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TYPE III FACTOR STATES ON O_2 WHICH EXTEND THE TRACE ON CHOI'S ALGEBRA

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INTRODUCTION

A fundamental fact in the theory of C^*-algebras is that the restriction map from the state space of a C^*-algebra to the state space of a C^*-subalgebra is surjective (where for simplicity we assume that the two algebras share the unit element). Recently it has been proved in the separable case that the same is true if we write "factor state space" instead of "state space" in the above. A natural endeavor, then, is to investigate the richness of the preimage of a fixed factor state of the subalgebra (see [2] for a more general discussion of this problem). A very beautiful example is that of Choi's algebra, viewed as a subalgebra of the Cuntz algebra O_2 ([3], [9]). There is a unique tracial state on Choi's algebra, which is a type II_1 factor state. Evans showed that a type $\mathrm{III}_{\frac{1}{2}}$ factor state of O_2 can be constructed by using the crossed-product decomposition of O_2, and Lance showed that this state extends the tracial state of Choi's algebra (see [2], Section 4). In [9], pure states of O_2 are constructed which extend the trace on Choi's algebra. In this paper we construct, for each $\lambda \in [0,1]$, a type III_λ factor state of O_2 which extends the trace on Choi's algebra. In the case $\lambda = 0$ our construction yields uncountably many non-isomorphic factors.

Our method extends the method of Evans and Lance. The case $\lambda = \frac{1}{2}$ can be obtained easily by starting with the trace on a UHF algebra, and constructing the crossed product of a type II_∞ factor by an automorphism scaling the trace by $\frac{1}{2}$. However for general λ we must start with a general product state on a UHF algebra, and hence analyze the crossed product of a type III factor by an automorphism. The analysis in this case is considerably more difficult.

The paper is divided into three sections. In Section 1 we present the construction of the states on O_2 which are the basic objects of study. In Section 2 we show that the states so obtained extend the tracial state of Choi's algebra. The proof is combinatorial. It relies on the computation of the zeroeth "Fourier" coefficient in O_2 of certain elements in Choi's algebra. In Section 3 we show that the basic construction yields type III_λ factor states on O_2. We rely heavily on the structure theory of factors due to Connes and Takesaki, et. al.

As Section 3 is much longer then Sections 1 and 2, we briefly outline its contents here. It begins with an infinite tensor product factor constructed from what we call a weight sequence (which is equivalent to the eigenvalue list of [1]). We then show that two weight sequences which are close in an appropriate sense give equivalent results. We use this to give a sufficient condition on a weight sequence for the shift on the factors of the infinite tensor product to extend to an automorphism of the factor, i.e. for the weight to be quasi-invariant under the automorphism. This allows us to work exclusively with the von Neumann algebras, rather than with the C^*-algebra O_2. Restricting to the non-type I case, we prove that the automorphism is outer, and from this that the crossed product is always a type III factor. The heart of the section is in Propositions 3.21 and 3.23, where we compute the modular spectrum and modular period of the crossed product from the same invariants for the original infinite tensor product factor. The key tool is a certain central sequence with a very strong property relative to the shift automorphism (Lemma 3.16).

We remark that in Section 3 of this paper we work only with weights on the underlying AF-algebra which are quasi-invariant under the shift automorphism. We have a paper in preparation, in which we study the factors obtained from non-quasi--invariant weights on the AF-algebra. In this case, which is easier than the quasi--invariant case, our results provide examples of type II_∞ factor states of O_2 which extend the trace on the Choi's algebra, as well as other examples of type III factor state extensions. These, together with the results of the present paper, solve the problem posed in the introduction of [9].

We mention that for $\lambda = 1$ or λ a root of an equation $\lambda^n + \lambda^{n+1} - 1 = 0$ for some integer n, the result has been obtained independently by Archbold, Lazar, Tsui and Wright ([15]). Finaly, we wish to thank John Phillips and Ian Putnam for many helpful discussions on this problem, and Rob Archbold for pointing out several errors in the preprint of this paper.

WEIGHT SEQUENCES AND STATES ON O_2

For $n = 0, 1, 2, \ldots$ let $A_n = \bigotimes_{j=-n}^{\infty} M_2^{(j)}$, where $M_2^{(j)} = M_2(\mathbb{C})$ for each j. For an element b in $M_2(\mathbb{C})$ we write $b^{(j)}$ for the same element viewed in $M_2^{(j)}$. Let $i_n : A_n \to A_{n+1}$ be given by: $i_n(x) = e_{11}^{(-n-1)} \otimes x$. Via the $\{i_n\}$ we view the $\{A_n\}$ as nested. Put $A = \left(\bigcup_n A_n\right)^-$. If $k \leqslant l$ with $|k| \leqslant n$, and $b_k, b_{k+1}, \dots, b_l \in M_2(\mathbb{C})$ we write $\bigotimes_{j=k}^{l} b_j^{(j)}$ for the element

$$\left(\bigotimes_{j=-n}^{k-1} e_{11}^{(j)}\right) \otimes \left(\bigotimes_{j=k}^{l} b_j^{(j)}\right) \otimes \left(\bigotimes_{j=l+1}^{\infty} 1^{(j)}\right)$$

in A_n. Let $B_n = \bigotimes_{j=-n}^{n} M_2^{(j)}$ for $n = 0, 1, 2, \dots$. Then B_n is a unital subalgebra of A_n, and $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$. Let $B = \bigcup_n B_n$. Note that $A = \overline{B}$, and for each n, $A_n = (A_n \cap B)^-$. Let e_n be the identity element of A_n. (We remark that in the above notation, $e_n = 1^{(-n)}$.) Then $\{e_n\}$ is an approximate unit for A consisting of projections. Define $\alpha_n : B_n \to B_{n+1}$ by $\alpha_n\left(\bigotimes_j b_j^{(j)}\right) = \bigotimes_j b_{j-1}^{(j)}$. As $B_{n-1} \subseteq \alpha_n(B_n)$, it is clear that $\{\alpha_n\}$ extends to an automorphism of A. Note that $\alpha(B) = B$. We will let $E : A \times_\alpha \mathbb{Z} \to A$ denote the canonical conditional expectation. Let $D = e_0(A \times_\alpha \mathbb{Z})e_0$. As in [6], D is isomorphic to O_2: D is generated by isometries $S_1 = Ue_0$, $S_2 = e_{21}^{(0)} S_1$ satisfying $S_1S_1^* + S_2S_2^* = 1$, where $U \in A \times_\alpha \mathbb{Z}$ is the unitary element implementing α. Elements in the finite dimensional subalgebras $\{A_0 \cap B_n\}$ approximating A_0 are obtained by

$$e_{p_0,q_0}^{(0)} \otimes \dots \otimes e_{p_n,q_n}^{(n)} = S_{p_0}S_{p_1}\dots S_{p_n}S_{q_n}^* \dots S_{q_1}^* S_{q_0}^*.$$

Every element z in the $*$-algebra generated by S_1 and S_2 has a unique expression of the form

$$z = \sum_{n<0} (S_1^*)^{-n} F_n(z) + F_0(z) + \sum_{n>0} F_n(z) S_1^n,$$

where $F_n(z) \in A_0$ for all n. The $\{F_n\}$ extend to bounded linear maps $F_n : D \to A_0$, and F_0 is a faithful conditional expectation ([6]). Notice that

$$F_0 = E \mid D. \tag{1.1}$$

We now give the basic construction of the paper.

1.2. DEFINITION. By a *weight sequence* we mean an integer j_0 and a sequence $\{t_j\}_{j \geqslant j_0}$ with $t_j \in [0,1]$ for each j.

Given a weight sequence $\{t_j\}_{j \geqslant j_0}$ we define for $j \in \mathbb{Z}$ elements $\Lambda_j \in M_2(\mathbb{C})$ by

$$\Lambda_j = \begin{cases} \operatorname{diag}\{t_j, 1 - t_j\}, & j \geqslant j_0 \\ 1, & j < j_0. \end{cases}$$

For $n \geqslant 0$ define $f_n \in (A_n)^*_+$ by

$$f_n\Big(\bigotimes_{j=-n}^{\infty} b_j^{(j)}\Big) = \prod_{j=-n}^{\infty} \operatorname{Tr}(A_j b_j),$$

for $x = \otimes b_j^{(j)}$ in $A_n \cap B$, and where Tr is the usual trace on $M_2(\mathbb{C})$, taking value 1 on rank-one projections. Then for $n \geqslant |j_0|$ and for $x \in A_n \cap B$ we have $f_{n+1}(x) = f_n(x)$. It follows that

$$f_{n+1} \mid A_n = f_n, \text{ for } n \geqslant j_0. \tag{1.3}$$

1.4. LEMMA. *For $x \in A_+$ and $n \geqslant |j_0|$, $f_n(e_n x e_n) \leqslant f_{n+1}(e_{n+1} x e_{n+1})$.*

Proof. This follows from the Cauchy-Schwartz inequality, the fact that for $y \in A_{n+1}$, we have $f_{n+1}(e_n y) = f_{n+1}(y e_n)$, and (1.3). ■

1.5. DEFINITION. Given a weight sequence $\{t_j\}_{j \geqslant j_0}$, define a weight f on A by $f(x) = \sup_n f_n(e_n x e_n)$, $x \in A_+$.

1.6. LEMMA. *The weight f defined in 1.5 is densely defined and lower semi--continuous. If additionally, $t_j \in (0,1)$ for each j, then f is faithful.*

Proof. Dense definition follows from (1.3). Lower semi-continuity follows from the definition of f. If $t_j \in (0,1)$ for each j, then faithfulness follows from the facts that each f_n is faithful on A_n ([10], 6.5.9), that $\{e_n\}$ is an approximate unit, and Lemma 1.4. ■

We note that if $x = \bigotimes_{j=-\infty}^{\infty} x_j^{(j)}$ is in B, then it follows from the definition of f that

$$f(x) = \prod_{j=-\infty}^{\infty} \operatorname{Tr}(A_j x_j). \tag{1.7}$$

Given a weight sequence, $\{t_j\}_{j \geqslant j_0}$, and hence a weight f on A, let $\tilde{f} = f \circ E$ and $g = \tilde{f} \mid D$. Then $\tilde{f}$ is a densely defined lower semi-continuous weight on $A \times_\alpha \mathbf{Z}$, and g is a positive linear functional on D. It is clear that g is a state, i.e. $g(e_0) = 1$, if $j_0 = 0$.

THE RESTRICTION TO CHOI'S ALGEBRA

In this section we show that if g is a state of O_2 constructed from a weight sequence as in Section 1, then the restriction of g to Choi's algebra is the (unique) tracial state of Choi's algebra.

NOTATION. Throughout this section we will write

$$(p_0, q_0) \otimes \ldots \otimes (p_n, q_n)$$

for the element $\bigotimes_{j=0}^{n} e^{(j)}_{p_j,q_j}$ in A_0.

Recall from [3] that Choi's algebra $C = C^*_r(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a subalgebra of O_2 as follows: C is isomorphic to $C^*(u,v)$ where u and v are unitary elements of orders 2 and 3, respectively, given by

$$u = S_1S_2^* + S_2S_1^* = (1,2) + (2,1)$$

(2.1) $$v = S_1S_2^*S_2^* + S_2S_2S_1^*S_2^* + S_2S_1S_1^* =$$

$$= S_1^*((1,2)\otimes(1,2)) + (2,2)\otimes(2,1) + ((2,1)\otimes(1,1))S_1.$$

2.2. PROPOSITION. *If $z \neq 1$ is an element of the group generated by u and v, and g is a state of O_2 obtained from a weight sequence as in Section 1, then $g(z) = 0$.*

The proof is divided into a number of lemmas. We remark that the only group-theoretic relations satisfied by u and v are $u^2 = 1$, $v^3 = 1$.

NOTATION. In this section the letters m, p and q will take values in the set $\{1,2\}$. We will write $\hat{}$ for the non-trivial permutation of $\{1,2\}$.

2.3. LEMMA. *Let $a \in A_0$. Then*

(i) $aS_1^* = S_1^*((1,1)\otimes a)$,

(ii) $S_1 a = ((1,1)\otimes a)S_1$,

(iii) $S_1 a S_1^* = (1,1)\otimes a$,

(iv) $S_1^* a S_1 = \begin{cases} a', & \text{if } a = (1,1)\otimes a' \\ 0, & \text{if } a = (p,q)\otimes a' \text{ with } p \text{ or } q \text{ equal to } 2. \end{cases}$

Proof. This is easily checked using the fact that $(1,1)\otimes a = \alpha(a)$. ■

2.4. LEMMA. *Let $z = uv^m$. Then $z = S_1^*F_{-1}(z) + F_0(z) + F_1(z)S_1$, where*

$$F_{-1}(z) = (1,2)\otimes(2,\hat{m}),$$
$$F_0(z) = (1,2)\otimes(\hat{m},m),$$
$$F_1(z) = (1,1)\otimes(m,1).$$

Proof. This is a straightforward verification using the equations (2.1) and Lemma 2.3. ■

2.5. LEMMA. *Let* $z_1 = uv^{m_i}$, $i = 1, 2$. *Then*

$$F_1(z_1)S_1 \cdot S_1^* F_{-1}(z_2) = 0,$$
$$F_0(z_1) \cdot F_1(z_2)S_1 = 0,$$
$$F_0(z_1) \cdot F_0(z_2) = 0,$$
$$S_1^* F_{-1}(z_1) \cdot F_0(z_2) = 0,$$
$$S_1^* F_{-1}(z_1) \cdot F_1(z_2)S_1 = 0.$$

Proof. This follows immediately from Lemma 2.4. ■

2.6. LEMMA. *Let* $z = uv^{m_1}uv^{m_2}\dots uv^{m_r}$, $r > 0$. *Then* $F_0(z) = 0$ *if* r *is even, and if* $r = 2d-1$, *then*

$$F_0(z) = (1,2) \otimes (m_1, \hat{m}_r) \otimes (m_2, \hat{m}_{r-1}) \otimes \cdots \otimes (m_{d-1}, \hat{m}_{d+1}) \otimes (\hat{m}_d, m_d).$$

Proof. By Lemma 2.4 we have

$$z = \prod_{i=1}^{r} [S_1^* F_{-1}(uv^{m_i}) + F_0(uv^{m_i}) + F_1(uv^{m_i})S_1].$$

Lemmas 2.5 and 2.3 imply that upon expanding the product, only one term contributes to $F_0(z)$. We obtain $F_0(z) = 0$ if r is even, and if $r = 2d-1$ then

$$F_0(z) = \left[\prod_{i=1}^{d-1} (F_1(uv^{m_i})S_1)\right] [F_0(uv^{m_d})] \left[\prod_{i=d+1}^{r} (S_1^* F_{-1}(uv^{m_i}))\right].$$

We proceed by induction on d. Lemma 2.4 establishes the case $d = 1$. Suppose the result is true for $d-1$. Then

$$F_0\left(\prod_{i=2}^{r-1} uv^{m_i}\right) = (1,2) \otimes (m_2, \hat{m}_{r-1}) \otimes \cdots \otimes (m_{d-1}, \hat{m}_{d+1}) \otimes (\hat{m}_d, m_d).$$

Hence Lemmas 2.5 and 2.3 imply that

$$F_0(z) = F_1(uv^{m_1})S_1 \cdot F_0\left(\prod_{i=2}^{r-1} uv^{m_i}\right) \cdot S_1^* F_{-1}(uv^{m_r}) =$$

$$= [(1,1) \otimes (m_1, 1)]S_1[(1,2) \otimes (m_2, \hat{m}_{r-1}) \otimes \cdots \otimes (\hat{m}_d, m_d)]S_1^*[(1,2) \otimes (2, \hat{m}_r)] =$$

$$= [(1,1) \otimes (m_1, 1)][(1,1) \otimes (1,2) \otimes \cdots \otimes (\hat{m}_d, m_d)][(1,2) \otimes (2, \hat{m}_r)] =$$

$$= (1,2) \otimes (m_1, \hat{m}_r) \otimes \cdots \otimes (m_{d-1}, \hat{m}_{d+1}) \otimes (\hat{m}_d, m_d).$$ ■

2.7. LEMMA. *Let $z = u^p v^{m_1} u v^{m_2} u \dots u v^{m_r} u^q$, $r > 0$. If r is even then $F_0(z) = 0$, and if $r = 2d-1$, then*

$$F_0(z) = (p,q) \otimes (m_1, \hat{m}_r) \otimes \cdots \otimes (m_{d-1}, \hat{m}_{d+1}) \otimes (\hat{m}_d, m_d).$$

Proof. Let $z' = \prod_{i=1}^{r} u v^{m_i}$. Then $z = u^{\hat{p}} z' u^q$. Since $u, u^2 \in A_0$, it follows that $F_0(z) = u^{\hat{p}} F_0(z') u^q$. The lemma now follows from Lemma 2.6 and the computation: $u^{\hat{p}}(1,2)u^q = u^p(2,2)u^q = (p,q)$. ■

Proof of Proposition 2.2. Since $u \in A_0$,

$$g(u) = f(e_{12}^{(0)} + e_{21}^{(0)}) = \mathrm{Tr}(\Lambda_0(e_{12}^{(0)} + e_{21}^{(0)})) = 0.$$

If $z \neq u, 1$ then $z = u^p v^{m_1} u \dots u v^{m_r} u^q$ with $r > 0$. By (1.1) and the definition of g we have

$$g(z) = f \circ F_0(z).$$

By Lemma 2.7 we have $g(z) = 0$ if r is even, and if $r = 2d-1$,

$$g(z) = f((p,q) \otimes \cdots \otimes (\hat{m}_d, m_d)) =$$

$$= \mathrm{Tr}(\Lambda_0 e_{pq}) \dots \mathrm{Tr}(\Lambda_{d+1} e_{\hat{m}_d, m_d}) = 0,$$

since the last factor is zero. ■

2.8. THEOREM. *If g is a state of O_2 obtained from a weight sequence as in Section 1, then $g \mid \mathbf{C}$ is the (unique) trace of Choi's algebra.*

Proof. This follows from Proposition 2.2 and the representation of C as the reduced group C^*-algebra of $\mathbb{Z}_2 * \mathbb{Z}_3$. ■

WEIGHT SEQUENCES AND TYPE III FACTORS

Throughout this section we will assume that all weight sequences consists of numbers in the open interval $(0,1)$, and hence that the weights obtained from them are faithful. Given a weight sequence $\{t_j\}_{j \geqslant j_0}$ and associated weight f, as in Section 1, let $M = \pi_f(A)''$. Let ω_n be the vector functional on $L(H_f)$ defined by $\eta_f(e_n)$. Then Lemma 1.4 implies that

$$\omega_n \mid \pi_f(A) \leqslant \omega_{n+1} \mid \pi_f(A).$$

By the strong continuity of vector functionals it follows that

$$\omega_n \mid M \leqslant \omega_{n+1} \mid M. \tag{3.1}$$

Thus we may define a semifinite normal weight φ on M by $\varphi = \sup_n(\omega_n \mid M_+)$. Since $\omega_n \mid \pi_f(e_n)M\pi_f(e_n)$ is faithful for each n, and $\pi_f(e_n)$ tends strongly to 1, it follows that φ is faithful. In what follows we will identify $H_f = H_\varphi$, and if there is no potential source of confusion we will drop the subscript f or φ from H and η, and will omit reference to π_f and π_φ. We remark that B is σ-weakly dense in M, and $\eta(B)$ is norm dense in H.

3.2. LEMMA. *Let M and φ be constructed from a weight sequence as above. If $x = \otimes x_j^{(j)}$ is an element of B, then*

$$\Delta_\varphi^{it}\eta(x) = \eta[\otimes(\Lambda_j^{it}x_j\Lambda_j^{-it})^{(j)}],$$

and hence

$$\sigma_t^\varphi(x) = \otimes(\Lambda_j^{it}x_j\Lambda_j^{-it})^{(j)}.$$

Proof. This follows easily from the construction of φ. ∎

In Lemma 3.9 we will give a sufficient condition on the weight sequence to ensure that α extend to an automorphism of M. First we give a series of three lemmas which allow us to compare the constructions associated to two weight sequences which are close in the appropiate sense.

3.3. LEMMA. *Let $\{t_j\}_{j\geqslant j_0}$ be a weight sequence, let $\{\Lambda_j\}_{j\in\mathbb{Z}}$ and f be as in Section 1, and let M and φ be as above. Let $\{r_j\}_{j\geqslant j_1}$ be another weight sequence, and for $j \in \mathbb{Z}$ let $\Omega_j \in M_2(\mathbb{C})$ be given by*

$$\Omega_j = \begin{cases} \operatorname{diag}\{r_j, 1-r_j\}, & j \geqslant j_1 \\ 1, & j < j_1. \end{cases}$$

Suppose that $\sum^{\infty}(t_j - r_j)^2 t_j^{-1} < \infty$. Put $h_n = \bigotimes_{j=-n}^{n}(\Omega_j\Lambda_j^{-1})^{(j)}$ in B_n. Then for each $x \in B$, the sequences $\{\eta(h_nx)\}_{n=0}^\infty$ and $\{\eta(xh_n)\}_{n=0}^\infty$ are Cauchy in H. Moreover, letting h and h' be the linear operators in H with domain $\eta(B)$ defined by

$$h\eta(x) = \lim_{n\to\infty}\eta(h_nx)$$

$$h'\eta(x) = \lim_{n\to\infty}\eta(xh_n),$$

the closures of h and h' are nonsingular positive self-adjoint operators, and $\overline{h}$ and $\overline{h}'$ are affiliated with M_φ and M', respectively.

Proof. We will prove the results for h. The proof for h' is similar, with one exception which we will indicate. Let $x \in B_k$ and let $k \leqslant l \leqslant n$, with $l \geqslant \max(j_0, j_1)$. We have

$$\|\eta(h_l x) - \eta(h_n x)\|^2 = \varphi(x^*(h_l - h_n)^2 x) =$$

$$= \varphi\left[(x^* h_k^2 x) \otimes \left(\bigotimes_{j=k+1}^{l} (\Omega_j^2 \Lambda_j^{-2})^{(j)}\right) \otimes \left(\bigotimes_{j=l+1}^{n} 1^{(j)} - \bigotimes_{j=l+1}^{n} (\Omega_j \Lambda_j^{-1})^{(j)}\right)^2\right] =$$

$$= \varphi(x^* h_k^2 x) \left(\prod_{j=k+1}^{l} \mathrm{Tr}(\Omega_j^2 \Lambda_j^{-1})\right) \left(-1 + \prod_{j=l+1}^{n} \mathrm{Tr}(\Omega_j^2 \Lambda_j^{-1})\right).$$

Thus to show that $\{\eta(h_n x)\}$ is Cauchy, it suffices to show that the infinite product $\prod^{\infty} \mathrm{Tr}(\Omega_j^2 \Lambda_j^{-1})$ converges. An easy computation yields

$$\mathrm{Tr}(\Omega_j^2 \Lambda_j^{-1}) = r_j^2 t_j^{-1} + (1 - r_j)^2 (1 - t_j)^{-1} =$$

$$= (r_j - t_j)^2 t_j^{-1} (1 - t_j)^{-1} + 1.$$

Hence the convergence of the infinite product is equivalent to the convergence of the series $\sum^{\infty} (t_j - r_j)^2 t_j^{-1} (1 - t_j)^{-1}$.

To show that $\overline{h}$ is self-adjoint it suffices to show that $h + \mathrm{i}$ and $h - \mathrm{i}$ have dense range ([12], 13.20). We will show that $R(h + \mathrm{i})^-$ contains $\eta(B)$, the proof for $h - \mathrm{i}$ being similar. For $n = 0, 1, 2, \ldots$ let $b_n = (h_n + \mathrm{i} e_n)^{-1}$, where the inverse is taken in B_n. We note that $\|b_n\| \leqslant 1$, and that b_n commutes with h_l for all n, l. Let $x \in B_k$. Then for $l \geqslant k$ we have

$$(h_l + \mathrm{i} e_l) b_l x = x.$$

Then

$$(h + \mathrm{i})\eta(b_l x) - \eta(x) = h\eta(b_l x) - h_l \eta(b_l x) =$$

$$= \lim_{n \to \infty} \eta((h_n - h_l) b_l x).$$

We have

$$\|\eta((h_n - h_l) b_l x)\|^2 = \|b_l \eta((h_n - h_l) x)\|^2 \leqslant$$

$$\leqslant \|\eta((h_n - h_l) x)\|^2 \to 0 \quad \text{as } l, n \to \infty.$$

(We mention that at this point in the proof for h', one uses the fact that b_l commutes with $\bigotimes_{j=-l}^{l} \Lambda_j^{(j)}$.)

Notice that we have shown that for $\xi \in R(h + \mathrm{i})$,

$$\lim_{l \to \infty} b_l \xi = (\overline{h} + \mathrm{i})^{-1} \xi.$$

Since $R(h+\mathrm{i})$ is dense and $\{b_l\}$ is bounded, we have that $b_l \to (\overline{h}+\mathrm{i})^{-1}$ strongly. Thus $(\overline{h}+\mathrm{i})^{-1} \in M_\varphi$ since $\{b_l\} \subset M_\varphi$. This implies that the spectral resolution of $(\overline{h}+\mathrm{i}^{-1})$, and hence that of $\overline{h}$, is contained in M_φ. It follows that $\overline{h}$ is affiliated with M_φ. To see that $\overline{h}$ is positive, let $F = \{z \in \mathbb{C} : |z + \frac{1}{2}\mathrm{i}| = \frac{1}{2},\ \mathrm{Re}\, z \geqslant 0\}$. Note that F is the image of $[0,\infty]$ under the map $(t+\mathrm{i})^{-1}$. Hence $\mathrm{sp}(b_l) \subset F$ for all l. Since $b_l \to (\overline{h}+\mathrm{i})^{-1}$ strongly, $\{b_l\}$ is bounded, and $\{b_l\}$ and $(\overline{h}+\mathrm{i})^{-1}$ are normal operators, a routine argument shows that $\mathrm{sp}((\overline{h}+\mathrm{i})^{-1}) \subset F$. It then follows by the functional calculus that $\mathrm{sp}(\overline{h}) \subset [0,\infty)$.

To prove that $\overline{h}$ is nonsingular, we show that $R(h)$, and hence $R(\overline{h})$, is dense in H. Let h_n^{-1} denote the inverse of h_n as an element of B_n. Let $x \in B_k$. Then $h\eta(h_l^{-1}x) = \lim_{n\to\infty} \eta(h_n h_l^{-1}x)$. For $k \leqslant l \leqslant n$ and $k \geqslant \max(j_0, j_1)$, we have

$$h_n h_l^{-1}x - x = x \otimes \left(\bigotimes_{j=k+1}^{l} 1^{(j)}\right) \otimes \left(\bigotimes_{j=l+1}^{n} (\Omega_j \Lambda_j^{-1})^{(j)} - \bigotimes_{j=l+1}^{n} 1^{(j)}\right).$$

Hence

$$\|\eta(h_n h_l^{-1}x) - \eta(x)\|^2 = \varphi(x^*x)\left(-1 + \prod_{j=l+1}^{n} \mathrm{Tr}(\Omega_j^2 \Lambda_j^{-1})\right) \to 0 \quad \text{as } l \to \infty.$$

Thus $\eta(x) \in \overline{R(h)}$. ∎

3.4. LEMMA. *We maintain all notations, hypotheses, and conclusions of Lemma 3.3. Then for $x \in B$ we have*

$$\varphi(\overline{h}x) = \lim_{n\to\infty} \varphi(h_n x),$$

where $\varphi(\overline{h}\,\cdot\,)$ is as in [11].

Proof. Let $L_k = \bigotimes_{j=-k}^{k} \Lambda_j^{(j)}$ in B_k. By checking on elementary tensors one can easily verify that for x, $y \in B$ with $x \in B_k$ we have

$$\varphi(L_k^{-1} x L_k y) = \varphi(yx), \tag{3.5}$$

where L_k^{-1} denotes the inverse of L_k in B_k. Since $\varphi \mid e_k M e_k$ is a bounded normal functional, it follows that for fixed $x \in B_k$, (3.5) is true for any $y \in M$ commuting with e_k.

For any positive self-adjoint operator a, and any $\varepsilon > 0$, let $a_\varepsilon = a(1+\varepsilon a)^{-1}$, as in [11]. We claim that for each $\varepsilon > 0$,

$$(h_n)_\varepsilon \to \overline{h_\varepsilon} \text{ strongly} \tag{3.6}$$

$$(3.7) \qquad ((h_n)_\varepsilon)' \to \overline{h'_\varepsilon} \quad \text{strongly},$$

where for x and y in B we let $x'\eta(y) = \eta(yx)$, and note that since h_n commutes with L_n, $((h_n)_\varepsilon)'$ extends to a bounded operator on H.

To verify (3.6), note first that $a_\varepsilon = \varepsilon^{-1} - \varepsilon^{-1}(1+\varepsilon a)^{-1}$, so that it suffices to show that $(1+\varepsilon h_n)^{-1} \to (1+\overline{\varepsilon h})^{-1}$ strongly. This follows from the equation

$$(1+\overline{\varepsilon h})^{-1} - (1+\varepsilon h_n)^{-1} = \varepsilon(1+\varepsilon h_n)^{-1}(h_n - \overline{h})(1+\overline{\varepsilon h})^{-1},$$

and the fact that $R(1+\varepsilon h)$ is dense in H (which is proved in the same way that was used to show the density of $R(h+\mathrm{i})$ in Lemma 3.3). Statement (3.7) is verified in a similar way. (We thank John Phillips for showing us this argument.)

Now let x and y be elements of B_k. Then

$$\begin{aligned}
\varphi(\overline{h}yx) &= \lim_{\varepsilon\to 0} \varphi(\overline{h}_\varepsilon yx) = \\
&= \lim_\varepsilon \varphi(L_k^{-1}xL_k\overline{h}_\varepsilon y) = && \text{by (3.5)},\\
&= \lim_\varepsilon \langle \overline{h}_\varepsilon \eta(y), \eta(L_k x^* L_k^{-1})\rangle = \\
&= \lim_\varepsilon \lim_{n\to\infty} \langle (h_n)_\varepsilon \eta(y), \eta(L_k x^* L_k^{-1})\rangle = && \text{by (3.6)},\\
&= \lim_\varepsilon \lim_n \varphi((h_n)_\varepsilon yx) = \\
&= \lim_\varepsilon \lim_n \langle \eta(x), \eta(y^*(h_n)_\varepsilon)\rangle = \\
&= \lim_\varepsilon \langle \eta(x), \overline{h}'_\varepsilon \eta(y^*)\rangle = && \text{by (3.7)},\\
&= \langle \eta(x), \overline{h}' \eta(y^*)\rangle = \\
&= \lim_n \langle \eta(x), \eta(y^* h_n)\rangle = \\
&= \lim_n \varphi(h_n yx).
\end{aligned}$$

■

3.8. LEMMA. *We maintain all notations, hypotheses, and conclusions from Lemmas 3.3 and 3.4. In addition, let g be the weight on A constructed from $\{r_j\}$ as in Section 1, let $N = \pi_g(A)''$, and let ψ be the faithful normal semi-finite weight on N obtained from g. Then*

(i) *π_f and π_g are unitarily equivalent.*

(ii) *If $\theta : M \to N$ is the unitary equivalence in* (i), *then $\varphi(\overline{h}\,\cdot\,) = \psi\circ\theta$.*

Proof. (i) Define a linear operator $T : D(T) \subset H_f \to H_g$, with $D(T) = \eta_f(B)$, by $T\eta_f(x) = \eta_g(x)$. We will show that T has a densely defined adjoint, that T^*T has dense range, and that $T^*\overline{T}$ is affiliated with M'. Then since T also has dense range, the partial isometry in the polar decomposition of $\overline{T}$ will be a unitary operator intertwining π_f and π_g.

Let $x = \otimes x_j^{(j)}$ and $y = \otimes y_j^{(j)}$ be elementary tensors in B. Then

$$\begin{aligned}\langle \eta_g(x), T\eta_f(y)\rangle &= g(y^* x) = \\ &= \prod_j \mathrm{Tr}(\Omega_j y_j^* x_j) = &&\text{. by (1.7)}\\ &= \prod_j \mathrm{Tr}(\Lambda_j y_j^* x_j \Omega_j \Lambda_j^{-1}) = \\ &= \lim_{n\to\infty} f(y^* x h_n) = \\ &= \lim_n \langle \eta_f(xh_n), \eta_f(y)\rangle = \\ &= \langle h'\eta_f(x), \eta_f(y)\rangle .\end{aligned}$$

Hence $\eta_g(B) \subset D(T^*)$ and $T^*T = h'$. Part (i) now follows from Lemma 3.3.

(ii) It follows from (1.7) and Lemma 3.4 that $\varphi(\overline{h}x) = \psi\circ\theta(x)$ for $x \in \pi_f(B)$. By Lemma 3.2, $\pi_f(B)$ is globally invariant under $\sigma^{\psi\circ\theta}$. Hence by [11], Proposition 5.9, it suffices to prove that $\psi\circ\theta$ is invariant under $\sigma_t^{\varphi(\overline{h}\,\cdot\,)}$. Since $\sigma_t^{\varphi(\overline{h}\,\cdot\,)} = \mathrm{Ad}(\overline{h}^{it})\circ\sigma_t^{\varphi}$ ([4]), it suffices to prove that $\overline{h}$ is affiliated with $M_{\psi\circ\theta}$ and that $\psi\circ\theta$ is invariant under σ_t^{φ}.

Applying Lemma 3.2 to ψ, it follows that $\pi_g(e_n)$ and $\pi_g(h_n)$ belong to M_ψ. Hence $\theta\circ\pi_f(b_n)$ belongs to M_ψ, where b_n is as in the proof of Lemma 3.3. From [4], 1.2.10, it follows that $\pi_f(b_n) \in M_{\psi\circ\theta}$. Then as in the proof of Lemma 3.3, it follows that $\overline{h}$ is affiliated with $M_{\psi\circ\theta}$.

To see that $\psi\circ\theta$ is σ^φ-invariant , let T be as in the proof of (i), and let $\overline{T} = W|\overline{T}|$ be the polar decomposition of $\overline{T}$. Then $\theta = \mathrm{Ad}(W)$. If $x \in M_+$ we have

$$\begin{aligned}\psi\circ\theta\circ\sigma_t^\varphi(x) &= \sup_n \langle \theta\circ\sigma_t^\varphi(x)\eta_g(e_n), \eta_g(e_n)\rangle = \\ &= \sup_n \langle \sigma_t^\varphi(x) W^*\overline{T}\eta_f(e_n), W^*\overline{T}\eta_f(e_n)\rangle = \\ &= \sup_n \langle \sigma_t^\varphi(x)|\overline{T}|\eta_f(e_n), |\overline{T}|\eta_f(e_n)\rangle = \\ &= \sup_n \langle \sigma_t^\varphi(x)\eta_f(e_n), \overline{h'}\eta_f(e_n)\rangle = \\ &= \sup_n \lim_{k\to\infty} \langle \sigma_t^\varphi(x)\eta_f(e_n), \eta_f(e_n h_k)\rangle = \\ &= \sup_n \lim_k \langle x\Delta_\varphi^{-it}\eta_f(e_n), \Delta_\varphi^{-it}\eta_f(e_n h_k)\rangle =\end{aligned}$$

$$= \sup_n \lim_k \langle x\eta_f(e_n), \eta_f(e_n h_k)\rangle = \qquad \text{by Lemma 3.2,}$$

$$= \psi \circ \theta(x). \qquad \blacksquare$$

3.9. LEMMA. *Let $\{t_j\}_{j \geqslant j_0}$ be a weight sequence that $\sum^{\infty}(t_j - t_{j+1})^2 t_j^{-1}(1-t_j)^{-1} < < \infty$. Then α is implemented in $L(H)$, and hence α extends to an automorphism of M, which we also denote by α. Moreover, letting $r_j = t_{j+1}$, $j \geqslant j_0 - 1$, and letting h be as in Lemma 3.3, we have*

$$\varphi \circ \alpha = \varphi(\overline{h} \cdot\,).$$

Proof. We will freely use the notation of Lemmas 3.3 and 3.8. Let g be the weight on A constructed from $\{r_j\}$ as in Section 1. A straightforward calculation shows that $f \circ \alpha \mid B = g \mid B$. (In fact $f \circ \alpha = g$, but we do not need this fact.) Using this, it is easily verified that the map $\eta_f(x) \mapsto \eta_g(\alpha^{-1}(x))$, $x \in B$, extends by continuity to a unitary operator V intertwining $\pi_f \circ \alpha$ and π_g. Since π_f and π_g are unitarily equivalent by Lemma 3.8 (i), it follows that $\pi_{f\circ\alpha}$ and π_f are unitarily equivalent (the equivalence is $\mathrm{Ad}(V^*) \circ \theta$). Next we claim that $\varphi = \psi \circ \mathrm{Ad}(V)$. To see this let $x \in M_+$. Then

$$\psi(VxV^*) = \sup_n \langle VxV^* \eta_g(e_n), \eta_g(e_n)\rangle =$$

$$= \sup_n \langle x\eta_f(e_{n-1}))\rangle = \varphi(x).$$

Then we have

$$\varphi(\overline{h} \cdot\,) = \psi \circ \theta = \qquad \text{by Lemma 3.8 (ii),}$$

$$= \psi \circ \mathrm{Ad}(V) \circ \alpha =$$

$$= \varphi \circ \alpha. \qquad \blacksquare$$

Our next goal is to show that under mild hypotheses on the weight sequence, namely that α extend to M and $\sum t_j(1 - t_j) \doteq \infty$, the states of O_2 obtained from our construction are type III factor states (Lemmas 3.14 and 3.15).

3.10. LEMMA. *Let M be a von Neumann algebra, let B be a σ-weakly dense *-subalgebra of M, and let Γ be a norm dense subset of M_*. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in the unit ball of M, and suppose that $\lim_{n\to\infty} \|[x_n, g] \mid B\| = 0$ for each $g \in \Gamma$. Then $\lim_{n\to\infty} \|[x_n, g]\| = 0$ for each $g \in M_*$.*

Proof. This follows from Kaplansky's density theorem. $\blacksquare$

3.11. LEMMA. *Let $\{x_n\}$ be as in Lemma 3.10. Then $[x_n, y] \to 0$ strongly for each $y \in M$.*

Proof. [5], 2.8. ■

3.12. LEMMA. *Let f be a weight on A constructed from a weight sequence as in section 1. Let $x \in B$ have the form $x = \otimes x_j^{(j)}$ where each x_j is a diagonal matrix in $M_2(\mathbb{C})$. Then $f([y, x]) = 0$ for all $y \in B$.*

Proof. This follows from the fact that x_j commutes with Λ_j. ■

3.13. PROPOSITION. *Let $\{t_j\}_{j \geqslant j_0}$ be a weight sequence, with associated von Neumann algebra and weight (M, φ), such that*

(i) *α extends to an automorphism of M.*

(ii) $\sum t_j(1 - t_j) = \infty$.

Then for each $k \neq 0$, α^k is an outer automorphism of M.

Proof. We will give the proof for the case $k = 1$. The general case $k \neq 0$ is similar.

Case (i). $t_j \nrightarrow 0,1$. Let s be a limit point of $\{t_j\}$ in $(0, 1)$. Then $0 < s - s^2$. Let $\{t_{j_n}\}_{n \geqslant 1}$ be a subsequence converging to s. Let x_n be the following element of B:

$$x_n = \left(\bigotimes_{p=-j_n}^{j_n - 1} 1^{(p)} \right) \otimes e_{11}^{(j_n)}.$$

We will verify the hypotheses of Lemma 3.10. Let $\Gamma = \{g_{a,b} : a, b \in B\}$, where $g_{a,b}(x) = \varphi(axb)$. Since φ is a faithful semi-finite normal weight and $\eta(B)$ is dense in $H = H_\varphi$, Γ is norm dense in M_*. If $a, b \in B_l$, then for any n with $j_n > l$, and any $c \in B$, we have $a[x_n, c]b = [x_n, acb]$. Hence $[x_n, g_{a,b}](c) = \varphi(a[x_n, c]b) = 0$, by Lemma 3.12. Thus $\{x_n\}$ satisfies the hypotheses of Lemma 3.10, and so by Lemma 3.11 we have that $[x_n, y] \to 0$ strongly for each $y \in M$. Hence $x_n - \beta(x_n) \to 0$ strongly for each inner automorphism β of M. But we claim that $x_n - \alpha(x_n) \nrightarrow 0$ strongly. Note first that x_n and $\alpha(x_n)$ are commuting projections. Then for n such that $j_n \geqslant |j_0|$,

$$\begin{aligned} \|(x_n - \alpha(x_n))\eta(e_{j_0})\|^2 &= \varphi(e_{j_0}(x_n + \alpha(x_n) - 2x_n\alpha(x_n))e_{j_0}) = \\ &= t_{j_n} + t_{j_n+1} - 2t_{j_n}t_{j_n+1} = \\ &= (t_{j_n} - t_{j_n}^2) + (t_{j_n+1} - t_{j_n+1}^2) + (t_{j_n} - t_{j_n+1})^2 \geqslant \\ &\geqslant t_{j_n} - t_{j_n}^2 \to s - s^2 > 0. \end{aligned}$$

Case (ii). $\lim_{j \to \infty} t_j = 0$. (The case $\lim_{j \to \infty} t_j = 1$ is handled analogously.) Choose ε with $0 < \varepsilon < \frac{1}{6}$. For $n > 0$ choose positive integers k_n and l_n so that

(a) $$k_n \geqslant n, \quad k_n \geqslant |j_0|$$

(b) $$\frac{1}{2} - \varepsilon < \prod_{j=0}^{l_n} (1 - t_{k_n+2j}) < \frac{1}{2} + \varepsilon.$$

This is possible because $\lim t_j = 0$ and $\sum t_j = \infty$. Let $x_n \in B$ be given by

$$x_n = \left(\bigotimes_{j=-k_n}^{k_n-1} 1^{(j)} \right) \otimes \left(\bigotimes_{j=0}^{l_n} \left(e_{22}^{(k_n+2j)} \otimes 1^{(k_n+2j+1)} \right) \right).$$

Exactly as in case (i) we have that $x_n - \beta(x_n) \to 0$ strongly for each inner automorphism β of M. We will show that $x_n - \alpha(x_n) \not\to 0$ strongly. Let

$$a_n = \langle x_n \eta(e_{j_0}), \eta(e_{j_0}) \rangle = \varphi(e_{j_0} x_n e_{j_0}) = \prod_{j=0}^{l_n} (1 - t_{k_n+2j})$$

$$b_n = \langle \alpha(x_n) \eta(e_{j_0}), \eta(e_{j_0}) \rangle.$$

Then $\langle x_n \alpha(x_n) \eta(e_{j_0}), \eta(e_{j_0}) \rangle = a_n b_n$. Since x_n and $\alpha(x_n)$ are commuting projections we have

$$\begin{aligned} \|(x_n - \alpha(x_n))\eta(e_{j_0})\|^2 &= a_n + b_n - 2a_n b_n > \\ &> \frac{1}{2} - \varepsilon + b_n - 2\left(\frac{1}{2} + \varepsilon\right) b_n = \\ &= \frac{1}{2} - \varepsilon - 2\varepsilon b_n > \frac{1}{2} - 3\varepsilon > 0. \end{aligned}$$ ∎

DEFINITION. A weight sequence satisfying (i) and (ii) in the statement of Proposition 3.13 will be called *admissible*.

3.14. COROLLARY. *Let M be constructed from an admissible weight sequence. Then $M \times_\alpha \mathbb{Z}$ is a type* III *factor.*

Proof. Note that $M = N \otimes e_{j_0} M e_{j_0}$ where N is a type I_∞ factor and $e_{j_0} M e_{j_0}$ is a factor ([1]), and hence M is a factor. From Proposition 3.13 and [4], 4.1.1, it follows that $M \times_\alpha \mathbb{Z}$ is a factor. If M is of type III then so is $M \times_\alpha \mathbb{Z}$ ([8], 13.4.2). Using [1], 2.14, it is easy to show that the only other possibility is that M is of type II_∞ with $\sum (t_j - \frac{1}{2})^2 < \infty$. Then by Lemma 3.8 we may assume that $t_j = \frac{1}{2}$ for all j; i.e. that φ is tracial. Then α scales φ by $\frac{1}{2}$, and so $M \times_\alpha \mathbb{Z}$ is of type $\mathrm{III}_{\frac{1}{2}}$ ([4]). ∎

3.15. LEMMA. *Let $\{t_j\}_{j \geqslant j_0}$ be an admissible weight sequence. Let g be the positive linear functional on D obtained as in Section 1. Then $\pi_g(D)''$ is isomorphic to $M \times_\alpha \mathbb{Z}$.*

Proof. Let $\tilde{E} : M \times_\alpha \mathbb{Z} \to M$ be the canonical conditional expectation, and put $\tilde{\varphi} = \varphi \circ \tilde{E}$. Let $M \times_\alpha \mathbb{Z}$ act on $H_{\tilde{\varphi}}$. Then $\pi_g(d)''$ is unitarily equivalent to

$e_0(M \times_\alpha \mathbb{Z})e_0 P$, where $P \in (e_0(M \times_\alpha \mathbb{Z})e_0)'$ is the projection onto $\overline{\eta_{\tilde{\varphi}}(D)}$. The lemma now follows from Corollary 3.14. ∎

The sequence $\{x_n\}$ constructed in Proposition 3.13 has the following stronger property.

3.16. LEMMA. *Let $\{t_j\}$ be an admissible weight sequence. There is a positive number δ such that the sequence $\{x_n\}$ constructed in Proposition 3.13 has the following property: if π is a normal nondegenerate representation of M, then for each $k \neq 0$ and each vector ξ we have*

$$\liminf_{n\to\infty} \|\pi(x_n - \alpha^k(x_n))\xi\| \geqslant \delta\|\xi\|.$$

Proof. It is clear that if the result is true for π then it is also true for the cutdown of π by a projection in $\pi(M)'$. Thus it suffices to prove the result for $\pi = \mathrm{id} \otimes 1$, acting on $H \otimes H_1$. Let $y_n = x_n - \alpha^k(x_n)$. Let $\xi = \sum \eta(a_i) \otimes \xi_i$, where $a_i \in B$, $\xi_i \in H_1$, and the sum is finite. For some l, all of the $\{a_i\}$ belong to B_l. Then for all large enough n,

$$\|\pi(y_n)\xi\|^2 = \sum_{i,j} \langle y_n\eta(a_i),\ y_n\eta(a_j)\rangle\langle \xi_i, \xi_j\rangle =$$

$$= \sum_{i,j} \varphi(y_n^2 a_j^* a_i)\langle \xi_i, \xi_j\rangle =$$

$$\varphi(e_{j_0} y_n^2)\sum_{i,j} \varphi(a_j^* a_i)\langle \xi_i, \xi_j\rangle =$$

$$= \|y_n\eta(e_{j_0})\|^2\|\xi\|^2.$$

Since such vectors ξ are dense in $H \otimes H_1$, we may take $\delta^2 = \frac{1}{2}(s - s^2)$ if $\{t_j\}$ falls under case (i) of Proposition 3.13, and $\delta^2 = \frac{1}{2}\left(\frac{1}{2} - 3\varepsilon\right)$ in case (ii). ∎

Our final task is to show that for each $\lambda \in [0,1]$ there is an admissible weight sequence $\{t_j\}$ for which $M \times_\alpha \mathbb{Z}$ is a type III_λ factor. We will use the continous decomposition of M and $M \times_\alpha \mathbb{Z}$ ([13]) and the modular period group T ([4]). Let $\tilde{E}$ and $\tilde{\varphi}$ be as in the proof of Lemma 3.15. We now fix some notation. Let $\tilde{H} = H \otimes \ell^2(\mathbb{Z}) \otimes L^2(\mathbb{R})$. We let $\{\delta_n\}_{n\in\mathbb{Z}}$ be both the usual orthonormal basis for $\ell^2(\mathbb{Z})$ and the rank-one projections with ranges spanned by those vectors. We let R be the shift on $\ell^2(\mathbb{Z})$: $R\delta_n = \delta_{n+1}$. We let $\{L_t\}_{t\in\mathbb{R}}$ be the translation group on $L^2(\mathbb{R})$: $(L_t g)(s) = g(s-t)$, $g \in L^2(\mathbb{R})$. Let $\pi : M \to L(\tilde{H})$ be given by

$$\pi(x) = \sum_{n\in\mathbb{Z}} \alpha^{-n}(x) \otimes \delta_n \otimes 1,$$

let $U = 1 \otimes R \otimes 1$, and let

$$v_t = \sum_{n \in \mathbb{Z}} (D\varphi \circ \alpha^n : D\varphi)_t \Delta^{it} \otimes \delta_n \otimes L_t, \quad t \in \mathbb{R},$$

where Δ is the modular operator for φ.

3.17. LEMMA. *With the above notation,*

(i) $$(\pi(M) \cup \{U\})'' \doteq M \times_\alpha \mathbb{Z}$$

(ii) $$(\pi(M) \cup \{v_t\}_{t \in \mathbb{R}})'' \doteq M \times_{\sigma^\varphi} \mathbb{R}$$

(iii) $$(\pi(M) \cup \{U\} \cup \{v_t\}_{t \in \mathbb{R}})'' \doteq (M \times_\alpha \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R} \doteq$$

$$\doteq (M \times_{\sigma^\varphi} \mathbb{R}) \times_{\tilde{\alpha}} \mathbb{Z},$$

where $\tilde{\alpha} = \mathrm{Ad}(U) \mid (\pi(M) \cup \{v_t\}_{t \in \mathbb{R}})''$.

Proof. (i) is clear. (ii) follows from a straightforward computation using [4], 1.2.10. To prove (iii), we claim that $v_t U v_{-t} = U\pi((D\varphi \circ \alpha : D\varphi)_t)$. Then (iii) follows from [7], 3.2. To prove the claim, note first that if φ and ψ are faithful normal semifinite weights on a von Neumann algebra, and θ is an automorphism, then

(3.18) $$\theta((D\psi \circ \theta : D\varphi \circ \theta)_t) = (D\psi : D\varphi)_t.$$

This follows from the proof of [4], 1.2.2. The claim now follows by a straightforward calculation. ∎

We note that the dual actions $(\sigma^\varphi)\hat{}$ and $(\sigma^{\tilde{\varphi}})\hat{}$ are both implemented on $\tilde{H}$ by $\{1 \otimes 1 \otimes w_t\}_{t \in \mathbb{R}}$, where $(w_t g)(s) = e^{ist} g(s)$, $g \in L^2(\mathbb{R})$. The dual actions $\hat{\alpha}$ and $(\tilde{\alpha})\hat{}$ are both implemented by $\{1 \otimes Q_\zeta \otimes 1\}_{\zeta \in \mathbb{T}}$, where $Q_\zeta \delta_n = \zeta^n \delta_n$, $n \in \mathbb{Z}$. In what follows we will omit reference to the representation π, and will use 3.17 (i), (ii), (iii) as definitions of the various crossed-products.

3.19. LEMMA. *Let* (M, φ) *be constructed from an admissible weight sequence, and let* $\{x_n\}$ *be the sequence constructed in Proposition* 3.13. *Then* $[x_n, y] \to 0$ *strongly for each* y *in* $M \times_{\sigma^\varphi} \mathbb{R}$ *under any normal representation of* $M \times_{\sigma^\varphi} \mathbb{R}$.

Proof. It suffices to prove the lemma for the semi-cyclic representation associated to a faithful normal semi-finite weight. Let γ be the weight on $M \times_{\sigma^\varphi} \mathbb{R}$ which is dual to φ ([7], 3.1 and 3.2). Let $\tilde{B} = \bigcup_l C_c(\mathbb{R}, B_l) \subset M \times_{\sigma^\varphi} \mathbb{R}$. Then $\tilde{B}$ is contained in the domain of definition of γ. For $x \in B$, $a \in \tilde{B}$, we have $\gamma(a) = \varphi(a(0))$, and xa, $ax \in \tilde{B}$

with $(xa)(t) = \sigma^{\varphi}_{-t}(x) \cdot a(t)$, $(ax)(t) = a(t) \cdot x$. Let $\tilde{\Gamma} = \{\tilde{g}_{a,b} : a, b \in B\}$, where $\tilde{g}_{a,b}(c) = \gamma(acb)$. Since each x_n is σ^{φ}-invariant, (Lemma 3.2), the lemma follows from Lemmas 3.10, 3.11, and 3.12. ■

3.20. LEMMA: *Let M be constructed from an admissible weight sequence. Then*

$$M' \cap [(M \times_{\alpha} \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}] = M' \cap (M \times_{\sigma^{\varphi}} \mathbb{R}).$$

Proof. $\supset$ is clear. To prove $\subset$, let

$$z \in M' \cap [(M \times_{\alpha} \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}] = M' \cap [(M \times_{\sigma^{\varphi}} \mathbb{R}) \times_{\tilde{\alpha}} \mathbb{Z}].$$

Let $E : (M \times_{\sigma^{\varphi}} \mathbb{R}) \times_{\tilde{\alpha}} \mathbb{Z} \to M \times_{\sigma^{\varphi}} \mathbb{R}$ be the canonical conditional expectation. Let $z_k = E(zU^{-k})$ (see [14, V.7.5]). Then for $x \in M$ and $k \in \mathbb{Z}$ we have

$$xz_k = xE(zU^{-k}) = E(xzU^{-k}) = E(zU^{-k}\alpha^k(x)) = z_k\alpha^k(x).$$

Letting $z_k = y_k|z_k|$ be the polar decomposition, we obtain

$$(y_k x y_k^* - \alpha^{-k}(x))y_k y_k^* = 0,$$

for all $x \in M$, $k \in \mathbb{Z}$. Let $\{x_n\}$ be the sequence constructed in Proposition 3.13, and let δ be the positive number obtained in Lemma 3.16. Then for any vector ξ_k in the range of y_k we have

$$0 = \liminf_{n\to\infty} \|(y_k x_n y_k^* - \alpha^{-k}(x_n))\xi_k\| =$$

$$= \liminf_{n\to\infty} \|(x_n - \alpha^{-k}(x_n))\xi_k\| \geqslant \qquad \text{by Lemma 3.19.}$$

$$\geqslant \delta\|\xi_k\| \quad \text{, if } k \neq 0 \qquad \text{by Lemma 3.16.}$$

Thus $z_k = 0$ for $k \neq 0$, and so $z = z_0 \in M \times_{\sigma^{\varphi}} \mathbb{R}$. ■

3.21. PROPOSITION. *Let M be constructed from an admissible weight sequence. Then*

$$Z((M \times_{\alpha} \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}) \subset Z(M \times_{\sigma^{\varphi}} \mathbb{R}).$$

Proof. From Lemma 3.20 we have

$$Z((M \times_{\alpha} \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}) \subset M' \cap [(M \times_{\alpha} \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}] \cap (M \times_{\sigma^{\varphi}} \mathbb{R})' =$$

$$= M' \cap (M \times_{\sigma^\varphi} \mathbb{R}) \cap (M \times_{\sigma^\varphi} \mathbb{R})' =$$

$$= Z(M \times_{\sigma^\varphi} \mathbb{R}). \qquad \blacksquare$$

3.22. COROLLARY. *Let (M, φ) be constructed from an admissible weight sequence. If M is of type* III_1, *then $M \times_\alpha \mathbb{Z}$ is of type* III_1. *If M is not of type* III_0 *then $M \times_\alpha \mathbb{Z}$ is not of type* III_0.

Proof. This follows immediately from [13] and Proposition 3.21. ■

We will now show that for certain weight sequences the above result can be sharpened.

3.23. PROPOSITION. *Let $\{t_j\}_{j \geqslant j_0}$ be a weight sequence such that $t_j \geqslant t_{j+1}$ for all $j \geqslant j_0$, $\lim t_j = 0$, and $\sum t_j = \infty$. Then $\{t_j\}$ is admissible, and $T(M) = T(M \times_\alpha \mathbb{Z})$.*

Proof. Since $\{t_j\}$ decreases to zero, the hypothesis of Lemma 3.9 is satisfied, and so $\{t_j\}$ is admissible.

Proof of $\supset$: We remark that this containment is true for any admissible weight sequence. Let $s \in T(M \times_\alpha \mathbb{Z})$. Then there is a unitary $z \in M \times_\alpha \mathbb{Z}$ implementing $\sigma_s^{\tilde{\varphi}}$. Then

$$v_s^* z \in (M \times_\alpha \mathbb{Z})' \cap [(M \times_\alpha \mathbb{Z}) \times_{\sigma^{\tilde{\varphi}}} \mathbb{R}] \subset$$

$$\subset M \times_{\sigma^\varphi} \mathbb{R} \qquad \text{by Lemma 3.20.}$$

Hence $z \in M \times_{\sigma^\varphi} \mathbb{R}$. It follows that z is invariant under $(\tilde{\alpha})\hat{}$. Since $z \in M \times_\alpha \mathbb{Z}$, $(\tilde{\alpha})\hat{}(z) = \hat{\alpha}(z)$, and hence z is invariant under $\hat{\alpha}$. It follows that $z \in M$, and thus that $s \in T(M)$.

Proof of $\subset$: Let $s \in T(M)$. Since $T(M \times_\alpha \mathbb{Z})$ is a group we may assume that $s \geqslant 1$. Let $\lambda_j = t_j(1 - t_j)^{-1}$ for $j \geqslant j_0$. Then $\lambda_j > 0$, $\{\lambda_j\}$ decreases to zero, and $\sum \lambda_j = \infty$. Since $s \in T(M)$ we have by [4], 1.3.9, that

$$\sum_j [1 - |t_j^{1+is} + (1 - t_j)^{1+is}|] < \infty.$$

An easy computation shows that this is equivalent to

$$\sum_j \lambda_j (1 - \cos \log \lambda_j^s) < \infty. \tag{3.24}$$

It is a simple calculus exercise to show that there is a constant c such that

$$1 - \cos \log t \geqslant c(1 - t)^2, \quad \text{for } e^{-\pi} \leqslant t \leqslant e^{\pi}.$$

For each j there is unique $n_j \in \mathbb{Z}$ such that

$$(2n_j - 1)\pi \leqslant \log \lambda_j^s < (2n_j + 1)\pi.$$

Let $\mu_j = e^{2\pi n_j s^{-1}}$, for $j \geqslant j_0$. Notice that $\mu_j > 0$, $\{\mu_j\}$ decreases to zero, and $\sum \mu_j = \infty$, since $\{\lambda_j\}$ has these properties. We have

$$e^{-\pi} \leqslant (\lambda_j \mu_j^{-1})^s \leqslant e^{\pi}, \quad j \geqslant j_0.$$

We then have

$$\begin{aligned} 1 - \cos\log\lambda_j^s = 1 - \cos\log(\mu_j\lambda_j^{-1})^s &\geqslant \\ \geqslant c[1 - (\mu_j\lambda_j^{-1})^s]^2 &\geqslant \\ \geqslant c(1 - \mu_j\lambda_j^{-1})^2, &\qquad \text{since } s \geqslant 1. \end{aligned}$$

It then follows from (3.24) that

$$\sum_j \lambda_j (1 - \mu_j \lambda_j^{-1})^2 < \infty. \tag{3.25}$$

Let $r_j = \mu_j(1 + \mu_j)^{-1}$. Then $r_j \in (0, 1)$, $\{r_j\}$ decreases to zero, and $\sum r_j = \infty$. A straightforward calculation shows that (3.25) is equivalent to $\sum (t_j - r_j)^2 t_j^{-1} < \infty$. Since $\lim t_j = 0$, this is equivalent to $\sum (t_j - r_j)^2 t_j^{-1}(1 - t_j)^{-1} < \infty$. It now follows from Lemma 3.8 that the weight sequences $\{t_j\}$ and $\{r_j\}$ give unitarily equivalent results. Thus by replacing $\{t_j\}$ with $\{r_j\}$, we may assume without loss of generality that $\lambda_j^{is} = 1$ for all $j \geqslant j_0$.

It then follows from Lemma 3.2 that $\Delta_\varphi^{is} = 1$. We claim that $(D\varphi \circ \alpha^n : D\varphi)_s = 1$ for all n. This will imply that $v_s = 1 \otimes 1 \otimes L_s$ and hence that $\sigma_s^\varphi = \mathrm{id}$, concluding the proof of the proposition.

Note first that for n positive,

$$(D\varphi \circ \alpha^n : D\varphi) = (D\varphi \circ \alpha^n : D\varphi \circ \alpha^{n-1}) \cdots (D\varphi \circ \alpha : D_\varphi),$$

and that by (3.18), $(D\varphi\circ\alpha^{j+1} : D\varphi\circ\alpha^j)_s = \alpha^{-j}((D\varphi\circ\alpha : D\varphi)_s)$. A similar argument for negative j shows that it suffices to prove that $(D\varphi \circ \alpha : D\varphi)_s = 1$. By Lemma 3.9 and [4], 1.2.3(b), $(D\varphi \circ \alpha : D\varphi)_t = (\bar{h})^{it}$, where h is obtained as a limit of $\{h_n\}$ as in Lemma 3.9. Since $\{t_j\}$ is decreasing, we may compute $\|h_n\|$ from the largest eigenvalue of h_n: for $n > |j_0|$ we have

$$\|h_n\| = (1 - t_{j_0}) \prod_{j=j_0}^{n} (1 - t_{j+1})(1 - t_j)^{-1} =$$

$$= 1 - t_{n+1} \leqslant 1.$$

Hence $\{h_n\}$ is bounded. It follows that $\overline{h}$ is bounded and that $h_n \to \overline{h}$ strongly. It then follows that $h_n^{is} \to \overline{h}^{is}$ strongly. Note that we have

$$\Lambda_{j+1}\Lambda_j^{-1} = \begin{cases} \mathrm{diag}\{t_{j+1}t_j^{-1}, (1-t_{j+1})(1-t_j)^{-1}\}, & j \geqslant j_0 \\ \mathrm{diag}\{t_{j_0}, 1-t_{j_0}\}, & j = j_0 - 1 \\ 1, & j \leqslant j_0 - 2, \end{cases}$$

or,

$$\Lambda_{j+1}\Lambda_j^{-1} = \begin{cases} (1+\lambda_{j+1})^{-1}(1+\lambda_j)\mathrm{diag}\{\lambda_{j+1}\lambda_j^{-1}, 1\}, & j \geqslant j_0 \\ (1+\lambda_{j_0})^{-1}\mathrm{diag}\{\lambda_{j_0}, 1\}, & j = j_0 - 1 \\ 1, & j \leqslant j_0 - 2. \end{cases}$$

Hence for $n \geqslant |j_0|$ we have $h_n^{is} = (1+\lambda_{n+1})^{-is} \cdot 1$. Thus $h_n^{is} \to 1$ is norm, and it follows that $\overline{h}^{is} = 1$. ■

3.26. THEOREM. *For each $\lambda \in [0,1]$ there is an admissible weight sequence for which $M \times_\alpha \mathbb{Z}$ is a type III_λ factor. In the case $\lambda = 0$, an uncountable family of non-isomorphic type III_0 factors can be so obtained.*

Proof. $\lambda = 1$: By Corollary 3.22 it suffices to choose $\{t_j\}$ admissible so that M is of type III_1. It follows easily from [1], 5.8, [4], 3.6.1, and Lemma 3.8(i) that M will be of type III_1 if the weight sequence has two distinct limit points in $(0,1)$. Using Lemma 3.9 this can easily be done.

$0 < \lambda < 1$: By [1], 9.4, a weight sequence $\{t_j\}$ can be chosen with $\lim t_j = 0$ and M of type III_λ. Since the isomorphism class of M is unaffected by the ordering of $\{t_j\}$, we may assume that $\{t_j\}$ is decreasing, and hence admissible $\left(\sum t_j = \infty\right.$ since M is not of type $\left.\mathrm{I}\right)$. By Corollary 3.22, $M \times_\alpha \mathbb{Z}$ is not of type III_0. Therefore the modular spectrum of $M \times_\alpha \mathbb{Z}$ is uniquely determined by $T(M \times_\alpha \mathbb{Z})$. By Proposition 3.23, $T(M \times_\alpha \mathbb{Z}) = T(M)$, and hence $M \times_\alpha \mathbb{Z}$ is of type III_λ also.

$\lambda = 0$: Note that any factor constructed as in [1], 10.2 can be obtained from a weight sequence with $\lim t_j = 0$. By [1], 11.10 and [4], 3.6.2, uncountably many non-isomorphic type III_0 factors may be obtained this way, which are identifiable as type III_0, and pairwise distinguishable, by the invariant T. The result now follows from Proposition 3.23. ■

3.27. THEOREM. *For each $\lambda \in [0,1]$ there is a factor state on O_2 of type III_λ which extends the trace on C. In the case $\lambda = 0$, uncountably many non-isomorphic factors of type III_0 can be so obtained.*

Proof. This follows from Lemma 3.15 and Theorems 2.8 and 3.26. ■

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