J. OPERATOR THEORY
26(1991), 53-71

APPROXIMATIONS OF POSITIVE OPERARTORS AND CONTINUITY OF THE SPECTRAL RADIUS

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1. INTRODUCTION

Let $T = (a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix acting on some ℓ^p space, $1 \leqslant p \leqslant \infty$. Let $T_n = (a_{i,j}^n)_{i,j=1}^{\infty}$ where $a_{i,j}^n = a_{ij}$ if $1 \leqslant i,j \leqslant n$, $a_{i,j}^n = 0$ otherwise. We would like the following formula: $\lim_n r(T_n) = r(T)$ to be true. Morever, if v_n, v are vectors in ℓ^p such that $T_n v_n = r(T_n)v_n$, $Tv = r(T)v$ when v converges to v in some suitable topology (coordinatewise or the strong topology)? These questions have been adressed for stochastic matrices and many results can be found in [9]. We want to adress this kind of problems in a more general setting. To be more precise, we set the following framework: Let E be a Banach lattice and let T be a positive operator on E, $0 \leqslant T \in \mathcal{L}(E)$. Let $0 \leqslant T_n \in \mathcal{L}(E)$ be such that $0 \leqslant T_n \uparrow T$ i.e. T_n is an increasing sequence with supremum T. Let $r(T_n)$, $r(T)$ be spectral radius of T_n, T respectively. Let $v_n, v \in E$ be such that $T_n v_n = r(T_n)v_n$, $Tv = r(T)v$. Can we give some general results saying that $r(T) = \lim_n r(T_n)$ and $v = \lim_n v_n$ (in a suitable topology)? Our purpose is to give some positive results to these questions under some technical assumptions, the most essential one being "$r(T)$ is a Riesz point of $\sigma(T)$". We remark that T_n need not be taken increasing. The precise assumptions and further consequences of our results are discussed in the following sections. Finally, let us mention that the continuity of the spectral radius in the order topology when T is a compact operator was proved by A. R. Schep in [8].

2. PRELIMINARIES

A Banach lattice E is a Banach space with a lattice structure both compatible in the sense that if $x, y \in E$, $|x| \leqslant |y|$, then $\|x\| \leqslant \|y\|$. We write $0 \leqslant x \in E$ to say that x is a positive element of E. The notation $0 \leqslant x_n \uparrow$ $(0 \leqslant x_n \downarrow)$ is used as an abbreviation for "x is an increasing (decreasing) sequence of positive elements of E". $0 \leqslant x_n \uparrow x$ $(0 \leqslant x_n \downarrow x)$ means that x_n is an increasing (decreasing) sequence in E with supremum (infimum) x. We say that the sequence $\{x_n\} \subset E$ order converges to $x \in E$ if there exists $0 \leqslant u_n \downarrow 0$ such that $|x - x_m| \leqslant u_n$ for all $m \geqslant n$. A subset of the Banach lattice E is called *solid* if $x \in A$, $y \in E$ and $|y| \leqslant |x|$ implies $y \in A$. A solid vector subspace I of E is called *an ideal* of E. Given a Banach lattice E, a bounded linear operator T on E is called *positive* if $Tx \geqslant 0$ for all $0 \leqslant x \in E$. This is denoted by $0 \leqslant T \in \mathcal{L}(E)$. A positive operator $0 \leqslant E \in \mathcal{L}(E)$ is called *irreducible* if T has no invariant ideals except $\{0\}$ and E, i.e. if I is a closed ideal of E such that $T(I) \subset I$ then $I = \{0\}$ or $I = E$. Let us mention that the Banach lattice E is called *Dedekind-complete* if any majorized subset of E has a supremum $\sup A \in E$. If E is Dedekind-complete, then the set $\mathcal{L}^r(E) := \{T \in \in \mathcal{L}(E) : T = = T_1 - T_2$ for some $0 \leqslant T_1, T_2 \in \mathcal{L}(E)\}$ is a Banach lattice which is also Dedekind-complete. Finally, we say that a Banach lattice E has order continuous norm if the order intervals of E (sets of type $[x,y] = \{z \in E : x \leqslant z \leqslant y\}, x, y \in E$) are $\sigma(E, E')$-compact. Equivalenty, if $0 \leqslant x_\alpha \uparrow x \in E$ implies that $\lim_\alpha \|x_\alpha - x\| = 0$ ([7], Theorem II.5.10). More information about the structure of Banach lattices and positive operators can be found in [7] or [10].

Now we recall some definitions from spectral theory. Let be a bounded linear operator on the Banach space $E, T \in \mathcal{L}(E)$. The spectrum of T, the set of $z \in \mathbb{C}$ such that $z \cdot I - T$ is not invertible in $\mathcal{L}(E)$ will be denoted by $\sigma(T)$. The spectral radius of T, $r(T)$ is the number $\sup\{|z| : z \in \sigma(T)\}$ $(= \lim_n \|T_n\|^{1/n})$. If $z \in \mathbb{C} - \sigma(T) =: \rho(T)$, the resolvent of T, $R(z,T) := (z - T)^{-1}$ is an analytic function on $\rho(T)$. $\lambda \in \sigma(T)$ is called a *Riesz point* of $\sigma(T)$ if λ is a pole of the resolvent $R(z,T)$ with a residuum $P := \dfrac{1}{2\pi i} \displaystyle\int_C R(z,T)dz$ of finite rank (where C is a curve in the complex plane around λ containing λ as the only singularity of $R(z,T)$).

To finish these preliminaries, let us recall the following construction. Let us fixe an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the Fréchet filter and let E be a Banach space. The ultrapower of E with respect to $\mathcal{U}$, denoted by $\widehat{E}_{\mathcal{U}}$ or simply by $\widehat{E}$, is defined by $\ell^\infty(E)/c_{\mathcal{U}}(E)$ where $\ell^\infty(E) := \{(x_n)_{n=1} : x_n \in E, \sup_n \|x_n\| < \infty\}$ and $c_{\mathcal{U}}(E) := := \{(x_n) \in \ell^\infty(E) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. If E is a Banach lattice, then $\widehat{E}$ is again a Banach lattice. It is easy to construct a projection $m_{E''}$ from $\widehat{E}$ onto E''. Let $\widehat{x} \in \widehat{E}$, $\varphi \in E$. Then $\langle m_{E''}(\widehat{x}), \varphi\rangle := \lim_{\mathcal{U}} \langle x_n, \varphi\rangle$ defines the desired projection $m_{E''}$. If E is a dual Banach lattice $E = F'$, we can define $m_E : \widehat{E} \to E$ by $\langle m_E(\widehat{x}), \varphi\rangle := \lim_{\mathcal{U}} \langle x_n, \varphi\rangle$, $\widehat{x} \in \in \widehat{E}$, $\varphi \in F$. If E is a Banach lattice, $m_{E''}$, m_E are positive projections. Operators on E can be lifted to operators on $\widehat{E}$ by $\widehat{T}\widehat{x} = (Tx_n)_{\mathcal{U}}$, $\widehat{x} \in \widehat{E}$, $T \in \mathcal{L}(E)$, in such a way that $\sigma(\widehat{T}) = \sigma(T)$ ([7], Theorem V.1.4). Notice that the approximate spectrum of T is converted into the point spectrum of $\widehat{T}$. A basic idea which is exploited below was elaborated in [1] and is contained in the following result ([1], Theorem 3.4).

THEOREM 2.1. *Let E be a Banach space and let $T \in \mathcal{L}(E)$. Let $\partial_\infty\sigma(T)$ be the exterior boundary of $\sigma(T)$ (= the boundary of the unbounded connected component of $\rho(T)$). Then:*

$$\partial_\infty\sigma(T) \cap \sigma_{\text{ess}}(T) = \partial_\infty\sigma(T) \cap \{z \in \mathbf{C} : \dim\operatorname{Ker}(z - \widehat{T}) \text{ is infinite}\}.$$

If E is a dual Banach space and T is a dual operator, both sets coincide with $\partial_\infty\sigma(T) \cap \{z \in \mathbf{C} :$ there exists $\widehat{y} \in \widehat{E}$, $\widehat{y} \neq 0$, $m_E(\widehat{y}) = 0$ and $\widehat{T}\widehat{y} = z\widehat{y}\}$.

This result means that the eigenspace associated to a Riesz point in the exterior boundary of $\sigma(T)$ is contained in E and cannot be enlarged by going to $\widehat{E}$. This is the main idea which is exploited below. Let us mention that a special case of Theorem 2.1 can be found in ([6], D-III, Proposition 2.3). The precise statement of Theorem 2.1 is taken from [1].

3. THE MAIN RESULTS

First we prove the following result.

THEOREM 3.1. *Let E be a Banach lattice. Let $0 \leqslant T_n$, $T \in \mathcal{L}(E)$ be such that $T_n x \to Tx$ for all $x \in E$ and $\|(T_n - T)^+\| \to 0$. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Then $r(T_n) \to r(T)$.*

Proof. First, we prove our claim under the assumption that $0 \leqslant T_n \leqslant T$. Then it follows that $0 \leqslant r(T_n) \leqslant r(T)$. If $r(T) = 0$ we are done. Thus, suppose that $r(T) > 0$ and $r(T_n)$ does not converge to $r(T)$. Passing to a subsequence, if necessary, we can find $\lambda, \varepsilon > 0$ such that $r(T_n) \leqslant \lambda - \varepsilon < \lambda < r(T)$ and $\lambda \notin \sigma(T)$.

Let $0 \leqslant x \in E'$. We claim that $R(\lambda, T_n')x$ is bounded in E'. Otherwise, $\|R(\lambda, T_n')x\| \to +\infty$. Let $\rho_n := \|R(\lambda, T_n')x\|$, $z_n := \rho_n^{-1}R(\lambda, T_n')x$. Then $0 \leqslant z_n \in E'$. $\|z_n\| = 1$. From $T_n'R(\lambda, T_n') = \lambda R(\lambda, T_n') - I$ we get:

$$T_n'z_n = \lambda z_n - \rho_n^{-1}x. \tag{1}$$

Let $\mathcal{U}$ be a fixed ultrafilter on $\mathbf{N}$ containing the Fréchet filter. Let $\widehat{E'}$ be the ultrapower of E' with respect to $\mathcal{U}$. Let $\widehat{z} = (z_n)_{\mathcal{U}}$. Then, since $\rho_n \to +\infty$, from (1) it follows:

$$(T_n'z_n)_{\mathcal{U}} = \lambda\widehat{z}. \tag{2}$$

Hence $\lambda\widehat{z} \leqslant \widehat{T'}\widehat{z}$. Let $m_{E'} : \widehat{E'} \to E'$ be the canonical projection defined in Section 2. Since $r(T)$ is a Riesz point of $\sigma(T)$, $r(T')$ is a Riesz point of $\sigma(T')$. Then, using [7], Theorem 5.5 and the refinement in [1], Lemma 4.4, all the points in $\pi\sigma(T') := \{z \in \sigma(T') : |z| = r(T)\}$ are Riesz points of $\sigma(T')$. (Here we have used the posivity of T!) Let P be the spectral projection associated to the spectral set $\pi\sigma(T)$. P' maps E' onto the finite dimensional space $\bigoplus_{\lambda\in\sigma(T')} \mathrm{Ker}(\lambda - T')$. Let us suppose that $m_{E'}(\widehat{z}) \neq 0$. Then, $\widehat{P'T'}\widehat{z} = 0$ and

(3) $$(\widehat{T'} - \widehat{P'T'})\widehat{z} = \widehat{T'} - \widehat{P'T'}\widehat{z} = \widehat{T'}\widehat{z} \geqslant \lambda\widehat{z}.$$

This implies $r(T' - P'T') \geqslant \lambda$. Since we have freedom to choose λ from the beggining such that $\lambda < r(T - PT)$ we get a contradiction. Therefore, $m_{E'}(\widehat{z}) \neq 0$. Let $0 \leqslant \varphi \in E$. Then:

$$\langle m_{E'}(T'_n z_n),\ \varphi\rangle = \lim_{\mathcal{U}} \langle T'_n z_n,\ \varphi\rangle = \lim_{\mathcal{U}} \langle z_n, T_n\varphi\rangle.$$

Since $T_n\varphi \to T\varphi$ in the norm of E, we continue the chain of equalites to get:

(4) $$\langle m_{E'}(T'_n z_n),\ \varphi\rangle = \langle m_{E'}(z_n),\ T\varphi\rangle = \langle T' m_{E'}(z_n),\ \varphi\rangle.$$

Applying the projection $m_{E'}$ in (2) and using (4) we get:

(5) $$T' m_{E'}(\widehat{z}) = \lambda m_{E'}(\widehat{z}) \quad \text{where } m_{E'}(\widehat{z}) \neq 0.$$

This is a contradiction with our assumption that $\lambda \notin \sigma(T)$. Therefore, $R(\lambda, T'_n)x$ is bounded in E'. It follows that for all $x \in E'$, $R(\lambda, T'_n)x$ is bounded in E'. We can define the operator $R'(\lambda) : E' \to E'$ by $\langle R'(\lambda)x,\ \varphi\rangle = \lim_{\mathcal{U}} \langle R(\lambda, T'_n)x,\ \varphi\rangle$ $x \in E'$, $\varphi \in E$. Recall that

(6) $$\lambda R(\lambda, T'_n)x = x + T'_n R(\lambda, T'_n)x \quad x \in E.$$

Since $T_n\varphi \to T\varphi$ in E for all $\varphi \in E$, then

(7) $$\langle T'_n R(\lambda, T'_n)x,\ \varphi\rangle \to \langle R'(\lambda)x,\ T\varphi\rangle = \langle T' R'(\lambda)x,\ \varphi\rangle.$$

Letting $n \to \infty$ in (6) and using (7) we get $\lambda R'(\lambda)x = x + T'R'(\lambda)x$, i.e., $(\lambda - T')R'(\lambda)x = x$, for all $x \in E$. Since $\lambda \notin \sigma(T')$, $R'(\lambda) = (\lambda - T')^{-1}$. But, notice that $R'(\lambda) \geqslant 0$. Hence $(\lambda - T')^{-1} \geqslant 0$. This is contradiction ([7], Chapter V. Exercise 5).

Now, let $0 \geqslant T_n$, T be such that $T_n x \to Tx$ for all $x \in E$, $\|(T_n - T)^+\| \to 0$. From $T_n = T_n \wedge T + (T_n - T)^+$ it follows that $(T_n \wedge T)x \to Tx$ for all $x \in E$. The above proof says that $r(T_n \wedge T) \to r(T)$. Now write:

$$T_n \wedge T \leqslant T_n \leqslant V_n := T + (T_n - T)^+. \tag{8}$$

Notice that $V_n \to T$ in norm and $r(V_n) \geqslant r(T)$. Let $\lambda > r(T)$. Then $\mu - T$ is invertible for all $\mu \in [\lambda, \sup_n \|V_n\|]$. Since $\mu - V_n \to \mu - T$ uniformly and the set of invertible elements of $\mathcal{L}(E)$ is open, there exists $n_0 \in \mathbf{N}$ such that $\mu - V_n$ is invertible for all $n \geqslant n_0$ and for all $\mu \in [\lambda, \sup_n \|V_n\|]$. This implies that $r(V_n) \leqslant \lambda$ for all $n \geqslant n_0$. It follows that $r(V_n) \to r(T)$. From (8) and the above, we conclude that $r(T_n) \to r(T)$.

Remarks. (a) A particular case of Theorem 3.1 follows when we suppose that E is a Banach lattice with order continuous norm and T_n order converges to T with $\|(T_n - T)^+\| \to 0$. In this case, $T_n x \to Tx$ in order for all $x \in E$. Since E has order continuous norm this implies that $T_n x \to Tx$ in the norm of E.

(b) Theorem 3.1 is not true without the assumption that $r(T)$ is a Riesz point of $\sigma(T)$. A counterexample is easy to construct using the following operator ([7]. Chapter V, Exercise 9(c)). Let $E = L^p(\mu, \mathbf{T})$ where μ is the Haar measure on the circle $\mathbf{T}$ and $1 \leqslant p \leqslant \infty$. Let $S : E \to E$ be defined by $Sf(z) := g(z)f(\alpha z)$, $z \in \mathbf{T}$, $f \in E$, where α is not a root of unity and $g \in C(\mathbf{T})$ is such that $0 < g(z) \leqslant 1$, $z \neq 1$ and vanishes sufficiently fast as $z \to 1$ (tipically like $\exp(-1/(z-1)^2)$). Then S is an irreducible operator on $L^p(\mu, \mathbf{T})$ $1 \leqslant p < \infty$ (band-irreducible on $L^\infty(\mu, \mathbf{T})$). Let $T : E \to E$ be defined by $Tf(z) := f(\alpha z)$, $z \in \mathbf{T}$, $f \in E$ and α as above. We take a sequence g_n of functions having the properties of g above and such that $g_n(z) \uparrow 1$ for all $z \neq 1$. Let $T_n : E \to E$ be given by $T_n f(z) := g_n(z) f(\alpha z)$, $z \in \mathbf{T}$, $f \in E$, and α as above. Then $r(T_n) = 0 \not\to r(T) = 1$.

(c) The assumption $T_n x \to Tx$ alone is not sufficient to guarantee that $r(T_n) \to$ $\to r(T)$. For instance, let $E = \ell^2(\mathbf{N})$ and let $T = [t_{ij}]$ be any compact positive matrix with $r(T) > 0$. Let the approximations T_n be defined by $T_n = [t_{ij}^{(n)}]$, $t_{ij}^{(n)} =$ $= t_{ij}$, $1 \leqslant i, j \leqslant n$, $t_{2n,2n}^{(n)} = 2r(T)$, 0 otherwise. Then $\|T_n x - Tx\|_E \to 0$ but $r(T_n) =$ $= 2r(T) \not\to r(T)$.

Before continuing whit the next theorems let us fixe some notation. Given two sequences $\{x_n\}$, $\{y_n\}$ of E, we write $x_n = y_n \pmod{C_0(E)}$ if $\|x_n - y_n\| \to 0$.

Given an operator $T : E \to E$, the peripheral spectrum of T, denoted by $\pi\sigma(T)$, is defined as $\{z \in \sigma(T) : |z| = r(T)\}$. Finally recall that a positive operator T defined from the Banach lattice E into the Banach lattice F is called *AM-compact* if T maps order intervals of E into relatively compact subsets of F. A detailed study of them can be found in ([10], Chapter 18). Let us prepare the proof of the next results with the following lemma.

LEMMA 3.2 (Dini's Lemma, [7], Theorem II.5.9). *Let K be a compact Hausdorff space. Let E be a Banach lattice with order continuous norm. Let f_n, $f : K \to K$ be continuous functions. Suppose that "f_n order converges to f"*, i.e. *there exist $\Delta_n \in C(K,E)$, $\Delta_n(x) \downarrow 0$ on E for all $x \in K$ such that $|f(x) - f_m(x)| \leqslant \Delta_n(x)$, $m \geqslant n$, $x \in K$. Then f_n converges to f uniformly on K.*

Proof. By assumption, there exist some $\Delta_n \in C(K,E)$, $\Delta_n(x) \downarrow 0$ on E such that $0 \leqslant |f(x) - f_m(x)| \leqslant \Delta_n(x)$, for all $m \geqslant n$, $x \in K$. Since E has order continuous norm $\Delta_n(x) \to 0$ in the norm of E. Let $K_1 = K \times (U^+_{E'},\ \sigma(E',E))$. Let $F_n : K_1 \to \mathbb{N}$ be defined by $F_n(x,u) = \langle \Delta_n(x),\ u\rangle$, $x \in K$, $u \in U^+_{E'}$. First of all, notice that K_1 is a compact space. Second, F_n are continuous functions. Moreover $F_n(x,u) \downarrow 0$ for all $(x,u) \in K_1$. Using the classical Dini's lemma, $F_n \downarrow 0$ uniformly on K_1. Thus, given $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ $|F_n(x,u)| \leqslant \varepsilon$ for all $(x,u) \in K \times U^+_{E'}$. Hence $\|\Delta_n(x)\| \leqslant \varepsilon$ for all $x \in K$. It follow that f_n converges to f uniformly on K.

THEOREM 3.3. *Let E be a dual Banach lattice with order continuous norm. Let $0 \leqslant T \in \mathcal{L}(E)$ be a dual operator which is ireducible and can be decomposed as $T = T_1 + T_2$ with $r(T_1) < r(T)$ and T_2 is AM-compact. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \leqslant T_n \in \mathcal{L}(E)$ be such that T_n order converges to T and $\|(T_n - T)^+\| \to 0$. Then for n sufficiently big, $r(T_n)$ is a Riesz point of $\sigma(T_n)$.*

REMARK. The assumption $r(T_1) < r(T)$ follows as a consequence of Theorem 4.3 in [1] and the rest of assumptions in the statment above.

Proof. We fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the Fréchet filter. Let $\widehat{F}_n = \operatorname{Ker}(r(T_n) - \widehat{T}_n)$. Suppose that there is a subsequence of $\widehat{F}_n$ such that $\dim \widehat{F}_n = \infty$. Call it again $\widehat{F}_n$. Let $\widehat{\widehat{F}} = \ell^\infty(\widehat{F}_n)/C_{\mathcal{U}}(\widehat{F}_n)$ where $\ell^\infty(\widehat{F}_n) = \{(x_n) : x_n \in \widehat{F}_n : \sup_n \|x_n\| < \infty\}$, $C_{\mathcal{U}}(\widehat{F}_n) = \{(x_n) \in \ell^\infty(\widehat{F}_n) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. Then, one easily checks that $\dim \widehat{\widehat{F}} = \infty$. Since $\dim \widehat{\widehat{F}} = \infty$, there exists some $\widehat{\widehat{v}} = (\widehat{v}_n) \in \widehat{\widehat{F}}$ such that $\widehat{\widehat{v}}^+ = (\widehat{v}_n^+) \neq 0$, $\widehat{\widehat{v}}^- = (\widehat{v}_n^-) \neq 0$, ([7]. Proposition II.3.4). Let $\alpha > 0$ be such that $\lim_{\mathcal{U}} \|\widehat{v}_n^+\|$, $\lim_{\mathcal{U}} \|\widehat{v}_n^-\| > \alpha > 0$. Take a subsequence such that $\|\widehat{v}_n^+\|$, $\|\widehat{v}_n^-\| \geqslant \alpha > 0$. Write $\widehat{v}_n = (v_{nk})_k$ and choose for each n, $v_{n_{i_n}}$ such that $\|v^+_{n_{i_n}}\|$, $\|v^-_{n_{i_n}}\| \geqslant \alpha/2 > 0$. Let $v_n = v_{n_{i_n}}$. Therefore, we can choose $v_n \in E_n$, $\|v_n\| = 1$, $\|v_n^+\|$, $\|v_n^-\| \geqslant \alpha/2 > 0$, and such that $\|T_n v_n - r(T_n) v_n\| < \varepsilon_n$, $\varepsilon_n \downarrow 0$. Let $\rho_n := T_n v_n - r(T_n) v_n$. Hence

$$(r(T_n)v_n)^+ = (T_n v_n + \rho_n)^+ \leqslant T_n v_n^+ + \rho_n^+$$
$$(r(T_n)v_n)^- = (-T_n v_n - \rho_n)^+) \leqslant T_n(-v_n)^+ + (-\rho_n)^+ = T_n v_n^- + \rho_n^- .$$

Let $w_n \in \{v_n^+, v_n^-\}$. In any case:

$$r(T_n)w_n \leqslant T_n w_n + |\rho_n| \leqslant T_n w_n + |\rho_n| + (T_n - T)^+ w_n. \tag{9}$$

Let $\widehat{w} = (w_n)_{\mathcal{U}}$. Since $\| |\rho_n| + (T_n - T)^+ w_n \| \to 0$ it follows from (9) that $r(T)\widehat{w} \leqslant \widehat{T}\widehat{w}$. Let P be the spectral projection onto $\bigoplus_{\lambda \in \sigma(T)} \mathrm{Ker}(\lambda - T)$. If $m_E(\widehat{w}) = 0$ then $(\widehat{T} - \widehat{PT})^k \widehat{w} = \widehat{T}^k \widehat{w} \geqslant r(T)^k \widehat{w}$ for all $k \in \mathbb{N}$. Since $\widehat{w} \neq 0$, it follows that $r(T - PT) \geqslant r(T)$ which is a contradiction. Therefore, $m_E(\widehat{w}) \neq 0$. Apply m_E on the inequality $r(T)\widehat{w} \leqslant \widehat{T}\widehat{w}$ to get $r(T)m_E(\widehat{w}) \leqslant Tm_E(\widehat{w})$. Being T irreducible and $r(T)$ a Riesz point of $\sigma(T)$, there exist $0 \leqslant w \in E$ a quasi interior point of E and a strictly positive linear form $\varphi \in E'$ such that $Tw = r(T)w$, $T'\varphi = r(T)\varphi$. It follows that $Tm_E(\widehat{w}) = r(T)m_E(\widehat{w})$. Hence, $m_E(\widehat{w}) \in \langle w \rangle_+ = \{\lambda w : \lambda \geqslant 0\}$. Let $\widehat{w}_+ := (v_n^+)$, $\widehat{w}_- := (v_n^-)$. By construction $\widehat{w}_+ \neq 0$, $\widehat{w}_- \neq 0$. We have proved that $m_E(\widehat{w}_+)$, $m_E(\widehat{w}_-) \in \langle w \rangle_+$. Let β, $\gamma > 0$ be such that $m_E(\widehat{w}_+) = \beta w$, $m_E(\widehat{w}_-) = \gamma w$. Let $\widehat{v} = (v_n)$. Then $m_E(\widehat{v}) = (\beta - \gamma)w$. Let $\widehat{z} = \widehat{v} - m_E(\widehat{v})$. Notice that

$$(T_n z_n)_{\mathcal{U}} = (T_n v_n - T_n m_E(\widehat{v}))_{\mathcal{U}} = (r(T_n)v_n - (\beta - \gamma)T_n w)_{\mathcal{U}}.$$

Since E has order continuous norm and $T_n \to T$ in order, $T_n w \to Tw$ in the norm of E. Hence:

$$(T_n z_n)_{\mathcal{U}} = (r(T_n)v_n - (\beta - \gamma)Tw)_{\mathcal{U}} = r(T)(v_n - (\beta - \gamma)w)_{\mathcal{U}} = r(T)\widehat{z} \tag{10}$$

It follows that $r(T)\widehat{z}^+ \leqslant (T_n z_n^+) \leqslant (T_n z_n^+) = \widehat{T}\widehat{z}^+$. Suppose that $\widehat{z}^+ \neq 0$. If $m_E(\widehat{z}^+) = 0$, then $(\widehat{T} - \widehat{PT})^k \widehat{z}^+ = \widehat{T}^k \widehat{z}^+ \geqslant r(T)^k \widehat{z}^+$. Hence, $r(T - PT) \geqslant r(T)$, a contradiction. Therefore, $m_E(\widehat{z}^+) \neq 0$. From $r(T)\widehat{z}^+ \leqslant \widehat{T}\widehat{z}^+$ it follows that $r(T)m_E(\widehat{z}^+) \leqslant Tm_E(\widehat{z}^+)$. Hence $r(T)m_E(\widehat{z}^+) = Tm_E(\widehat{z}^+)$ and $m_E(\widehat{z}^+) \in \langle w \rangle_+$. Let $\widehat{z}_1 = \widehat{z}^+ - m_E(\widehat{z}^+)$. Then

$$(\widehat{T} - \widehat{PT})^k \widehat{z}_1 = \widehat{T}^k \widehat{z}_1 = \widehat{T}^k \widehat{z}^+ - \widehat{T}^k m_E(\widehat{z}^+) \geqslant r(T)^k \widehat{z}_1.$$

If $\widehat{z}_1^+ \neq 0$ we get $r(T - PT) \geqslant r(T)$, a contradiction. Thus, $\widehat{z}_+^1 = 0$, i.e., $\widehat{z}_1 \leqslant 0$. This means that $0 \leqslant \widehat{z}^+ \leqslant m_E(\widehat{z}^+)$. If $\widehat{z}^+ = 0$ also hold these inequalities. In any case, $0 \leqslant \widehat{z}^+ \leqslant m_E(\widehat{z}^+)$. Similarly, $0 \leqslant \widehat{z}^- \leqslant m_E(\widehat{z}^-)$. Let $x := m_E(\widehat{z}^+) + m_E(\widehat{z}^-)$. Let $u_n^1, u_n \geqslant 0$ be such that $0 \leqslant \widehat{z}^+ = (u_n) \leqslant x$, $0 \leqslant \widehat{z}^- = (u_n^1) \leqslant x$. Since $u_n = u_n \wedge x +$

$+(u_n - x)^+$, $\|(u_n - x)^+\| \to 0$ we can take $0 \leqslant u_n \leqslant x$. Similarly, $0 \leqslant u_n^1 \leqslant x$. Let $S_n = T_n \wedge T$. Then $0 \leqslant S_n \leqslant T_1 + T_2$. Since E has order continuous norm, the AM-compact operators are a band of $\mathcal{L}^r(E)$, ([10], Chapter 18). Taking, if necessary, T_1 disjoint with this band, we descompose $S_n = S_{n1} + S_{n2}$ by taking $S_{ni} := S_n \wedge \wedge T_i$ $i = 1,2$. Then $0 \leqslant S_{ni} \leqslant T_i$ and $S_{ni} \to T_i$ in order, $i = 1,2$. Moreover S_{n2} is also AM-compact. Since S_{n2}, T_2 are AM-compact, they define continous functions from $([0,x], \sigma(E,E')) \to (E, \|\ \|)$. Since S_{n2} order converges to T_n, by Dini's lemma $S_{n2} \to \to T_2$ uniformly on $([0,x], \sigma(E,E'))$. Using the ultrafilter $\mathcal{U}$ given at the beggining of this proof and since $[0,x]$ is $\sigma(E,E')$-compact, the ultrafilter of $[0,x]$ with basis $\{(u_n : n \in U) : U \in \mathcal{U}\}$ converges to some point $u \in [0,x]$. Similarly, $\{(u_n^1 : n \in U) : : U \in \mathcal{U}\}$ converges to $u^1 \in [0,x]$. Let $\varepsilon > 0$. Since $T_2 : ([0,x], \sigma(E,E')) \to (E, \|\ \|)$ is continuous, there is a $\sigma(E,E')$-neighborhood V of u in $[0,x]$ such that $\|T_2\tilde{u} - -T_2 z\| \leqslant \varepsilon/2$ for all $\tilde{u} \in V$. Let $n_0 \in \mathbf{N}$ be such that $\|S_{n2}z - S_2 z\| \leqslant \varepsilon/2$ for all $z \in \in [0,x]$ and all $n \geqslant n_0$. Let $U \in \mathcal{U}$ be such that $u_n \in V$ for all $n \in U$. Now, $U_0 = U \cap \cap [n_0, \infty) \in \mathcal{U}$. Let $n \in U_0$. Then

$$\|S_{n2}u_n - T_2 u\| \leqslant \|S_{n2}u_n - T_2 u_n\| + \|T_2 u_n - T_2 u\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, $\lim_{\mathcal{U}} \|S_{n2}u_n - T_2 u\| = 0$. Similarly, $\lim_{\mathcal{U}} \|S_{n2}u_n^1 - T_2 u^1\| = 0$. It is easy to check that $u - u^1 = m_E(\widehat{z}) = 0$. Therefore, $\lim_{\mathcal{U}} \|S_{n2}z_n\| = \lim_{\mathcal{U}} \|S_{n2}(u_n - u_n^1)\| = 0$. Now:

$$T_n z_n = S_n z_n + (T_n - T)^+ z_n = S_{n1} z_n + S_{n2} z_n + (T_n - T)^+ z_n =$$

$$= S_{n1} z_n + S_{n2} z_n \pmod{C_0(E)},$$

i.e., $S_{n1} z_n + S_{n2} z_n = r(T) z_n \pmod{C_0(E)}$. Let $g_n := S_{n2} z_n$. Since

$$r(S_{n1}) \leqslant r(T_1) < r(T), \quad z_n = (r(T) - S_{n1})^{-1} g_n = \sum_{k=0}^{\infty} \frac{S_{n1}^k}{r(T)^{k+1}} g_n.$$

Now,

$$\|z_n\| \leqslant \sum_{k \geqslant 0} \frac{\|S_{n1}^k |g_n|\,\|}{r(T)^{k+1}} \leqslant \sum_{k \geqslant 0} \frac{\|T_1^k |g_n|\,\|}{r(T)^{k+1}} \leqslant \sum_{k \geqslant 0} \frac{\|T_1^k\|}{r(T)^{k+1}} \|g_n\| \underset{\mathcal{U}}{\to} 0.$$

This means that $\widehat{z} = 0$. Therefore, $\widehat{v} = m_E(\widehat{v}) = (\beta - \gamma) w$. If $\alpha \geqslant \beta$ then $\widehat{v} \geqslant 0$. If $\alpha < \beta$ then $\widehat{v} < 0$. Both cases contradict our assumptions on $\widehat{v}$. Thus, for some $n_0 \in \mathbf{N}$ and all $n \geqslant n_0$ $\dim \operatorname{Ker}(r(T_n) - \widehat{T}_n) < \infty$. Theorem 2.1 proves that $r(T_n)$ is a Riesz point of $\sigma(T_n)$.

Our purpose now is to show that under the assumptions of Theorem 3.3, $\pi\sigma(T)$ converges to $\pi\sigma(T)$. We prepare the proof of this result with the following lemma:

LEMMA 3.4. *Let E be a dual Banach lattice. Let $0 \leqslant T \in \mathcal{L}(E)$ be a dual operator. Suppose that T is irreducible and $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \leqslant T_n \in \mathcal{L}(E)$ be such that $T_n x \to Tx$ in the norm of E for all $x \in E$ and $\|(T_n - T)^+\| \to 0$. Let $0 \leqslant v_n \in E$, $\|v_n\| = 1$ be such that $r(T_n)v_n \leqslant T_n v_n$ $(\mathrm{mod} C_0(E))$. Let $\widehat{v} = (v_n)$. Then $m_E(\widehat{v}) \neq 0$ and $Tm_E(\widehat{v}) = r(T)m_E(\widehat{v})$.*

Proof. By Theorem 3.1, $r(T_n) \to r(T)$. It follows that

$$r(T)\widehat{v} = (r(T_n)v_n)_{\mathcal{U}} \leqslant (T_n v_n)_{\mathcal{U}} = ((T_n \wedge T)v_n)_{\mathcal{U}} \leqslant \widehat{T}\widehat{v}. \tag{11}$$

Let P be the spectral projection onto $\bigoplus_{\lambda \in \sigma(T)} \mathrm{Ker}(\lambda - T)$ as in the proof of Theorem 3.1. If $m_E(\widehat{v}) = 0$, $\widehat{PT}\widehat{v} = 0$. Hence, $(\widehat{T} - \widehat{PT})\widehat{v} = \widehat{T}\widehat{v}$. By induction, one easily checks that $(\widehat{T} - \widehat{PT})^k\widehat{v} = \widehat{T}^k\widehat{v} \geqslant r(T)^k\widehat{v}$ for all $k \in \mathbf{N}$. This implies that $r(T-$ $-TP) \geqslant r(T)$, which is a contradiction. Therefore $m_E(\widehat{v}) \neq 0$. Applying m_E to (11), we get $r(T)m_E(\widehat{v}) \leqslant Tm_E(\widehat{v})$. Since T is irreducible and $r(T)$ is a Riesz point of $\sigma(T')$, there exists a strictly positive linear form $\varphi \in E'$ such that $T'\varphi = r(T)\varphi$ ([7], Corollary to Theorem V.5.2). Now, $0 \leqslant \langle Tm_E(\widehat{v}) - r(T)m'_E(\widehat{v}), \varphi\rangle = 0$ implies that $Tm_E(\widehat{v}) = r(T)m_E(\widehat{v})$. The irreducibility of T implies that $\mathrm{Ker}(r(T) - T) = \langle v\rangle =$ $= \{\lambda v : \lambda \in \mathbf{R}\}$, where $v \geqslant 0$ is a quasiinterior point of E. Thus $m_E(\widehat{v}) = \lambda v$ for some $\lambda > 0$. Notice that $0 < \lambda = \lambda\|v\| = \|m_E(\widehat{v})\| \leqslant \|\widehat{v}\| = 1$, i.e. $0 < \lambda \leqslant 1$.

The next result will be used to prove Theorem 3.6 bellow. It is due to Moustakas ([5], Satz 3.2) in a more general form. Our statement is suitable for our purposes. Since the reference may not be easily available and for the sake of completeness, we include the main lines of the proof.

THEOREM 3.5. ([5], Satz 3.2). *Let E be a Banach lattice. Let $0 \leqslant T \in \mathcal{L}(E)$ be an irreducible operator with $r(T) > 0$ a Riesz point of $\sigma(T)$. Let $0 \leqslant S \in \mathcal{L}(E)$ be such that $0 \leqslant S \leqslant T$. Then $r(S)\dfrac{\pi\sigma(T)}{r(T)} \subset \pi\sigma(S)$.*

Proof. Without loss of generality we may suppose that $r(T) = 1$. Let $\alpha \in$ $\in \pi\sigma(T)$. Then $\dim \mathrm{Ker}(\alpha - T) = 1$ ([7], Theorem V 5.2, V 5.4). Let $0 \neq x \in E$ be such that $Tx = \alpha x$. Let $0 \leqslant \varphi \in E'$ be the strictly positive linear form such that $T'\varphi = \varphi$ ([7], Corollary to Theorem V.5.2). Since, $|x| \leqslant T|x|$, it follows from $0 \leqslant \langle T|x| - |x|, \varphi\rangle = 0$ that $T|x| = |x|$. Then $|x|$ is a quasiinterior point of E, i.e. $\{y \in E : |y| \leqslant nx$ for some $n \in \mathbf{N}\}$ is dense in E ([7], Theorem V 5.2). Then, extending Proposition V 5.1 in [7], Moustakas ([5], Lemma 3.1) shows that there exists a surjective isometry $V \in \mathcal{L}(E)$ such that $\alpha S = V^{-1}SV$. It follows that $\alpha\sigma(S) = \sigma(\alpha S) = \sigma(V^{-1}SV) = \sigma(S)$. Since $r(T) \in \sigma(S)$, it follows that $\alpha r(T) \in$ $\in \pi\sigma(S)$.

THEOREM 3.6. *Let E be a dual Banach lattice with order continuous norm. Let $0 \leqslant T \in \mathcal{L}(E)$ be a dual operator which is irreducible and can be decomposed as $T = T_1 + T_2$ with $r(T_1) < r(T)$ and T_2 is AM-compact. Suppose that $r(T)$ is a Riesz point of $\sigma(T)$. Let $0 \leqslant T_n \in \mathcal{L}(E)$ with $T_n \to T$ in order in $\mathcal{L}^r(E)$ and $\|(T_n - T)^+\| \to 0$. Then $\pi\sigma(T_n) \to \pi\sigma(T)$. Moreover, if $T_n z_n = \alpha z_n$ (mod$C_0(E)$), $\|z_n\| = 1$, $\alpha_n \to \alpha$, then $\alpha \in \pi\sigma(T)$ and z_n norm converges to the unique normalized solution of $Tz = \alpha z$.*

Proof. Let $\alpha \in \pi\sigma(T_n)$. Since α_n is bounded, there is a convergent subsequence. Call it $\{\alpha_n\}$ and let $\alpha = \lim_n \alpha_n$. We want to prove that $\alpha \in \pi\sigma(T)$. Theorem 3.3 tells us that for n sufficiently big, $r(T_n)$ is a Riesz point of $\sigma(T_n)$. Therefore, $\pi\sigma(T_n)$ consists of Riesz points of $\sigma(T_n)$ ([7], Theorem V.5.5. and [1], Lemma 4.4). Then $\alpha_n \in \pi\sigma(T_n)$ is a Riesz point of $\sigma(T_n)$ and there is $z_n \in E$, $\|z_n\| = 1$ satisfying the equation $T_n z_n = \alpha_n z_n$. We can even supose that $T_n z_n = \alpha_n z_n (\text{mod } C_0(E))$. (In this way we can avoid the use of Theorem 3.3 because α_n is in the approximate spectrum of T_n and we can always found z_n such that $\|T_n z_n - \alpha_n z_n\| < 1/n$). Since $\|(T_n - T)^+\| \to 0$, we have $(T_n \wedge T) z_n = \alpha_n z_n (\text{mod } C_0(E))$. Let $S_n := T_n \wedge T$. Let us denote by z_n any subsequence of $\{z_n\}$. Let $\widehat{z} = (z_n)_{\mathcal{U}}$. Let $\widehat{t} = |\widehat{z}| - m_E(|\widehat{z}|)$. From $r(T_n)|z_n| \leqslant T_n|z_n|$ and Lemma 3.4 it follows that $T m_E(|\widehat{z}|) = r(T) m_E(|\widehat{z}|)$. Let P be the spectral projection onto $\bigoplus_{\lambda \in \sigma(T)} \mathrm{Ker}(\lambda - T)$. Since $\widehat{P}\widehat{t} = 0$, $(\widehat{T} - \widehat{PT})^k \widehat{t} \geqslant r(T)^k \widehat{t}$ for all $k \geqslant 1$. If $\widehat{t}^+ \neq 0$, this implies that $r(T - PT) \geqslant r(T)$, a contradiction. Therefore, $\widehat{t}^+ = 0$, i.e., $\widehat{t} \leqslant 0$. It follows that $0 \leqslant |\widehat{z}| \leqslant m_E(|\widehat{z}|) \leqslant v$ where $0 \leqslant v \in E$, $Tv = r(T)v$, $\|v\| = 1$. Thus, there exists $z_n^1 \in [-v, v]$ such that $\widehat{z} = (z_n^1)_{\mathcal{U}}$. As in Theorem 3.3, we decompose $S_n = S_{n1} + S_{n2}$, $0 \leqslant S_{ni} \leqslant T_i$, S_{ni} order converges to T_i, $i = 1, 2$ and S_{n2} are AM-compact. Also, S_{n2}, T_2 define continuous functions from $([-v, v], \sigma(E, E'))$ into $(E, \|\ \|)$. Again, using Dini's Lemma, Lemma 3.2 above, $S_{n2} \to T_2$ uniformly on $[-v, v]$. Using again the ultrafilter $\mathcal{U}$, the ultrafilter or $[-v, v]$ with basis $\{(z_n^1 : n \in U) : U \in \mathcal{U}\}$ $\sigma(E, E')$-converges to some element of $[-v, v]$. Since $([-v, v], \sigma(E, F))$, $([-v, v], \sigma(E, E'))$ are compact Hausdorff spaces and $\sigma(E, F)|_{[-v,v]} \subset \sigma(E, E')|_{[-v,v]}$ both topologies coincide. Therefore, we can identify the limit of $\{(z_n^1 : n \in U) : U \in \mathcal{U}\}$ as $m_E(\widehat{z})$. Prove as in Theorem 3.3 that $\lim_{\mathcal{U}} \|S_{n2} z_n^1 - T_2 m_E(\widehat{z})\| = 0$. Then the following equalities hold modulo $C_0(E)$:

$$\alpha z_n = \alpha_n z_n = T_n z_n = S_n z_n + (T_n - T)^+ z_n = S_{n1} z_n + S_{n2} z_n.$$

Since $\lim_{\mathcal{U}} \|z_n - z_n^1\| \to 0$, $\lim_{\mathcal{U}} \|S_{n2} z_n - T_2 m_E(\widehat{z})\| = 0$. Let $u_n := S_{n2} z_n$. Since

$$r(S_{n1}) \leqslant r(T_1) < r(T) \text{ and } |\alpha| = r(T), \ z_n = (\alpha - S_{n1})^{-1} u_n = \sum_{k \geqslant 0} \frac{S_{n1}^k}{\alpha^{k+1}} u_n,$$

$$\|z_n - (\alpha - T_1)^{-1}T_2 m_E(\hat{z})\| = \left\| \sum_{k\geqslant 0} \frac{S_{n1}^k}{\alpha^{k+1}} u_n - \sum_{k\geqslant 0} \frac{T_1^k}{\alpha^{k+1}} T_2 m_E(\hat{z}) \right\| \leqslant$$

$$\leqslant \left\| \sum_{k\geqslant 0} \frac{S_{n1}^k u_n - S_{n1}^k T_2 m_E(\hat{z})}{\alpha^{k+1}} \right\| + \left\| \sum_{k\geqslant 0} \frac{S_{n1}^k T_2 m_E(\hat{z}) - T_1^k T_2 m_E(\hat{z})}{\alpha^{k+1}} \right\| \leqslant$$

$$\leqslant \sum_{k\geqslant 0} \frac{\|T_1^k\|}{|\alpha|^{k+1}} \|u_n - T_2 m_E(\hat{z})\| + \sum_{k=0}^{N} \frac{\|S_{n1}^k T_2 m_E(\hat{z}) - T_1^k T_2 m_E(\hat{z})\|}{|\alpha|^{k+1}} +$$

$$+ \sum_{k=N+1}^{\infty} \frac{\|S_{n1}^k T_2 m_E(\hat{z}) - T_1^k T_2 m_E(\hat{z})\|}{|\alpha|^{k+1}}.$$

The last term can be majorized by $\sum_{k\geqslant N+1} \frac{2\|T_1^k\|}{|\alpha|^{k+1}} \|T_2 m_E(\hat{z})\| \leqslant \varepsilon$ for N sufficiently big. Moreover, $S_{n1}^k u \to T_1^k u$ for all $k \in \mathbf{N}$. This is easily proved by induction. Let us do first step.

$$\|T_1^2 u - S_{n1}^2 u\| \leqslant \|T_1^2 u - T_1 S_{n1} u\| + \|T_1 S_{n1} u - S_{n1}^2 u\| \leqslant \|T_1\| \, \|T_1 u - S_{n1} u\| + \\ + \|(T_1 - S_{n1}) T_1 u\| \to 0.$$

This implies that, for n sufficiently big, the second term of the right hand side in the above inequality is also $\leqslant \varepsilon$. Taking a suitable $U \in \mathcal{U}$, the first term is also $\leqslant \varepsilon$ for all $n \in U$. Putting all this together, it follows that $\lim_{\mathcal{U}} \|z_n - (\alpha - T_1)^{-1} T_2 m_E(\hat{z})\| = 0$. This implies that $\hat{z} = m_E(\hat{z}) = (\alpha - T_1)^{-1} T_2 m_E(\hat{z})$ and $(\hat{T}_1 + \hat{T}_2)\hat{z} = \alpha \hat{z}$. This implies that there is a subsequence z_n' of $\{z_n\}$, $z_n' \to z$ where z is the unique normalized solution of $Tz = \alpha z$. We have proved that any subsequence of the original $\{z_n\}$ contains a further subsequence converging to z. Therefore, the whole sequence $z_n \to z$ in E. We have also proved that $\alpha \in \pi\sigma(T)$. Now, let $\alpha \in \pi\sigma(T)$. If α is not an accumulation point of a sequence $\alpha_n \in \pi\sigma(T_n)$, then there is an open disc $\mathbf{D}$ around α in $\mathbf{C}$ such that $z - T_n$ is invertible for all $z \in \mathbf{D}$ and all $n \geqslant n_0$ (for some $n_0 \in \mathbf{N}$). Since $(T_n - T)^+ \to 0$ and the invertible operators are an open subset of $\mathcal{L}(E)$, $z - T_n \wedge T$ is invertible for all $z \in D$ and all $n \geqslant n_1$ for some n_1 sufficiently big. But $0 \leqslant T_n \wedge T \leqslant T$. Using Moustakas' result, Theorem 3.5 above, we know that $r(T_n \wedge T) \frac{\pi\sigma(T)}{r(T)} \subset \pi\sigma(T_n \wedge T)$. Therefore, $\frac{\alpha}{r(T)} r(T_n \wedge T) \in \pi\sigma(T_n \wedge T)$. But $\frac{r(T_n \wedge T)}{r(T)} \to 1$ (use Theorem 3.1). Hence $\lambda_n \alpha \in \pi\sigma(T_n \wedge T) \cap \mathbf{D}$ for n sufficiently big. For such $n \geqslant n_1$, $\lambda_n \alpha - T_n \wedge T$ is not invertible, contradicting our assertion above. It follows that, given $\alpha \in \pi\sigma(T)$, there exists a sequence $\alpha_n \in \pi\sigma(T_n)$ such that $\alpha_n \to \alpha$. The theorem is proved.

COROLLARY 3.7. *Let E, T, T_n be as above. Let $0 \leqslant v_n \in E$ be such that $T_n v_n =$*

$= r(T_n)v_n(\mathit{mod}\ C_0(E))$, $\|v_n\| = 1$. *Then,* v_n *norm converges to the unique normalized positive eigenvector of* T *associated to* $r(T)$, $Tv = r(T)v$.

REMARKS. (a) We used that T_n order converges to T to guarantee that $(T_n \wedge \wedge T_2)x$ converges to T_2x uniformly on order intervals of E. Thus, both Theorems 3.3 and 3.6 are still true if we suppose that $(T_n \wedge T_2)x \to T_2x$ uniformly on order intervals of E and $\|(T_n - T)^+\| \to 0$ instead of T_n order converges to T and $\|(T_n - T)^+\| \to 0$. The assumption that E has order continuous norm has been used to guarantee that the set of AM-compact operators of E is a band of $\mathcal{L}^r(E)$ and order intervals of E are $\sigma(E', E)$-compact. Modifying suitably our assumptions we could avoid to suppose that E has order continuous norm but we do not wish to take more complicated our statments.

(b) The assumption that E is a dual Banach lattice excludes the interesting case in applications when $E = \mathsf{L}^1(X, \Sigma, \mu)$, (X, Σ, μ) is a σ-finite measure space. To include it with our type of proof we need to suppose that $T : \mathsf{L}^1(\mu) \to \mathsf{L}^1(\mu)$ is a positive irreducible operator with $r(T)$ a Riesz point of T and $T'\varphi = r(T)\varphi$ for some $0 < \delta \leqslant \varphi(s) \in \mathsf{L}^\infty(\mu)$ (for instance, if T is stochastic, $T'1 = 1$, $r(T) = 1$). Then, the proofs of Theorems 3.3 and 3.6 can be adapted to this case. Now, one works with the projection $m_{E''} : \widehat{E} \to E''$ instead of $m_E : \widehat{E} \to E$ as above. For instance, if $0 \leqslant v_n \in \in E$, $\|v_N\| = 1$ are such that $T_n v_n = r(T_n)v_n$, then $m_{E''}(v_n)$ is again an eigenvector of T associated to $r(T)$. Let $\widehat{v} = (v_n)_{\mathcal{U}}$. Then $r(T)\widehat{v} = (T_n v_n) = ((T_n \wedge T)v_n) \leqslant \widehat{T}\widehat{v}$. Let P be the spectral projection onto $\bigoplus_{\lambda \in \sigma(T)} \mathrm{Ker}(\lambda - T)$. If $m_{E''}(\widehat{v}) = 0$, then $\widehat{PT}\widehat{v} = 0$. Thus, $(\widehat{T} - \widehat{PT})^k\widehat{v} \geqslant r(T)^k\widehat{v}$ for all $k \geqslant 1$. It follows that $r(T - PT) \geqslant r(T)$, a contradiction. Therefore, $m_{E''}(\widehat{v}) \neq 0$ and $r(T)m_{E''}(\widehat{v}) \leqslant T'' m_{E''}(\widehat{v})$. But $\langle T'' m_{E''}(\widehat{v}) - -r(T)m_{E''}(\widehat{v}), \varphi\rangle = 0$. Since φ is bounded away from zero, $T'' m_{E''}(\widehat{v}) = r(T)m_{E''}(\widehat{v})$. But $\dim \mathrm{Ker}(r(T) - T) = \dim \mathrm{Ker}(r(T) - T'')$. Therefore, $m_{E''}(\widehat{v}) \in E$ and $\|m_{E''}(\widehat{v})\| = \lim_{\mathcal{U}}\langle v_n, 1\rangle = 1$. Thus, $m_{E''}(\widehat{v}) = v$ where v is the unique normalized solution of $Tv = r(T)v$. Also, inequalities like $0 \leqslant \widehat{z}^+ \leqslant m_{E''}(\widehat{z}^+)$ (or $0 \leqslant |\widehat{z}| \leqslant m_{E''}(|\widehat{z}|)$) which appear in the proof of Theorem 3.3 (Theorem 3.6) are not an obstruction since $m_{E''}(\widehat{z}^+) \in \langle w\rangle_+ \subset E$. Since E has order continuous norm, E is an ideal of E'' ([7], Theorem II.5.10). Then, the inequality $0 \leqslant \widehat{z}^+ \leqslant m_{E''}(\widehat{z}^+)$ holds in fact in E. Going to E'' is not a solution because we loose the irreducibility of T and, probably, the AM-compactness of T_2''.

(c) One could drop the assumption of irreducibility of T in favour of the assumption that Q is strictly positive, where Q is the spectral projection associated to $r(T)$, i.e. if $Tv = r(T)v$, $T'\varphi = r(T)\varphi$ where v and φ are strictly positive, then $Q = \varphi \otimes v$.

4. APPLICATIONS

In this section we give three possible applications of our results in Section 3. We start with an application to the transport equation. Then, we study the case of the approximation of the leading eigenvalue of integral operators in $\mathbf{L}^p$-spaces. The last one concerns the approximation of eigenvalues of infinite stochastic matrices and we recover the results of [9], Chapters VI and VII.

I. TRANSPORT THEORY. We situate ourselves in the context of [4]. Suppose that $1 < p < \infty$. Let D be an open, convex subset of $\mathbf{R}^3$ and let V be an open subset of $\mathbf{R}^3$ symmetric with respect to the origin. Let A_p be the transport operator defined by

$$D(A_p) := \{\psi(x,v) \in \mathbf{L}^p(D \times V) : v\frac{\partial \psi}{\partial x} \in \mathbf{L}^p(D \times V),\ \psi|D_- = 0\}$$

where $D_- = \{(x,v) \in \partial D \times V : n(x) \cdot v < 0\}$ where n denotes the outward pointing normal to ∂D at the point x and

$$A_p\psi := -v \nabla \psi - \sigma(v)\psi + \int_V k(v,v')\psi(x,v')\mathrm{d}v' =: T\psi + K\psi$$

where $\sigma \in \mathbf{L}^\infty(V)$ and $K \in \mathcal{L}(\mathbf{L}^p(D \times V))$, $k(v,v') \geqslant 0$ almost everwhere. We call A_p the transport operator on $\mathbf{L}^p(D \times V)$ associated to σ and k. It is well known that A_p is the generator of a strongly continuous semigroup $T_p(t)$ on $\mathbf{L}^p(D \times V)$ which is positivity preserving. Moreover, if $k(v,v') > 0$ a.e., then $T_p(t)$ is irreducible ([6], C-III. Exercise 3.4).

Let $G(t)$ be the semigroup with generator $T = -v \nabla - \sigma(v)$. Suppose that $KG(t)K$ is compact $\forall t > 0$ and $t \to KG(t)K$ is continuous from $(0,\infty)$ into $\mathcal{L}(\mathbf{L}^p(D\times \times V))$. Let $\lambda^* = \inf \sigma(v)$. The set $\Sigma' = \sigma(A_p) \cap \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > -\lambda^*\}$ is a set of eigenvalues of A_p of finite multiplicity (eventually empty) independent of p with no accumulation point except on $\operatorname{Re}\lambda = -\lambda^*$. In [4] conditions are given which imply that Σ' is not empty. In this case $r(T_p(t))$ is a Riesz point of $\sigma(T_p(t))$

Let $\mathbf{S}$ be a fixed ball of $\mathbf{R}^3$. Suppose that

(α) $\sigma_n \to \sigma$ in $\mathbf{L}^\infty(V)$

(β) $K_n \to K$ in $\mathcal{L}(\mathbf{L}^p(S \times V))$

(γ) D_n is a sequence of convex subsets of $\mathbf{S}$ such that $D_n \to D$ in the sense that $\chi_{(D_n - D)\cup(D - D_n)} \to 0$ pointwise almost everywhere. Then,

PROPOSITION 4.1 ([4], Corollaire 2). *The leading eigenvalue of the transport operator depends continuously on* (σ, K, D), i.e., *if* $(\sigma_n, K_n, D_n) \to (\sigma, K, D)$ *in the sense given above then* $r(T_{p,n}(t)) \to r(T_p(t))$ *where* $T_{p,n}(t)$, $T_p(t)$ *are the semigroups* *generated by $A_{p,n}$, A_p are the transport operators associated to (σ_n, K_n, D_n) and (σ, K, D) respectively.*

We want to mention that using the results of Section 3, the assumptions (α), (β) above could be weakened. For instance, they could be substituded by:

(α') $\sigma_n \to \sigma$ pointwise on V and $(\sigma - \sigma_N)^+ \to 0$ in $\mathsf{L}^\infty(V)$.

(β') $K_n x \to Kx$ $\forall x \in E$ and $(K_n - K)^+ \to 0$ in $\mathcal{L}(\mathsf{L}^p(S \times V))$.

A result similar to Proposition 4.1 holds. This result could be complemented with the corresponding result for the leading eigenvectors (using Theorem 3.6 above). We leave the details of this to the interested reader.

II. Approximation of the spectral radius of kernel operators. Let (X, Σ, μ) be a finite measure space. Let $E = \mathsf{L}^p(X, \Sigma, \mu)$, $1 \leqslant p < \infty$. Let $0 \leqslant T : E \to E$ be an abstract kernel operator, hence AM-compact ([10], Theorem 123.5). If $p = 1$, we consider that $T'\varphi = r(T)\varphi$ for some $\varphi \in \mathsf{L}^\infty(\mu)$, $\varphi \geqslant \delta > 0$ and some $\delta > 0$. Let $k(s,t)$ be the integral kernel of T, i.e., $Tf(s) = \int\limits_X k(s,t)f(t)\mathrm{d}t$, $s \in X$. Let $\mathcal{P} = \{A_1, \dots, A_n\}$ be a finite partition of X with $a_i = \mu(A_i) > 0$. Let $e_i = \chi_{A_i}/a_i$. Let $a_{ij} = \langle Ke_i,\ e_j\rangle$. Let $k_{\mathcal{P}}(s,t) = \sum\limits_{ij \geqslant 1} a_{ij}\chi_{A_i \times A_j}$. Let

$$T_{\mathcal{P}}f(s) = \int\limits_X k_{\mathcal{P}}(s,t)f(t)\mathrm{d}t = \sum_{ij=1}^{n} a_{ij} \int\limits_{A_j} f(t)\mathrm{d}t\chi_{A_i}, \qquad s \in X.$$

Let $P_{\mathcal{P}}f(s) = \sum\limits_{j=1}^{n} \int\limits_{A_j} f(t)\mathrm{d}t/a_j\chi_{A_j}$. Notice that $T_{\mathcal{P}}P_{\mathcal{P}} = T_{\mathcal{P}}$. Thus, $T_{\mathcal{P}} : E \to E$. If $\|(T_{\mathcal{P}} - T)^+\| \to 0$, $T_{\mathcal{P}} \wedge T \to T$ in order (which is the case) and $r(T)$ is a Riesz point of $\sigma(T)$, then $r(T_{\mathcal{P}}) \to r(T)$. To be sure that these conditions hold, we first approximate K from below by $k_n = k \wedge n$ and then we approximate k_n by troncatures like above: $\overline{k}_n$. If $\|(\overline{K}_n - K_n)^+\| \to 0$ then $\|(\overline{K}_n - T)^+\| \to 0$ and $\overline{K}_n \to T$ in order at the same time.

Let us transform the calculation of $r(T_{\mathcal{P}})$ in a problem for matrices. Let us represent $T_{\mathcal{P}}$ as a matrix. Let $E_{\mathcal{P}} := \left\{\sum\limits_{j=1}^{n} \langle e_{j,f}\rangle\chi_{A_j} : f \in E\right\}$. Since

$$\|\chi_{A_j}\|_p = \mu(A_j)^{1/p} =: \lambda_j, \quad \sum_{j=1}^{n}\langle e_j, f\rangle\chi_{A_j} = \sum_{j=1}^{n}\lambda_j\langle e_j, f\rangle h_j$$

where

$$h_j := \lambda_j^{-1}\chi_{A_j}, \quad E_{\mathcal{P}} = \left\{\sum_{j=1}^{n} x_j h_j : x_j \in \mathbf{R}\right\} \quad \text{and} \quad \left\|\sum_{j=1}^{n} x_j h_j\right\|_p = \left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p},$$

$$T(E_{\mathcal{P}}) \subset E_{\mathcal{P}}.$$

Notice that $\int_{A_j} h_k = \mu(A_j)^{1-1/p}\delta_{jk}$. Thus

$$T_{\mathcal{P}}\left(\sum_{j=1}^{n} x_j h_j\right)(s) = \sum_{i=1}^{n}\left(\lambda_i \sum_{j=1}^{n} a_{ij} x_j \lambda_j^{p-1}\right) h_j.$$

Let $t_{\mathcal{P}} : \ell_p^n \to \ell_p^n$ be given by the matrix $t_{\mathcal{P}} = (a_{ij}\lambda_i\lambda_j^{p-1})$. Let $h_i^* := \chi_{A_i}/\mu(A_i)^{1/q}$. Then $a_{ij}\lambda_i\lambda_j^{p-1} = \langle Th_j, h_i^*\rangle$ $(p^{-1}+q^{-1}=1)$. Notice that

$$h_j h_j^* = e_j, \quad P_{\mathcal{P}} h_j = h_j, \quad P_{\mathcal{P}} f = \sum_{j=1}^{n} (h_j^* \otimes h_j)(f), \quad P_{\mathcal{P}} T_{\mathcal{P}} f = T_{\mathcal{P}} f.$$

It follows that $P_{\mathcal{P}} T_{\mathcal{P}} = t_{\mathcal{P}} P_{\mathcal{P}}$. Check that $\|T_{\mathcal{P}}^k\| = \|t_{\mathcal{P}}^k\|$ for all $k \in \mathbb{N}$. It follows that $r(T_{\mathcal{P}}) = r(t_{\mathcal{P}})$. To compute $r(t_{\mathcal{P}})$, suppose that T is irreducible. Then, $t_{\mathcal{P}}$ is also irreducible and we can compute $r(t_{\mathcal{P}})$ with the min-max formula: let

$$t_{ij,\mathcal{P}} = a_{ij}\lambda_i\lambda_j^{p-1},$$

$$r(t_{\mathcal{P}}) = \max_{\substack{x \geqslant 0 \\ x \neq 0}} \min_i \left\{\sum_{j=1}^{n} t_{ij,\mathcal{P}} x_j / x_i\right\} =$$

$$= \inf_{x \gg 0} \max_i \left\{\sum_{j=1}^{n} t_{ij,\mathcal{P}} x_j / x_i\right\}.$$

As we said before, if $r(T)$ is a Riesz point of $\sigma(T)$, then $r(T_{\mathcal{P}}) \to r(T)$.

III. TRUNCATIONS OF STOCHASTIC MATRICES. Now, we give Seneta's results ([9]) in our framework. Let us start with an easy application of our results.

THEOREM 4.2. *Let* $E = \ell^p(\mathbb{N})$, $1 \leqslant p < \infty$. *Let* $0 \leqslant T = (t_{ij})_{i,j\geqslant 1} : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ *be a positive matrix. Suppose that* T *is irreducible and* $r(T)$ *is a Riesz point of* $\sigma(T)$. *Let* $T_n = (t_{ij}(n))_{i,j\geqslant 1}$ *where* $t_{ij}(n) = t_{ij}$ *for* $1 \leqslant i,j \leqslant n$, 0 *otherwise. Then* $r(T_n) \to r(T)$ *and if* $T_n v_n = r(T_n) v_n$, $v_n \geqslant 0$, $\|v_n\| = 1$, *then* $v_n \to v$ *in the norm of* E *where* v *is the unique positive normalized solution of* $Tv = r(T)v$.

If we want to approximate the eigenvectors of $T : C_0 \to C_0$ we can go to $\ell^1(\mathbb{N})$ as we did with $\mathbf{L}^1 - \mathbf{L}^\infty$. We need only that $T'\varphi = r(T)\varphi$ with $\varphi \gg 0$. This is the case if T is irreducible and $r(T)$ is a Riesz point of $\sigma(T)$.

We are going to use our techniques developped here to approach some results in Chapters VI and VII of [9] relative to the truncation of infinite stochastic matrices. First of all we give our approach to

THEOREM 4.3 ([9], Theorem 7.3). *Let $P = (p_{ij})_{i,j\in\mathbb{N}}$ be an stochastic matrix. Let us suppose that all elements of at least one column are uniformly bounded away from zero. Let $P(n)$ be the $n \times n$ truncation of P and let the stochastic $\overline{P(n)}$ be formed from $P(n)$ by augmentation of $P(n)$, if necessary. Let $\pi(n)P(n) = r(P(n))\pi(n)$, $\overline{\pi(n)}\,\overline{P(n)} = \overline{\pi(n)}$, $\pi P = \pi$. Then, if we normalize $\|\pi(n)\| = \|\overline{\pi(n)}\| = 1$, then $\pi(n) \to \pi$ in ℓ^1 and $\overline{\pi(n)} \to \pi$ in $\sigma(\ell^1, C_0)$.*

Proof. We suppose without loss of generality that $p_{i_1} \geqslant \delta > 0$ for all $i \in \mathbb{N}$ and some $\delta > 0$. First of all, let us prove that 1 is a Riesz point of $\sigma(P)$. Let us write P as $P = \delta e_1 \otimes e + R$ where $e_1 = (1,0,0,\ldots)$, $e = (1,1,\ldots)$ and $e_1 \otimes e(f) = \langle e_1, f\rangle e$. From $\mathbf{1} = P\mathbf{1} = \delta\mathbf{1} + R\mathbf{1}$ we see that $R\mathbf{1} = (1-\delta)\mathbf{1}$. Hence $r(R) = 1-\delta < r(P) = 1$. Since $P - R = \delta e_1 \otimes e$ is a compact operator, $r(P) = 1$ is a Riesz point of $\sigma(P)$. Let T be the operator on $\ell^1(\mathbb{N})$ such that $T' = P$. Let us prove that dim $\mathrm{Ker}(I-T) = 1$. First of all, we remark that $\mathrm{Ker}(I-T)$ is a sublattice of $\ell^1(\mathbb{N})$, i.e., if $Tx = x$, then $T|x| = |x|$.

Second, since the first raw of T is strictly positive, given $x \geqslant 0$, $x \neq 0$, $Tx = x$ then $x_1 > 0$. Now, let $0 \leqslant x, y \in \mathrm{Ker}(I-T)$ both not null, then $x_1, y_1 > 0$. Multiplying x by $\alpha > 0$, if necessary, we may assume without loss of generality that $x_1 \geqslant y_1$. Since $T(y-x) = y-x$, $T(y-x)^+ = (y-x)^+$ and $((y-x)^+)_1 = (y_1-x_1)^+ = 0$. Hence, $(y-x)_+ = 0$, i.e., $y \leqslant x$, i.e. $\mathrm{Ker}(I-T)$ is totally ordered. Hence, $\mathrm{Ker}(I-T)$ is one-dimensional ([7], Proposition II.3.4). Let $T\pi = \pi$ be its stationary distribution normalized so that $\|\pi\|_1 = 1$. With $E = \ell^1(\mathbb{N})$ and T as above, T is not necessarily a dual operator. Moreover $r(T) = 1$ is a Riesz point of $\sigma(T)$. Let T_n be the operator on $\ell^1(\mathbb{N})$ such that $T_n' = P(n)$. $0 \leqslant T_n \leqslant T$, $T_n \uparrow T$, T is AM-compact. Two remarks: T is not irreducible and is not a dual operator, in general. This is not a problem because the only thing we need is dim $\mathrm{Ker}(I-T) = 1$ and $\mathrm{Ker}(I-T') = \langle \mathbf{1}\rangle$ and $\langle \mathbf{1}, x\rangle = 0$, $0 \leqslant x \in E''$ implies that $x = 0$. In consequence, as in Theorem 3.6, if $T_n\pi(n) = r(T_n)\pi(n)$ then $\pi(n) \to \pi$ in $\ell^1(\mathbb{N})$. Let $\overline{P(n)} = (\overline{p_{ij}}(n))$ be constructed by adding to $P(n)$ certain amounts so that $\overline{P(n)}$ is stochastic. Thus $0 \leqslant P(n) \leqslant \overline{P(n)}$ and $\overline{P(n)} \wedge P = P(n)$. Let $\overline{T}_n = (\overline{t}_{ij}(n))$ be the operator on $\ell^1(\mathbb{N})$ such that $\overline{T}_n' = \overline{P(n)}$. Let $\overline{\pi(n)} \geqslant 0$, $\|\overline{\pi(n)}\| = 1$, $\overline{T}_n\overline{\pi(n)} = \overline{\pi(n)}$. Then $\overline{T}_n = T_n + R_n$ and it may happen that $\|R_n\| \not\to 0$. Notice that $\overline{t}_{1j}(n) = \overline{p}_{1j}(n) \geqslant p_{1j} \geqslant \delta > 0$. It follows that $\overline{\pi(n)}_1 \geqslant \delta > 0$. Thus, $\left\|\dfrac{\overline{\pi(n)}}{\overline{\pi(n)}_1}\right\| \leqslant \dfrac{1}{\delta}$. It follows that $m_E(\overline{\pi(n)}) \neq 0$. From $\overline{T}_n\overline{\pi(n)} = \overline{\pi(n)}$ it follows that $T_n\overline{\pi(n)} \leqslant \overline{\pi(n)}$. Apply m_E on this inequality to get $(*)$ $m_E(T_n\overline{\pi(n)}) \leqslant m_E(\overline{\pi(n)})$. Let $0 \leqslant \varphi \in C_0$. Check that for all $N \in \mathbb{N}$,

$$\langle m_E(T_n\overline{\pi(n)}),\ \varphi\rangle \geqslant \sum_{j=1}^{N} m_E(\overline{\pi(n)})(j)T'\varphi(j).$$

Since $m_E(\overline{\pi(n)}) \in \ell^1(\mathbb{N})$ and $T'\varphi \in \ell^\infty(\mathbb{N})$, letting $N \to \infty$ and using the inequality (*) we get $\langle Tm_E(\overline{\pi(n)}),\ \varphi\rangle \leqslant \langle m_E(\overline{\pi(n)}),\ \varphi\rangle$ for all $0 \leqslant \varphi \in C_0$. Therefore, $Tm_E(\overline{\pi(n)}) \leqslant m_E(\overline{\pi(n)})$. It follows that $Tm_E(\overline{\pi(n)}) = m_E(\overline{\pi(n)})$. Since $\|m_E(\overline{\pi(n)})\| = \lim_{\mathcal{U}}\langle\overline{\pi(n)}, \mathbf{1}\rangle = 1$, then $m_E(\overline{\pi(n)}) = \pi$. Taking subsequences, if necessary, we see that any $\sigma(\ell^1, C_0)$-accumulation point of $\overline{\pi(n)}$ is π. Therefore, $\overline{\pi(n)} \to \pi$ in $\sigma(\ell^1(\mathbb{N}), C_0(\mathbb{N}))$. The theorem is proved.

THEOREM 4.4 ([9], Theorem 6.9). *Let E be an atomic Banach lattice on $\mathbb{N}$ with order continuous norm. Let $0 \leqslant T : E \to E$ be an irreducible operator with $r(T) = 1$. Let $0 < u \in E$, $0 < \varphi \in E'$ be such that $Tu = u$, $T'\varphi = \varphi$. We suppose that u and φ are normalized in the sense that $u(1) = 1$, $\varphi(1) = 1$. Let $0 \leqslant T_n$ be the truncation matrices of T. Let u_n, $\varphi(n)$ be positive, normalized,* i.e., *$u_n(1) = \varphi_n(1) = 1$, vectors such that $T_n u_n = r(T_n)u_n$, $T_n'\varphi_n = r(T_n)\varphi_n$. Then $r(T_n) \uparrow r(T)$, $u_n \to u$ coordinatewise and $\varphi_n \to \varphi$ coordinatewise.*

Proof. Let $k > 0$. Let $z_n^k = u_n \wedge ku$. Then $T_n z_n^k \leqslant (r(T_n)u_n) \wedge ku$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing the Fréchet filter. Let $0 \leqslant \varphi \in E'$. Since the order intervals of E are compact and $T_n'\varphi \to T'\varphi$ $\sigma(E', E)$, letting $z^k := \lim_{\mathcal{U}} z_n^k$, we know that for each $\varepsilon > 0$ there exists $U \in \mathcal{U}$ and $n_0 \in \mathbb{N}$ such that if $n \in U \cap [n_0, \infty)$ then $\|z_n^k - z^k\| \leqslant \varepsilon$ and $|\langle z^k, T'\varphi - T_n'\varphi\rangle| \leqslant \varepsilon$. It follows that for all $n \in U \cap [n_0, \infty)$, $|\langle z^k, T'\varphi\rangle - \langle z_n^k, T_n'\varphi\rangle| \leqslant \varepsilon\|T'\varphi\| + \varepsilon$. Hence

$$\langle Tz^k, \varphi\rangle = \lim_{\mathcal{U}}\langle T_n z_n^k, \varphi\rangle = \langle m_{E''}(T_n z_n^k), \varphi\rangle \leqslant \langle m_{E''}(z_n^k), \varphi\rangle = \langle z^k, \varphi\rangle.$$

Therefore, $Tz^k \leqslant z^k$ and because of the normalization condition, $z^k \neq 0$. Since $\varphi \gg\gg 0$, $Tz^k = z^k$. Hence, $z^k \in \langle u\rangle$. Since $z^k(1) = 1$, $\forall k \geqslant 1$, $z^k = u$ $\forall k \geqslant 1$. Since $u(i) > 0$ for all i, taking $k \geqslant 2$, this means that $\{u_n(i)\}$ is bounded for each $i \in \mathbb{N}$. In fact, if for some i, some subsequence $u_{nj}(i)$ is unbounded, using u_{nj} as u_n above and from $m_{E''}(u_{nj} \wedge 2u) = u$ we get a contradiction. Fix i, take k such that $u_n(i) \leqslant ku(i)$ to conclude from $z^k = u$ that $u_n(i) \to u(i)$. Therefore, $u_n \to u$ coordinatewise. Now let $0 \leqslant \varphi_n \in E'$ be such that $T_n'\varphi_n = r(T_n)\varphi_n$. Let $k \geqslant 1$ and let $v_n^k = \varphi_n \wedge k\varphi$. Let $0 \leqslant \psi \in E$. Notice that $T_n\psi \to T\psi$ in the norm of E. It follows that

$$\langle T'm_{E'}(v_n^k), \psi\rangle = \langle m_{E'}(T_n'v_n^k)\psi\rangle \leqslant \langle m_{E'}(v_n^k), \psi\rangle.$$

Therefore, $T'm_{E'}(v_n^k) \leqslant m_{E'}(v_n^k)$. Since $\varphi(1) = 1$, $m_{E'}(v_n^k) \neq 0$. It follows that $T'm_{E'}(v_n^k) = m_{E'}(v_n^k)$, i.e., $m_{E'}(v_n^k) \in \langle\varphi\rangle$, Since $v_n^k(1) = 1$ $\forall k \geqslant 1$, $m_{E'}(v_n^k) = \varphi$. Again $\varphi_n(i)$ is bounded for each i and it follows from $m_{E'}(v_n^k) = \varphi$ that $\varphi_n(i) \to \varphi(i)$ for each i. Let $k \geqslant 1$. Let $e_1(i) = \delta_{1i}$, $e_1 \in E$. Let $r_0 = \lim_n r(T_n) \leqslant 1$. Then

$$1 = \langle\varphi, e_1\rangle = \langle T'\varphi, e_1\rangle = \langle T'm_{E'}(v_n^k), e_1\rangle = \langle m_{E'}(T_n'v_n^k), e_1\rangle \leqslant$$
$$\leqslant \langle m_{E'}(r(T_n)\varphi_n \wedge k\varphi), e_1\rangle = \lim_n r(T_n) \wedge k = r_0 \wedge k \leqslant 1.$$

Therefore, $r_0 = 1$.

Let us finish with the following:

THEOREM 4.5 ([9], Theorem 7.4). *Let* $0 \leqslant T : \ell^1(\mathbf{N}) \to \ell^1(\mathbf{N})$ *be irreducible and stochastic* (i.e. $T'\mathbf{1} = \mathbf{1}$). *Let* $0 \leqslant v_n \in \ell^1(\mathbf{N})$ *be such that* $T_n v_n = r(T_n)v_n$ *with* v_n *tight* (i.e. $\|v_n\|_1 = 1$, $\langle v_n, e_i\rangle \to \langle v, e_i\rangle$ $\forall i \in \mathbf{N}$ *with* $v \in \ell^1(\mathbf{N})$, $\|v\|_1 = 1$) *where* T_n *are the truncation matrices of* T. *Then* $r(T_n) \to r(T)$ *and* $Tv = v$.

Proof. Let $r_0 := \lim_n r(T_n)$. Since v_n is tight, $\langle v_n, e_i\rangle \to \langle v, e_i\rangle$ $\forall i \in \mathbf{N}$. Fixe an ultrafilter $\mathcal{U}$ on $\mathbf{N}$ containing the Fréchet filter. Write $m = m_{\ell^1(\mathbf{N})}$. Let $\widehat{v} = (v_n)_{\mathcal{U}}$. Then $m(\widehat{v}) = v$. From $(T_n v_n)_{\mathcal{U}} = (r(T_n)v_n)_{\mathcal{U}} = r_0\widehat{v}$ we get for all $0 \leqslant \varphi \in C_0$:

$$
\begin{aligned}
r_0\langle m(\widehat{v}), \varphi\rangle &= \lim_{\mathcal{U}}\langle r(T_n)v_n, \varphi\rangle = \lim_{\mathcal{U}}\langle T_n v_n, \varphi\rangle = \lim_{\mathcal{U}}\langle v_n, T_n'\varphi\rangle = \\
&= \lim_{\mathcal{U}}\sum_{j=1} v_n(j)T_n'\varphi(j) \geqslant \lim_{\mathcal{U}}\sum_{j=1}^{N} v_n(j)T_n'\varphi(j) = \sum_{j=1}^{N}\lim_{\mathcal{U}} v_n(j)T_n'\varphi(j) = \\
&= \sum_{j=1}^{N} v(j)T'\varphi(j).
\end{aligned}
$$

Letting $N \to \infty$ we get $r_0\langle m(\widehat{v}), \varphi\rangle \geqslant \langle v, T'\varphi\rangle$. But $m(\widehat{v}) = v$. Therefore, $Tv \leqslant r_0 v$. Since T is irreducible and $v \neq 0$, then $v \gg 0$. Notice that $r_0 \leqslant 1$ and $r_0\mathbf{1} \leqslant T'\mathbf{1}$. Then:

$$0 \leqslant \langle T'\mathbf{1} - r_0\mathbf{1} v\rangle = \langle \mathbf{1}, Tv\rangle - r_0\langle \mathbf{1}, v\rangle = r_0\langle \mathbf{1}, v\rangle - r_0\langle \mathbf{1}, v\rangle = 0.$$

This implies that $T'\mathbf{1} = r_0\mathbf{1}$. Therefore $r_0 = 1$ and $Tv = v$. The theorem is proved.

REFERENCES

1. CASELLES, V., On the peripheral spectrum of positive operators, *Israel J. Math.*, **58** (1987), 144–160.
2. CASELLES, V., *The Perron-Frobenius theorem for positive operators in Banach lattices*, Publ. Mathem. de Besançon, 1987.
3. GREINER, G., Spectral properties and assymptotic behaviour of the linear transport equation. *Math. Z.*, **185**(1984) 167–177.
4. MOKHTAR-KARROUBI, M., Existence, inexistence, majoration, stabilité et invariance du spectre de l'opérateur de transport, *C.R. Acad. Sci. Paris Ser. I.*, **297** (1983), 331–334.
5. MOUSTAKAS, U., Majorisierung und Spektraleigenschaften positiver Operatoren auf Banachverbaenden, Dissertation, Univ. Tuebingen, 1984.
6. NAGEL, R.(ed), *One-parameter semigroups of positive operators*, Springer LNS, **1184**, Springer Verlag, 1986.
7. SCHAEFER, H. H., *Banach lattices and positive operators*, Springer Verlag, 1974.

8. SCHEP, A. R., Positive diagonal and triangular operators, J. *Operator Theory*, **3**(1980) 165–178.
9. SENETA, E., *Non-negative matrices and Markov chains*, Springer Verlag, 1981.
10. ZAANEN, A.C., *Riesz spaces. II*, North Holland, Amsterdam, 1983.

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Received September 28, 1989.