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ON COMPACT OPERATORS IN CERTAIN REFLEXIVE OPERATOR ALGEBRAS

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One of the most interesting questions involving C.S.L. algebras is the question of when a C.S.L. algebra contains a non-zero operator. Since it seems difficult to have a complete answer to this question, many authors have restricted their attention either to concrete classes of compact operators (namely the well known von Neumann--Schatten classes $(C_p, \|\cdot\|_p)$, $1 \leqslant p < +\infty$; we denote by C_∞ the colection of all compact operators) or to concrete classes of C.S.L. algebras. In this direction, many examples have been obtained, describing the quantity and the quality of compact operators contained in such algebras (see [3], [6], [7]).

The purpose of this article is to add some more flavor to the general problem by showing that for a certain class of C.S.L. algebras the appearance of compact operators is closely related to the abundance of rank-one operators. Moreover, some related results are obtained.

In this paper, will use the following notation. If $(X, \|\ \|)$ is a Banach space, then the set of all bounded operators on X will be denoted by $B(X)$. If $(X, \|\ \|)$ is a Banach space and Y a subset of its dual X^*, then the weakest topology τ wich makes every element of Y a continuous linear functional on (X, τ) will be denoted by $\sigma(X, Y)$.

The terms Hilbert space and subspace will be used to mean separable Hilbert space and closed subspace respectively. A commutative subspace lattice (abr. C.S.L.) L, acting on a Hilbert space H, is a lattice of commuting projections in H, containing 0 and I, which is closed in the strong operator topology of $B(H)$. By $\mathrm{Alg}\, L$, we will denote the (strongly closed, unital) subalgebra of $B(H)$, consisting of all operators A leaving invariant the range of each projection P in L, i.e., $AP = PAP$ for all P in L. By $R_1(L)$ we will denote the algebra of all finite rank operators which can be written as a finite sum of rank one operators belonging to $\mathrm{Alg}\, L$. A nest algebra is an algebra of the form $\operatorname{Alg} N$, where N, is a totally ordered C.S.L., that is, a nest.

Let L a C.S.L. and let P be a projection in L. We define

$$P_- = \vee\{Q \in L : P \not\subseteq Q\}$$

$$P_* = \wedge\{Q_- : Q \in L,\ Q \not\subseteq P\}.$$

A C.S.L. L is called *completely distributive* if $P_* = P$ for every P in L. There is a standard lattice-theoretic definition of complete distributivity which Longstaff has shown equivalent to this one (see [9]).

If N is a nest and F an arbitrary finite sublattice of N, we write

$$U_F(A) = \sum_{P \in F} PA\Delta_F(P)$$

$$D_F(A) = \sum_{P \in F} \Delta_F(P)A\Delta_F(P)$$

$$L_F(A) = \sum_{P \in F} (P - \Delta_F(P))A\Delta_F(P), \quad A \in B(H)$$

where

$$\Delta_F(P) = P - (\vee\{Q \in F : Q \subset P \text{ but } Q \neq P\}).$$

As F varies over the finite sublattices of N, the sets $\{U_F\}_F$, $\{D_F\}_F$, $\{L_F\}_F$ form nets of idempotents from $B(H)$ onto $\operatorname{Alg} F$, F' and $\operatorname{Rad} \operatorname{Alg} F$ respectively. We now quote two theorems about the properties of these nests to be found in [4].

THEOREM 1. *Let N be a nest. Then, for every operator K in C, $1 < p < +\infty$, the nets $\{U_F(K)\}_F$, $\{D_F(K)\}_F$, and $\{L_F(K)\}_F$ are $\|\cdot\|_p$-convergent to operators $U_N(K)$, $D_N(K)$ and $L_N(K)$ respectively. Moreover,*

$$D_N(K) = \sum_{E \in N} (E - E_-)K(E - E_-).$$

THEOREM 2. *Let N be a nest and let K be a compact operator in* $\operatorname{Alg} N$. *Then*

$$K = L_N(K) + D_N(K).$$

In this note we will consider only commutative subspace lattices which are generated by a completely distributive C.S.L. L_0 and finitely many commuting nests N_i, $1 \leqslant i \leqslant k$ (for such lattices L we write $L = L_0 \vee N_1 \vee .. \vee N_k$). We begin our exposition by giving a short transparent proof of results of Davidson and Pitts [1]. First we need a proposition that explores the question of whether such algebras can fail to contain rank-one operators.

PROPOSITION 3. *If L is a completely distributive lattice and N a nest commuting with L, then $L \vee N$ contains a non-atomic Boolean algebra if and only if* $\mathrm{Alg}(L \vee N)$ *contains no rank-one operators.*

Proof. One direction is trivial. Let us assume that Alg $(L \vee N)$ does not contain any rank-one operators. We will show that, for every $E \in N$, we have $E^{\perp} \in L$.

Suppose that $Q \in L$ and $E \in N$. The definition of E_- implies that, for any operator A in $B(H)$, the operator $EQAQ_-^{\perp}E_-^{\perp}$ is in both $\mathrm{Alg}\,L$ and $\mathrm{Alg}\,N$, and thus lies in $\mathrm{Alg}(L \vee N)$. (A remark: the subscript "$-$" is defined with reference to a particular lattice. We mean for E_- to be computed in the lattice N and Q_- in the lattice L.) Consequently, for any rank-one operator R, we have $EQRQ_-^{\perp}E_-^{\perp} = 0$ (because $\mathrm{Alg}(L \vee N)$ contains no rank-one operators). This means that, whenever $E \in N$ and $Q \in L$, either $EQ = 0$ or $(E_- \vee Q_-) = I$.

Let E be a projection in N, and let P be the greatest projection in L that is orthogonal to E, that is, $P = \vee\{Q \in L : EQ\}$. Then $E^{\perp} \supseteq P$, but we also have

$$P = P_* = \wedge\{Q_- : Q \in L \text{ and } Q \not\subseteq P\} =$$

$$= \wedge\{Q_- : Q \in L \text{ and } QE \neq 0\} \supseteq$$

$$\supseteq \wedge\{Q_- : Q \in L \text{ and } E_-^{\perp} \subseteq Q_-\} \supseteq E_-^{\perp} \supseteq E^{\perp}.$$

Thus, $E^{\perp} = P$ and the conclusion follows. ■

THEOREM A [1]. *Let L be a commutative subspace lattice generated by a completely distributive* C.S.L. *and finitely many commuting nests. Then L is compact in the strong operator topology of $B(H)$ if and only if L is completely distributive.*

Proof. One direction is well known [11, Corollary 2.6]. Let us assume that $L = L_0 \vee N$ (where L_0 is completely distributive and N is a nest) and, moreover, that L is compact without being completely distributive. Theorem 3.1 in [10] implies the existance of a semi-invariant projection P in L'' such that the algebra $\mathrm{Alg}(PL) = P\mathrm{Alg}\,L|_{P(H)}$ contains no rank one operators. Hence, by Proposition 1, the lattice $PL = PL_0 \vee PN$ contains a non-atomic Boolean algebra, i.e., PL is not compact, a conclusion which contradicts the fact that a compression of a compact lattice must be compact.

The general case follows inductively from the fact that complete distributivity and compactness are hereditary for complete sublattices. ■

Proposition 3 implies immediately that for lattices of the form $L = L_0 \vee N$, where L_0 is a completely distributive C.S.L. and N is a nest, the appearance of non-zero compact operators in $\operatorname{Alg} L$ is equivalent to the existence of non-zero rank one operators. This fact can be also deduced from the following, more general result.

THEOREM B. *Let L be a commutative subspace lattice generated by a completely distributive* C.S.L. *and finitely many commuting nests N_i, $1 \leqslant i \leqslant k$. Then every compact operator in* $\operatorname{Alg} L$ *is the norm limit of finite sums of rank one operators belonging to* $\operatorname{Alg} L$.

Proof. First we ensure that the statement is true for completely distributive lattices. Indeed, it is known (see [8]) that for every completely distributive lattice L_0, the algebra $R_1(L_0)$ is w*-dense in $\operatorname{Alg} L_0$. Hence, using elementary Banach space theory, we have:

$$\overline{R_1(L_0)}^{\|\ \|} = \overline{R_1(L_0)}^{\sigma(C_\infty, C_1)} =$$
$$= \overline{R_1(L_0)}^{\sigma(B(H), C_1)} \cap C_\infty = \operatorname{Alg} L \cap C_\infty,$$

and the conclusion follows.

We now proceed to the general case; the proof follows by induction on k.

If $k = 0$, we have just proved the result. Let us assume that the statement is true for $k = n-1$ and let L be a C.S.L. generated be a completely distributive C.S.L. L_0 and finitely many commuting nests N_i, $1 \leqslant i \leqslant n$.

If K is an arbitrary compact operator in $\operatorname{Alg} L$, K belongs to $\operatorname{Alg} N_n$ and so, by Theorem 2, we have:

$$K = \sum_{E \in N_n} (E - E_-)K(E - E_-) + L_{N_n}(K).$$

The induction hypothesis now implies that every operator of the form $(E - -E_-)K(E - E_-)$ can be approximated by elements of the set $R_1(L)$; hence the same is true for $D_{N_n}(K)$.

It remains to prove that $L_{N_n}(K)$ belongs to $\overline{R_1(L)}^{\|\ \|}$. Let ε be a positive number. Since $L_{N_n}(K)$ is the norm limit of the net $\{L_F(K)\}_F$, where F ranges over all finite sublattices of N_n, there exists a finite nest $F_\varepsilon \subseteq N_n$ such that:

$$\| L_{N_n}(K) - L_{F_\varepsilon}(K) \| < \varepsilon/2.$$

On the other hand, L_{F_ε} defines a bounded idempotent on $B(H)$ and hence there exists an element K_ε of $R_1(L_0 \vee N_1 \cdots \vee N_{n-1})$ such that

$$\| L_{F_\varepsilon}(K) - L_{F_\varepsilon}(K_\varepsilon) \| < \varepsilon/2.$$

Thus, the finite rank operator $L_{F_\varepsilon}(K_\varepsilon)$ is ε-close to $L_{N_n}(K)$. The rest of the proof follows from the fact that

$$L_{F_\varepsilon}(K_\varepsilon) \in \mathrm{Alg}(L_0 \vee N_1 \vee \cdots \vee N_{n-1}) \cap \mathrm{Rad}\,\mathrm{Alg}\, F_\varepsilon \subseteq \mathrm{Alg}\, L \ ;$$

the last inclusion holds because $\mathrm{Rad}\,\mathrm{Alg}\, F_\varepsilon$ consists of strictly upper triangular operators and is therefore a subset of $\mathrm{Alg}\, N_n$. ■

REMARKS. 1. We observe that Theorem B gives the finishing touch to the work in [6]; in a finite width C.S.L. algebra $\mathrm{Alg}\, L$ every finite rank operator can be approximated by finite rank operators belonging to $\mathrm{Alg}\, L$.

2. The theorem above is not true for other commutative subspace lattices; see the work of Froelich ([3], for example), or Example 4 in [5]. In fact, our theorem shows that the lattices in these examples are not of the type described in Theorem B.

COROLLARY 4. *If L is a commutative subspace lattice generated by a completely distributive* C.S.L. *L_0 and finitely many commuting nests N_i, $1 \leqslant i \leqslant k$, then every C_p-operator in* $\mathrm{Alg}\, L$, $1 < p < +\infty$, can be approximated, in the corresponding $\|\cdot\|_p$-norm, by finite rank operators belonging to $\mathrm{Alg}\, L$.

Proof. The proof follows the lines of the proof of Theorem B except for the fact that at one point has to use a result of Davidson and Power (Proposition 2.6 in [2]). ■

The following gives a complete answer to the question that motivates the work of Laurie in [7].

COROLLARY 5. *Let L be as in Corollary 4. Then L is contained in the invariant subspace lattice of a non zero compact operator if and only if there exists a projection P in L such $P_- \neq I$.*

Proof. The conclusion follows immediately from Theorem B and a well known Proposition due to Longstaff (Lemma 3.1 in [9]). ■

Can Theorems A and B be extended to commutative subspace lattices that are generated by finitely many completely distributive lattices (instead of nests)? The answer is not known, but a proof would surely shed on the structure of such lattices.

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