J. OPERATOR THEORY
25(1991), 331-345

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# NEW CLASSES OF SPACES ON WHICH COMPACT OPERATORS SATISFY THE DAUGAVET EQUATION

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## 1. INTRODUCTION

We use the standard terminology on Banach spaces and Banach lattices (see [3], [17]).

In 1963 Daugavet [7] proved that each compact operator in $C[0,1]$ satisfies the following equation:

$$\text{(DE)} \qquad \|I+T\| = 1 + \|T\|,$$

where $I$ denotes, as usual, the identity operator.

In 1956 Foiaş and Singer [10] extended this result to arbitrary atomless $C(K)$--spaces, and in 1966 Lozanovski [16] discovered that compact operators on $L_1[0,1]$ enjoy the same property. Apart from a couple of generalizations of these results obtained in [15] and [19], there was no special attention given to this subject until Babenko and Pichugov [5] recently rediscovered the Lozanovski result and pointed out its applications to approximation theory. Since then the interest in this subject has begun to grow and a number of publication has followed (see [6], [12–14], [21]), the main result of which may be summarized as follows:

THEOREM 1. (Daugavet, Lozanovski, Foiaş and Singer, Krasnoselski, Babenko and Pichugov, Holub, Kamowitz, Chauveheid, Schmidt). *Let $X$ be an arbitrary atomless $L_1(\mu)$- or $L_\infty(\mu)$-space and $T$ be an arbitrary weakly compact operator on $X$. Then $T$ satisfies the Daugavet equation* (DE).

We would like to stress that the classical spaces $L_1(\mu)$ and $L_\infty(\mu)$ often play an exceptional role in many questions, and initially it seemed quite natural to expect that the (DE)-property might be characteristic for these spaces. But (and it is our main result here) that is not the case. The purpose of this work is to present new classes of spaces (different from $L_1(\mu)$ and $L_\infty(\mu)$) for which the (DE)-property still holds.

Let us remark incidentally that throghout our work we use the isometric point of view, and it is the only reasonable point of view when one deals with (DE), since the property (DE) is isometric by its nature.

In the concluding section we will present several related results, comments and questions. This section can be viewed as a brief survey on the (DE)-property.

## MAIN RESULTS

In this section our main results, Theorem 3 and 4, will be established. We start with an auxiliary approximation lemma that is of interest in its own right.

APPROXIMATION LEMMA 2. *Let $T$ be a weakly compact operator on the space $C(Q)$, where $Q$ is a compact Hausdorff space without isolated points. Let two functions $u$, $v \in C(Q)$ and two different points $p_0$, $q_0 \in Q$ be fixed. Then for each $\varepsilon > 0$ and for each neighborhood $V(q_0)$ of $q_0$ there is a function $\tilde{u} \in C(Q)$ and a point $q_0' \in V(q_0)$ such that*

(i) $\|\tilde{u}\| \leqslant \|u\|$,

(ii) $|\tilde{u}(q_0') + T\tilde{u}(q_0') + v(q_0')| \geqslant |u(p_0)| + |Tu(q_0) + v(q_0)| - \varepsilon$.

*Proof.* We will assume that the neighborhood $V(q_0)$ is small enough to guarantee that

(1) $$|v(q) - v(q_0)| \leqslant \varepsilon/4$$

and

(2) $$|Tu(q) - Tu(q_0)| \leqslant \varepsilon/4$$

for each $q \in V(q_0)$, otherwise we may choose a smaller $V'(q_0) \subset V(q_0)$ satisfying these conditions. Since $q_0$ is not an isolated point we can easily find in $C(Q)$ a bounded sequence $\{x_n\}$ satisfying the following conditions:

(a) $|x_n| \wedge |x_m| = 0, \quad n \neq m$;

(b) $\operatorname{supp}(x_n) \subset V(q_0)$;

(c) $|u(q) + x_n(q)| \leqslant \|u\|$ for each $q \in V(q_0)$;

(d) For each $n \in \mathbb{N}$ there exists a $q_n \in V(q_0)$, such that

(3) $$(u(q_n) + x_n(q_n))(Tu(q_0) + v(q_0)) \geqslant 0,$$

that is, the signs of the numbers $u(q_n) + x_n(q_n)$ and $Tu(q_0) + v(q_0)$ coincide and

$$|u(q_n) + x_n(q_n)| \geqslant |u(p_0)| - \varepsilon/4. \tag{4}$$

At this point we will apply the following important result due to P. Dodds (see [9, Theorem 4.2] or [3, Theorem 18.6]): Any weakly compact operator from a Banach lattice $G$ into a Banach space $E$ sends an arbitrary order bounded sequence of pairwise disjoint elements in $G$ to a norm convergent sequence in the space $E$. Therefore, since $\{x_n\}$ is exactly such a sequence and since $T$ is weakly compact, we have $\|Tx_n\| \to 0$. Fix an arbitrary number $n$ for which

$$\|Tx_n\| < \varepsilon/4 \tag{5}$$

and let

$$\tilde{u} := u + x_n \tag{6}$$

and

$$q_0' := q_n. \tag{7}$$

Using (6) and (7) we can rewrite (4) as follows:

$$|\tilde{u}(q_0')| \geqslant |u(p_0)| - \varepsilon/4. \tag{8}$$

We claim that the function $\tilde{u}$ and the point $q_0'$ are as required.

Indeed, (6), (a), and (c) clearly imply (i). To verify (ii) we have the following chain of estimates:

$$|\tilde{u}(q_0') + T\tilde{u}(q_0') + v(q_0')| = |\tilde{u}(q_0') + Tu(q_0') + Tx_n(q_0') + v(q_0')| \overset{\text{by}(5)}{\geqslant}$$

$$\overset{\text{by}(5)}{\geqslant} |\tilde{u}(q_0') + Tu(q_0') + v(q_0')| - \varepsilon/4 \overset{\text{by }(1),(2)}{\geqslant} |\tilde{u}(q_0') + Tu(q_0) + v(q_0)| - 3\varepsilon/4 \overset{\text{by}(3)}{=}$$

$$\overset{\text{by}(3)}{=} |\tilde{u}(q_0')| + |Tu(q_0) + v(q_0)| - 3\varepsilon/4 \overset{\text{by }(8)}{\geqslant} |u(p_0)| + |Tu(q_0) + v(q_0)| - \varepsilon.$$

Q. E. D.

REMARKS. 1) Let us notice that if (after $\tilde{u}$ and $q_0'$ have been found and fixed) we slightly perturb the function $v$ (up to a $\delta$), then the inequality in (ii) will not be affected by more than $2\delta$. We will use this simple observation later on.

2) The conclusion of the lemma becomes trivial if $|u(q_0)| \geqslant |u(p_0)|$ and the signs of $u(q_0)$ and $Tu(q_0) + v(q_0)$ coincide, since in this case the function $u$ itself can be

taken as $\tilde{u}$. But, apart from this trivial case, the change of $u$ to $\tilde{u}$ implies the change in $Tu$ and the conclusion is no longer trivial. Moreover, taking a normed $u$ for which $Tu$ almost attains norm $\|T\|$ and setting $v = 0$ this lemma immediately gives a new proof of Theorem 1.

Recall that if $X$ and $Y$ are arbitrary Banach spaces, then $Z = X \oplus_\infty Y$ (resp. $X \oplus_1 Y$ ) is the standard Banach space of all pairs $z = (x, y) = x \oplus y$ with $x \in X$, $y \in Y$ and with the norm $\|z\| = \|x\|_X \vee \|y\|_Y$ (resp. $\|z\| = \|x\|_X + \|y\|_Y$). It is well known that $(X \oplus_\infty Y)^* = X^* \oplus_1 Y^*$ and $(X \oplus_1 Y)^* = X^* \oplus_\infty Y^*$.

Throughout this section $\mu$ and $\nu$ are arbitrary atomless measures and we let

$$U := L_\infty(\mu) \oplus_1 L_\infty(\nu)$$

and

$$V := L_1(\mu) \oplus_\infty L_1(\nu).$$

We will denote by $Q_1$ and $Q_2$ the Stone spaces of $L_\infty(\mu)$ and $L_\infty(\nu)$, respectively. Recall that $L_\infty(\mu)$ can be identified with $C(Q_1)$ and $L_\infty(\nu)$ can be identified with $C(Q_2)$ (see, for example, [22]).

In what follows we will deal with an operator $T : X \oplus Y \to X \oplus Y$ and for our purposes it will be convenient to treat $T$ as a matrix operator

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{9}$$

where $T_1 : X \to X$, $T_2 : Y \to X$, $T_3 : X \to Y$, and $T_4 : Y \to Y$.

Accordingly, for each $z = (x, y) \in X \oplus Y$ one has

$$Tz = (T_1x + T_2y) \oplus (T_3x + T_4y) \tag{10}$$

and

$$(I + T)z = (x + T_1x + T_2y) \oplus (y + T_3x + T_4y). \tag{11}$$

It is worth mentioning that neither of the spaces $U$ or $V$ is an AL- or AM--space. Nevertheless, and it is our first main result in this article, all weakly compact operators on $U$ and $V$ satisfy (DE).

THEOREM 3. *Let $Z$ be either of the spaces $U$ or $V$ introduced above and let $T : Z \to Z$ be an arbitrary weakly compact operator. Then $T$ satisfies the Daugavet equation*

$$\|I + T\| = 1 + \|T\|. \tag{DE}$$

*Proof.* We start with the case $Z = U = X \oplus_1 Y = L_\infty(\mu) \oplus_1 L_\infty(\nu)$. The second case will be reduced to this one. As we have agreed above, we represent a given weakly compact operator $T : Z \to Z$ in the matrix form (9).

In view of (10) we have

$$\|T\| = \sup_{\|z\| \leqslant 1} \|Tz\| = \sup_{\|x\|+\|y\| \leqslant 1} \left\{ \|T_1 x + T_2 y\|_\infty + \|T_3 x + T_4 y\|_\infty \right\}. \tag{12}$$

Hence, for each $\varepsilon > 0$ we can find a $z_0 = (x_0, y_0) \in Z$ such that

$$\|Tz_0\| \geqslant \|T\| - \varepsilon, \tag{13}$$

where

$$\|z_0\| = \|x_0\|_\infty + \|y_0\|_\infty = 1. \tag{14}$$

This implies that there exist points $\overline{p}_0 \in Q_1$ and $\overline{\overline{p}}_0 \in Q_2$ such that $\|x_0\|_\infty = = |x_0(\overline{p}_0)|$, $\|y_0\|_\infty = |y_0(\overline{\overline{p}}_0)|$ and therefore,

$$|x_0(\overline{p}_0)| + |y_0(\overline{\overline{p}}_0)| = 1. \tag{15}$$

In view of (13) and (12) we have

$$\|T_1 x_0 + T_2 y_0\|_\infty + \|T_3 x_0 + T_4 y_0\|_\infty \geqslant \|T\| - \varepsilon$$

and consequently, there are points $\overline{q}_0 \in Q_1$, $\overline{\overline{q}}_0 \in Q_2$ such that

$$|(T_1 x_0 + T_2 y_0)(\overline{q}_0)| + |(T_3 x_0 + T_4 y_0)(\overline{\overline{q}}_0)| \geqslant \|T\| - \varepsilon. \tag{16}$$

Since $Q_1$ and $Q_2$ have no isolated points, it is plain to see that without loss of generality we can assume that $\overline{p}_0 \neq \overline{q}_0$ and $\overline{\overline{p}}_0 \neq \overline{\overline{q}}_0$.

Let $V(\overline{q}_0)$ be a clopen neighborhood of $\overline{q}_0$ on which the oscillation of $T_2 y_0$ is less than $\varepsilon/4$.

To the functions $x_0$ and $v = T_2 y_0$, to the operator $T_1$, and to the points $\overline{p}_0$ and $\overline{q}_0$ we apply the Approximation Lemma. By this lemma there is a function $\tilde{x}_0$ and a point $\overline{q}_0'$ in $V(\overline{q}_0)$ such that

$$\|\tilde{x}_0\| \leqslant \|x_0\| \tag{17}$$

and

$$|\tilde{x}_0(\overline{q}_0') + T_1 \tilde{x}_0(\overline{q}_0') + T_2 y_0(\overline{q}_0')| \geqslant |x_0(\overline{p}_0)| + |T_1 x_0(\bar{q}_0) + T_2 y_0(\bar{q}_0)| - \varepsilon. \tag{18}$$

Since $\tilde{x}_0$ is obtained from $x_0$ by a perturbation generated by a bounded sequence of pairwise disjoint functions, a second application of the Dodds theorem guarantees that we can additionally assume that

$$\|T_3\tilde{x}_0 - T_3x_0\|_\infty \leqslant \varepsilon/4. \tag{19}$$

Now we apply the Approximation Lemma for the second time to the functions $u = y_0$ and $v = T_3\tilde{x}_0$, to the operator $T_4$, and to the points $\overline{\overline{p}}_0 \in Q_2$and $\overline{\overline{q}}_0 \in Q_2$. By this lemma there exist a function $\tilde{y}_0$ and a point $\overline{\overline{q}}_0'$ in a vicinity of $\overline{\overline{q}}_0$ such that

$$\|\tilde{y}_0\| \leqslant \|y_0\| \tag{20}$$

and

$$\left|\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_4\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_3\tilde{x}_0\left(\overline{\overline{q}}_0'\right)\right| \geqslant \left|y_0\left(\overline{\overline{p}}_0\right)\right| + \left|T_4y_0\left(\overline{\overline{q}}_0\right) + T_3\tilde{x}_0\left(\overline{\overline{q}}_0\right)\right| - \varepsilon. \tag{21}$$

In view of (19) we can rewrite (21) as follows:

$$\begin{aligned} &\left|\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_4\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_3\tilde{x}_0\left(\overline{\overline{q}}_0'\right)\right| \geqslant \\ &\geqslant \left|y_0\left(\overline{\overline{p}}_0\right)\right| + \left|T_4y_0\left(\overline{\overline{q}}_0\right) + T_3x_0\left(\overline{\overline{q}}_0\right)\right| - 5\varepsilon/4. \end{aligned} \tag{22}$$

Finally, again using Remark 1 following the Approximation Lemma, we note that $\tilde{y}_0$ was obtained from $y_0$ by a perturbation generated by a bounded sequence of pairwise disjoint elements and hence, again by the Dodds theorem, $\|T_2\tilde{y}_0 - T_2y_0\|_\infty$ can be made as small as we wish; in particular, it will suffice if

$$\|T_2\tilde{y}_0 - T_2y_0\|_\infty \leqslant \varepsilon/4.$$

Consequently, inequality (18) can be rewritten as follows:

$$|\tilde{x}_0(q_0') + T_1\tilde{x}_0(q_0') + T_2\tilde{y}_0(q_0')| \geqslant |x_0(\overline{p}_0)| + |T_1x_0(\overline{q}_0) + T_2y_0(\overline{q}_0)| - 5\varepsilon/4. \tag{23}$$

Summing (23) and (22) we obtain

$$\begin{aligned} &|\tilde{x}_0(\overline{q}_0') + T_1\tilde{x}_0(\overline{q}_0') + T_2\tilde{y}_0(\overline{q}_0')| + \left|\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_4\tilde{y}_0\left(\overline{\overline{q}}_0'\right) + T_3\tilde{x}_0\left(\overline{\overline{q}}_0'\right)\right| \geqslant \\ &\geqslant |x_0(\overline{p}_0)| + \left|y_0\left(\overline{\overline{p}}_0\right)\right| + |T_1x_0(\overline{q}_0) + T_2y_0(\overline{q}_0)| + \\ &+ \left|T_4y_0\left(\overline{\overline{q}}_0\right) + T_3x_0\left(\overline{\overline{q}}_0\right)\right| - 5\varepsilon/2. \end{aligned} \tag{24}$$

Our next step is to estimate both sides in (24). Let $R$ (resp. $L$) be the right (resp. left) side in (24). Using (15) and (16) we can estimate $R$ from below in the following manner:

$$R \geqslant 1 + \|T\| - 7\varepsilon/2. \tag{25}$$

An estimation for $L$ from above goes this way:

$$L \leqslant \|\tilde{x}_0 + T_1\tilde{x}_0 + T_2\tilde{y}_0\|_\infty + \|\tilde{y}_0 + T_3\tilde{x}_0 + T_4\tilde{y}_0\|_\infty \overset{(11)}{=}$$

$$= \|(I+T)\,\tilde{z}_0\| \leqslant \|I+T\|\,\|\tilde{z}_0\|,$$

and since

$$\|\tilde{z}_0\| = \|\tilde{x}_0\| + \|\tilde{y}_0\| \overset{(17),(20)}{\leqslant} \|x_0\| + \|y_0\| \overset{(14)}{=} 1$$

we have

$$L \leqslant \|I+T\|. \tag{26}$$

From (25) and (26) we finally obtain the desired inequality

$$\|I+T\| \geqslant 1 + \|T\| - 7\varepsilon/2.$$

Since $\varepsilon$ is arbitrary and since the converse inequality

$$\|I+T\| \leqslant 1 + \|T\|$$

is trivially correct, the proof of the identity $\|I+T\| = 1+\|T\|$ in the case $T\colon L_\infty(\mu)\oplus_1 \oplus_1 L_\infty(\nu) \to L_\infty(\mu)\oplus_1 L_\infty(\nu)$ is finished.

Now we can easily prove the remaining case when $T$ is a weakly compact operator on the space $V = L_1(\mu)\oplus_\infty L_1(\nu)$. Indeed, then the conjugate operator $T^*$ is likewise a weakly compact operator on $L_1^*(\mu)\oplus_1 L_1^*(\nu) = L_\infty(\mu)\oplus_1 L_\infty(\nu)$, and hence, we can apply the part already proved:

$$\|I+T\| = \left\|(I+T)^*\right\| = \|I+T^*\| = 1 + \|T^*\| = 1 + \|T\|$$

Q. E. D.

In our search for spaces different from the classical AL- and AM-spaces, the choice of the spaces $U$ and $V$ introduced above may seem quite natural, since these spaces are the "closest relatives" of the classical ones, and $U$ is order isomorphic to an $L_\infty$-space and $V$ to an $L_1$-space. However, our next result shows that we can construct a Banach lattice which is not isomorphic to either of the classical spaces, and nevertheless each compact operator on which satisfies (DE).

To this end we introduce the following two (types of) spaces

$$Z := L_\infty(\mu)\oplus_\infty L_1(\nu),$$

and

$$W := L_\infty(\mu)\oplus_1 L_1(\nu)$$

where $\mu$ and $\nu$ are, as before, arbitrary atomless measures. That is, as opposed to the case of the spaces $U$and $V$, the "components" of $Z$ and $W$ are no longer of the same type, and therefore neither $Z$ nor $W$ is isomorphic to a classical space.

THEOREM 4. *Each compact operator on the spaces $Z$ and $W$ satisfies* (DE).

*Proof.* Let $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ be an arbitrary compact operator on the space $Z = X \oplus_\infty Y$, where $X := L_\infty(\mu)$, $Y := L_1(\nu)$, $T_1 : X \to X$, $T_2 : Y \to X$, $T_3 : X \to Y$, and $T_4 : Y \to Y$. That is, we again use the matrix form (9) to represent a given operator $T : Z \to Z$. Hence by (10) we have

$$\|T\| = \sup_{\|z\| \leqslant 1} \|Tz\| = \sup_{\|x\|_\infty \vee \|y\|_1 \leqslant 1} \{\|T_1x + T_2y\|_\infty \vee \|T_3x + T_4y\|_1\}. \tag{27}$$

It is easy to show (see Corollary to Lemma 5 below) that the supremum in (27) may be evaluated by considering only $\|x\|_\infty = 1$ and $\|y\|_1 = 1$, i. e.,

$$\|T\| = \sup_{\|x\|_\infty = \|y\|_1 = 1} \{\|T_1x + T_2y\|_\infty \vee \|T_3x + T_4y\|_1\}. \tag{28}$$

The rest of the proof may be splitted into the following two cases:

$$\text{(I)} \quad \|T\| = \sup_{\|x\|_\infty = \|y\|_1 = 1} \|T_1x + T_2y\|_\infty \tag{29}$$

or

$$\text{(II)} \quad \|T\| = \sup_{\|x\|_\infty = \|y\|_1 = 1} \|T_3x + T_4y\|_1. \tag{30}$$

The proof in Case (I) is similar to that of Theorem 3 and we will only outline it. By (29) there exists $z_0 = (x_0, y_0) \in Z$ such that

$$\|z_0\|_Z = \|x_0\|_\infty = \|y_0\|_1 = 1 \tag{31}$$

and

$$\|T_1x_0 + T_2y_0\|_\infty \geqslant \|T\| - \varepsilon,$$

where $\varepsilon$ is an arbitrary fixed positive number. Therefore we can find distinct points $p_0$ and $q_0$ in $\mathbf{Q}_1$, the Stone space on which $X := L_\infty(\mu)$ is represented as $C(Q_1)$, such that

$$|x_0(p_0)| = 1 \tag{31'}$$

and

$$|(T_1x_0 + T_2y_0)(q_0)| \geqslant \|T\| - \varepsilon. \tag{32}$$

Fix also any neighborhood $V(q_0)$ of the point $q_0$. Then, applying the approximation lemma, we can find a function $\tilde{x}_0$ and a point $q_0' \in V(q_0)$, such that $\|\tilde{x}_0\| \leqslant \|x_0\|$ and

$$(33) \qquad |(\tilde{x}_0 + T_1\tilde{x}_0 + T_2 y_0)(q_0')| \geqslant |x_0(p_0)| + |(T_1 x_0 + T_2 y_0)(q_0)| - \varepsilon.$$

In view of (32), (31′) and (31) we can rewrite (33) as follows:

$$|(\tilde{x}_0 + T_1\tilde{x}_0 + T_2 y_0)(q_0')| \geqslant 1 + \|T\| - 2\varepsilon,$$

and by (11) this implies that

$$\|I + T\| \geqslant 1 + \|T\|.$$

This finishes Case (I).

Now we will adress the harder case (II). Again fix $\varepsilon > 0$ and find $z_0 = (x_0, y_0) \in Z$ satisfying (31) and such that

$$(34) \qquad \|T_3 x_0 + T_4 y_0\|_1 \geqslant \|T\| - \varepsilon.$$

Of course we cannot expect to have $\|y_0 + T_3 x_0 + T_4 y_0\|_1 \geqslant 1 + \|T\| - \varepsilon$, but assume that we can find a function $\overline{y}_0 \in Y$ with the following properties:

(a) $\|\overline{y}_0\|_1 = 1$,

(b) $\|T_4\overline{y}_0 - T_4 y_0\|_1 < \delta$, where $\delta > 0$ is an arbitrary fixed number, and

(c) $\nu(\operatorname{supp}(\overline{y}_0)) < \delta$, i. e., the measure of the support of this function $\overline{y}_0$ is as small as we wish.

Consider the function $v = T_3 x_0 + T_4 y_0 \in L_1(\nu)$. By virtue of the absolute continuity of the integral there exists $\delta > 0$ such that whenever $\nu(D) < \delta$ we have

$$\left\|v\chi_D\right\|_1 < \varepsilon.$$

In view of (c) this implies that for $D = \operatorname{supp}(\overline{y}_0)$ we have

$$(35) \qquad \begin{aligned} \|\overline{y}_0 + v\|_1 &= \left\|\overline{y}_0 + (v - v\chi_D) + v\chi_D\right\|_1 \geqslant \left\|\overline{y}_0 + (v - v\chi_D)\right\|_1 - \varepsilon = \\ &= \|\overline{y}_0\|_1 + \left\|v - v\chi_D\right\|_1 - \varepsilon \geqslant \|\overline{y}_0\|_1 + \|v\|_1 - 2\varepsilon = 1 + \|v\|_1 - 2\varepsilon. \end{aligned}$$

Finally, let us consider the element $\overline{z} = (x_0, \overline{y}_0) \in Z$. By (a) and (31) we have $\|\overline{z}\|_Z = 1$. Using this element we can estimate from below the norm $\|I + T\|$.

$$(36) \qquad \begin{aligned} \|I + T\| &\geqslant \|(I + T)(\overline{z})\|_Z = \|x_0 + T_1 x_0 + T_2\overline{y}_0\|_\infty \vee \|\overline{y}_0 + T_3 x_0 + T_4\overline{y}_0\|_1 \geqslant \\ &\geqslant \|\overline{y}_0 + T_3 x_0 + T_4\,\overline{y}_0\|_1 = \|\overline{y}_0 + T_3 x_0 + T_4 y_0 + T_4\overline{y}_0 - T_4 y_0\|_1 \overset{(b)}{\geqslant} \\ &\geqslant \|\overline{y}_0 + T_3 x_0 + T_4\overline{y}_0\|_1 - \delta = \|\overline{y}_0 + v\| - \delta \overset{(35)}{=} \\ &1 + \|v\|_1 - 2\varepsilon - \delta \overset{(34)}{\geqslant} 1 + \|T\| - 3\varepsilon - \delta. \end{aligned}$$

Since $\varepsilon$ and $\delta$ are arbitrary, (36) implies that

$$\|I+T\| \geqslant 1+\|T\|,$$

and this, apart from the existence of the function $\overline{y}_0$ satisfying (a)–(c), finishes the proof in Case (II). The proof of the existence of $\overline{y}_0$ will be given below in Lemma 6. Thus, we have proved that each compact operator on the space $Z$ satisfies (DE).

We omit a more or less similar (but rather cumbersome) proof for the space $W$. The proof is a combination of the technique used above and that of the "consecutive corrections" used in the proof of Theorem 3. Q. E. D.

REMARKS. 1) We do not know whether Theorem 4 remains valid for arbitrary weakly compact operators.

2) Since $Z^*$ and $W^*$ both have atoms, it is impossible to reduce the proof for one of these spaces to that for another as it is done in Theorem 3. (Compare Remark 5 in the last section.)

Now we are ready to justify the two assumptions made in the course of our proof above.

LEMMA 5. *Let $u$, $v$ be two arbitrary elements of a Banach space $E$ and let $\gamma \in [0,1]$. Then*

$$\|u+\gamma v\| \leqslant \max\{\|u+v\|, \|u-v\|\}.$$

*Proof.* It is easily seen that the following identity

$$u+\gamma v = \frac{(1+\gamma)}{2}(u+v) + \frac{(1-\gamma)}{2}(u-v)$$

holds. Applying the triangle inequality we come to the desired conclusion. Q. E. D.

COROLLARY. Let $T: X \to Y$ and $S: Y \to Y$ be two operators, where $X$ and $Y$ are Banach spaces. Then

$$\sup_{\|x\| \leqslant 1, \|y\| \leqslant 1} \|Tx+Sy\| = \sup_{\|x\|=1, \|y\|=1} \|Tx+Sy\|.$$

LEMMA 6. *Let $T$ be a compact operator on the space $L_1(\nu)$, where $\nu$ is an atomless measure. Then for each $\delta > 0$ and for each $y \in L_1(\nu)$ there exists $\overline{y} \in L_1(\nu)$ such that*

(a) $\|y\|_1 = \|\overline{y}\|_1$,

(b) $\|Ty - T\overline{y}\|_1 < \delta$, and

(c) $\nu(\mathrm{supp}(\overline{y})) < \delta$.

*Proof.* Since $T$ is compact we may assume without loss of generality that $T$ is a finite rank operator, i. e., there exist $\{x_i\}_{i=1}^n \in L_1(\nu)$ and $\{f_i\}_{i=1}^n \in L_\infty(\nu)$ such that for any $x \in L_1(\nu)$,

$$Tx = \sum_{i=1}^{n} f_i(x)x_i.$$

Moreover, we may additionally assume that $x_i \in A$ and $f_i \in B$, where $A$ is a dense subspace of $L_1(\nu)$ and $B$ is a dense subspace of $L_\infty(\nu)$. Let $A$ and $B$ be the subspaces of step functions. Let $\{x_i\}$ and $\{f_i\}$ be fixed. Then we can find scalars $a_k^{(i)}$ and $b_k^{(i)}$, and pairwise disjoint measurable sets $\{E_k\}_{k=1}^l$, that partition the underlying measure space, such that

$$x_i = \sum_{k=1}^{l} a_k^{(i)} \chi_{E_k} \quad \text{and} \quad f_i = \sum_{k=1}^{l} b_k^{(i)} \chi_{E_k}.$$

Now fix an arbitrary $y \in L_1(\nu)$. Without loss of generality we may assume that on each set $E_k$ the function $y$ does not change its sign, otherwise we can go to a finer partition with this property. Set $\delta_k = \int_{E_k} y\, d\nu$. In view of the previous assumption about the partition we have $\left|\int_{E_k} y\, d\nu\right| = \int_{E_k} |y|\, d\nu$ for all $k = 1, \cdots, l$. Further, for each $k$ let us fix any measurable subset $e_k \subset E_k$ and finally, let $\overline{y} = \sum_{k=1}^{l} \frac{\delta_k}{\nu(e_k)} \chi_{E_k}$. (Of course we are assuming that $\nu(e_k) > 0$.) Obviously $\overline{y}$ satisfies (a) and also (c), provided $\sum_{k=1}^{l} \nu(e_k) < \delta$. To see that $\overline{y}$ satisfies (b) it is enough to notice that $f_i(\overline{y}) = f_i(y)$ for all $i$. We omit the straightforward verification. Q. E. D.

REMARK. To prove the omitted part of Theorem 4 we need the following extra property of the function $\overline{y}$: If $S$ is a second compact operator on $L_1(\nu)$ with values in $L_\infty(\mu)$, then (d) $\|S\overline{y} - Sy\| < \delta$. The proof that there exists an $\overline{y}$ satisfying (a)–(d) is exactly the same.

## CONCLUDING REMARKS AND COMMENTS

1. Both the Approximation Lemma 2 and Theorems 3 and 4 still hold if one replaces $L_\infty$-spaces by arbitrary atomless AM-spaces. The proof remains basically the same.

2. It is easy to see that Theorem 3 includes Theorem 1 as a special case. Indeed, if $S$ is a weakly compact operator on an atomless $L_\infty(\mu)$ or on $L_1(\mu)$, then

$$T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

is a weakly compact operator on $U$ or $V$, respectively, and thus

$$\|I + S\| = \|I + T\| = 1 + \|T\| = 1 + \|S\|;$$

i. e., we obtain all the generalization of both Daugavet's and Lozanovsky's results.

3. Answering a question posed to the author by Professor C. Foiaş, we can introduce spaces even more general then $U$ and $V$, for which the conclusion of Theorem 3 still holds. To define them, let $\mu_i$ $(i = 1, 2, \cdots, n)$ be an atomless measure an let

$$\tilde{U} := L_\infty(\mu_1) \oplus_1 L_\infty(\mu_2) \oplus_1 \cdots \oplus_1 L_\infty(\mu_n)$$

and

$$\tilde{V} := L_1(\mu_1) \oplus_\infty L_1(\mu_2) \oplus_\infty \cdots \oplus_\infty L_1(\mu_n).$$

Then we have the following result.

THEOREM 7. *Each weakly compact operator on $\tilde{U}$ and $\tilde{V}$ satisfies* (DE).

The proof of this theorem is a slight refinement of that of Theorem 3 and is omitted.

4. In this remark we want to mention a simple explanation of why we deal with atomless spaces. The counterexamples to Theorem 1 when spaces have atoms are well known (see [6], [10], [12], [14]), but it is worth giving a rule-of-thumb reason for the existence of this counterexamples. In any finite dimensional Banach space $E$ the identity operator $I$ is compact and thus, putting $T := -I$, we get $\|I + T\| \neq 1 + \|T\|$ in $E$. Basically the same idea works when the spaces have atoms.

5. We would like to stress that (no matter how strange it may seem while dealing with AL- and AM-spaces) the order of "treating" spaces in our proof of Theorem 3, i. e., first $U$ and then $V$ cannot be reversed for the following reason. The conjugate to $L_\infty(\mu) \oplus_1 L_\infty(\nu)$ is, of course, $L^*_\infty(\mu) \oplus_\infty L^*_\infty(\nu) = L_1(\tilde{\mu}) \oplus_\infty L_1(\tilde{\nu})$, but in this case the new measures $\tilde{\mu}$ and $\tilde{\nu}$ are not atomless and this prevents one from reducing the second case to the first one.

Since the proof of Theorem 3 is not simple and since Theorem 1 has attracted so much attention lately, we remark here that Theorem 1 admits a rather simple proof based on the following two propositions. The idea of this approach goes back to Lozanovski [16].

Let $X$ be any AL- or AM-space and $T : X \to X$ be a bounded operator.

PROPOSITION 8. *If the identity operator $I$ and $T$ are disjoint, then $T$ satisfies* (DE)

PROPOSITION 9. *If in addition $X$ is atomless, then each weakly compact operator $T$ is disjoint from $I$.*

The detailed proofs of these propositions are presented in [2]. Here we mention only that the proof of Proposition 8 is more or less straightforward and that the proof of Proposition 9 depends on the fact that in the spaces under consideration, any operator dominated by an absolute value of a weakly compact operator is again weakly compact [4], [20]. Obviously, Theorem 1 is a direct corollary of these two propositions. It would be interesting to find a simpler proof along the same lines for Theorems 3 and 4, too. Obviously, Proposition 9 remains valid for spaces $\tilde{U}$ and $\tilde{V}$, but with Proposition 8 the situation is more complicated.

Still another proof of Theorem 1, based on the theory of ortomorphisms, has been found by K. Schmidt [21].

6. In [1] and [21] generalising an interesting result due to J. Holub [12] the authors have proved the following theorem.

THEOREM 10. *Let $T$ be an arbitrary continuous operator on an arbitrary AL- or AM-space. Then either $\|I+T\|$ or $\|I-T\|$ equals $1+\|T\|$; that is, either $T$ or $-T$ or both satisfy* (DE).

It would be interesting to find out whether or not this result can be generalized to the spaces $U$ and $V$, $Z$ and $W$, and $\tilde{U}$ and $\tilde{V}$ introduced above.

7. Throughout the paper we have restricted our attention to weakly compact operators only. But it is worthwile to mention that there are several results in which the (DE)-property is obtained for some other operators, too.

For instance, M. Krasnoselski [15] introduced a more general class of operators, called rammers, on any $C(K)$-space on a compact metric space $K$, which still satisfy (DE). Generalizing the concept of rammers, Diallo and Zabreiko [8] introduced a broader class of operators (called inframmers), and this class allows one to capture both Daugavet's and Lozanovski's cases simultaneously.

Now let $T$ be an almost integral operator [16], [18] on an arbitrary atomless $L_\infty(\mu)$-space. Then, as was shown by J. Synnatzschke [18], $T$ is disjoint from $I$ and hence, by Proposition 8, satisfies (DE). Both this and a similar result may be found in [19].

It is also worth mentioning that quite recently K. Schmidt [21] showed that Dunford-Pettis operators on classical spaces also satisfy (DE). It would be interesting to generalize this result to the spaces introduced in the present article.

8. Our final question seems to be rather hard because a positive answer to it will imply Theorems 3, 4, and 7 and also positive answers to the questions posed above.

Let $X$ and $Y$ be two arbitrary Banach spaces such that in each of them the Daugavet equation holds for every weakly compact (or compact) operator.

It is an open question whether or not the (DE)-property will hold in $X \oplus_\infty Y$ and $X \oplus_1 Y$.

If $X$ and $Y$ are both atomless AM- or AL-spaces, then the answer is "yes" (Theorem 3). If $X$ is an atomless AM-space, $Y$ is an atomless AL-space, and we deal with compact operators on $X \oplus_\infty Y$ or $X \oplus_1 Y$, then again the answer is "yes" (Theorem 4). But in general, we do not know the structure of the components $X$ and $Y$ and that makes the problem difficult.

9. The following two results are due to Professor T. Ando and they are included here (with his kind permission) since they are related to the problems under consideration.

PROPOSITION 11. *If $T$ is a lattice isomorphism on a Banach lattice $E$ such that* $\sup\limits_{n=\pm 1,\pm 2,\cdots} \|T^n\| < \infty$, *then* $\|I - T\| \geqslant 1$.

PROPOSITION 12. *If $G$ is a group of positive operators on a Banach lattice $E$ and* $\sup\limits_{T \in G} \|T\| = \gamma < \infty$, *then* $\|T_1 - T_2\| \geqslant 1/\gamma$ *for each distinct* $T_1, T_2 \in G$.

10. In conclusion we refer to a paper by C. Franchetti and E. Cheney [11], where there may be found several results suggesting possible connections between the (DE)--property and some geometric properties of Banach spaces, and to a paper by C. D. Aliprantis, O. Burkinshaw and the author [2] where, among other things, a complete description of operators satisfying (DE) on arbitrary $L_p(\mu)$-spaces ($1 < p < \infty$) is given.

*Added in proof.* Professor T. Ando has answered in the affirmative the question posed after Theorem 10. His elegant proof is based on a finite-dimensional evaluation of the norm of $T$ using extreme points and a subsequent approximation by conditional expectations.

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Received October 26, 1989.