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SUBNORMAL WEIGHTED SHIFTS AND SPECTRA

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1. Let S be a purely subnormal operator on a separable Hilbert space H. If N denotes its minimal normal extension on $K \supset H$ then

(1.1) $$N = \begin{pmatrix} S & X \\ 0 & T^* \end{pmatrix} \quad \text{on } K = H \oplus H^{\perp};$$

see Conway [1] and [2], pp. 129 ff. Here, T denotes the dual of S and is purely subnormal on $H^{\perp}$ with minimal normal extension N^* on K. It is known that $\sigma(T) = \{\overline{z} : z \in \sigma(S)\}$ and that S is the dual of T. It is easily verified that $S^*S - SS^* \equiv D = XX^*$, $T^*T - TT^* = X^*X$ and that

(1.2) $$\mathrm{Re}(N) = \begin{pmatrix} \mathrm{Re}(S) & 0 \\ 0 & \mathrm{Re}(T) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \quad \text{on } K = H \oplus H^{\perp}.$$

In case

(1.3) $$X \text{ is of trace class (that is, } D^{\frac{1}{2}} \text{ is of trace class)},$$

an application to (1.2) of the Rosenblum-Kato perturbation theory and the fact that $\mathrm{Re}(S)$ and $\mathrm{Re}(T)$ are absolutely continuous (see [8]) yield

(1.4) $$(\mathrm{Re}(N))_{\mathrm{a}} \cong \mathrm{Re}(S) \oplus \mathrm{Re}(T).$$

(Since, for any operator B, $\mathrm{Im}(B) = \mathrm{Re}(-\mathrm{i}B)$, one also has, of course, $(\mathrm{Im}(N))_{\mathrm{a}} \cong \mathrm{Im}(S) \oplus \mathrm{Im}(T)$.) Here, for any selfadjoint operator A, $(A)_{\mathrm{a}}$ denotes the absolutely continuous part of A and the symbol $\cong$ denotes unitary equivalence.

More generally, if $p(B, B^*)$ denotes a selfadjoint polynomial in B and B^* (B an arbitrary bounded operator on a Hilbert space) then $p(B, B^*)$ can be assumed to be a finite sum of terms of the form $aC + \bar{a}C^*$ where $C = B^{j_1}B^{*k_1}\dots B^{j_n}B^{*k_n}$ and $j_1, k_1, \dots, j_n, k_n$ are nonnegative integers. It is easy to see from (1.1) that

$$p(N, N^*) = \begin{pmatrix} p(S, S^*) & 0 \\ 0 & p(T^*, T) \end{pmatrix} + (X), \tag{1.5}$$

where (X) denotes an element of the ideal generated by $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and its adjoint in the space of bounded operators on K. In particular, if (1.3) holds, then (X) of (1.5) is of trace class and, by the Rosenblum-Kato theory,

$$(p(N, N^*))_{\rm a} \cong (p(S, S^*))_{\rm a} \oplus (p(T^*, T))_{\rm a}. \tag{1.6}$$

(It is understood that some of the operators in (1.6) may be absent.) However, if, for instace, $p(B, B^*)$ is a polynomial in $\mathrm{Re}(B) = \frac{1}{2}(B + B^*)$ ($\neq$ constant multiple of the identity) then since $\mathrm{Re}(S)$ and $\mathrm{Re}(T) = \mathrm{Re}(T^*)$ are absolutely continuous, $(p(B, B^*))_{\rm a} = p(B, B^*)$ with $B = S$ or T^*.

Next, let S be a unilateral subnormal weighted shift of norm 1. If $\{e_0, e_1, e_2, \dots\}$ is an orthonormal basis for H then

$$Se_n = \alpha_n e_{n+1} \quad (n \geqslant 0), \quad \text{where } 0 < \alpha_0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \dots \to 1. \tag{1.7}$$

(The monotonicity of the α_n-sequence is a consequence of hyponormality, that is, $S^*S - SS^* \geqslant 0$. Also, it is known that S is irreducible and, in particular, pure.) The requirement that S be subnormal is equivalent to the existence of a probability measure $\nu = \nu(r)$ on $[0, 1]$ with 1 in the support of ν and with the property that if

$$\mathrm{d}\mu(r\mathrm{e}^{\mathrm{i}\theta}) = (2\pi)^{-1}\mathrm{d}\theta\mathrm{d}\nu(r), \tag{1.8}$$

then

$$S \cong S_\mu, \quad \text{where } (S_\mu f)(z) = zf(z), \tag{1.9}$$

and $f(z)$ belongs to the closure $H^2(\mu)$ of the polynomials in $L^2(\mu)$. Moreover, the probability measure ν satisfies

$$(\alpha_0\alpha_1\dots\alpha_{n-1})^2 = \int_0^1 r^{2n}\mathrm{d}\nu(r) \equiv c_n \quad (n \geqslant 1), \tag{1.10}$$

and the correspondence between the operators S and such probability functions ν is one to one. (See [2], pp. 159 ff. and the references given there.) In the usual matrix representation of S of (1.7) on ℓ^2 one has

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ \alpha_0 & 0 & 0 & 0 & \\ 0 & \alpha_1 & 0 & 0 & \\ 0 & 0 & \alpha_2 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{1.11}$$

and $S^*S - SS^* = D \doteq \operatorname{diag}\{\alpha_0^2, \alpha_1^2 - \alpha_0^2, \alpha_2^2 - \alpha_1^2, \ldots\}$, so that $\operatorname{tr}(D)(= \|D\|) = 1$. Further,

$$\operatorname{tr}(D^{\frac{1}{2}}) = \alpha_0 + \sum_1^\infty (\alpha_n^2 - \alpha_{n-1}^2)^{\frac{1}{2}}. \tag{1.12}$$

Thus, $D^{\frac{1}{2}}$ is of trace class if and only if the series of (1.12) is convergent. A general necessary and sufficient condition on the probability function $\nu(r)$ ensuring this convergence will be obtained below (see Theorem 3.2).

In case δ_0 and δ_1 are unit point masses at 0 and 1 and $\nu(r) = (1+\alpha)^{-1}(\delta_1 + \alpha\delta_0)$, where $\alpha \geqslant 0$, then the corresponding S has weights $\{(1+\alpha)^{-\frac{1}{2}}, 1, 1, \ldots\}$; see [2], p. 161. In particular, the cases $\alpha = 0$ and $\alpha = 1$ correspond respectively to the simple shift and an example of Sarason cited in Halmos [4], pp. 307–308. It is clear that $D = S^*S - SS^*$ is then of rank 1 if S is the shift and is otherwise of rank 2.

Moreover, in view of a result of Stampfli [11], p. 377, cited in [2], p. 162, even for an arbitrary subnormal unilateral weighted shift with weights as in (1.7), either the weight sequence is strictly monotone, or $\alpha_n = \alpha_1$ for all $n \geqslant 1$. Thus, for S given by (1.8) and (1.9), $S^*S - SS^* = D$ can have ranks 1, 2 or ∞.

Further ([2], p. 161), if $d\nu(r) = 2r dr$ on $[0,1]$ the associated shift is the Bergman operator on $|z| < 1$, that is, the operator S_μ of (1.9) on the Hilbert space of analytic functions on the disk belonging to $L^2(dxdy)$. In this case the weight sequence is $\{(n/(n+1))^{\frac{1}{2}}\}$ for $n = 1, 2, \ldots$, and the series of (1.12) is divergent, so that $D^{\frac{1}{2}}$ is not of trace class. In fact, it will be shown later that a necessary (but not sufficient) condition in order that $\operatorname{tr}(D^{\frac{1}{2}}) < \infty$ is that 1 belong to the point spectrum of $\nu(r)$; see Theorem 3.1.

In Section 2, the role of the condition $d\nu(1) > 0$ will be discussed. In Section 3, it will be shown that $d\nu(1) > 0$ is a necessary condition in order that the square root of the self-comutator of a unilateral subnormal weighted shift satisfy (1.3). A necessary and sufficient condition for (1.3) involving $\nu(r)$ will also be obtained. In Section 4, it will be shown that when (1.3) is satisfied, one can obtain the spectral multiplicity function for any selfadjoint polynomial operator in T and T^* (T = dual of S). Some examples are also given. In Section 5, a simpler necessary and sufficient condition for

(1.3) will be obtained when $\nu(r)$ is a sufficiently well behaved absolutely continuous function near $r = 1$. Some consequences of this will be discussed. In Section 6, there will be considered the case where $\nu(r)$ is arbitrary and not necessarily absolutely continuous near $r = 1$.

2. THEOREM 2.1. *Let S be a unilateral subnormal shift defined by* (1.8) *and* (1.9) *and for which* 1 *belongs to the point spectrum of $\nu(r)$, that is*

$$\mathrm{d}\nu(1) > 0. \tag{2.1}$$

Then $S - S_0$ is of trace class, where S_0 is a unilateral shift.

Proof. Clearly, $\{z^n\}_{n=0,1,2,\ldots}$ is an orthogonal system on $H^2(\mu)$ and

$$\int |z^n|^2 \mathrm{d}\mu = \int_0^1 r^{2n} \mathrm{d}\nu = c_n. \tag{2.2}$$

Then $\{e_n = (c_n)^{-\frac{1}{2}} z^n\}$ is a complete orthonormal system on $H^2(\mu)$ and the usual matrix representation on ℓ^2 of S is given by (1.11) with $\alpha_n = (c_{n+1}/c_n)^{\frac{1}{2}}$ $(n = 0, 1, 2, \ldots)$. The matrix representation for $S - S_0$ is given by

$$S - S_0 = \begin{pmatrix} 0 & 0 & 0 & \\ \alpha_0 - 1 & 0 & 0 & \\ 0 & \alpha_1 - 1 & 0 & \\ & & & \ddots \end{pmatrix}.$$

Since $(S - S_0)^*(S - S_0) = \mathrm{diag}\{(\alpha_0 - 1)^2, (\alpha_1 - 1)^2, \ldots\}$, it is clear that $S - S_0$ is of trace class if and only if $\sum(1 - \alpha_n) < \infty$. Since $1 - \alpha_n^2 = (1 - \alpha_n)(1 + \alpha_n)$ and $1 \leqslant 1 + \alpha_n \leqslant 2$, it is clear that $S - S_0$ is of trace class if and only if

$$\sum(1 - \alpha_n^2) < \infty. \tag{2.3}$$

For convenience, here and also later, the following notation will be used. If $A_n > 0$, $B_n > 0$ then $A_n \sim B_n$ will be defined by

$$A_n \sim B_n \Longleftrightarrow 0 < a \leqslant A_n/B_n \leqslant b < \infty \quad (n = 0, 1, 2, \ldots), \tag{2.4}$$

where a and b are positive constants. (Thus, e.g., $1 - \alpha_n^2 \sim 1 - \alpha_n$ with $a = 1$ and $b = 2$.) It is easily shown that $A_n \sim B_n \Leftrightarrow B_n \sim A_n$ and that

$$A_n \sim B_n \text{ and } C_n \sim D_n \Rightarrow A_n + C_n \sim B_n + D_n \text{ and } A_nC_n \sim B_nD_n. \tag{2.5}$$

In order to prove (2.3) note that since $d\nu(1) > 0$ then $c_n \geqslant$ const > 0 and so

$$1 - \alpha_n^2 = \left(\int r^{2n} d\nu - \int r^{2n+2} d\nu\right) \Big/ \int r^{2n} d\nu \sim c_n - c_{n+1}.$$

Thus, for some $K =$ const > 0,

$$\sum_0^\infty (1-\alpha_n^2) \leqslant K \sum_0^\infty \int_0^{1-} r^{2n}(1-r^2) d\nu = K \int_0^{1-} \sum_0^\infty (r^2)^n (1-r^2) d\nu = K \int_0^{1-} d\nu \leqslant K < \infty.$$

This proves Theorem 2.1.

REMARKS. Since $d\nu(1) > 0 \Rightarrow S - S_0$ is of trace class, it is clear that if $p(B, B^*)$ is a selfadjoint polynomial in B and B^* (B =bounded operator on a Hilbert space) then $p(S, S^*) - p(S_0, S_0^*)$ is of trace class. It then follows from the Rosenblum-Kato theory that

(2.6) $$(p(S,S^*))_a \cong (p(S_0, S_0^*))_a \quad \text{if } d\nu(1) > 0.$$

It may be noted that $d\nu(1) > 0$ if and only if S is similar to S_0. See [2], p. 168, Problem 7. (See also [4], Problem 199, pp. 307–308 and Problem 90, p. 50.)

A special case of (2.6) is that $d\nu(1) > 0 \Rightarrow \mathrm{Re}(S)$ $(= (\mathrm{Re}(S))_a) \cong \mathrm{Re}(S_0)$ $(= (\mathrm{Re}(S_0))_a)$. Since $\mathrm{Re}(S_0)$ is known to have multiplicity 1 on $[-1,1]$ then so also does $\mathrm{Re}(S)$. Thus,

(2.7) $$\mathrm{Re}(S) \cong \text{multiplication by } t \text{ on } [-1,1].$$

Actually, the result (2.7) is true even without the hypothesis (2.1). In fact, more generally, one has the following

THEOREM 2.2. *If A is a hyponormal, unilateral weighted shift with $0 < \alpha_0 \leqslant \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \to 1$ then*

(2.8) $$\mathrm{Re}(A) \cong \textit{multiplication by } t \textit{ on } [-1,1].$$

Proof. One may suppose that

$$A = \begin{pmatrix} 0 & 0 & 0 & \\ \alpha_0 & 0 & 0 & \\ 0 & \alpha_1 & 0 & \\ & & & \ddots \end{pmatrix}.$$

If $e_0 = (1,0,0,\dots)^{\mathrm{T}}$, then a direct calculation shows that $\{(\mathrm{Re}(A))^n e_0\}_{n=0,1,2,\dots}$ generates ℓ^2. Thus $\mathrm{Re}(A)$ is absolutely continuous with simple spectrum (that is, its spectral multiplicity function $n(t) \equiv 1$ on its absolutely continuous support α). Since $A^*A - AA^* = D = \mathrm{diag}\{\alpha_0^2, \alpha_1^2 - \alpha_0^2, \alpha_2^2 - \alpha_1^2, \dots\}$, then $\|D\| = 1$ and $\pi = \pi\|D\| \leqslant \leqslant \int_\alpha F(t)\mathrm{d}t$, where $F(t)$ is the (linear) measure of the cross section $\sigma(A) \cap \{z : \mathrm{Re}(z) = t\}$; see [6]. But $\sigma(A)$ is the unit disk and it follows that $\alpha = [-1,1]$ (modulo a zero set). Q.E.D.

REMARK. A slightly different argument is that $\mathrm{tr}(D)(= \|D\|) = 1$ and $\pi = = \pi\mathrm{tr}(D) \leqslant \int_\alpha n(t)F(t)\mathrm{d}t = \int_\alpha F(t)\mathrm{d}t$; see [11]. (Here $n(t)$ is the spectral multiplicity function of $\mathrm{Re}(A)$.)

THEOREM 2.3. *Let S be a unilateral subnormal weighted shift defined by* (1.8) *and* (1.9). *Then*

$$\sum_0^\infty (1 - \alpha_n^2) < \infty \iff \mathrm{d}\nu(1) > 0. \tag{2.9}$$

Proof. Let $\sum = \sum_0^\infty$. That $\mathrm{d}\nu(1) > 0 \Rightarrow \sum(1-\alpha_n^2) < \infty$ was shown in the proof of Theorem 2.1 above. To prove the reverse implication suppose that $\sum(1-\alpha_n^2) < \infty$. It will be shown that $\mathrm{d}\nu(1) > 0$.

Now $\sum(1 - \alpha_n^2) = \sum c_n^{-1} \int r^{2n}(1 - r^2)\mathrm{d}\nu$ $\left(\text{clearly } \int = \int_0^1 = \int_0^{1-}\right)$. Since $\sum(1-\alpha_n^2) < \infty$ then $\sum_N^\infty (1-\alpha_n^2) \to 0$ as $N \to \infty$. Also the sequence $\{c_n = \int_0^1 r^{2n}\mathrm{d}\nu\}$ is monotone nonincreasing and hence

$$\sum_N^\infty (1 - \alpha_n^2) = \sum_N^\infty c_n^{-1} \int_0^1 r^{2n}(1-r^2)\mathrm{d}\nu \geqslant$$

$$\geqslant c_N^{-1} \int_0^{1-} r^{2N}(1 + r^2 + r^4 + \cdots)(1 - r^2)\mathrm{d}\nu \geqslant c_N^{-1} \int_0^{1-} r^{2N}\mathrm{d}\nu.$$

Now if $\mathrm{d}\nu(1) = 0$ then $c_N^{-1} \int_0^{1-} r^{2N}\mathrm{d}\nu = 1$, so that $\sum_N^\infty (1 - \alpha_n^2) \not\to 0$, a contradiction. Thus $\mathrm{d}\nu(1) > 0$, as was to be shown.

3. Before stating the next theorem it will be convenient to prove the following

LEMMA. *Let c_n be defined by (1.10) and for a fixed constant a, $0 < a < 1$, let*

$$c_n = \int_0^1 r^{2n} \mathrm{d}\nu(r) \quad \text{and} \quad c_n^* = \int_a^1 r^{2n} \mathrm{d}\nu(r) \tag{3.1}$$

and

$$\alpha_n^2 = c_{n+1}/c_n \quad \text{and} \quad \alpha_n^{*2} = c_{n+1}^*/c_n^*.$$

Then, for $b = \text{const}$, $0 < b < 1$,

$$\alpha_n^2 = \alpha_n^{*2}(1 + [b^{2n}]), \tag{3.2}$$

where $[b^{2n}]$ *denotes a term satisfying* $|[b^{2n}]| \leqslant \text{const}(b^{2n})$. *In particular,*

$$1 - \alpha_n^2 = 1 - \alpha_n^{*2} + [b^{2n}] \tag{3.3}$$

and

$$\alpha_{n+1}^2 - \alpha_n^2 = \alpha_{n+1}^{*2} - \alpha_n^{*2} + [b^{2n}]. \tag{3.4}$$

Proof. Clearly, $c_n = c_n^* + \int_0^a r^{2n} \mathrm{d}\nu \leqslant c_n^* + a^{2n}$, and if $a < a_1 < 1$, also $c_n^* \geqslant$ $\geqslant \int_{a_1}^1 r^{2n} \mathrm{d}\nu \geqslant k a_1^{2n}$ with $k = \int_{a_1}^1 \mathrm{d}\nu > 0$. Hence $c_n^* \leqslant c_n \leqslant c_n^* + a^{2n} = c_n^*(1 + a^{2n}/c_n^*) \leqslant$ $\leqslant c_n^*(1 + [b^{2n}])$ with $b = a/a_1$. So $c_n/c_n^* = 1 + [b^{2n}]$. But

$$\alpha_n^2 = c_{n+1}/c_n = (c_{n+1}^*/c_n^*)(c_n^*/c_n)(c_{n+1}/c_{n+1}^*) = \alpha_n^{*2}(1 + [b^{2n}]),$$

that is, (3.2).

It may be noted that $c_n^* / \int_a^1 \mathrm{d}\nu = \int_0^1 r^{2n} \mathrm{d}\nu_a(r)$, where $\nu_a(r)$ is the probability function on $[0,1]$ defined by $\nu_a(r) = 0$ for $0 \leqslant r < a$ and $\nu_a(r) = \nu(r) / \int_a^1 \mathrm{d}\nu$ for $a \leqslant r \leqslant 1$, so that the α_n^* are the weights of the unilateral subnormal weighted shift associated with $\nu_a(r)$.

The following theorem is crucial for the remainder of this paper.

THEOREM 3.1. *Let S be a unilateral subnormal weighted shift defined by* (1.8) *and* (1.9) *with the self-commutator $S^*S - SS^* = D$. Then*

$$\mathrm{tr}(D^{\frac{1}{2}}) < \infty \Rightarrow \mathrm{d}\nu(1) > 0. \tag{3.5}$$

Proof. In view of (1.12) and Theorem 2.3, relation (3.5) is equivalent to

$$\sum(\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}} < \infty \Rightarrow \sum(1 - \alpha_n^2) < \infty, \tag{3.5$'$}$$

which, in view of (3.3) and (3.4), is equivalent to

$$\sum(\alpha_{n+1}^{*2} - \alpha_n^{*2})^{\frac{1}{2}} < \infty \Rightarrow \sum(1 - \alpha_n^{*2}) < \infty. \tag{3.5$''$}$$

What this means is that there is no loss of generality in supposing that the given probability function $\nu(r)$ already satisfies

$$\nu(r) = 0 \quad \text{on } [0, a] \text{ for some constant } a,\ 0 < a < 1. \tag{3.6}$$

Henceforth, then, (3.6) will be assumed and the implication (3.5)$'$ will be proved.

Recall the identity $\left(\text{with } \int = \int_0^1 = \int_a^1\right)$

$$\left(\int fg\mathrm{d}\nu\right)^2 + \frac{1}{2}\int\int(f(r)g(s) - f(s)g(r))^2\mathrm{d}\nu(r)\mathrm{d}\nu(s) = \int f^2\mathrm{d}\nu\int g^2\mathrm{d}\nu. \tag{3.7}$$

Now,

$$1 - \alpha_n^2 = 1 - c_{n+1}/c_n = (c_n - c_{n+1})/c_n = c_n^{-1}\int r^{2n}(1 - r^2)\mathrm{d}\nu,$$

so that, since $1 - r^2 = (1 + r)(1 - r)$ and $0 \leqslant r \leqslant 1$, one has

$$1 - \alpha_n^2 \sim c_n^{-1}\int r^{2n}(1 - r)\mathrm{d}\nu. \tag{3.8}$$

(See relation (2.4) for notation.)

If $f(r) = r^n$ and $g(r) = (1 - r)r^n$, then, on dividing by c_n^2 in (3.7), one has

$$\begin{aligned}&\left(\int r^{2n}(1 - r)\mathrm{d}\nu/c_n\right)^2 + \frac{1}{2}\left(\int\int r^{2n}s^{2n}(r - s)^2\mathrm{d}\nu\mathrm{d}\nu/c_n\right)^2 = \\ &= \int r^{2n}(1 - r)^2\mathrm{d}\nu/c_n.\end{aligned} \tag{3.9}$$

Next,

$$\alpha_{n+1}^2 - \alpha_n^2 = c_{n+2}/c_{n+1} - c_{n+1}/c_n = (c_{n+2}c_n - c_{n+1}^2)/c_nc_{n+1},$$

that is,

$$\alpha_{n+1}^2 - \alpha_n^2 = \frac{1}{2}\int\int r^{2n}s^{2n}(r^2 - s^2)^2 \mathrm{d}\nu\mathrm{d}\nu/c_n c_{n+1}.$$

Since $c_{n+1}/c_n \sim 1$ (in fact, $c_{n+1}/c_n \to 1$ as $n \to \infty$) and $r^2 - s^2 = (r-s)(r+s)$ with $0 < a \leqslant r,\ s \leqslant 1$, it is clear that

$$\alpha_{n+1}^2 - \alpha_n^2 \sim \int\int r^{2n}s^{2n}(r-s)^2 \mathrm{d}\nu\mathrm{d}\nu/c_n^2. \tag{3.10}$$

Consequently, in view of (3.8) and (3.10), relation (3.9) implies

$$(1-\alpha_n^2)^2 + \alpha_{n+1}^2 - \alpha_n^2 \sim d_n/c_n \quad \text{where } d_n = \int r^{2n}(1-r)^2\mathrm{d}\nu. \tag{3.11}$$

Now

$$1 - \alpha_N^2 = \sum_N^\infty(\alpha_{n+1}^2 - \alpha_n^n) = \sum_N^\infty(\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}}(\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}},$$

so that, by (3.11),

$$1 - \alpha_N^2 \leqslant \text{ const } \sum_N^\infty (d_n/c_n)^{\frac{1}{2}}(\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}}. \tag{3.12}$$

Let $c(x)$ be defined by

$$c(x) = \int_0^1 r^x \mathrm{d}\nu(r). \tag{3.13}$$

Then

$$c'(x) = \int r^x \log r\,\mathrm{d}\nu(r) \ \text{ and } \ c''(x) = \int r^x \log^2 r\,\mathrm{d}\nu, \tag{3.14}$$

and hence, in view of the Schwarz inequality, $(c'/c)' = (c''c - c'^2)/c^2 \geqslant 0$. Thus c'/c is negative and increasing (as x increases), so that $|c'|/c$ is decreasing. Since $|c'| = \int r^x \mathrm{d}\nu_1(r)$ with $\mathrm{d}\nu_1 = |\log r|\mathrm{d}\nu$, it is clear that $c''/|c'|$ is also decreasing and hence $(c''/|c'|)(|c'|/c) = c''/c$ is decreasing. Since $|\log r|/(1-r) \to 1$ as $r \to 1\text{-}0$, it is clear that

$$d_n = \int r^{2n}(1-r)^2\mathrm{d}\nu \sim \int r^{2n}\log^2 r\,\mathrm{d}\nu = c''(2n).$$

Consequently, $d_n/c_n = (d_n/c''(2n))(c''(2n)/c_n) \sim c''(2n)/c_n$. But $c''(2n)/c(2n) = c''(2n)/c_n$ is monotone decreasing as $n \to \infty$ and hence (3.12) implies that

$$1 - \alpha_N^2 \leqslant \text{ const } (d_N/c_N)^{\frac{1}{2}} \sum_N^\infty (\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}}. \tag{3.15}$$

Since, for $A, B \geqslant 0$, $(A+B)^{\frac{1}{2}} \leqslant A^{\frac{1}{2}} + B^{\frac{1}{2}}$, it follows from (3.11) that $(d_N/c_N)^{\frac{1}{2}} \leqslant \leqslant \text{const}(1-\alpha_N^2 + (\alpha_{N+1}^2 - \alpha_N^2)^{\frac{1}{2}})$ and hence, by (3.15), $1-\alpha_N^2 \leqslant \text{const}(1-\alpha_N^2 + +(\alpha_{N+1}^2-\alpha_N^2)^{\frac{1}{2}})p_N$, where $p_N = \sum_N^\infty (\alpha_{n+1}^2 - \alpha_n^2)^{\frac{1}{2}} \to 0$ as $N \to \infty$. Hence, for N large,

$$(1-\alpha_N^2)(1-\text{const}p_N) \leqslant \text{const}(\alpha_{N+1}^2-\alpha_N^2)^{\frac{1}{2}}p_N \leqslant \text{const}(\alpha_{N+1}^2-\alpha_N^2)^{\frac{1}{2}},$$

and therefore (3.5)′. This completes the proof of Theorem 3.1.

Since, by Theorem 3.1, $\text{tr}(D^{\frac{1}{2}}) < \infty \Rightarrow \mathrm{d}\nu(1) > 0$ and since this latter condition is the same as $c_n \geqslant \text{const} > 0$ (for all n), it is clear from the argument given above (see (3.11) in particular) that $\text{tr}(D^{\frac{1}{2}}) < \infty \Leftrightarrow \mathrm{d}\nu(1) > 0$ and $\sum (d_n)^{\frac{1}{2}} < \infty$. But $\sum (d_n)^{\frac{1}{2}} < \infty \Leftrightarrow \sum (c''(2n))^{\frac{1}{2}} < \infty$. Since $c''(x) \downarrow$ as $x \to \infty$ this can be stated in terms of the function $c(x)$ of (3.13) as

THEOREM 3.2. *Under the hypothesis of Theorem* 3.1,

$$\text{tr}(D^{\frac{1}{2}}) < \infty \Longleftrightarrow \mathrm{d}\nu(1) > 0 \text{ and } \int^\infty (c''(x))^{\frac{1}{2}}\mathrm{d}x < \infty, \tag{3.16}$$

where $c(x)$ *is defined by* (3.13).

4. THEOREM 4.1. *Suppose that* S *is a unilateral subnormal weighted shift defined by* (1.8) *and* (1.9) *satisfying*

$$\text{tr}(D^{\frac{1}{2}}) < \infty. \tag{4.1}$$

Then its minimal normal extension, N, *satisfies*

$$N = \begin{pmatrix} S_0 & 0 \\ 0 & T^* \end{pmatrix} + \textit{trace class operator}, \tag{4.2}$$

where S_0 *is a unilateral shift, so that, in particular,*

$$(\text{Re}(N))_a \cong \text{Re}(S_0) \oplus \text{Re}(T) \tag{4.3}$$

and, more generally,

$$(p(N,N^*))_a \cong (p(S_0,S_0^*))_a \oplus (p(T^*,T))_a, \tag{4.4}$$

where $p(B,B^*)$ *denotes a selfadjoint polynomial in a bounded operator* B.

Proof. One need only note that, by Theorem 3.1, relation (4.1) implies (2.1). An application of Theorem 2.1 (see also (1.5) and (2.6)) then implies (4.2), hence also (4.3) and (4.4).

Since $N \cong N_\mu$ where $(N_\mu f)(z) = zf(z)$, $f(z)$ in $L^2(\mu)$, the spectral multiplicity of $(p(N, N^*))_{\mathrm{a}}$ is known. In addition, so is that of $(p(S_0, S_0^*))_{\mathrm{a}}$. In fact, since the self-commutator of S_0 (as that of S in general) is of trace class, it is clear that $p(S_0, S_0^*) = $ $= \mathrm{Re}(Q)+$trace class operator, where $Q = a_0 I + \sum_1^N a_n S_0^n$. In view of Rosenblum-Kato perturbation theory one has

$$(4.5) \qquad (p(S_0, S_0^*))_{\mathrm{a}} \cong (\mathrm{Re}(Q))_{\mathrm{a}}.$$

However, Q is an analytic (even polynomial) Toeplitz operator. Unless Q is a multiple of the identity, $\mathrm{Re}(Q)$ is absolutely continuous. Moreover, its spectral multiplicity can be read off from the Rosenblum-Ismagilov multiplicity theory. (See [10], Corollary 2, p. 716 and [5].)

In the present case, let $f(\mathrm{e}^{\mathrm{i}t}) = a_0 + \sum_1^N a_n \mathrm{e}^{\mathrm{i}nt} = u + \mathrm{i}v$, where $u(t) = \mathrm{Re}(f)$ on $[0, 2\pi]$. If $n(\lambda)$ denotes the Banach indicatrix of $u(t)$, that is, the number of times the line $y = \lambda$ hits the graph of $u(t)$ on $[0, 2\pi]$, then, except for a finite number of values λ, $n(\lambda)$ is even and $N(\lambda) = (\frac{1}{2})n(\lambda)$ is the required multiplicity function of the operator of multiplication by u on H^2, that is, of $(p(S_0, S_0^*))_{\mathrm{a}}$. (In the trivial case where $u \equiv$ constant, $(p(S_0, S_0^*))_{\mathrm{a}}$ is absent.)

As a consequence of Theorem 4.1 and relation (4.5) the spectral multiplicity function of $(p(T^*, T))_{\mathrm{a}}$ can then be read off from the known spectral multiplicity functions of $(p(N, N^*))_{\mathrm{a}}$ and $(p(S_0, S_0^*))_{\mathrm{a}}$.

EXAMPLE 1. Suppose that $\nu(r)$ has a pure point spectrum consisting of $r = \frac{1}{2}$ and $r = 1$. Since 1 is an isolated point of the spectrum of $\nu(r)$ then $c''(x) = $ $= \int_0^1 r^x \log^2 r \mathrm{d}\nu(r) \doteq 2^{-x} \log^2 2 \mathrm{d}\nu(\frac{1}{2})$ and $\mathrm{tr}(D^{\frac{1}{2}}) < \infty$ by (3.16). It follows from (4.3) that the multiplicity function $m(\lambda)$ of $\mathrm{Re}(T)$ is defined (a.e.) by $m(\lambda) = 3$ on $(-\frac{1}{2}, \frac{1}{2})$, $m(\lambda) = 1$ on $(\frac{1}{2}, 1)$ and $(-1, -\frac{1}{2})$, and $m(\lambda) = 0$ for $|\lambda| > 1$.

EXAMPLE 2. Suppose that $\nu(r)$ is given by

$$(4.6) \qquad \mathrm{d}\nu(r) = r\mathrm{d}r \text{ on } [0, 1) \text{ and } \mathrm{d}\nu(1) = \frac{1}{2}.$$

Then $\mathrm{tr}(D^{\frac{1}{2}}) < \infty$ (see, e. g., (5.8) of Theorem 5.2 below). Then $\mathrm{Re}(T)$ has spectral multiplicity function $m(\lambda) = \infty$ or 0 according as $|\lambda| < 1$ or $|\lambda| > 1$. Also, $(T^*T)_{\mathrm{a}}$

(or $(\tilde{T}T^*)_a$) has a multiplicity function given by $m(\lambda) = \infty$ on $(0,1)$ and $m(\lambda) = 0$ elsewhere.

Since, by Theorem 3.1, $\mathrm{tr}(D^{\frac{1}{2}}) < \infty \Rightarrow \mathrm{d}\nu(1) > 0$, it is clear from Theorem 4.1 that whenever $\mathrm{tr}(D^{\frac{1}{2}}) < \infty$, the spectral multiplicity of $(p(S_0, S_0^*))_a$ is determined from the contribution to $(p(N, N^*))_a$ of the discontinuity of $\nu(r)$ at $r = 1$. The spectral multiplicity of $(p(T^*, T))_a$ is determined not only by $\mathrm{d}\nu(1)$ but by the behavior of $\nu(r)$ at all points of its spectrum.

5. A necessary and sufficient condition in order that $\mathrm{tr}(D^{\frac{1}{2}}) < \infty$ was given in (3.16). In general, however, it is not easy to establish the convergence of the integral appearing there. On the other hand, if the probability function $\nu(r)$ is absolutely continuous and "well behaved" on some interval $(1-a, 1)$ one can obtain a more readily verifiable integral condition.

THEOREM 5.1. *Let S be a unilateral subnormal weighted shift defined by* (1.8) *and* (1.9) *satisfying*

$$\mathrm{d}\nu(1) > 0. \tag{5.1}$$

Suppose that $\nu(t)$ is absolutely continuous on some interval $(1-a,1)$ ($a =$ const > 0), so that $\mathrm{d}\nu = \nu'(t)\mathrm{d}t$ on this interval. In addition, suppose that

$$\nu'(t)(1-t)^{-p} \uparrow \text{ on } (1-a, 1) \tag{5.2}$$

for some constant $p \geqslant 0$ and that

$$\nu'(t)(1-t)^{q} \downarrow \text{ on } (1-a, 1) \tag{5.3}$$

for some constant q, $0 \leqslant q < 3$. Then

$$\mathrm{tr}(D^{\frac{1}{2}}) < \infty \Longleftrightarrow \int^{1-} (\nu'(t)/(1-t))^{\frac{1}{2}}\mathrm{d}t < \infty. \tag{5.4}$$

Proof. Since $\mathrm{d}\nu(1) > 0$, it is clear from (3.16) that (5.4) will be proved if it is shown that

$$\int^{\infty} (c''(x))^{\frac{1}{2}}\mathrm{d}x < \infty \Longleftrightarrow \int^{1-} (\nu'(t)/(1-t))^{\frac{1}{2}}\mathrm{d}t < \infty. \tag{5.5}$$

First, the implication $\Rightarrow$ of (5.5) will be proved. One has $c''(x) \geqslant \int_r^1 t^x \log^2 t\, d\nu(t)$ with $r = 1 - 1/x$ for x sufficiently large. Since $\log^2 t/(1-t)^2 \to 1$ as $t \to 1$-, then for appropriate generic "const" one has (for large x)

$$c''(x) \geqslant \text{const} \int_r^1 t^x (1-t)^2 \nu'(t)\mathrm{d}t \geqslant \text{const}\, r^x \int_r^1 \nu'(t)(1-t)^{-p}(1-t)^{2+p}\mathrm{d}t.$$

Since $r^x = (1-1/x)^x \to \mathrm{e}^{-1}$ as $x \to \infty$, one has by (5.2),

$$c''(x) \geqslant \text{const}\, \nu'(r)(1-r)^{-p} \int_r^1 (1-t)^{2+p}\mathrm{d}t \geqslant \text{const}\, \nu'(r)(1-r)^3.$$

Since $x = (1-r)^{-1}$ then $\mathrm{d}x = (1-r)^{-2}\mathrm{d}r$ and hence $(c''(x))^{\frac{1}{2}}\mathrm{d}x \geqslant \text{const}(\nu'(r)/(1-r))^{\frac{1}{2}}\mathrm{d}r$. This proves $\Rightarrow$ in (5.5).

So far, the hypothesis (5.3) has not been used. It will however be needed, along with (5.2), in the proof of $\Leftarrow$ in (5.5).

First, we note that, in general, both

$$\int_0^1 t^x \log^2 t\, \mathrm{d}\nu \Big/ \int_0^1 t^x (1-t)^2 \mathrm{d}\nu \quad \text{and} \quad \int_0^1 t^x \log^2 t\, \mathrm{d}\nu \Big/ \int_a^1 t^x \log^2 t\, \mathrm{d}\nu$$

tend to 1 as $x \to \infty$. This follows from an argument similar to that used in the proof of Lemma in Section 3. The idea is that $c''(x)$ for large x is determined essentially only by the behavior of $\nu(t)$ as $t \to 1$-0. Consequently, there is no loss of generality in supposing that (5.2) holds on (0,1), and, for conveniece, this hypothesis will be made.

For x sufficiently large,

$$c''(x) \leqslant \text{const} \int_0^1 t^x (1-t)^2 \nu'(t)\mathrm{d}t$$

where

$$\int = \int_0^r + \int_r^1 \quad \text{with } r = 1 - 1/x.$$

But

$$\int_0^r = \int_0^r t^x (1-t)^{2+p} \nu'(t)(1-t)^{-p}\mathrm{d}t \leqslant \nu'(r)(1-r)^{-p} \int_0^r t^x (1-t)^{2+p}\mathrm{d}t \leqslant$$

$$\leqslant \nu'(r)(1-r)^{-p}\int_0^1 t^x(1-t)^{2+p}\mathrm{d}t.$$

However, the last integral is the Beta function $B(x+1,3+p)$, which can be expressed in terms of Gamma functions and then asymptotically estimated via Stirling's formula. Thus, $B(x+1,3+p)=\Gamma(x+1)\Gamma(3+p)/\Gamma(x+4+p)\sim \mathrm{const}/x^{3+p}$; see, e.g., Greenberg [2], pp. 63, 66. Hence, for large x,

$$\int_0^r \leqslant \mathrm{const}\,\nu'(r)(1-r)^{-p}/x^{3+p} \leqslant \mathrm{const}\,\nu'(r)(1-r)^3. \tag{5.6}$$

Next,

$$\int_r^1 \leqslant \mathrm{const}\int_r^1 t^x(1-t)^{2-q}\nu'(t)(1-t)^q\mathrm{d}t \leqslant$$

$$\leqslant \nu'(r)(1-r)^q\int_r^1(1-t)^{2-q}\mathrm{d}t = \mathrm{const}\,\nu'(r)(1-r)^3,$$

in view of (5.3). Thus, for large x,

$$\int_r^1 \leqslant \mathrm{const}\,\nu'(r)(1-r)^3. \tag{5.7}$$

In view of (5.6) and (5.7) one has by an argument similar to that occurring earlier in this section, for large x, $(c''(x))^{\frac{1}{2}}\mathrm{d}x \leqslant \mathrm{const}(\nu'(r)/(1-r))^{\frac{1}{2}}\mathrm{d}r$ and $\Leftarrow$ of (5.5) follows.

As a corollary of Theorem 5.1 one has

THEOREM 5.2. *Suppose that* $\mathrm{d}\nu(1)>0$ *and that* $\nu(t)$ *is absolutely continuous on* $(1-a,1)$ *for some constant* $a>0$. *If* $\varepsilon=\mathrm{const}>0$ *then*

$$\nu'(t)\leqslant \mathrm{const}/(1-t)|\log(1-t)|^{2+\varepsilon} \text{ near } t=1 \Rightarrow \mathrm{tr}(D^{\frac{1}{2}})<\infty \tag{5.8}$$

and

$$\nu'(t)\leqslant \mathrm{const}/(1-t)|\log(1-t)|^{2} \text{ near } t=1 \not\Rightarrow \mathrm{tr}(D^{\frac{1}{2}})<\infty \tag{5.9}$$

Proof of (5.8). Let $\nu_1(t)$ be defined on $(0,1)$ so that

$$\nu_1'(t)=A/(1-t)|\log(1-t)|^{2+\varepsilon} \text{ near } t=1 \text{ and } \mathrm{d}\nu_1(1)>0, \tag{5.10}$$

where the constant A is chosen so that $\int_0^1 \mathrm{d}\nu_1 = 1$. A change of variable $u = |\log(1-t)|$ shows that

$$\int^{1-} (\nu_1'(t)/(1-t))^{\frac{1}{2}} \mathrm{d}t < \infty. \tag{5.11}$$

However, in view of (5.8),

$$\mathrm{d}\nu(t) = \nu'(t)\mathrm{d}t \leqslant \text{const}\,\nu_1'(t)\mathrm{d}t = \text{const}\,\mathrm{d}\nu_1(t) \tag{5.12}$$

near $t = 1$, so that, for x large,

$$c''(x) \leqslant \text{const}\, c_1''(x); \tag{5.13}$$

where $c(x)$ is given by (3.13) and $c_1(x) = \int_0^1 t^x \mathrm{d}\nu_1(t)$. Clearly the function $\nu_1(t)$ of (5.10) satisfies the hypotheses of Theorem 5.1 with $p = 0$ in (5.2) and $q = 1$ in (5.3), and so, by (5.11) and (5.13), $\int^{\infty} (c''(x))^{\frac{1}{2}} \mathrm{d}x \leqslant \text{const} \int^{\infty} (c_1''(x))^{\frac{1}{2}} \mathrm{d}x < \infty$. Since $\mathrm{d}\nu(1) > 0$, relation (5.8) follows.

In order to prove (5.9), let $\nu'(t) = A/(1-t)\log^2(1-t)$ near $t = 1$ and $\mathrm{d}\nu(1) > 0$ in such a way that $\int_0^1 \mathrm{d}\nu = 1$. Again, if one makes the change of variable $u = |\log(t-1)|$, it is seen that $\int^{1-} (\nu'(t)/(1-t))^{\frac{1}{2}} \mathrm{d}t = \infty$ and so, by Theorem 5.1, $\mathrm{tr}(D^{\frac{1}{2}}) = \infty$.

Similarly, one can sharpen Theorem 5.2 to obtain

$$\mathrm{d}\nu(1) > 0 \text{ and, for some } \varepsilon = \text{const} > 0,$$

$$\nu'(t) \leqslant \text{const}/(1-t)\log^2(1-t)\Big|\log|\log(1-t)|\Big|^{2+\varepsilon} \tag{5.8$_1$}$$

$$\text{near } t = 1 \Rightarrow \mathrm{tr}(D^{\frac{1}{2}}) < \infty,$$

while

$$\mathrm{d}\nu(1) > 0 \text{ and } \nu'(t) \leqslant \text{const}/(1-t)\log^2(1-t)\Big|\log|\log(1-t)|\Big|^{2} \tag{5.9$_1$}$$

$$\text{near } t = 1 \not\Rightarrow \mathrm{tr}(D^{\frac{1}{2}}) < \infty.$$

More generally, if $L_1(t) = |\log(1-t)|$ and, for $n = 2,3,\ldots$, $L_n = L_n(t) = |\log L_{n-1}(t)|$, then with (5.8) and (5.9) corresponding to $n = 0$ in $(5.8)_n$ and $(5.9)_n$ below,

$$d\nu(1) > 0 \text{ and, for some } \varepsilon = \text{const} > 0 \text{ and some fixed } n = 0,1,2,\ldots,$$

$$(5.8)_n \qquad \nu'(t) \leqslant \text{const}/(1-t)L_1^2L_2^2\ldots L_n^2L_{n+1}^{2+\varepsilon}$$
$$\text{near } t = 1 \Rightarrow \text{tr}(D^{\frac{1}{2}}) < \infty;$$

while

$$(5.9)_n \qquad \begin{gathered} d\nu(1) > 0 \text{ and, for some fixed } n = 0,1,2,\ldots, \\ \nu'(t) \leqslant \text{const}/(1-t)L_1^2L_2^2\ldots L_n^2L_{n+1}^2 \text{ near } t = 1 \not\Rightarrow \text{tr}(D^{\frac{1}{2}}) < \infty. \end{gathered}$$

The proof of $(5.8)_n$ and $(5.9)_n$ for $n = 1,2,\ldots$, is similar to that for the case $n = 0$ if one makes the change of variabile $u = L_{n+1}(t)$.

6. In this section there will be obtained conditions on a general $\nu(t)$ (not necessarily absolutely continuous near $t = 1$) which assure that $\text{tr}(D^{\frac{1}{2}}) < \infty$.

THEOREM 6.1. *Let* $d\nu(1) > 0$. *If* $\varepsilon = \text{const} > 0$, *then*

$$(6.1) \qquad \nu(1\text{-}) - \nu(t) \leqslant \text{const}/|\log(1-t)|^{2+\varepsilon} \text{ near } t = 1 \Rightarrow \text{tr}(D^{\frac{1}{2}}) < \infty.$$

Proof. If $\int = \int_0^{1-}$ then

$$c''(x) = \int t^x \log^2 t \, d\nu = -\int t^x \log^2 t \, dA(t), \qquad \text{where } A(t) = \nu(1\text{-}) - \nu(t).$$

Thus

$$c''(x) = -t^x \log^2 t \, A(t)\Big|_0^{1-} + \int A(t)(t^x \log^2 t)' dt.$$

Since $(\ldots)\Big|_0^{1-} = 0$, then, in view of the hypothesis, it must be shown that if

$$(6.2) \qquad \begin{gathered} c''(x) = \int A(t)(t^x \log^2 t)' dt, \text{ with } |A(t)| \leqslant \text{const}/|\log(1-t)|^p \\ \text{and } p = \text{const} > 2, \end{gathered}$$

then $(c''(x))^{\frac{1}{2}} \in L(1,\infty)$. Since $(t^x \log^2 t)' = xt^{x-1}\log^2 t + 2t^{x-1}\log t$, it is enough to show that

$$(6.3) \qquad (a(x))^{\frac{1}{2}} \quad \text{and} \quad (b(x))^{\frac{1}{2}} \in L(1,\infty),$$

where

$$(6.4)\qquad \begin{aligned} a(x) &= x\int t^{x-1}(\log^2 t/|\log(1-t)|^p)\mathrm{d}t \quad \text{and} \\ b(x) &= \int t^{x-1}(|\log t|/|\log(1-t)|^p)\mathrm{d}t. \end{aligned}$$

An integration by parts leads to

$$a(x) = -\int t^x(\log^2 t/|\log(1-t)|^p)'\mathrm{d}t,$$

so that

$$(6.5)\quad a(x) = -2\int t^{x-1}(\log t/|\log(1-t)|^p)\mathrm{d}t + p\int t^x \log^2 t/(1-t)|\log(1-t)|^{p+1})\mathrm{d}t.$$

Since $|\log t|/(1-t) \to 1$ as $t \to 1$ the integrand of the second integral of (6.5) is majorized by that of the first integral near $t = 1$. Since the first integral is just $2b(x)$, relation (6.3) will be proved if it is shown that $b(x)$ of (6.4) satisfies

$$(6.6)\qquad (b(x))^{\frac{1}{2}} \in L(1,\infty).$$

An integration by parts of the integral for $b(x)$ then yields

$$b(x) = x^{-1}t^x|\log t|/|\log(1-t)|^p\Big|_0^1 - x^{-1}\int t^x(\ldots)'\mathrm{d}t.$$

On noting that $(\ldots)\Big|_0^1 = 0$ and that $|\log t|/(1-t) \to 1$ as $t \to 1-$, one obtains

$$(6.7)\qquad \begin{aligned} b(x) &= x^{-1}\int (t^{x-1}/|\log(1-t)|^p)\mathrm{d}t + d(x), \\ &\text{where } d(x) \leqslant (\mathrm{const})px^{-1}\int (t^x/|\log(1-t)|^{p+1})\mathrm{d}t. \end{aligned}$$

The integrand of the second integral of (6.7) is majorized by that of the first near $t = 1$ and so, in order to prove (6.6), it is sufficient to show that

$$(6.8)\qquad (b_1(x))^{\frac{1}{2}} \in L(1,\infty) \quad \text{where } b_1(x) = x^{-1}\int_0^1 (t^{x-1}/|\log(1-t)|^p)\mathrm{d}t.$$

An integration by parts of the integral for $b_1(x)$ then gives

$$b_1(x) = -x^{-2}\int_0^1 t^x(1/|\log(1-t)|^p)'\mathrm{d}t = -x^{-2}\int_0^{\frac{1}{2}} - x^{-2}\int_{\frac{1}{2}}^1.$$

Since $\left|x^{-2}\int_0^{\frac{1}{2}}\cdot\right| \leqslant \text{const}/x^2 2^x$, which has a square root in $L(1,\infty)$, then, in order to prove (6.8), it is enough to show that

$$(6.9) \qquad |b_2(x)|^{\frac{1}{2}} \in L(1,\infty) \text{ where } b_2(x) = -x^{-2}\int_{\frac{1}{2}}^{1} t^x(1/|\log(1-t)|^p)'\mathrm{d}t.$$

Now $b_2(x) = -x^{-2}\left(\int_{\frac{1}{2}}^{1-x^{-\frac{1}{2}}} + \int_{1-x^{-\frac{1}{2}}}^{1}\right)$, for x sufficiently large. Since $(1/|\log(1-t)|^p)' < 0$, one has

$$0 \leqslant -x^{-2}\int_{\frac{1}{2}}^{1-x^{-\frac{1}{2}}} \leqslant -x^{-2}(1-x^{-\frac{1}{2}})^x(1/|\log(1-t)|^p)\Big|_{\frac{1}{2}}^{1-x^{-\frac{1}{2}}} \leqslant$$

$$\leqslant \text{const}\, x^{-2}\mathrm{e}^{-x^{\frac{1}{2}}} \text{ (for } x \text{ large)},$$

and so $-x^{-2}\int_{\frac{1}{2}}^{1-x^{-\frac{1}{2}}}$ has a square root in $L(1,\infty)$.

Thus, one need only show that $-x^{-2}\int_{1-x^{-\frac{1}{2}}}^{1}(1/|\log(1-t)|^p)'\mathrm{d}t$ has a square root in $L(1,\infty)$. But this last expression $= \text{const}/x^2|\log x|^p$ and, since $p > 2$, its square root is in $L(1,\infty)$. This completes the proof of Theorem 6.1.

Assertion (6.1) can be sharpened in a manner analogous to the improvement $(5.8)_n$ of (5.8). Thus, for a fixed $n = 0,1,2,\ldots$,

$$(6.10) \qquad \begin{gathered}\mathrm{d}\nu(1) > 0 \text{ and } \nu(1\text{-}) - \nu(t) \leqslant \text{const}(L_1^2 L_2^2 \ldots L_n^2 L_{n+1}^{2+\varepsilon}) \\ \text{near } t = 1 \Rightarrow \operatorname{tr}(D^{\frac{1}{2}}) < \infty.\end{gathered}$$

The proof is similar to that given above and will be omitted.

The results of (6.1) can be stated in an equivalent form as follows:

Theorem 6.2. *Let $\mathrm{d}\nu(1) > 0$. Then, for $\varepsilon = \text{const} > 0$,*

$$(6.11) \qquad \int_0^{1\text{-}} |\log(1-t)|^{2+\varepsilon}\mathrm{d}\nu < \infty \Rightarrow \operatorname{tr}(D^{\frac{1}{2}}) < \infty.$$

Proof. It is sufficient to show that the hypotheses of (6.1) and (6.11) are equivalent. First, assume the hypotesis of the left side of the implication (6.11). Then

$$|\log(1-t)|^{2+\varepsilon}(\nu(1\text{-})-\nu(t)) = |\log(1-t)|^{2+\varepsilon}\int_t^{1-} \mathrm{d}\nu(u) \leqslant$$

$$\leqslant \int_t^{1-} |\log(1-u)|^{2+\varepsilon}\mathrm{d}\nu(u)$$

and hence (6.1) holds (with the same value of ε).

Next, assume the hypothesis of (6.1). It will be shown that

$$\int_0^{1-} |\log(1-t)|^{2+\delta}\mathrm{d}\nu < \infty, \quad \text{for any } \delta = \text{const}, 0 < \delta < \varepsilon. \tag{6.13}$$

To see this, let $A(t) = \nu(1\text{-}) - \nu(t)$. Then, for t near 1, a partial integration yields

$$\int_t^{1-} |\log(1-u)|^{2+\delta}\mathrm{d}\nu(u) = -\int_t^{1} |\log(1-u)|^{2+\delta}\mathrm{d}A(u) = -|\log(1-u)|^{2+\delta}A(u)\Big|_t^{1-} +$$

$$+\int_t^{1-}(A(u)(2+\delta)|\log(1-u)|^{1+\delta}/(1-u))\mathrm{d}u.$$

In view of (6.1), both $(\ldots)\Big|_t^{1-}$ and the last integral are finite and (6.13) follows. This completes the proof of Theorem 6.2.

Whether the equivalent implications (6.1) or (6.11) become false if $\varepsilon = 0$ will remain undecided. That both implications are false if $2+\varepsilon$ is replaced by 1 is easy to see, however, by choosing $\nu(t)$ so that, near $t = 1$, $\nu'(t) = \text{const}/(1-t)|\log(1-t)|^2$. Then $\nu(1\text{-}) - \nu(t) = \text{const}/|\log(1-t)|$ and, by (5.9), $\text{tr}(D^{\frac{1}{2}}) = \infty$. Thus, we pose the following

QUESTION. *Do the implications of* (6.1) *and* (6.11) *remain valid if* $2+\varepsilon$ *is replaced by* $1+\varepsilon$ *?*

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