $\mathcal{D}$.*

(i) *$\mathcal{L}$ contains a sublattice isomorphic to the double triangle.*

(ii) *There are closed and densely defined linear transformations F and G from $K^\perp$ into K which satisfy Theorem 3.2(i) and such that* $\mathrm{Alg}\,\mathcal{L} = \mathcal{A}(F,G)$.

(iii) *$S_t = \mathrm{cl}(F+tG) = R_t$ for all $t \in \mathbb{C}$, so that every $\mathcal{K}_t$ consists of only one subspace M_{S_t}. $\mathcal{L} = \left\{\{0\}, H, \bigcup_{t\in S^2} M_{S_t}\right\}$ where $M_{S_\infty} = K$. For all distinct t and u in S^2, $M_{S_t} \wedge M_{S_u} = \{0\}$ and $M_{S_t} \vee M_{S_u} = H$.*

Proof. Since non-trivial subspaces in $\mathcal{L}$ are not comparable, it follows from

Theorem 2.6 that $\mathcal{L}$ contains a sublattice isomorphic to the double triangle. By Theorem 3.4, there are linear transformations F and G which satisfy Theorem 3.2(i) and such that $\operatorname{Alg}\mathcal{L}$ is a subalgebra of $\mathcal{A}(F,G)$. Therefore $\operatorname{Lat}\mathcal{A}(F,G) \subseteq \mathcal{L}$.

By Theorems 3.2 and 3.3,

$$\operatorname{Lat}\mathcal{A}(F,G) = \left\{ \{0\}, H, \bigcup_{t \in S^2} \mathcal{K}'_t \right\} \quad \text{and} \quad \mathcal{L} = \left\{ \{0\}, H, \bigcup_{t \in S^2} \mathcal{K}_t \right\}$$

where for $t \in \mathbb{C}$,

$$\mathcal{K}'_t = [M_{S_t}, M_{R_t}] = \{M_T : S_t \subseteq T \subseteq R_t \text{ and } A_{22}D(T) \subseteq D(T) \text{ for all } A \in \mathcal{A}(F,G)\}$$

and

$$\mathcal{K}_t = [M_{P_t}, M_{Q_t}] = \{M_T : P_t \subseteq T \subseteq Q_t \text{ and } A_{22}D(T) \subseteq D(T) \text{ for all } A \in \operatorname{Alg}\mathcal{L}\}.$$

It also follows from Theorem 3.2(ii) that M_{S_t} and M_{R_t} belong to $\mathcal{K}_t$. Since we have not yet proved that $\mathcal{A}(F,G) = \operatorname{Alg}\mathcal{L}$, other subspaces from $\mathcal{K}'_t$ do not necessarily belong to $\mathcal{K}_t$. Since non-trivial subspaces in $\mathcal{L}$ are not comparable, it follows from (6) that $M_{P_t} = M_{S_t} = M_{R_t} = M_{Q_t}$. Therefore $\mathcal{K}_t = \mathcal{K}'_t$ and consists of only one subspace M_{S_t}. Thus $\operatorname{Lat}\mathcal{A}(F,G) = \mathcal{L}$, so that $\operatorname{Alg}\mathcal{L} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}(F,G)$. Since $\operatorname{Alg}\mathcal{L} \subseteq \mathcal{A}(F,G) \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A}(F,G)$, we obtain that $\operatorname{Alg}\mathcal{L} = \mathcal{A}(F,G)$. The theorem is proved.

Finally, we shall consider a sufficient topological condition for an s-lattice to contain a sublattice isomorphic to the double triangle.

THEOREM 4.3. *Let a subspace K in an s-lattice $\mathcal{L}$ be not comparable to any other non-trivial subspace in $\mathcal{L}$. If K is not isolated in $\mathcal{L}$, then $\mathcal{L}$ has a sublattice isomorphic to the double triangle.*

Proof. $\mathcal{L}$ contains at least five subspaces, since K is not isolated in $\mathcal{L}$. If $\mathcal{L}$ does not contain $\mathcal{T}$, then $\mathcal{L}$ satisfies Theorem 2.6(ii) or (iii) or (iv). Assume that $\mathcal{L}$ satisfies Theorem 2.6(ii). Then $\mathcal{L} = \left\{ \{0\}, K, [\mathcal{M}, H] \right\}$ and there exist $\{M_n\}$ in $[\mathcal{M}, H]$ such that $\|P_K - P_{M_n}\| \to 0$. Since $(1 - P_{M_n})P_{\mathcal{M}} = 0$,

$$\|(1 - P_K)P_{\mathcal{M}}\| = \|(1 - P_K)P_{\mathcal{M}} - (1 - P_{M_n})P_{\mathcal{M}}\| =$$

$$= \|(P_{M_n} - P_K)P_{\mathcal{M}}\| < \|P_{M_n} - P_K\| \to 0.$$

Therefore $P_{\mathcal{M}} = P_K P_{\mathcal{M}}$, so that $\mathcal{M} \subset K$. We obtain similar contradictions if we assume that $\mathcal{L}$ satisfies Theorem 2.6(iii) or (iv). Thus the theorem is proved.

5. CHAINS OF s-LATTICES

In this section we investigate reflexive lattices $\mathcal{L}$ which contain many segments $[M, N]$ isomorphic to s-lattices. These segments are situated in $\mathcal{L}$ in such a way that it becomes possible to obtain a description of the structure of $\mathcal{L}$. First we shall study a particular class of such lattices by imposing certain conditions on $\mathcal{L}$. At the end of the section we shall consider a much larger class of lattices by weakening one of the conditions.

We first assume that $\mathcal{L}$ contains a *complete nest* $\mathcal{N}$ ($\mathcal{N}$ is a totally ordered subspace lattice) as a sublattice and

1) every non-trivial element in $\mathcal{N}$ is *not isolated* in $\mathcal{L}$ with respect to the metric distance;

2) every element L in $\mathcal{L} \setminus \mathcal{N}$ is *comparable to all elements in $\mathcal{N}$ apart from one element* $N(L)$ which depends on L.

From condition 1) it follows that $\mathcal{L} \neq \mathcal{N}$. For every N in $\mathcal{N}$ set

$$N_- = \bigvee\{M \in \mathcal{N} : M \subset N\} \text{ and } N_+ = \bigwedge\{M \in \mathcal{N} : N \subset M\},$$

so that N_- is the immediate predecessor of N in $\mathcal{N}$ and N_+ is the immediate successor of N in $\mathcal{N}$, if $N_- \neq N \neq N_+$.

As in the case of s-lattices we also impose the following important condition:

3) for every N in $\mathcal{N}$ such that $N_- \neq N \neq N_+$, the algebra $\operatorname{Alg}\mathcal{L}(N_-, N_+)$ has a non-zero finite rank operator ($\mathcal{L}(N_-, N_+) = \{K \subseteq N_+ \ominus N_- : N_- \oplus K \in \mathcal{L}\}$).

THEOREM 5.1. *For every $L \in \mathcal{L} \setminus \mathcal{N}$, $N_-(L) \neq N(L) \neq N_+(L)$ and $N_-(L)$ is the immediate predecessor of $N(L)$ in $\mathcal{L}$ and $N_+(L)$ is the immediate successor of $N(L)$ in $\mathcal{L}$. $\mathcal{L}(N_-(L), N_+(L))$ is an s-lattice which contains $L \ominus N_-(L)$ and which contains $\mathcal{T}$ as a sublattice.*

Proof. Set $N_- = N_-(L)$ and $N_+ = N_+(L)$. Since L is comparable to all elements in $\mathcal{N}$ apart from $N(L)$, $N \subset L$ for all N in $\mathcal{N}$ such that $N \subset N(L)$. Therefore $N_- \subseteq L$. Since L is not comparable to $N(L)$, $N_- \neq N(L)$. Similarly, $L \subseteq N_+$ and $N(L) \neq N_+$. Thus $L \ominus N_-$ is contained in $\mathcal{L}(N_-, N_+)$.

Suppose that there is M in $\mathcal{L}$ such that $N_- \subset M \subset N(L)$. Then $M \in \mathcal{L} \setminus \mathcal{N}$ and $N_- \subseteq N(M) \subseteq N(L)$. Since N_- is the immediate predecessor of $N(L)$ in $\mathcal{N}$, $N(M)$ is either N_- or $N(L)$ and M is comparable to $N(M)$. This contradiction shows that N_- is also the immediate predecessor of $N(L)$ in $\mathcal{L}$. Similarly, N_+ is the immediate successor of $N(L)$ in $\mathcal{L}$. Therefore $N(L) \ominus N_-$ is not comparable to any other non-trivial subspace in $\mathcal{L}(N_-, N_+)$. Since the algebra $\operatorname{Alg}\mathcal{L}(N_-, N_+)$ has a non-zero finite rank operator, $\mathcal{L}(N_-, N_+)$ is an s-lattice.

Since $N(L)$ is not isolated in $\mathcal{L}$, for every $1 > r > 0$ there is M in $\mathcal{L}$ such that $d(N(L), M) < r$. It is clear that $M \in \mathcal{L} \setminus \mathcal{N}$. Since $N_-(M) \subset M \subset N_+(M)$,

$N_-(M) \subset N(L) \subset N_+(M)$. Therefore $M \in [N_-, N_+]$ and $d(N(L), M) = d(N(L) \ominus \ominus N_-, M \ominus N_-) < r$. Hence $N(L) \ominus N_-$ is not isolated in $\mathcal{L}(N_-, N_+)$. By Theorem 4.3, $\mathcal{L}(N_-, N_+)$ has $\mathcal{T}$ as a sublattice. The theorem is proved.

DEFINITION. We say that subspaces L and M in $\mathcal{L} \setminus \mathcal{N}$ are *linked* if either $N(L) = N_-(M)$ (so that $N_+(L) = N(M)$) or $N(M) = N_-(L)$ (so that $N_+(M) = = N(L)$). We say that subspaces L and M in $\mathcal{L} \setminus \mathcal{N}$ are *related* if either $N(L) = N(M)$ or there are subspaces $\{L_i\}_{i=1}^n$ in $\mathcal{L} \setminus \mathcal{N}$, $L = L_1$ and $M = L_n$, such that L_i and L_{i+1} are linked.

It is easy to see that this relation on $\mathcal{L} \setminus \mathcal{N}$ is in fact an equivalence relation.

We denote the set of all integers by $\mathbb{Z}$, all positive integers by $\mathbb{Z}_+$, all nonpositive by $\mathbb{Z}_-$ and all integers m such that $1 \leqslant m \leqslant k$, by $\mathbb{Z}_k$.

THEOREM 5.2. (i) *All elements in $\mathcal{L} \setminus \mathcal{N}$ are related. $\mathcal{N}$ is a discrete nest and $\mathcal{N} = \Big\{\{0\}, H\Big\} \cup \{N^i\}_{i \in I}$ where I is either $\mathbb{Z}$, or $\mathbb{Z}_+$, or $\mathbb{Z}_-$ or $\mathbb{Z}_k$ for some k. For every i in I such that $i+1 \in I$, $N^i = N_-^{i+1}$ and $N_+^i = N^{i+1}$. If I is $\mathbb{Z}_+$ or $\mathbb{Z}_k$, then $N_-^1 = \{0\}$. If $I = \mathbb{Z}_-$, then $N_+^0 = H$ and if $I = Z_k$, then $N_+^k = H$.*

(ii) *For every $i \in I$, the segment $[N_-^i, N_+^i] = S^i$ is isomorphic to an s-lattice which contains $\mathcal{T}$ as a sublattice, $S^i \cap S^{i+1} = \{N^i, N^{i+1}\}$ and $\mathcal{L} = \Big\{\{0\}, H\Big\} \cup \cup \Big(\bigcup_{i \in I} S^i\Big)$.*

Proof. Fix L in $\mathcal{L} \setminus \mathcal{N}$ and let $R(L)$ be the set of all segments in $\mathcal{L} \setminus \mathcal{N}$ related to L. Set

$$P = \bigwedge\{N(M) : M \in R(L)\} \quad \text{and} \quad Q = \bigvee\{N(M) : M \in R(L)\}.$$

Then P and Q belong to $\mathcal{N}$. If $P \neq \{0\}$, then P is not isolated in $\mathcal{L}$. There is L^1 in $\mathcal{L} \setminus \mathcal{N}$ such that $d(P, L^1) < 1$. Since $N(L^1)$ is the only element in $\mathcal{N}$ which is not comparable to L^1, $N(L^1) = P$. By Theorem 5.1, $P_+ \neq P$. Therefore $L^1 \in R(L)$.

If $P_- \neq \{0\}$, there is L^2 in $\mathcal{L} \setminus \mathcal{N}$ such that $N(L^2) = P_-$. Therefore L^2 is related to L^1 and, hence, is related to L. Hence $P \subseteq N(L^2) \subseteq P_-$. This contradiction shows that either $P = \{0\}$ or $P_- = \{0\}$. Similarly, either $Q = H$ or $Q_+ = H$.

Let $P_- = \{0\}$ and $Q_+ = H$. Then $P \neq \{0\}$ and $Q \neq H$. As above, L^1 and L are related and $N(L^1) = P$. Similarly, since $Q \neq H$, there is L^3 in $\mathcal{L} \setminus \mathcal{N}$ such that $N(L^3) = Q$ and that L and L^3 are related. Thus L^1 and L^3 are related and there are subspaces $\{M_j\}_{j=1}^k$ in $\mathcal{L} \setminus \mathcal{N}$ such that $L^1 = M_1$, $L^3 = M_k$ and that M_j and M_{j+1} are linked. Thus in this case all elements in $\mathcal{L} \setminus \mathcal{N}$ are related and

$$\mathcal{N} = \Big\{\{0\}, H\Big\} \cup \{N^i\}_{i \in Z_k}, \quad N^i = N(M_i).$$

Let now $P = \{0\}$ and $Q = H$. Since $N(M)$ is never $\{0\}$, the set $\{N(M) : : N(M) \subseteq N(L)\}$ is not finite. Therefore there is a countable sequence $\{L_i\}_{i=0}^{-\infty}$ of subspaces in $\mathcal{L} \setminus \mathcal{N}$ such that $L_0 = L$, such that all L_{i-1} and L_i are linked and such that $N(L_{i-1}) = N_-(L_i)$. If $N = \wedge N(L_i) \neq \{0\}$, then, since $P = \{0\}$, there is M related to L such that $N(M) \subset N$. Therefore there is only a finite number of distinct subspaces in $\mathcal{N}$ between $N(M)$ and $N(L)$. This contradicts the fact that all subspaces $\{N(L_i)\}_{i=0}^{-\infty}$ are distinct and lie between $N(M)$ and $N(L)$. Therefore $N = \{0\}$. Similarly, there is a countable sequence $\{L_i\}_{i=0}^{\infty}$, $L_0 = L$, of subspaces in $\mathcal{L} \setminus \mathcal{N}$ such that all L_i and L_{i+1} are linked and such that $\wedge N(L_i) = H$.

If M is a subspace in $\mathcal{L} \setminus \mathcal{N}$, there is i such that $N(M) = N(L_i)$. Therefore M is related to L and

$$\mathcal{N} = \left\{ \{0\}, H \right\} \cup \{N^i\}_{i \in \mathbf{Z}}, \quad N^i = N(L_i).$$

Similarly, one can consider two other cases:

1) $P_- = \{0\}$ and $Q = H$,

2) $P = \{0\}$ and $Q_+ = H$.

Thus part (i) is proved.

Part (ii) follows easily from part (i) and from Theorem 5.1.

REMARK. The structure of every segment $[N_-^i, N_+^i]$ was obtained in Theorem 3.2. Therefore Theorem 5.2 gives a full description of the structure of the lattices which satisfy conditions 1), 2) and 3).

DEFINITION. We shall call the lattices whose structure was described in Theorem 5.2 *chains of s-lattices*.

Let $\mathcal{L}$ be a chain of s-lattices, let $\mathcal{N} = \left\{ \{0\}, H \right\} \cup \{N^i\}_{i \in I}$ be the discrete nest in $\mathcal{L}$. Set $H_i = N^{i+1} \ominus N^i$, $H_0 = N^1$ if I is $\mathbf{Z}_+$ or $\mathbf{Z}_k$, $H_0 = H \ominus N^0$ if $I = \mathbf{Z}_-$ and $H_k = H \ominus N^k$ if $I = \mathbf{Z}_k$. Then

$$H = \sum_{j \in J}{}^{\oplus} H_j$$

where $J = I$ if I is $\mathbf{Z}$ or $\mathbf{Z}_-$ and $J = \{0\} \cup I$ if I is $\mathbf{Z}_+$ or $\mathbf{Z}_k$.

We denote by P_j the projection onto H_j and by $\mathcal{A}_j$ the algebra $P_j \operatorname{Alg} \mathcal{L} P_j$. Set

$$\mathcal{B} = \{B = (B_{ij}) \in B(H),\ i, j \in J : B_{ij} = 0 \text{ if } j - 1 \leqslant i\}$$

and

$$\mathcal{V} = \{V = (V_{ij}) \in B(H),\ i, j \in J : V_{ij} = 0 \text{ if } i > j \text{ or } j - 1 > i\},$$

i.e.,

$$V = \begin{pmatrix} \ddots & \ddots & & 0 \\ & V_{ii} & V_{ii+1} & \\ 0 & & \ddots & \ddots \end{pmatrix}.$$

From Theorems 3.2 and 5.2 we immediately obtain the following description of $\operatorname{Alg}\mathcal{L}$.

THEOREM 5.3. *Let $\mathcal{L}$ be a chain of s-lattices. Then*

(i) *$\mathcal{B} \subset \operatorname{Alg}\mathcal{L}$, $\mathcal{A}_i$ are transitive algebras on H_i and for every A in $\operatorname{Alg}\mathcal{L}$, $A = V + B$, $V \in \mathcal{V}$ and $B \in \mathcal{B}$;*

(ii) *there are closed and densely defined transformations F_i and G_i from H_{i+1} into H_i which satisfy Theorem 3.2(i) and such that for every $V \in \operatorname{Alg}\mathcal{L} \cap \mathcal{V}$*

$$V_{i+1i+1}D(F_i) \subseteq D(F_i) \text{ and } V_{ii+1} \mid D(F_i) = (F_iV_{i+1i+1} - V_{ii}F_i) \mid D(F_i)$$

and

$$V_{i+1i+1}D(G_i) \subseteq D(G_i) \text{ and } V_{ii}G_i \mid D(G_i) = G_iV_{i+1i+1} \mid D(G_i).$$

Let us now weaken the first condition on $\mathcal{L}$:

1′) every element of the form $N(L)$ in $\mathcal{N}$ is not isolated in $\mathcal{L}$ with respect to the metric distance.

Now not all elements in $\mathcal{L} \setminus \mathcal{N}$ are necessarily related to each other. Therefore $\mathcal{L} \setminus \mathcal{N}$ can be partitioned into a family of sets $\{C_u\}_{u \in U}$ which consist of related elements. For every set C_u put

$$N_-(C_u) = \wedge\{N(L) : L \in C_u\} \text{ and } N_+(C_u) = \vee\{N(L) : L \in C_u\}.$$

Using Theorem 5.1 and repeating the argument of Theorem 5.2 we obtain the following description of $\mathcal{L}$.

THEOREM 5.4. (i) *The elements $N_-(C_u)$ are comparable to all elements in $\mathcal{L}$ and the segments $[N_-(C_u), N_+(C_u)]$ are isomorphic to chains of s-lattices.*

(ii) *If $u \neq v$, then either $N_+(C_u) \subseteq N_-(C_v)$ or $N_+(C_v) \subseteq N_-(C_u)$. The set $\mathcal{L} \setminus \bigcup_{u \in U} [N_-(C_u), N_+(C_u)]$ is contained in $\mathcal{N}$ and consists of elements comparable to all elements in $\mathcal{L}$.*

REFERENCES

1. DAVIDSON, K. R.; HARRISON, K. J., Distance formulae for subspace lattices, *J. London Math. Soc.*, **39**(1989), 309–323.
2. FILLMORE, P.; WILLIAMS, J., On operator ranges, *Adv. in Math.*, **7**(1971), 254–281.
3. HALMOS, P. R., Reflexive lattices of subspaces, *J. London Math. Soc. (2)*, **4**(1971), 257–263.
4. KISSIN, E. V., On some reflexive operator algebras constructed from two sets of closed

operators and from a set of reflexive algebras, *Pacific J. Math.*, **126**(1987), 125–143.
5. KISSIN, E. V., On lattices of invariant subspaces of operator algebras, *J. Operator Theory*, **18**(1987), 373–392.
6. KISSIN, E. V., On derivations of operator algebras implemented by bounded operators, in *Proceedings AMS Summer Research Institute on Operator Theory/Operator Algebras*, 1989, to appear.
7. LONGSTAFF, W. E.; ROSENTHAL, P., On two questions of Halmos concerning subspace lattices, *Proc. Amer. Math. Soc.*, **75**(1979), 85–86.
8. LONGSTAFF, W. E., Strongly reflexive lattices, *J. London Math. Soc. (2)*, **11**(1975), 491–498.
9. LONGSTAFF, W. E., Non-reflexive double triangles, *J. Austral. Math. Soc. Ser. A*, **35**(1983), 349–356.
10. RADJAVI, H.; ROSENTHAL, P., *Invariant subspaces*, Springer-Verlag, Berlin, 1973.

EDWARD KISSIN
Department of Mathematics, Statistics and Computing,
The Polytechnic of North London,
Holloway, London N7 8DB,
Great Britain.

Received July 27, 1989; revised December 20, 1989 and March 6, 1990.