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A CONVERSE OF THE KATO-ROSENBLUM THEOREM

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1. INTRODUCTION

Let $\mathfrak{H}$ be a separable Hilbert space. We denote by $\mathcal{S}(\mathfrak{H})$, $\mathcal{B}_{\rm self}(\mathfrak{H})$ and $\mathcal{L}_1(\mathfrak{H})$ the set of self-adjoint operators, the set of bounded self-adjoint operators and the set of trace class operators on $\mathfrak{H}$, respectively. Further $P^{\rm ac}(\cdot)$ denotes the orthogonal projection onto the absolutely continuous subspace of a self-adjoint operator (see e.g. [1]). Then the famous trace class existence theorem of Kato and Rosenblum ([5], [11]) states that for $H_0 \in \mathcal{S}(\mathfrak{H})$ and $V \in \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\rm self}(\mathfrak{H})$ the wave operators $W_\pm(H_0+V, H_0)$,

$$W_\pm(H_0+V,H_0) = \text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t(H_0+V)}\mathrm{e}^{-\mathrm{i}tH_0}P^{\rm ac}(H_0), \tag{1.1}$$

exist and give a unitary equivalence between the absolutely continuous parts of H_0+V and H_0. Note that the operator H_0+V defined by

$$(H_0+V)f = H_0f + Vf, \quad f \in \mathrm{dom}(H_0+V) = \mathrm{dom}(H_0) \tag{1.2}$$

is self-adjoint (see e.g. [6]).

Naturally, the question arises whether there is a wider class of operators from $\mathcal{B}_{\rm self}(\mathfrak{H})$ such that for any V of this class and $H_0 \in \mathcal{S}(\mathfrak{H})$ the wave operators $W_\pm(H_0+$ $+V, H_0)$ exist. More precisely, we can introduce the sets $\mathcal{W}_\pm(\mathfrak{H}) \subseteq \mathcal{B}_{\rm self}(\mathfrak{H})$ defined by

$$\mathcal{W}_\pm(\mathfrak{H}) = \{V \in \mathcal{B}_{\rm self}(\mathfrak{H}) : \forall H_0 \in \mathcal{S}(\mathfrak{H})\ \exists\ W_\pm(H_0+V,H_0)\}. \tag{1.3}$$

On account of the mentioned Kato-Rosenblum theorem we have

$$\mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\rm self}(\mathfrak{H}) \subseteq \mathcal{W}_\pm(\mathfrak{H}). \tag{1.4}$$

Now the question is whether the equality

$$\mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H}) = \mathcal{W}_\pm(\mathfrak{H}) \tag{1.5}$$

holds. In other words (using some results of Section 3), does there exist an operator $V \in \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$, $V \notin \mathcal{L}_1(\mathfrak{H})$, such that for all $H_0 \in \mathcal{S}(\mathfrak{H})$ the absolutely continuous parts of $H_0 + V$ and H_0 are unitary equivalent via the wave operators? The problem of the validity of (1.5) has been posed by M.M. Skriganov from the Leningrad State University during a stay of the authors at this university.

It is expected that equality (1.5) is valid. The expectation is based on the famous Weyl-von Neumann theorem and its generalization by Kuroda (see [6], [7]). In the generalized version this theorem says that for any self-adjoint operator H_0, any $\varepsilon > 0$, and any cross norm γ not equivalent to the trace norm there is a self-adjoint compact perturbation V such that H_0+V is a diagonal operator (i.e. an operator with pure point spectrum) and $\gamma(V) < \varepsilon$. However, the quoted theorem does not prove (1.5). For this, it would be necessary to have a converse of the generalized Weyl-von Neumann theorem. Namely, for any compact self-adjoint operator V, $V \notin \mathcal{L}_1(\mathfrak{H})$, there is an absolutely continuous self-adjoint operator H_0 such that $H_0 + V$ becomes a diagonal operator. But such a theorem is unknown for the authors. In this paper we show that the equality (1.5) is valid. This means that the class of self-adjoint trace class operators is the largest class of self-adjoint perturbations which implies the existence of the wave operators independent of the chosen unperturbed self-adjoint operators. More precisely, we prove a stronger result, namely that (1.5) is also true if we use a weaker definition of the wave operators.

The proof of (1.5) is done as follows. First we shortly introduce the generalized wave operators and some notions in the next section. In Section 3 we prove some consequences of the assumption that (1.5) is not true. In particular, it follows that for the multiplication operator H_0 by the independent variable λ on $L^2([-\pi, \pi])$ there is a special self-adjoint operator V such that the generalized wave operators exist but $V \notin \mathcal{L}_1(\mathfrak{H})$. Using the special form of the operators V and H_0 from Section 3, we prove in Section 4 that the wave operators $W_\pm(H_0 + V, H_0)$ cannot exist. This gives the desired contradiction for the assumption that (1.5) is not true.

2. GENERALIZED WAVE OPERATORS

Generalized wave operators were systematically investigated in [8] and [9]. They are defined with the help of more general limiting processes. We only present generalized wave operators defined with the help of the Cesaro mean. We say, the generalized wave operators $\tilde{W}_\pm(H_0 + V, H_0)$ exist if the limits

$$(2.1) \qquad \tilde{W}_\pm(H_0 + V, H_0) = \text{s-}\lim_{T\to+\infty} \frac{1}{T}\int_0^T dt e^{\pm it(H_0+V)} e^{\mp itH_0} P^{ac}(H_0)$$

exist and are partial isometries satisfying

$$(2.2) \qquad \tilde{W}_\pm(H_0 + V, H_0)^* \tilde{W}_\pm(H_0 + V, H_0) = P^{ac}(H_0)$$

For brevity and in accordance with [1] we will use the notation

$$(2.3) \qquad \tilde{W}_\pm(H_0 + V, H_0) = |A|\text{-}\lim_{t\to\pm\infty} e^{it(H_0+V)} e^{-itH_0} P^{ac}(H_0).$$

if (2.1) and (2.2) are fulfilled. Note that the existence of the wave operators in the Cesaro mean is equivalent for the existence of the wave operators in a whole class of limiting processes (for details see [9]). Thus, the Cesaro mean is only a concrete description for a more general definition of wave operators. Further note that the existence of $W_\pm(H_0+V, H_0)$ implies the existence of $\tilde{W}_\pm(H_0+V, H_0)$ but the converse is not true in general.

Now we introduce the sets $\tilde{\mathcal{W}}_\pm(\mathfrak{H}) \subseteq \mathcal{B}_{self}(\mathfrak{H})$ by

$$(2.4) \qquad \tilde{\mathcal{W}}_\pm(\mathfrak{H}) = \{V \in \mathcal{B}_{self}(\mathfrak{H}) : \tilde{W}_\pm(H_0 + V, H_0) \text{ exist for all } H_0 \in \mathcal{S}(\mathfrak{H})\}$$

which obviously obey the relations

$$(2.5) \qquad \mathcal{W}_\pm(\mathfrak{H}) \subseteq \tilde{\mathcal{W}}_\pm(\mathfrak{H}).$$

In the following sections we show that

$$(2.6) \qquad \tilde{\mathcal{W}}_\pm(\mathfrak{H}) = \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{self}(\mathfrak{H})$$

First, we remark that the introduction of the generalized wave operators has clarified some problems, e.g. the invariance principle in a quite satisfactory manner (see [10], [14], [15]) (The invariance principle in its strong form is true for generalized wave operators but not for the usual wave operators.) Thus, it is not clear whether the validity of (1.5) implies the validity of (2.6). This is the reason why we have solved a converse of the Kato-Rosenblum theorem in a more general setting.

Let us finish this section with another remark. The proposed problem is meaningful only if $\dim(\mathfrak{H}) = +\infty$. In a finite dimensional Hilbert space $P^{ac}(\cdot)$ is always the null operator. Thus we obtain $P^{ac}(H_0) = P^{ac}(H_0 + V) = 0$ and $\tilde{\mathcal{W}}_\pm(\mathfrak{H}) = \mathcal{B}_{self}(\mathfrak{H})$.

Since every linear operator on a finite dimensional Hilbert space is a trace class operator, i.e. $\mathcal{L}_1(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$, the equality (2.6) is true for every finite dimensional Hilbert space. Hence the problem is trivial.

3. PROPERTIES OF $\tilde{W}_{\pm}(\mathfrak{H})$

In this section we establish some properties of the sets $\tilde{W}_{\pm}(\mathfrak{H})$ which will be useful in the next one. We agree to call $\hat{A}$ a part of a self-adjoint operator A on $\mathfrak{H}$ if there is a reducing subspace $\hat{\mathfrak{H}}$ of A such that $\hat{A} = A \mid \hat{\mathfrak{H}} \cap \mathrm{dom}(A)$. In particular, A can be regarded as a part of itself.

LEMMA 3.1.

(i) $\tilde{W}_+(\mathfrak{H}) = \tilde{W}_-(\mathfrak{H}) \overset{\mathrm{def}}{=} \tilde{W}(\mathfrak{H})$.

(ii) *If $V \in \tilde{W}(\mathfrak{H})$, then $-V \in \tilde{W}(\mathfrak{H})$.*

(iii) *If $\hat{\mathfrak{H}}$ is a reducing subspace of $V \in \tilde{W}(\mathfrak{H})$, then*

$$\hat{V} = V \mid \hat{\mathfrak{H}} \in \tilde{W}(\hat{\mathfrak{H}}).$$

(iv) *If $V \in \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ is unitarily equivalent to some $V' \in \tilde{W}(\mathfrak{H}')$, then $V \in \tilde{W}(\mathfrak{H})$.*

(v) *If $K \in \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ and $V \in \tilde{W}(\mathfrak{H})$, then $K + V \in \tilde{W}(\mathfrak{H})$.*

(vi) *If $V \in \tilde{W}(\mathfrak{H})$, then $V = D + K$ where D is a diagonal operator and $K \in \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$.*

(vii) $D \in \mathcal{L}_\infty(\mathfrak{H})$.

Proof. (i) Let J be a conjugation on $\mathfrak{H}$ commuting with $V \in \tilde{W}_+(\mathfrak{H})$ (such a conjugation always exist, see [13], p.223]). If $H_0 \in \mathcal{S}(\mathfrak{H})$, then

$$Je^{-it(H_0+V)}J = e^{it(JH_0J+V)}, \quad t \in \mathbb{R}^1, \tag{3.1}$$

where the operator JH_0J defined on $\mathrm{dom}(JH_0J) = J\mathrm{dom}(H_0)$ is obviously self-adjoint, i.e. $JH_0J \in \mathcal{S}(\mathfrak{H})$. Since $V \in \tilde{W}_+(\mathfrak{H})$ the wave operator $\tilde{W}_+(JH_0J+V, JH_0J)$, $H_0 \in \mathcal{S}(\mathfrak{H})$, exists. Furthermore, $P^{\mathrm{ac}}(\cdot)$ is a spectral projection [1]. Thus, $P^{\mathrm{ac}}(JH_0J) = JP^{\mathrm{ac}}(H_0)J$. Using (3.1) we get

$$J\tilde{W}_+(JH_0J + V, JH_0J)J =$$

$$= |A|\text{-}\lim_{t\to+\infty} Je^{it(JH_0J+V)}e^{-itJH_0J}P^{\mathrm{ac}}(JH_0J)J = \tag{3.2}$$

$$= |A|\text{-}\lim_{t\to+\infty} e^{-it(H_0+V)}e^{itH_0}P^{\mathrm{ac}}(H_0) = \tilde{W}_-(H_0 + V, H_0),$$

$H_0 \in \mathcal{S}(\mathfrak{H})$. Hence, $\tilde{\mathcal{W}}_+(\mathfrak{H}) \subseteq \tilde{\mathcal{W}}_-(\mathfrak{H})$. Similarly, we find $\tilde{\mathcal{W}}_-(\mathfrak{H}) \subseteq \tilde{\mathcal{W}}_+(\mathfrak{H})$ giving the desired relation.

(ii) We note that $V \in \tilde{\mathcal{W}}(\mathfrak{H})$ implies the existence of $\tilde{W}_-(-H_0+V, -H_0)$ for every $H_0 \in \mathcal{S}(\mathfrak{H})$. But the existence of $\tilde{W}_-(-H_0 + V, -H_0)$ is equivalent to the existence of $\tilde{W}_+(H_0 - V, H_0)$, $H_0 \in \mathcal{S}(\mathfrak{H})$. Hence $-V \in \tilde{\mathcal{W}}(\mathfrak{H})$.

(iii) Every $\hat{H}_0 \in \mathcal{S}(\hat{\mathfrak{H}})$ can be extended to some operator $H_0 \in \mathcal{S}(\mathfrak{H})$ such that $\hat{H}_0$ is a part of H_0, i.e. $H_0 \mid \hat{\mathfrak{H}} \cap \mathrm{dom}(H_0) = \hat{H}_0$. Obviously, we have $(H_0 + V) \mid \mid (\hat{\mathfrak{H}} \cap \mathrm{dom}(H_0 + V)) = \hat{H}_0 + \hat{V}$ and $P^{\mathrm{ac}}(H_0) \mid \hat{\mathfrak{H}} = P^{\mathrm{ac}}(\hat{H}_0)$. The last property is again based on the fact that $P^{\mathrm{ac}}(\cdot)$ is a spectral projection. Since $V \in \tilde{\mathcal{W}}(\mathfrak{H})$, the wave operator $\tilde{W}_+(H_0 + V, H_0)$ exists and we have

$$
\begin{aligned}
\tilde{W}_+(H_0 + V, H_0) \mid \hat{\mathfrak{H}} &= |A|\text{-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}t(H_0+V)} \mathrm{e}^{-\mathrm{i}tH_0} P^{\mathrm{ac}}(H_0) \mid \hat{\mathfrak{H}} = \\
&= |A|\text{-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}t(\hat{H}_0+\hat{V})} \mathrm{e}^{-\mathrm{i}t\hat{H}_0} P^{\mathrm{ac}}(\hat{H}_0) = \tilde{W}_+(\hat{H}_0 + \hat{V}, \hat{H}_0)
\end{aligned} \tag{3.3}
$$

for every $\hat{H}_0 \in \mathcal{S}(\hat{\mathfrak{H}})$. Hence $\hat{V} \in \tilde{\mathcal{W}}(\hat{\mathfrak{H}})$.

(iv) Since $V' \in \tilde{\mathcal{W}}(\mathfrak{H}')$ and $V \in \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ are unitarily equivalent, there is an isometry U from $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $V = U^{-1}V'U$. If $H_0 \in \mathcal{S}(\mathfrak{H})$, then $UH_0U^{-1} \in \in \mathcal{S}(\mathfrak{H}')$. By $V' \in \tilde{\mathcal{W}}(\mathfrak{H}')$ the wave operator $\tilde{W}_+(UH_0U^{-1} + V', UH_0U^{-1})$ exists for every $H_0 \in \mathcal{S}(\mathfrak{H})$. Hence, we get

$$
\begin{aligned}
& U^{-1}\tilde{W}_+(UH_0U^{-1} + V', UH_0U^{-1})U = \\
&= |A|\text{-}\lim_{t \to +\infty} U^{-1} \mathrm{e}^{\mathrm{i}t(UH_0U^{-1}+V')} \mathrm{e}^{-\mathrm{i}tUH_0U^{-1}} P^{\mathrm{ac}}(UH_0U^{-1})U = \\
&= |A|\text{-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}t(H_0+U^{-1}V'U)} \mathrm{e}^{-\mathrm{i}tH_0} P^{\mathrm{ac}}(H_0) = \tilde{W}_+(H_0 + V, H_0)
\end{aligned} \tag{3.4}
$$

for every $H_0 \in \mathcal{S}(\mathfrak{H})$. Thus, we have proved (iv).

(v) Since $V \in \tilde{\mathcal{W}}(\mathfrak{H})$ $\tilde{W}_+(H_0 + K + V,\ H_0 + K)$ exists for every $K \in \mathcal{L}_1(\mathfrak{H}) \cap \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ and $H_0 \in \mathcal{S}(\mathfrak{H})$. The trace class existence theorem implies the existence of $W_+(H_0+K, H_0)$ for every $K \in \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ and every $H_0 \in \mathcal{S}(\mathfrak{H})$. Using the chain rule [1] we get the existence of $\tilde{W}_+(H_0 + K + V,\ H_0)$ for every $K \in \mathcal{L}_1(\mathfrak{H}) \cap \mathcal{B}_{\mathrm{self}}(\mathfrak{H})$ and $H_0 \in \mathcal{S}(\mathfrak{H})$. Hence, $K + V \in \tilde{\mathcal{W}}(\mathfrak{H})$.

(vi) Assuming that $V \in \tilde{\mathcal{W}}(\mathfrak{H})$ has an absolutely continuous part. Since $0 = V - V$ we obtain by (ii) that the null operator has an absolutely continuous part, too. But this is oviously wrong. Hence V is a completely singular operator. Taking into account Lemma 1 of [3] (see also [2], [12]) we immediately prove the desired representation.

(vii) On account of (v) we find $D \in \tilde{\mathcal{W}}(\mathfrak{H})$. Denoting by $\{\lambda_l\}_{l=0}^{+\infty}$ the spectrum of D we introduce the direct sum $\mathfrak{H}_{\pm}^{q}$ of eigenspaces $\mathfrak{H}_l$ corresponding to eigenvalues λ_l

obeying $\pm\lambda_l \geqslant a > 0$. Assuming for a moment that there exist a number $a > 0$ such that $\dim(\mathfrak{H}_+^a) = +\infty$. Choosing an absolutely continuous operator $H_0 \geqslant 0$ defined on $\mathfrak{H}_+^a$ with $\mathrm{spec}(H_0) \subset (0, a)$ and setting $D_+^a = D \mid \mathfrak{H}_+^a$ we get that on account of (iii) the operator $H_0 + D_+^a$ possesses an absolutely continuous part spectrum of which contains that of H_0, i.e. $\mathrm{spec}(H_0) \subseteq \mathrm{spec}(H_0 + D_+^a)$. Since $D_+^a \geqslant aI$ we obviously find that $H_0 + D_+^a \geqslant aI$. But the last property contradicts $\mathrm{spec}(H_0) \subseteq \mathrm{spec}(H_0 + D_+^a)$. Thus $\dim(\mathfrak{H}_+^a) < +\infty$ for every $a > 0$. Similarly we show $\dim(\mathfrak{H}_-^a) < +\infty$ for every $a > 0$. But both facts immediately imply the compactness of D. ■

Notice that the properties (vi) and (vii) yield $\tilde{\mathcal{W}}(\mathfrak{H}) \subseteq \mathcal{L}_\infty(\mathfrak{H})$.

LEMMA 3.2. *Let* $\dim(\mathfrak{H}) = +\infty$. *If* $\tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H}) \neq 0$, *then there is a* $V \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ *such that*

(i) $V \in \mathcal{L}_\infty(\mathfrak{H})$,

(ii) $V \geqslant 0$,

(iii) $V \mid (\mathrm{ima}(V))^-$ *is simple*,

(iv) $\dim(\ker(V)) = +\infty$.

Proof. If $\tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H}) \neq 0$, then by Lemma 3.1(vi) we can assume the existence of a diagonal operator D such that $D \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$.

(i) This property is a consequence of Lemma 3.1 (vii).

(ii) Let $D \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$. Since D is self-adjoint there exist two reducing subspaces $\mathfrak{H}_\pm$ such that $\mathfrak{H} = \mathfrak{H}_+ \ominus \mathfrak{H}_-$, $D_+ = D \mid \mathfrak{H}_+ \geqslant 0$ and $D_- = D \mid \mathfrak{H}_- \leqslant 0$. Hence, we obtain either $D_+ \notin \mathcal{L}_1(\mathfrak{H}_+)$ or $D_- \notin \mathcal{L}_1(\mathfrak{H}_-)$. If $D_+ \notin \mathcal{L}_1(\mathfrak{H}_+)$, then $\dim(\mathfrak{H}_+) = +\infty$. Moreover, by Lemma 3.1 (iii) we find $D_+ \in \tilde{\mathcal{W}}(\mathfrak{H}_+)\backslash\mathcal{L}_1(\mathfrak{H}_+)$. If $D_- \notin \mathcal{L}_1(\mathfrak{H}_-)$, then $\dim(\mathfrak{H}_-) = +\infty$ and $D_- \in \tilde{\mathcal{W}}(\mathfrak{H}_-)\backslash\mathcal{L}_1(\mathfrak{H}_-)$. Applying Lemma 3.1 (ii) we get $0 \leqslant -D_- \in \tilde{\mathcal{W}}(\mathfrak{H}_-)\backslash\mathcal{L}_1(\mathfrak{H}_-)$. Thus, in every case there is a non-negative operator V' on some infinite dimensional Hilbert space $\mathfrak{H}'$ belonging to $\tilde{\mathcal{W}}(\mathfrak{H}')\backslash\mathcal{L}_1(\mathfrak{H}')$. Since $\dim(\mathfrak{H}') = +\infty$ there exists an isometry U from $\mathfrak{H}$ onto $\mathfrak{H}'$. Setting $V = U^{-1}V'U$ and applying Lemma 3.1 (iv) we find $V \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ and $V \geqslant 0$.

(iii) We note that always by a non-negative nuclear perturbation we can achieve that the spectrum of a non-negative compact operator becomes simple.

(iv) We assume that $D \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ satisfies the conditions (i), (ii) and (iii). Introducing the spectral decomposition $D = \sum_{l=0}^{\infty}\lambda_l P_l$, $0 < \lambda_l$, with $\dim(P_l\mathfrak{H}) = 1$, $l = 1, 2, \ldots$ we choose a monotonously decreasing infinite subsequence $\{\lambda_{l_j}\}_{j=0}^{+\infty}$ of $\{\lambda_l\}_{l=0}^{+\infty}$ obeying the condition

$$\sum_{j=0}^{+\infty} \lambda_{l_j} < +\infty. \tag{3.5}$$

Since $\lambda_l \searrow 0$ as $l \to +\infty$ such a subsequence always exists. We set

$$K = \sum_{j=0}^{+\infty} \lambda_{l_j} P_{l_j}. \tag{3.6}$$

Due to (3.5) K is a nuclear operator. Moreover, the subspace

$$\hat{\mathfrak{H}} = (\mathrm{ima}(K))^- = \bigoplus_{j=0}^{+\infty} \mathfrak{H}_{l_j} \tag{3.7}$$

is an infinite dimensional one and reduces D. Obviously, we have $K = D \mid \hat{\mathfrak{H}}$. Thus setting

$$V = D - K \tag{3.8}$$

we obtain a simple non-negative operator V such that $\ker(V) \supseteq \hat{\mathfrak{H}}$. Hence, $\dim(\ker(V)) = +\infty$. Since $D \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ we find $V \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ applying Lemma 3.1 (v). ∎

In the following we denote by $\{\varphi_l\}_{l=-\infty}^{+\infty}$ the orthonormal system $\varphi_l(x) = \frac{1}{\sqrt{2\pi}} e^{-ilx}$, $x \in [-\pi, \pi]$, in $L^2([-\pi,\pi])$.

LEMMA 3.3. *If for some infinite dimensional separable Hilbert space $\mathfrak{H}$ the set $\tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ is not empty, then there is a strongly decreasing infinite sequence of positive numbers $\{\lambda_l\}_{l=0}^{+\infty}$ tending to zero as $l \to +\infty$ and satisfying*

$$\sum_{l=0}^{+\infty} \lambda_l = +\infty \tag{3.9}$$

such that

$$V = \sum_{l=0}^{+\infty} \lambda_l (\cdot, \varphi_l)\varphi_l \tag{3.10}$$

belongs to $\tilde{\mathcal{W}}(L^2([-\pi,\pi]))\backslash\mathcal{L}_1(L^2([-\pi,\pi]))$.

Proof. Let $L \in \tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$. On account of Lemma 3.2 we can assume that in addition L satisfies the conditions (i)–(iv) of Lemma 3.2. Since $L \in \mathcal{L}_\infty(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H})$ and $L \geqslant 0$, the spectrum of L consists of $\{0\}$ and an infinite strongly decreasing sequence of positive numbers $\{\lambda_l\}_{l=0}^{+\infty}$ tending to zero as $l \to +\infty$. Moreover, by (iii) of Lemma 3.2 the sequence of eigenprojections $\{P_l\}_{l=0}^{+\infty}$ of L consists of one dimensional projections. Denoting by $\{\psi_l\}_{l=0}^{+\infty}$ the corresponding sequence of eigenfunctions of L we obtain the representation

$$L = \sum_{l=0}^{+\infty} \lambda_l(\cdot, \psi_l)\psi_l. \tag{3.11}$$

By Lemma 3.2 (iv) there is an isometry U from $\mathfrak{H}$ onto $L^2([-\pi, \pi])$ such that $U\ker(L) = \operatorname{clospa}\{\varphi_l : l = -1, -2, \ldots\}$ and $U\psi_l = \varphi_l$, $l = 0, 1, 2, \ldots$. Setting $V = ULU^{-1}$ by (3.11) V admits the representation (3.10). Applying Lemma 3.1 (iv) we find $V \in \tilde{\mathcal{W}}(L^2([-\pi, \pi]))\backslash\mathcal{L}_1(L^2([-\pi, \pi]))$. ■

4. MAIN THEOREM

For the proof of our main theorem we need two technical lemmas.

LEMMA 4.1. *If $V \in \tilde{\mathcal{W}}(\mathfrak{H})$, then*

$$\tilde{W}_+(H_0 + V,\ H_0) = \text{s-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nH} \mathrm{e}^{-\mathrm{i}nH_0} P^{\mathrm{ac}}(H_0). \tag{4.1}$$

Proof. Put $H = H_0 + V$ and define a bounded operator G by

$$G = \int_{-1}^{0} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} \mathrm{d}t. \tag{4.2}$$

From (2.1) we get the representation

$$\tilde{W}_+(H, H_0) = \text{s-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nH} G \mathrm{e}^{-\mathrm{i}nH_0} P^{\mathrm{ac}}(H_0). \tag{4.3}$$

Using the formula

$$\mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} = I + \mathrm{i} \int_0^t \mathrm{d}s \mathrm{e}^{\mathrm{i}sH} V \mathrm{e}^{-\mathrm{i}sH_0},$$

$t \in \mathbb{R}^1$, we obtain $G = I + C$ with

$$C = \mathrm{i} \int_{-1}^{0} \mathrm{d}t \int_0^t \mathrm{d}s \mathrm{e}^{\mathrm{i}sH} V \mathrm{e}^{-\mathrm{i}sH_0}.$$

By Lemma 3.2 we have $V \in \mathcal{L}_\infty(\mathfrak{H})$. Hence, we find $C \in \mathcal{L}_\infty(\mathfrak{H})$ as an easy calculation shows. Hence we have $\text{s-}\lim_{t\to\infty} C\mathrm{e}^{-\mathrm{i}tH_0} P^{\mathrm{ac}}(H_0) = 0$ (see e.g. [1], Theorem 6.45). Thus,

$$\text{s-}\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}nH} C \mathrm{e}^{-\mathrm{i}nH_0} P^{\mathrm{ac}}(H_0) = 0.$$

This relation together with $G = I + C$ and (4.3) gives (4.1). ■

The next lemma is essentially an application of a result from the theory of divergent series to our objects.

LEMMA 4.2. *Let $\{f_m\}_{m\geqslant 1}$ be a sequence of elements of $\mathfrak{H}$ converging in the Cesaro mean to $f \in \mathfrak{H}$,* i.e.

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{m=0}^{N} f_m = f. \tag{4.4}$$

Let $\{\lambda_m\}_{m=0}^{\infty}$ be a strongly decreasing sequence of positive numbers converging to zero as $m \to \infty$ and obeying

$$\lim_{m\to\infty} \sum_{n=1}^{m} \lambda_n = +\infty. \tag{4.5}$$

Set $0 = \lambda_{-1} = \lambda_{-2} = \dots$ and assume $\|f_m\| \leqslant 1$, $m \in \mathbb{N}$. Then,

$$\left\| \frac{1}{\sum\limits_{n=1}^{N}\sum\limits_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m \right\| \leqslant 1, \quad k \in \mathbb{Z}, \tag{4.6}$$

and

$$\lim_{N\to\infty} \frac{1}{\sum\limits_{n=1}^{N}\sum\limits_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m = f, \quad k \in \mathbb{Z}. \tag{4.7}$$

Proof. 1. We write the expression in (4.6) in another form. Let k be fixed. We put $v_m = \lambda_{m+k}$ for $m \in \mathbb{Z}$. Then $v_m \geqslant 0$, $\sum\limits_{m=-k}^{\infty} v_m = +\infty$, $\lim\limits_{m\to\infty} v_m = 0$. Further, we put

$$g_n = \frac{1}{n+1}\sum_{m=0}^{n} f_m. \tag{4.8}$$

This gives

$$\sum_{m=0}^{n-1} v_m f_m = n v_{n-1} g_{n-1} + \sum_{m=-1}^{n-2} (m+1)(v_m - v_{m+1}) g_m$$

with $g_{-1} = 0$. Using this connection between f_m and g_m we can write

$$\sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m = \sum_{n=1}^{N} \tilde{C}_{N,n} g_{n-1}. \tag{4.9}$$

where

$$\tilde{C}_{N,n} = n(N - n + 1)v_{n-1} - n(N - n)v_n. \tag{4.10}$$

The term $\sigma_N = \sum_{n=1}^{N} \sum_{m=0}^{n-1} \lambda_m$ can be written by

$$\sigma_N = \sum_{n=1}^{N} \sum_{m=0}^{n-1} v_{m-k} = \sum_{n=1}^{N} \sum_{j=-k}^{n-1-k} v_j =$$

$$= \sum_{n=1}^{N} \tilde{C}_{N,n} + \begin{cases} \sum_{n=1}^{N} \left\{ \sum_{j=-k}^{-1} v_j - \sum_{j=n-k}^{n-1} v_j \right\} & (k \geqslant 0) \\ \sum_{n=1}^{N} \left\{ - \sum_{j=0}^{-k-1} v_j + \sum_{j=n}^{n-k-1} v_j \right\} & (k < 0) \end{cases}$$

Since $v_j = \lambda_{j+k} = 0$ for $j < -k$ we note that $\sum_{j=0}^{-k-1} v_j = 0$. Setting $C_{N,n} = \tilde{C}_{N,n}\sigma_N^{-1}$ we obtain

$$\sum_{n=1}^{N} C_{N,n} g_{n-1} = \frac{1}{\sum_{n=1}^{N} \sum_{m=0}^{n-1} \lambda_m} \sum_{n=1}^{N} \sum_{m=0}^{n-1} \lambda_{m+k} f_m. \tag{4.11}$$

Thus, we have to prove that $\lim_{n \to \infty} g_n = f$ and $\|g_n\| \leqslant 1$ (this follows immediately from $\|f_n\| \leqslant 1$) imply

$$\left\| \sum_{n=1}^{N} C_{N,n} g_n \right\| \leqslant 1 \tag{4.12}$$

and

$$\lim_{N \to \infty} \sum_{n=1}^{N} C_{N,n} g_n = f. \tag{4.13}$$

2. We note some properties of the "transformation" $C_{N,n}$. It is a straightforward calculation to show that

$$\sum_{n=1}^{N} C_{N,n} \leqslant 1 \quad \text{and} \quad \lim_{N \to \infty} \sum_{n=1}^{N} C_{N,n} = 1.$$

Further, we also obtain that $\lim_{N \to \infty} C_{N,n} = \lim_{N \to \infty} \tilde{C}_{N,n}\sigma_N^{-1} = 0$ for each n. This is an easy consequence of $v_m \geqslant 0$, $\lim_{m \to \infty} v_m = 0$ and $\lim_{N \to \infty} \sum_{m=0}^{N} v_m = \infty$. These properties of the transformation $C_{N,n}$ imply that $C_{N,n}$ is a so-called regular transformation. This means that if S_n is a number sequence tending to S, then the sequence $\sum_{n=1}^{N} C_{N,n}S_n$ converges to S too (see [4, Theorem 1]).

3. The last step in our proof is now to extend the assertions about number sequences to sequences of Hilbert space vectors g_n. We have

$$\left\| \sum_{n=1}^{N} C_{N,n} g_n \right\| \leqslant \sum_{n=1}^{N} C_{N,n} \|g_n\| \leqslant \sum_{n=1}^{N} C_{N,n} \leqslant 1 \tag{4.14}$$

because of $\|g_n\| \leqslant 1$. This proves (4.12) and thus (4.6). Let $u \in \mathfrak{H}$ and set $a_n = (u, g_n)$. Obviously, a_n converges by assumption to $a = (u, f)$. Thus,

$$\lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n} a_n = a,$$

because of the considerations in 2. Thus $\sum_{n=1}^{N} C_{N,n} g_n = w_n$ converges weakly to f. On the other hand, since $\|g_n\|$ converges to $\|f\|$ we also get

$$\lim_{N\to\infty} \sum_{n=1}^{N} C_{N,n} \|g_n\| = \|f\|.$$

Using (4.14) we find that

$$\lim_{N\to\infty} \sup \|w_n\| \leqslant \|f\|. \tag{4.15}$$

Now, it is a standard conclusion to verify the strong convergence of w_n to f from its weak convergence to f and (4.15). This proves (4.13) and therefore (4.7). ■

THEOREM 4.3. *For every separable Hilbert space $\mathfrak{H}$ we have $\tilde{\mathcal{W}}(\mathfrak{H}) = \mathcal{L}_1(\mathfrak{H})$.*

Proof. As it was pointed out in Section 2, the problem is trivial if $\dim(\mathfrak{H}) <$ $< +\infty$. Thus, we assume $\dim(\mathfrak{H}) = +\infty$ in addition. Let $\tilde{\mathcal{W}}(\mathfrak{H}) \backslash \mathcal{L}_1(\mathfrak{H}) \neq 0$. By $V \in \mathcal{B}_{\text{self}}(L^2([-\pi, \pi]))$ we denote the operator characterized by Lemma 3.3. In the following, we will see that $V \in \tilde{\mathcal{W}}(L^2([-\pi, \pi]))$ is incompatible with all the other properties indicated in Lemma 3.3, in particular, with (3.9).

Let us introduce the self-adjoint operator H_0 on $L^2([-\pi, \pi])$ defined by $(H_0 f)(x) = x f(x)$, $f \in L^2([-\pi, \pi])$. Notice that

$$e^{-inH_0} \varphi_l = \varphi_{l+n}, \quad n, l \in \mathbf{Z}, \tag{4.16}$$

where $\{\varphi_l\}_{l=-\infty}^{+\infty}$ denotes the special orthonormal system used in Lemma 3.3. Let $h(\cdot) \neq 0$ be a C^∞-function defined on $[-\pi, \pi]$ which is zero in a neighbourhood of $-\pi$ and π. Setting $H = H_0 + V \in \mathcal{B}_{\text{self}}(L^2([-\pi, \pi]))$ we intend to show that

$$
\begin{aligned}
\lim_{N\to\infty} \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} e^{-i\tau H_0} h(H_0)\varphi_0 = \\
= \tilde{W}_+(H, H_0) e^{-i\tau H_0} h(H_0)\varphi_0,
\end{aligned}
\tag{4.17}
$$

$|\tau| \leqslant 1$. We remark that the existence of $\tilde{W}_+(H, H_0)$ follows from $V \in \tilde{\mathcal{W}}(L^2([-\pi, \pi]))$.

First of all, we note that $g(\tau) = e^{-i\tau H_0} h(H_0)\varphi_0$ admits a Fourier series given by

$$
(g(\tau))(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k(\tau) e^{-ikx} = \sum_{k=-\infty}^{+\infty} c_k(\tau)\varphi_k(x), \tag{4.18}
$$

$x \in [-\pi, \pi]$, for every $\tau \in [-\pi, \pi]$. A straightforward calculation shows that

$$
\sup_{|\tau|\leqslant 1} \sum_{k=-\infty}^{+\infty} |c_k(\tau)| < +\infty. \tag{4.19}
$$

We find

$$
\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} g(\tau) = \sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} \sum_{k=-\infty}^{+\infty} c_k(\tau)\varphi_k, \tag{4.20}
$$

$n \geqslant 1$, $\tau \in [-1, 1]$. Due to (4.19) the sums can be interchanded. Moreover, taking into account (3.10) and (4.16) we get

$$
\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} g(\tau) = \sum_{k=-\infty}^{+\infty} c_k(\tau) \sum_{m=0}^{n-1} \lambda_{m+k} e^{imH} e^{-imH_0} \varphi_k,
$$

$n \geqslant 1$, $\tau \in [-1, 1]$. Hence, we have to calculate

$$
\begin{aligned}
&\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} g(\tau) = \\
&\lim_{N\to+\infty} \sum_{k=-\infty}^{+\infty} c_k(\tau) \left\{ \frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} e^{imH} e^{-imH_0} \varphi_k \right\},
\end{aligned}
\tag{4.21}
$$

where we have interchanged $\sum_{n=1}^{N}$ and $\sum_{k=-\infty}^{+\infty}$ on account of (4.19). Introducing the sequence $\{f_m^{(k)}\}_{m=0}^{+\infty}$, $k \in \mathbf{Z}$, $f_m^{(k)} = e^{imH} e^{-imH_0} \varphi_k$, and applying Lemma 4.2 we see that the expression in the curved brackets is norm bounded by one. Consequently, because of (4.19) $\lim_{N\to\infty}$ and $\sum_{k=-\infty}^{+\infty}$ can be interchanded. Thus, we arrive at

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} g(\tau) =$$

$$= \sum_{k=-\infty}^{+\infty} c_k(\tau) \lim_{N\to+\infty} \left\{ \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} \lambda_{m+k} f_m^{(k)} \right\},$$

$\tau \in [-1,1]$. The sequence $\{f_m^{(k)}\}_{m=0}^{+\infty}$ converges in the Cesaro mean to $\tilde{W}_+(H,H_0)\varphi_k$ as $m \to +\infty$ for every $k \in \mathbb{Z}$. Applying Lemma 4.2 we obtain

$$\lim_{N\to+\infty} \frac{1}{\sum_{n=1}^{+\infty}\sum_{m=0}^{n-1}\lambda_m} \sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} e^{-i\tau H_0} h(H_0)\varphi_0 =$$

$$= \sum_{k=-\infty}^{+\infty} c_k(\tau)\tilde{W}_+(H,H_0)\varphi_k.$$

But taking into account the Fourier series (4.18) we find (4.17).

Using the representation

$$e^{inH} e^{-inH_0} f = f + i \int_0^1 d\tau e^{i\tau H} \sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} e^{-i\tau H_0} f,$$

$n \geqslant 1,\ f \in L^2([-\pi,\pi])$, we get

$$\left(\frac{1}{N}\sum_{n=1}^{N} e^{inH} e^{-inH_0} f,\ f'\right) = (f,f')+$$

$$+i\int_0^1 d\tau \left(\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1} e^{imH} V e^{-imH_0} e^{-i\tau H_0} f,\ e^{-i\tau H} f'\right),$$

$f, f' \in L^2([-\pi,\pi])$. Dividing by $\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m$ and setting $f = h(H_0)\varphi_0$ we find

$$\left(\frac{1}{\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\frac{1}{N}\sum_{n=1}^{N}\mathrm{e}^{\mathrm{i}nH}\mathrm{e}^{-\mathrm{i}nH_0}h(H_0)f,\ f'\right) =$$

$$(4.22)\qquad = \frac{1}{\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}(f,f')+$$

$$\mathrm{i}\int_0^1 \mathrm{d}\tau \left(\frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\mathrm{e}^{\mathrm{i}mH}V\mathrm{e}^{-\mathrm{i}mH_0}\mathrm{e}^{-\mathrm{i}\tau H_0}h(H_0)\varphi_0,\ \mathrm{e}^{-\mathrm{i}\tau H}f'\right),$$

$f' \in L^2([-\pi,\pi])$. Now, the left-hand side and the first term of the right-hand side tend to zero as $N \to +\infty$ since $\sum_{m=0}^{\infty}\lambda_m = \infty$ implies $\lim_{n\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m = \infty$ (Cesaro mean is a totally regular transformation. See [4]). Moreover, by the representation (4.20) and Lemma 4.2 we have

$$\left|\left(\frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\mathrm{e}^{\mathrm{i}mH}V\mathrm{e}^{-\mathrm{i}mH_0}\mathrm{e}^{-\mathrm{i}\tau H_0}h(H_0)\varphi_0,\ \mathrm{e}^{-\mathrm{i}\tau H}f'\right)\right| \leqslant$$

$$\leqslant \|f'\|\left\|\sum_{k=-\infty}^{+\infty}c_k(\tau)\frac{1}{\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_m}\sum_{n=1}^{N}\sum_{m=0}^{n-1}\lambda_{m+k}\mathrm{e}^{\mathrm{i}mH}\mathrm{e}^{-\mathrm{i}mH_0}\varphi_k\right\| \leqslant$$

$$\leqslant \|f'\|\sum_{k=-\infty}^{+\infty}|c_k(\tau)| \leqslant \|f'\|\sup_{\tau\in[-1,1]}\sum_{k=-\infty}^{+\infty}|c_k(\tau)| < +\infty,$$

$f' \in L^2([-\pi,\pi])$. Thus, applying the Lebesgue dominated convergence theorem to the second term of the right-hand side of (4.22) and taking into account (4.17) we obtain

$$(4.23)\qquad 0 = \mathrm{i}(\tilde{W}_+(H,H_0)h(H_0)\varphi_0,\ f'),$$

$f' \in L^2([-\pi,\pi])$. Since $f' \in L^2([-\pi,\pi])$ is an arbitrary element, (4.23) yields $\tilde{W}_+(H,H_0)h(H_0)\varphi_0 = 0$. But $\tilde{W}_+(H,H_0)$ is an isometry. Hence, $h(H_0)\varphi_0 = 0$ which implies $h(\cdot) \equiv 0$. But this fact contradicts $h(\cdot) \neq 0$. Consequently, $\tilde{\mathcal{W}}(\mathfrak{H})\backslash\mathcal{L}_1(\mathfrak{H}) \neq \emptyset$ is impossible for any infinite dimensional separable Hilbert space. ■

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