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FUNCTIONS OF GENERALIZED BOUNDED TYPE IN SPECTRAL THEORY OF NON-WEAK CONTRACTIONS

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To M. M. DJRBASHIAN

INTRODUCTION

The aim of the present paper is to show a way of application of the factorization theory of M. M. Djrbashian in the spectral theory of non-weak contractions. We mean the factorization theory of classes N_α of meromorhic functions, which have Nevanlinna characteristics of power growth near the boundary of the unit disk [5, Chapter IX] (for further development of this theory see [6, 7]). We shall consider analytic functions of classes N_α with $\alpha > 0$ — those analytic functions f in the unit disk, for which

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \left[D^{-\alpha} \log |f(re^{i\theta})| \right]^{+} d\theta < +\infty,$$

where

$$D^{-\alpha} \log |f(re^{i\theta})| = \frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} \log |f(te^{i\theta})| \, dt$$

and the standard notation $(u)^+ = \max\{u, 0\}$ for the function $u(r,\theta) = \log |f(re^{i\theta})|$ is accepted. Note that we avoid the notation A_α adopted for the subset of analytic functions of N_α [5, p.655].

It is important to point out that the application of M. M. Djrbashian's factorization results is a long standing interest for operator theory specialists. For example, soon after the monograph [5] was published, M. G. Kreĭn mentioned at Moscow Mathematical Congress the necessity to find spectral interpretations of its results. Further, M. S. Livšic [18] and his pupils (L. Megrabian, Do Kong Khan and others) realized the meromorphic functions of classes N_α as transfer functions of some special linear systems in a series of works. In this paper we find a different way to theory of classes N_α in the operator theory and we hope this way may turn out more reasonable. We start with the analytic function

$$\mathcal{D}_T(z) = \det_p W_T(z) W_T^*(\bar{z})$$

constructed for an invertible contraction T, for which $I - T^*T$ belongs to the ideal $\mathfrak{S}_p$ (p is natural). We denote this class of operators by C_p. Note that W_T is the characteristic operator-function of T and $\det_p$ is the regularized determinant [12, Chapter IV]. It is well known that if $p = 1$ the function $\mathcal{D}_T$ is bounded in the disk $|z| < 1$ and has a canonical multiplicative representation, which plays a significant role in the spectral theory of weak contractions [20]. The major problem we are connected with is: what factorization may be taken as an analogue of the canonical multiplicative representation of the function $\mathcal{D}_T$ in the case $p \geqslant 2$ and how such kind of factorization may be applied in the spectral theory of non-weak contractions?

In Section 1 it is shown that the regularized determinant $\mathcal{D}_T$ of the operator $T \in C_p$ belongs to N_α for any $\alpha > p - 1$. Further, zeros of the function $\mathcal{D}_T$ are taken off in succesive order by special elementary factors. Each of them turns out to be a regularized determinant of an operator-valued Blaschke dual factor connected with an eigenvalue of T. The factorizations of M. M. Djrbashian classes N_α are now applied for the function which do not vanish in the unit disk. As a result, in Thoerem 1.2 we come to the multiplicative representation

$$\mathcal{D}_T(z) = \mathcal{D}_0(z) \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{2}{(1-e^{-i\theta}z)^{1+\alpha}} - 1\right] d\psi_\alpha(0)\right\}; \quad \alpha > p - 1$$

and the full description of its parameters.

In Section 2 the previous results are used to illustrate a class of non-weak contractions in terms of the regularized determinant $\mathcal{D}_T$ and its factorization. To be precise, we mean the connection between the radial behaviour of the function $\mathcal{D}_T$ and the spectral properties of the operator $T \in C_p$. The formulas obtained may be considered as analogues of the trace formulas [15, 13] in the case of non-weak contractions.

In Section 3 it is considered in the region $\operatorname{Im} w < 0$ an analytic function $\mathcal{D}_A(w)$, connected with a dissipative operator A, which is the Cayley transform of an operator $T \in C_p$. The multiplicative representation of the function $\mathcal{D}_A$ is obtained now by the help of some factorization theorems for functions of generalized bounded type in the half-plane [8, 9]. It is established a criterion of representability of a dissipative operator A as a $\mathfrak{S}_p$-perturbation of some self-adjoint operator and the size of this perturbation is calculed by the parameters of the factorizaton of the function $\mathcal{D}_A$.

In Section 4 we discuss the possibility of the application of the factorization techniques, worked out for regularized determinants, in the completness problem of root subspaces of contractions and dissipative operators. Here a theorem is proved, which in case of dissipative operators essentially strenghthens the well-known result of M. V. Keldych [14], which states that an $\mathfrak{S}_p$-perturbations of a self-adjoint operator with discrete spectrum is complete.

The authors dedicate this paper to M. M. Djrbashian, whose fundamental works on the theory of functions find their applications in different fields of mathematics. We hope that the present paper will extend the sphere of such applications.

1. FACTORIZATION OF REGULARIZED DETERMINANTS OF CHARACTERISTIC OPERATOR-FUNCTIONS OF NON-WEAK CONTRACTIONS

1.1. For any $p \geqslant 1$ we denote by C_p the class of continuously invertible contractions T in a separable Hilbert space $\mathfrak{H}$, for which the operator $D_T^2 = I - T^*T$ belongs to Neumann-Schatten ideal $\mathfrak{S}_p$. Thus the set C_1 coincides with the class of invertible weak contractions [20]. The characteristic function W_T of the operator T will be defined as in [2]:

$$W_T(z)W_T(0) = \left[I - D_T(I - zT)^{-1}D_T\right] \mid \mathfrak{D}_T,$$
$$W_T(0) = (T^*T)^{1/2} \mid \mathfrak{D}_T, \quad \mathfrak{D}_T = \overline{D_T\mathfrak{H}}.$$

It is easy to verify, that the operator-function $W_T^*(\overline{z})$ differs from the characteristic function $\Theta_T(z)$ of B. Sz.-Nagy and C. Foiaş [20] only by a constant isometric factor. Let us recall from [20, 2], that $W_T(z)$ is analytic in $|z| < 1$ and its values are two-sided contractions in $\mathfrak{D}_T$:

$$W_T^*(z)W_T(z) \leqslant I, \quad W_T(z)W_T^*(z) \leqslant I \quad (|z| < 1).$$

Since

$$I - W_T^{-1}(0) = W_T^{-1}(0)\left[(T^*T)^{1/2} - I\right] \mid \mathfrak{D}_T =$$
$$= W_T^{-1}(0)\left[I + (T^*T)^{1/2}\right]^{-1}(T^*T - I) \mid \mathfrak{D}_T$$

and

$$I - W_T(z) = I - W_T^{-1}(0) + D_T(I - zT)^{-1}D_T W_T^{-1}(0), \quad D_T \in \mathfrak{S}_{2p},$$

the operator $I - W_T(z)$ belongs to $\mathfrak{S}_p$ for any $z \notin \sigma(T^{-1})$. Hence for each $p \geqslant 1$ the regularized determinant

$$d_T(z) = \det{}_p W_T(z) = \prod_k \lambda_k(z) \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j}[1 - \lambda_k(z)]^j\right\} \tag{1.1}$$

is analytic wherever the operator-function W_T is analytic [12, Chapter IV]. In the formula (1.1) $\{\lambda_k(z)\}$ is the set of eigenvalues of the operator $W_T(z)$. Further the functions $\mathcal{D}_T$ defined by

$$\mathcal{D}_T(z) = \det_p W_T(z) W_T^*(\bar{z}) \tag{1.2}$$

will play an important role.

Thus the formulas (1.1) and (1.2) state a correspondence between operators $T \in C_p$ (p is natural) and functions d_T and $\mathcal{D}_T$ analytic in $|z| < 1$.

1.2. The main result of this section states that functions d_T and $\mathcal{D}_T$ (analytic in $|z| < 1$) belong to classes N_α of M. M. Djrbashian [5, Chapter IX]. As it is well known, the functions d_T and $\mathcal{D}_T$ are bounded in $|z| < 1$, when $p = 1$. For the general case we have

THEOREM 1.1. *If $p \geqslant 2$ is an integer and $T \in C_p$, then the analytic functions d_T and $\mathcal{D}_T$ belong to the class $N_{p-1+\varepsilon}$ for any $\varepsilon > 0$.*

To prove this theorem we need the following

LEMMA 1.1. *If $p \geqslant 2$ is any integer, then*

$$\frac{1}{2\pi} \int\limits_{|z|=r} \|D_T(I - zT)^{-1} D_T\|_p^p \, |dz| \leqslant (1-r)^{-(p-1)} \|D_T^2\|_p^p \quad (0 < r < 1), \tag{1.3}$$

where $\|\cdot\|_p$ is the $\mathfrak{S}_p$-norm.

Proof. First suppose $p = 2^k$ ($k \geqslant 1$). It follows from the elementary properties of eigenvalues (λ_j) and singular values (s_j) of compact operators [12, Chapter II], that

$$\begin{aligned}
\|D_T(I - zT)^{-1} D_T\|_p^p &= \sum_j s_j^{p/2}(D_T(I - zT)^{-1} D_T^2 (I - \bar{z}T^*)^{-1} D_T) = \\
&= \sum_j \lambda_j^{p/2}(D_T(I - zT)^{-1} D_T^2 (I - \bar{z}T^*)^{-1} D_T) = \\
&= \sum_j \lambda_j^{p/2}(D_T^2(I - zT)^{-1} D_T^2 (I - \bar{z}T^*)^{-1}) \leqslant \\
&\leqslant \sum_j s_j^{p/2}(D_T^2(I - zT)^{-1} D_T^2 (I - \bar{z}T^*)^{-1}) \leqslant \\
&\leqslant \|(I - \bar{z}T^*)^{-1}\|^{p/2} \sum_j s_j^{p/2}(D_T^2(I - zT)^{-1} D_T^2) \leqslant \\
&\leqslant (1-r)^{-p/2} \|D_T^2(I - zT)^{-1} D_T^2\|_{p/2}^{p/2} \quad (|z| = r).
\end{aligned}$$

Now we use these relations $k - 1$ times and obtain

$$\begin{aligned}
\|D_T(I - zT)^{-1} D_T\|_p^p &\leqslant (1-r)^{-p\sum_{j=1}^{k-1} 2^{-j}} \left\|D_T^{2^{k-1}}(I - zT)^{-1} D_T^{2^{k-1}}\right\|_2^2 = \\
&= (1-r)^{-(p-2)} \|Q(I - zT)^{-1} Q\|_2^2,
\end{aligned}$$

where $Q = D_T^{p/2}$. Further, since

$$\|Q(I - zT)^{-1}Q\|_2^2 = \mathrm{Sp}(Q(I - zT)^{-1}Q^2(I - \bar{z}T^*)^{-1}Q) = \\ = \sum_{j,k \geqslant 0} \mathrm{Sp}(QT^jQ^2(T^*)^kQ)z^j\bar{z}^k,$$

we have

$$\int_{|z|=r} \|D_T(I - zT)^{-1}D_T\|_p^p \,|\mathrm{d}z| \leqslant$$

$$\leqslant (1-r)^{-(p-2)} \int_{-\pi}^{\pi} \sum_{j,k \geqslant 0} \mathrm{Sp}(QT^jQ^2(T^*)^kQ)r^{j+k}\mathrm{e}^{\mathrm{i}(j-k)\theta}\mathrm{d}\theta =$$

$$= 2\pi(1-r)^{-(p-2)} \sum_{j=0}^{\infty} \mathrm{Sp}(QT^jQ^2(T^*)^jQ)r^{2j}.$$

Recalling the definition of Q we get now

$$\mathrm{Sp}(QT^jQ^2(T^*)^jQ) = \mathrm{Sp}(Q^2T^jQ^2(T^*)^j) \leqslant \|Q^2T^j\|_2\|Q^2(T^*)^j\|_2 \leqslant \\ \leqslant \|Q^2\|_2^2 = \|D_T^p\|_2^2 = \|D_T^2\|_p^p$$

and we easily come to (1.3) in the case when $p = 2^k$ $(k \geqslant 1)$.

Let us note that the above presented proof will remain valid if we replace D_T by any normal operator Θ with spectral decomposition

$$\Theta = \sum_i \varphi_i(\cdot, e_k)e_k, \quad \{\varphi_i\} \subset \mathbb{C}, \ \sum_i |\varphi_i|^{2p} < +\infty.$$

This will be used further.

To prove (1.3) for any $p \geqslant 2$, we shall use the well known Hadamard theorem on three lines and some techniques similar to one worked out in the proof of the Riesz-Thorin theorem on interpolation of operators [1].

First note that if the values of the function $F(\varphi)$ are in $\mathfrak{S}_p$, then

$$(1.4) \qquad \sup_G \left| \int_{-\pi}^{\pi} \mathrm{Sp}\{F(\varphi)G(\varphi)\}\mathrm{d}\varphi \right| = \left\{ \int_{-\pi}^{\pi} \|F(\varphi)\|_p^p \mathrm{d}\varphi \right\}^{1/p}$$

where supremum is taken over all $\mathfrak{S}_q$-valued functions $G(\varphi)$ satisfying the condition

$$\int_{-\pi}^{\pi} \|G(\varphi)\|_q^q \mathrm{d}\varphi = 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Of course, we suppose that F and G satisfy the standard conditions in which the integrals of (1.4) exist. It is easy to check that (1.4) is an immediate consequence of the precise estimate $|\operatorname{Sp}\{FG\}| \leqslant \|F\|_p\|G\|_q$ (see [12]) and Hölder inequality.

So, the inequality (1.3) is proved for $p = 2^{k-1}, 2^k$. Now it will be interpolated for all $p \in [2^{k-1}, 2^k]$. We denote $p_0 = 2^{k-1}$, $p_1 = 2^k$, then we put, as in the proof of the Riesz-Thorin theorem,

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q(z)} = \frac{1-z}{q_0} + \frac{z}{q_1}, \quad \frac{1}{p_j} + \frac{1}{q_j} = 1 \quad (j = 0, 1)$$

and consider for any $n \geqslant 1$ the operator-function

$$\Phi_n(z) = \sum_{k=1}^{n} s_k^{\frac{p}{p(z)}}(\cdot, e_k)e_k,$$

$$G_n(\varphi, z) = \sum_{j=1}^{n} a_j^{\frac{q}{q(z)}}(\varphi)(\cdot, v_j(\varphi))u_j(\varphi), \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where $\{e_k\}$, $\{u_j(\varphi)\}$, $\{v_j(\varphi)\}$ are some weak measurable orthonormal sets of vectors, $s_k > 0$ and $a_j(\varphi)$, $[a_j(\varphi)]^{-1}$ are nonnegative bounded measurable functions. In addition, we suppose that for any fixed integer $n \geqslant 1$

$$\sum_{k=1}^{n} s_k^{2p} = 1, \quad \int_{-\pi}^{\pi} \sum_{j=1}^{n} a_j^q(\varphi)d\varphi = 1. \tag{1.5}$$

Now we introduce the entire function

$$f_r(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Sp}\{\Phi_n(z)(I - re^{i\varphi}T)^{-1}\Phi_n(z)G_n(\varphi, z)\}d\varphi =$$

$$= \frac{1}{2\pi}\sum_{k,m,j=1}^{n} s_k^{\frac{p}{p(z)}} s_m^{\frac{p}{p(z)}} \int_{-\pi}^{\pi} a_j^{\frac{q}{q(z)}}(\varphi)P_{k,m,j}(\varphi)d\varphi,$$

where $P_{k,m,j}(\varphi) = (e_k, v_j(\varphi))(u_j(\varphi), e_m)((I - re^{i\varphi}T)^{-1}e_m, e_k)$. Since (1.3) is true for $p = p_0$ and for the normal operator $\Phi_n(z)$,

$$|f_r(iy)| \leqslant (2\pi)^{\frac{1}{p_0}-1}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Phi_n(iy)(I - re^{i\varphi}T)^{-1}\Phi_n(iy)\|_{p_0}^{p_0}d\varphi\right\}^{1/p_0} \times$$

$$\times \left\{\int_{-\pi}^{\pi} \|G_n(\varphi, iy)\|_{q_0}^{q_0}\, d\varphi\right\}^{1/q_0} \leqslant$$

$$\leqslant (2\pi)^{-1/q_0}\|\Phi_n^2(\mathrm{i}y)\|_{p_0}(1-r)^{-1+1/p_0}\left\{\int_{-\pi}^{\pi}\|G_n(\varphi,\mathrm{i}y)\|_{q_0}^{q_0}\mathrm{d}\varphi\right\}^{1/q_0} =$$

$$= (2\pi)^{-\frac{1}{q_0}}\left\{\sum_{k=1}^{n}\left|s_k^{\frac{2p}{p(\mathrm{i}y)}}\right|^{p_0}\right\}^{1/p_0}(1-r)^{-\frac{1}{q_0}}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{n}\left|a_j^{\frac{q}{q(\mathrm{i}y)}}(\varphi)\right|^{q_0}\mathrm{d}\varphi\right\}^{1/q_0} =$$

$$= (2\pi)^{-\frac{1}{q_0}}\left\{\sum_{k=1}^{n}s_k^{2p}\right\}^{1/p_0}(1-r)^{-\frac{1}{q_0}}\left\{\int_{-\pi}^{\pi}\sum_{j=1}^{n}a_j^{q}(\varphi)\mathrm{d}\varphi\right\}^{1/q_0} =$$

$$= [2\pi(1-r)]^{-1/q_0}.$$

Similarly, we take $p = p_1$ and come to the inequality $|f_r(1+\mathrm{i}y)| \leqslant [2\pi(1-r)]^{-1/q_1}$. Consequently, it follows from Hadamard theorem that

$$|f_r(\theta)| \leqslant [2\pi(1-r)]^{-\frac{1-\theta}{q_0}-\frac{\theta}{q_1}} \quad (0 \leqslant \theta \leqslant 1).$$

Now let $p \in (2^{k-1}, 2^k)$ and the corresponding $\theta \in (0,1)$ is choosen by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \left(\frac{1}{p} + \frac{1}{q} = 1\right).$$

Then it is obvious, that

$$\Phi_n(\theta) = \sum_{k=1}^{n} s_k(\cdot, e_k)e_k, \quad G_n(\varphi,\theta) = \sum_{j=1}^{n} a_j(\varphi)(\cdot, v_j(\varphi))u_j(\varphi) \tag{1.6}$$

and therefore

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Sp}\left\{\Phi_n(\theta)(I - re^{\mathrm{i}\varphi}T)^{-1}\Phi_n(\theta)G_n(\varphi,\theta)\right\}\mathrm{d}\varphi\right| \leqslant [2\pi(1-r)]^{-1/q}$$

for any $n \geqslant 1$ and any functions $\Theta_n(\theta), G_n(\varphi,\theta)$ satisfying (1.5). If we use (1.4) instead of the first condition of (1.5), then it will follow from the last inequality that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|\Phi_n(\theta)(I - re^{\mathrm{i}\varphi}T)^{-1}\Phi_n(\theta)\right\|_p^p \mathrm{d}\varphi \leqslant \|\Phi_n^2(\theta)\|_p^p(1-r)^{-(p-1)}.$$

And if we take now $\Phi_n(\theta)$ to be the n-th sum of Schmidt series of the operator D_T and let $n \to \infty$, then we shall come to (1.3).

Proof of Theorem 1.1. We deduce from (1.1) by using a simple inequality for regularized determinants [10, Chapter XI, Section 22], that $\log^+|d_T(z)| \leqslant m_p\|I - W_T(z)\|_p^p$ $(|z| < 1)$. Therefore

$$\log^+|d_T(z)| \leqslant 2^p M_p\{\|I - W_T^{-1}(0)\|_p^p + \|W_T^{-1}(0)\|^p\|D_T(I - zT)^{-1}D_T\|_p^p\}.$$

Consequently, (1.3) gives

$$\int_{-\pi}^{\pi} \left[D^{-\alpha} \log\left|d_T(re^{i\theta})\right|\right]^+ d\theta \leqslant \int_{-\pi}^{\pi} D^{-\alpha} \log^+ |d_T(re^{i\theta})| d\theta =$$

$$= D^{-\alpha}\left\{\int_{-\pi}^{\pi} \log^+ |d_T(re^{i\theta})| d\theta\right\} \leqslant D^{-\alpha}\left[C_1 + C_2(1-r)^{-p+1}\right] \leqslant C < +\infty$$

for any $\alpha > p-1$. Hence $d_T \in N_\alpha$ $(\alpha > p-1)$. To prove that $\mathcal{D}_T \in N_\alpha$ for any $\alpha > p-1$, note that

$$\log^+ |\mathcal{D}_T(z)| \leqslant M_p \|I - W_T(z)W_T^*(\overline{z})\|_p^p \leqslant 2^p M_p \left\{\|I - W_T(z)\|_p^p + \right.$$
$$\left. + \|I - W_T^*(\overline{z})\|_p^p\right\} \leqslant C_1 + C_2\left\{\|D_T(I - zT)^{-1} D_T\|_p^p + \|D_T(I - \overline{z}T^*)^{-1} D_T\|_p^p\right\}.$$

Now it is enough to apply the inequality (1.3) again.

Corollary. *If $p \geqslant 3$ is an integer, $T \in C_p$ and $I - T^*T \in \mathfrak{S}_\alpha$ for any $\alpha \in [p-1, p)$, then $d_T, \mathcal{D}_T \in N_{p-1}$.*

Proof. Since Lemma 1.1 is true for every $p \geqslant 2$, which may be not an integer, we use the previous notations and obtain

$$\|D_T(I - zT)^{-1} D_T\|_p^p = \sum_j s_j^p = \sum_j s_j^{p-\alpha} s_j^\alpha \leqslant$$
$$\leqslant \|D_T(I - zT)^{-1} D_T\|^{p-\alpha} \sum_j s_j^\alpha \leqslant 2^{p-\alpha} \|D_T(I - zT)^{-1} D_T\|_\alpha^\alpha.$$

Thus

$$\frac{1}{2\pi} \int_{|z|=r} \|D_T(I - zT)^{-1} D_T\|_p^p |dz| \leqslant \frac{2^{p-\alpha}}{2\pi} \int_{|z|=r} \|D_T(I - zT)^{-1} D_T\|_\alpha^\alpha |dz| \leqslant$$
$$\leqslant 2^{p-\alpha} \|D_T^2\|_\alpha^\alpha (1-r)^{-(\alpha-1)}.$$

Now it is enough to recall the proof of Theorem 1.1 and take into account that the classes N_α are monotonly increasing [5, Chapter IX].

1.3. It is easy to come from Theorem 1.1 to the following, which will be used further.

Proposition 1.1. *Let $\{z_k\}$ be a sequence in $|z| < 1$ (here and further we accept a sequence of complex numbers enumerated in accordance with their multiplicities), such that for a given integer $p \geqslant 2$*

$$\sum_k (1 - |z_k|)^p < +\infty. \tag{1.7}$$

Then the functions d_0 and $\mathcal{D}_0$ defined by the formulas

$$(1.8)\qquad \begin{aligned} d_0(z) &= \prod_k b_{z_k}(z) \exp\left\{\sum_{k=1}^{p-1} \frac{1}{j}\,[1-b_{z_k}(z)]^j\right\},\\ \mathcal{D}_0(z) &= \prod_k B_k(z) \exp\left\{\sum_{k=1}^{p-1} \frac{1}{j}[1-B_k(z)]^j\right\},\\ b_{z_k}(z) &= \frac{z_k - z}{1-\overline{z}_k z}\cdot\frac{|z_k|}{z_k},\quad B_k(z) = b_{z_k}(z) b_{\overline{z}_k}(z) \end{aligned}$$

are analytic in $|z| < 1$ *and belong to the class* $N_{p-1+\varepsilon}$ *for every* $\varepsilon > 0$.

Proof. Consider in the Hilbert space $\mathfrak{H}$ the normal operator T with spectral decomposition

$$T = \sum_k \overline{z}_k(\cdot\,, e_k)e_k.$$

The condition (1.7) gives $T \in C_p$. For any $z \notin \sigma(T^{-1}) \cup \sigma(T^{*-1})$ the operators $W_T(z)$ and $W_T(z)W_T^*(\overline{z})$ are normal and the sequences $\{b_{z_k}(z)\}, \{B_k(z)\}$ are their eigenvalues. Thus $d_0(z) = d_T(z)$, $\mathcal{D}_0(z) = \mathcal{D}_T(z)$ and the proposition follows from Theorem 1.1.

Now we shall see how precise Theorem 1.1 is. First, it is easy to see that the discrete spectrum of the operator $T \in C_p$ satisfies the condition (1.7). Thus the functions d_T and $\mathcal{D}_T$ do not belong in general to M. M. Djrbashian class N_α with $\alpha < p-1$. To check it, note that the set of zeros of the function d_T is the discrete spectrum of the operator T^*. Indeed, it follows from (1.1) that $d_T(z) = 0$ if and only if the operator $W_T(z)$ has zero as an eigenvalue. Since $I - W_T(z) \in \mathfrak{S}_p$, it is equivalent to noninvertibility of $W_T(z)$. On the other hand, we may apply the equalities [20, 2]

$$W_T(z)W_T^*(\overline{z}^{-1}) = W_T^*(\overline{z}^{-1})W_T(z) = I; \quad z \notin \sigma(T^*),\ |z| < 1,$$

and conclude that the operator $W_T(z)$ ($|z| < 1$) is invertible if and only if $z \notin \sigma(T^*)$. Similarly, it is easy to verify that the set of zeros the function $\mathcal{D}_T$ is the sequence $\{z_k\} \cup \{\overline{z}_k\}$, where $\{z_k\}$ is the set of eigenvalues of T. Now if d_T (or $\mathcal{D}_T$) belongs to N_α for some $\alpha < p-1$, then, according to the property of zeros of functions of the class N_α [5, Chapter IX], we have

$$\sum_k (1-|z_k|)^{1+\alpha} < +\infty.$$

The last conclusion is obviously not true for arbitrary operators $T \in C_p$.

We do not know if the functions $d_T, \mathcal{D}_T$ belong to the class N_{p-1} for every operator $T \in C_p$. It is not known also if the products $d_0, \mathcal{D}_0$, constructed by any sequence which satisfy (1.7), belong to N_{p-1}. But we can state the existence of a sequence $\{z_k\}_1^\infty$, which satisfy (1.7), and for any $\varepsilon > 0$ also the condition

$$\sum_{k=1}^{\infty}(1-|z_k|)^{p-\varepsilon} = +\infty,$$

such that the corresponding products $d_0, \mathcal{D}_0 \in N_{p-1}$.

It is signifiant to note that the preciseness of Theorem 1.1 is not important for the problems considered further.

1.4. Now we shall find factorizations for the functions d_T and $\mathcal{D}_T$ and we shall pay the main attention to the function $\mathcal{D}_T$ as it shall play further an important role. Note the following simple property of this function

$$0 \leqslant \mathcal{D}_T(x) \leqslant 1; \quad -1 < x < 1, \tag{1.9}$$

which is an immediate consequence of $W_T(x)W_T^*(x) \leqslant I$.

THEOREM 1.2. *If $p \geqslant 2$ is an integer, $T \in C_p$ and $\{z_k\}$ is the discrete spectrum of this operator, then for any $\varepsilon > 0$*

$$\mathcal{D}_T(z) = \mathcal{D}_0(z)\exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}z)\mathrm{d}\psi_\varepsilon(\theta)\right\}; \quad |z| < 1. \tag{1.10}$$

Here ψ_ε is a real-valued continuous function of bounded variation in $[-\pi,\pi]$, $\mathcal{D}_0$ is defined by (1.8) and

$$S_{p-1+\varepsilon}(\zeta) = \Gamma(p+\varepsilon)\left\{\frac{2}{(1-\zeta)^{p+\varepsilon}} - 1\right\}.$$

Each factor of this representation satisfies the condition (1.9) and if it is normed $\psi_\varepsilon(-\pi) = 0$, then the factorization (1.10) is unique for any $\varepsilon > 0$.

To prove this theorem we need the following

LEMMA 1.2. *Let A and B be nonnegative contractions and moreover, let $A = I - \lambda P$, where $\lambda \in (0,1)$ and P is an one-dimensional orthogonal projector. If $I - B \in \mathfrak{S}_p$ for an integer $p \geqslant 2$, then*

$$\mathrm{Sp}\left\{\sum_{k=1}^{p-1}\frac{1}{k}\left[(I-A)^k + (I-B)^k - (I-AB)^k\right]\right\} \geqslant 0. \tag{1.11}$$

Proof. We shall first consider the case when A and B are acting in a finite-dimensional space and B is continuously invertible. Let $\{\lambda_k\}_1^n$, $\{\mu_k\}_1^n$ and $\{\nu_k\}_1^n$ be the sets of eigenvalues of the operators A, B and AB correspondingly, enumerated in decreasing order (note that the spectrum of the operator AB is positive). It is well known [11, addition], that

$$\prod_{j=1}^{m} \nu_j \leqslant \prod_{j=1}^{m} \lambda_j \mu_j \quad (1 \leqslant m < n-1), \ \prod_{j=1}^{n} \nu_j = \prod_{j=1}^{n} \lambda_j \mu_j .$$

In terms of these sequences the inequality (1.11) becomes

$$\sum_{j-1}^{n} \sum_{k=1}^{p-1} \frac{1}{k} \left\{ (1-\lambda_j)^k + (1-\mu_j)^k - (1-\nu_j)^k \right\} \geqslant 0.$$

Further, since

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{1}{k} \left\{ (1-\lambda_j)^k + (1-\mu_j)^k - (1-\nu_j)^k \right\} =$$

$$= \sum_{j=1}^{n} (\log \lambda_j + \log \mu_j - \log \nu_j) = -\log \prod_{j=1}^{n} \frac{\lambda_j \mu_j}{\nu_j} = 0,$$

the inequality we prove is equivalent to

$$\sum_{j=1}^{n} \sum_{k=p}^{\infty} \frac{1}{k} \left\{ (1-\lambda_j)^k + (1-\mu_j)^k - (1-\nu_j)^k \right\} \leqslant 0. \tag{1.12}$$

We put now

$$\Phi(t) = \int_0^t (1 - e^{-x})^{p-1} dx \quad (t \geqslant 0)$$

and note that

$$\sum_{k=p}^{\infty} \frac{(1-x)^k}{k} = \int_0^{1-x} \frac{t^{p-1}}{1-t} dt = \int_0^{-\log x} (1-e^{-y})^{p-1} dy = \Phi(-\log x).$$

So, if we denote $-\log \lambda_j = a_j$, $-\log \mu_j = b_j$, $-\log \nu_j = c_j$ $(1 \leqslant j \leqslant n)$, then the inequality (1.12) may be written as

$$\sum_{j=1}^{n} \{\Phi(a_j) + \Phi(b_j)\} \leqslant \sum_{j=1}^{n} \Phi(c_j).$$

Since $A = I - \lambda P$, where P is an one-dimensional orthogonal projector, we have $a_1 = a_2 = \ldots = a_{n-1} = 0$, $a_n = -\log(1-\lambda)$ and also

$$\sum_{j=1}^{m} b_j = -\log \left(\prod_{j=1}^{m} \mu_j \right) \leqslant -\log \left(\prod_{j=1}^{m} \nu_j \right) = \sum_{j=1}^{m} c_j \quad (1 \leqslant m \leqslant n-1),$$

$$a_n + \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} c_j .$$

Therefore

$$\sum_{j=1}^{n}\{\Phi(c_j)-\Phi(b_j)\}=\sum_{j=1}^{n}\int_{b_j}^{c_j}(1-\mathrm{e}^{-x})^{p-1}\mathrm{d}x,$$

where $c_j\geqslant b_j$ $(1\leqslant j\leqslant n)$ and $c_n\geqslant a_n$, since for each k $(1\leqslant k\leqslant n)$ we have $\nu_k\leqslant\lambda_k$ and $\nu_k\leqslant\mu_k$ [11, addition]. Let us note that at least the first $n-1$ intervals (b_j,c_j) $(1\leqslant j\leqslant \leqslant n-1)$ are disjoint. Indeed, let $\mathcal{L}$ be the linear hull of those eigenvectors of AB, which correspond to the eigenvalues $\nu_1,\nu_2,\ldots,\nu_{j-1}$, and $P=(\cdot,e)e$. Then, by help of minimaximal properties of eigenvalues [12, Chapter II], we obtain for any j $(1\leqslant j\leqslant n-1)$

$$\nu_j=\max_{x\perp\mathcal{L}}\frac{(ABx,x)}{(x,x)}\geqslant\max_{x\perp\mathcal{L},e}\frac{(Bx,Ax)}{(x,x)}=\max_{x\perp\mathcal{L},e}\frac{(Bx,x)}{(x,x)}\geqslant\mu_{j+1},$$

and so $c_j\leqslant b_{j+1}$. Further, since the intervals (b_j,c_j) $(1\leqslant j\leqslant n)$ are disjoint and the sum of their lengths is equal to a_n,

$$\sum_{j=1}^{n}\{\Phi(c_j)-\Phi(b_j)\}=\sum_{j=1}^{n}\int_{b_j}^{c_j}(1-\mathrm{e}^{-x})^{p-1}\mathrm{d}x\geqslant\int_{0}^{a_n}(1-\mathrm{e}^{-x})^{p-1}\mathrm{d}x=\Phi(a_n).$$

So, we come to the inequality (1.11) in the case when operators A and B are acting in a finite-dimensional space. It is clear that (1.11) is valid also in the case when B is not invertible.

It follows from the formula

$$\det\nolimits_p AB=(\det\nolimits_p A)(\det\nolimits_p B)\times$$
$$\times\exp\left\{-\mathrm{Sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left[(I-A)^k+(I-B)^k-(I-AB)^k\right]\right)\right\}\tag{1.13}$$

and other simple properties of regularized determinants [12, Chapter IV] that the left-hand side of (1.11) depends on operators $I-A$ and $I-B$ continuously in $\mathfrak{S}_p$ metric. Consider a monotonly increasing sequence of orthogonal projectors $\{P_n\}_1^\infty$, which strongly tend to I and $P_nA=AP_n$ $(n\geqslant 1)$. Since (1.11) is already proved for $A_n=P_nAP_n$ and $B_n=P_nBP_n$, we let $n\to\infty$ and come to (1.11) in the general case.

1.5. *Proof of Theorem 1.2.* As we have proved, the function $\mathcal{D}_T$ has zeros $\{z_k\}\cup\{\bar{z}_k\}$ and belongs to $N_{p-1+\varepsilon}$ for any $\varepsilon>0$. It follows from Proposition 1.1, that the product $\mathcal{D}_0$ constructed by the sequence $\{z_k\}$ also belongs to $N_{p-1+\varepsilon}$. Thus the function $\mathcal{D}_T(z)/\mathcal{D}_0(z)$, which does not vanish in $|z|<1$, also belongs to $N_{p-1+\varepsilon}$ and allows a representation

$$\mathcal{D}_T(z)/\mathcal{D}_0(z)=\mathrm{e}^{\mathrm{i}\gamma_\varepsilon}\exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}z)\mathrm{d}\psi_\varepsilon(\theta)\right\}\qquad(|z|<1),$$

where $\operatorname{Im}\gamma_\varepsilon = 0$ and ψ_ε is a real-valued function of bounded variation in $[-\pi,\pi]$ [5, Chapter IX]. But $\mathcal{D}_T(0), \mathcal{D}_0(0) > 0$, so $\gamma_\varepsilon = 0$ and the formula (1.10) is obtained.

We will show now that the function ψ_ε is continuous. Suppose the power series of the analytic function $\log(\mathcal{D}_T/\mathcal{D}_0)$ is

$$\log\left[\mathcal{D}_T(z)/\mathcal{D}_0(z)\right] = \sum_{k=0}^{\infty} d_k z^k \quad (|z|<1).$$

But the expansion of the kernel $S_{p-1+\varepsilon}$ gives

$$\log\left[\mathcal{D}_T(z)/\mathcal{D}_0(z)\right] = \frac{\Gamma(p+\varepsilon)}{\pi} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(p+\varepsilon+k)}{\Gamma(1+k)}\left(\int_{-\pi}^{\pi} e^{-ik\theta}\,d\psi_\varepsilon(\theta)\right) z^k.$$

Hence

$$-d_k\frac{\Gamma(1+k)}{\Gamma(p+\varepsilon+k)} = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\,d\psi_\varepsilon(\theta) \quad (k \geqslant 1).$$

Since the same equality is true for any ε_1 $(0<\varepsilon_1<\varepsilon)$, we have

$$|d_k|\frac{\Gamma(1+k)}{\Gamma(p+\varepsilon+k)} = |d_k|\frac{\Gamma(1+k)}{\Gamma(p+\varepsilon_1+k)}\frac{\Gamma(p+\varepsilon_1+k)}{\Gamma(p+\varepsilon+k)} =$$
$$= \frac{1}{\pi}\left|\int_{-\pi}^{\pi} e^{-ik\theta}\,d\psi_{\varepsilon_1}(\theta)\right| \frac{\Gamma(p+\varepsilon_1+k)}{(p+\varepsilon+k)} = o(1)$$

when $k\to\infty$, i.e. the Fourier coefficients of ψ_ε tend to zero and hence ψ_ε is continuous on $[-\pi,\pi]$ [21, Chapter III] (note that it may be said more about differential properties of ψ_ε). The uniqueness of the function ψ_ε immediately follows from the results of [5, Chapter IX]. Moreover, the following inversion formulas are true:

$$\psi_\varepsilon^{(\pm)}(\theta_2) - \psi_\varepsilon^{(\pm)}(\theta_1) = \lim_{r\to 1-0}\int_{\theta_1}^{\theta_2}\left[D^{-\alpha}\log\left|\frac{\mathcal{D}_T(re^{i\theta})}{\mathcal{D}_0(re^{i\theta})}\right|\right]^{\mp} d\theta, \tag{1.14}$$
$$\psi_\varepsilon(\theta) = \psi_\varepsilon^{(+)}(\theta) - \psi_\varepsilon^{(-)}(\theta) \quad (-\pi\leqslant\theta_1\leqslant\theta_2\leqslant\pi,\ -\pi\leqslant\theta\leqslant\pi,\ \alpha=p-1+\varepsilon).$$

Here $\psi_\varepsilon^{(\pm)}(\theta)$ are nonnegative continuous increasing functions defined as the positive and the negative variations of ψ_ε on $[-\pi,\theta]$:

$$\psi_\varepsilon^{(\pm)}(\theta) = \sup\sum_{j=0}^{n}\left[\psi_\varepsilon(a_{j+1}) - \psi_\varepsilon(a_j)\right]^{\pm},$$

where supremum is taken over all partitions $a_0 = -\pi\leqslant a_1\leqslant\ldots\leqslant a_{n-1}\leqslant a_n = \theta$ of $[-\pi,\theta]$.

Finally we shall show that each term of the factorization (1.10) satisfies the condition (1.9). It is well known [2] that to each invariant subspace $\mathfrak{H}_1$ of the operator T, on which T is invertible, corresponds a factorization of the characteristic function W_T

$$W_T(z) = W_1(z)W_2(z), \tag{1.15}$$

where W_1, W_2 are analytic contractions in $|z| < 1$. Now let z_1 be the first eigenvalue of T, $Te_1 = z_1e_1$, $\|e_1\| = 1$, and let $\mathfrak{H}_1$ be the one-dimensional invariant subspace born by e_1. We take into account (1.15) and (1.13) and obtain

$$\begin{aligned}\mathcal{D}_T(z) &= \det{}_p W_1(z)W_2(z)W_2^*(\overline{z})W_1^*(\overline{z}) = \\ &= \det{}_p W_1^*(\overline{z})W_1(z)W_2(z)W_2^*(\overline{z}) = \\ &= \det{}_p[W_1^*(\overline{z})W_1(z)]\det{}_p[W_2(z)W_2^*(\overline{z})]\times \\ &\times \exp\left\{-\mathrm{Sp}\left(\sum_{k=1}^{p-1}\frac{1}{k}\left[(I-A)^k+(I-B)^k-(I-AB)^k\right]\right)\right\},\end{aligned} \tag{1.16}$$

where $A = W_1^*(\overline{z})W_1(z)$, $B = W_2(z)W_2^*(\overline{z})$.

To calculate the elementary factor corresponding to the eigenvalue z_1, we note that

$$\det{}_p W_1^*(\overline{z})W_1(z) = \det{}_p W_1(z)W_1^*(\overline{z})$$

and use the formula for $W_1(z)W_1^*(\overline{z})$ [2]:

$$W_1(z)W_1^*(\overline{z}) = I - (1-z^2)D_TP_1(I-zT_1)^{-1}(I-\overline{z}T_1^*)^{-1}P_1D_T,$$

where P_1 is the orthogonal projection on $\mathfrak{H}_1$ and $T_1 = T\,|\,\mathfrak{H}_1$. Since for any $h \in \mathfrak{H}_1$

$$\begin{aligned}(1-|z_1|^2)h &= (I-T_1^*T_1)h = P_1(I-T^*T)P_1h = P_1D_T^2P_1h,\\ (I-zT_1)^{-1}h &= (1-zz_1)^{-1}h, \quad (I-zT_1^*)^{-1}h = (1-z\overline{z}_1)^{-1}h,\end{aligned}$$

the spectrum of the operator $W_1(z)W_1^*(\overline{z})$ coincides with the spectrum of the operator

$$\begin{aligned}&I-(1-z^2)(I-zT_1)^{-1}(I-\overline{z}T_1^*)^{-1}P_1D_T^2P_1 = \\ &= I-(1-z^2)(1-zz_1)^{-1}(1-z\overline{z}_1)^{-1}(1-|z_1|^2)P_1,\end{aligned}$$

which has only one eigenvalue $\nu_1(z)$, different from 1 and equal to

$$\begin{aligned}\nu_1(z) = 1-\frac{(1-z^2)(1-|z_1|^2)}{(1-zz_1)(1-z\overline{z}_1)} = \frac{(z-z_1)(z-\overline{z}_1)}{(1-zz_1)(1-z\overline{z}_1)} = \\ = b_{z_1}(z)b_{\overline{z}_1}(z) = B_1(z).\end{aligned}$$

It immediately follows that the elementary factor we need is

$$\det_p W_1(z)W_1^*(\bar{z}) = B_1(z)\exp\left\{\sum_{j=1}^{p-1}\frac{1}{j}[1-B_1(z)]^j\right\}.$$

Now we denote by $\Phi_k(z)$ the elementary factor corresponding to eigenvalue z_k. Then (1.16) gives

$$\mathcal{D}_T(x)/\Phi_1(x) \leqslant \det_p W_2(x)W_2^*(x) \quad (-1 < x < 1). \tag{1.17}$$

Here we used (1.11) with $A = W_1^*(x)W_1(x)$, $B = W_2(x)W_2^*(x)$. It is easy to see that in the considered case A and B satisfy the conditions of Lemma 1.2.

Since W_2 is also a characteristic function of a contraction from C_p [2], we may do the same with $\mathcal{D}_2(z) = \det_p W_2(z)W_2^*(\bar{z})$ and it will follow from (1.17), that

$$0 \leqslant \mathcal{D}_T(x)/[\Phi_1(x)\Phi_2(x)] \leqslant \det_p W_3(x)W_3^*(x) \leqslant 1 \quad (-1 < x < 1),$$

since W_3 is an analytic contraction. In such a way we take off zeros of $\mathcal{D}_T$ and come to the statement needed. Now the proof of Theorem 1.2 is complete.

The following question naturally arises: when the exponential factor in (1.10) is absent? The answer is given by

PROPOSITION 1.2. *Let T be a completely non-unitary contraction from $C_p (p \geqslant 2)$. Then its regularized determinant $\mathcal{D}_T$ is exactly the product $\mathcal{D}_0$ of* (1.10) *if and only if T is complete and normal.*

We omit the proof of this fact, since it will not be used in the future.

Note that, as it was stated in the proof of Theorem 1.2, each discrete factor of factorization (1.10) is a regularized determinant constructed by a divisor of characteristic operator-function W_T, corresponding to an eigenvalue of T. So the elementary Blaschke-type factors considered have a definite spectral interpretation and their application is reasonable in operator theory.

In [5] M. M. Djrbashian used Blaschke-type factors of different nature, which permit to obtain inversion formulas (1.14) without division by $\mathcal{D}_0$. Note that elementary factors of nearly the same type as in (1.10) have been introduced in early works of M. M. Djrbashian [3, 4] on factorization of meromorphic functions.

2. BOUNDARY BEHAVIOUR OF REGULARIZED DETERMINANTS OF CHARACTERISTIC OPERATOR-FUNCTIONS

In this section it is stated a connection between the behaviour of the function $\mathcal{D}_T$ near the boundary point $z = 1$ and spectral properties of the operator $T \in C_p$. Namely, it is proved

THEOREM 2.1. *Let $p \geqslant 2$ be an integer, let $T \in C_p$ for which 1 is not an eigenvalue and let $\mathcal{D}_T$ be the regularized determinant (1.2). Then the following conditions are equivalent:*

1) $D_T\mathfrak{H} \subset (I - T^*)\mathfrak{H}$ and $(I - T^*)^{-1}D_T \in \mathfrak{S}_{2p}$;

2) $\sup\limits_{0<r<1} (1 - r)^{-1}\|I - W_T(r)W_T^*(r)\|_p < +\infty$;

3) $\sup\limits_{0<r<1} (1 - r)^{-p}[1 - \mathcal{D}_T(r)] < +\infty$;

4) *If $\{z_k\}$ and ψ_ε are the parameters of the factorization (1.10), then*

$$(2.1)\qquad \sum_k \left(\frac{1 - |z_k|^2}{|1 - z_k|^2}\right)^p < +\infty, \quad \int_{-\pi}^{\pi} S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}r)\mathrm{d}\psi_\varepsilon(\theta) = \mathrm{O}((1 - r)^p)$$

when $r \to 1 - 0$;

5) *The following limits exist*

$$\lim_{r\to 1-0}(1 - r)^{-p}\|I - W_T(r)W_T^*(r)\|_p^p, \quad \lim_{r\to 1-0}(1 - r)^{-p}[1 - \mathcal{D}_T(r)],$$
$$\lim_{r\to 1-0}\int_{-\pi}^{\pi}(1 - r)^{-p}S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}r)\mathrm{d}\psi_\varepsilon(\theta).$$

Moreover, if any of conditions 1)–5) are satisfied, then the following equalities are true

$$(2.2)\qquad \begin{aligned} \|(I - T^*)^{-1}D_T\|_{2p}^{2p} &= 2^{-p}\lim_{r\to 1-0}(1 - r)^{-p}\|I - W_T(r)W_T^*(r)\|_p^p = \\ &= p2^{-p}\lim_{r\to 1-0}(1 - r)^{-p}[1 - \mathcal{D}_T(r)] = \\ &= \sum_k \left(\frac{1 - |z_k|^2}{|1 - z_k|^2}\right)^p + \lim_{r\to 1-0}\frac{p2^{-p-1}}{\pi}\int_{-\pi}^{\pi}(1 - r)^{-p}S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}r)\mathrm{d}\psi_\varepsilon(\theta). \end{aligned}$$

2.1. To prove Theorem 2.1 we need some simple lemmas.

LEMMA 2.1. *Let the condition 2) of Theorem 2.1 be satisfied. Then*

$$(2.3)\qquad |\log \mathcal{D}_T(r)| \leqslant M_p\|I - W_T(r)W_T^*(r)\|_p^p \quad (0 < r_0 < r < 1).$$

Proof. It follows immediately from (1.1), that

$$|\log \mathcal{D}_T(r)| = \sum_k \sum_{j=p}^{\infty} \frac{1}{j}[1-\lambda_k(r)]^j =$$
$$= \sum_k [1-\lambda_k(r)]^p \sum_{j=p}^{\infty} \frac{1}{j}[1-\lambda_k(r)]^{j-p},$$

where $\{\lambda_k(r)\}$ is the set of eigenvalues of the operator $W_T(r)W_T^*(r)$. Further, if r is sufficiently close to 1, then

$$\sum_k [1-\lambda_k(r)]^p = \|I - W_T(r)W_T^*(r)\|_p^p \leqslant M_p'(1-r)^p < \delta < 1.$$

Thus $1-\lambda_k(r) < \sqrt[p]{\delta}$ for each k, and we get

$$|\log \mathcal{D}_T(r)| \leqslant [p(1-\sqrt[p]{\delta})]^{-1} \sum_k [1-\lambda_k(r)]^p = M_p\|I - W_T(r)W_T^*(r)\|_p^p.$$

LEMMA 2.2. *If $\mathcal{D}_0$ is the product* (1.8), *then the following conditions are equivalent*

a) $\sup_{0\leqslant r<1} (1-r)^{-p}[1-\mathcal{D}_0(r)] < +\infty$;

b) $\sum_k \left(\frac{1-|z_k|^2}{|1-z_k|^2}\right)^p < +\infty.$

Proof. If a) is true and r is sufficiently close to 1, then clearly

$$\sum_k \frac{1}{p}[1-B_k(r)]^p \leqslant \left|\sum_k \sum_{j=p}^{\infty} \frac{1}{j}[1-B_k(r)]^j\right| = |\log \mathcal{D}_0(r)| \leqslant$$
$$\leqslant C[1-\mathcal{D}_0(r)] \leqslant C_1(1-r)^p.$$

Thus

$$\sum_{k=1}^{n} (1-r)^{-p}[1-B_k(r)]^p \leqslant C_2$$

for any $n \geqslant 1$. But

$$\frac{\mathrm{d}}{\mathrm{d}r}B_k(r)\bigg|_{r=1} = 2\frac{1-|z_k|^2}{|1-z_k|^2}$$

and so, we let $r \to 1-0$ and deduce

$$\sum_{k=1}^{n} \left(\frac{1-|z_k|^2}{|1-z_k|^2}\right)^p \leqslant C_3,$$

which implies b), since $n \geqslant 1$ is arbitrary. Conversely, since

$$1 - B_k(r) = \frac{(1-r^2)(1-|z_k|^2)}{|1-z_k|^2}$$

and $|1 - z_k r| \geqslant 2^{-1}|1 - z_k|$ ($|z_k| < 1$, $0 \leqslant r < 1$), we have

$$(2.4) \qquad (1-r)^{-1}[1 - B_k(r)] = (1+r)\frac{1-|z_k|^2}{|1-z_k|^2} \leqslant 8\frac{1-|z_k|^2}{|1-z_k|^2}.$$

Thus b) implies that

$$\sum_k [1 - B_k(r)]^p \leqslant C_4(1-r)^p$$

and therefore

$$|\log \mathcal{D}_0(r)| \leqslant M_p \sum_k [1 - B_k(r)]^p.$$

Finally

$$1 - \mathcal{D}_0(r) \leqslant |\log \mathcal{D}_0(r)| \leqslant C_4 M_p (1-r)^p.$$

LEMMA 2.3. *If the condition* 1) *of Theorem* 2.1 *is satisfied, then*

$$\operatorname*{s-lim}_{r\to 1-0}(I-T)(I-rT)^{-1} = I, \quad \operatorname*{s-lim}_{r\to 1-0}(I-T^*)(I-rT^*)^{-1} = I.$$

Proof. It is easy to see that, for example, the left relation is a consequence of

$$(2.5) \qquad \operatorname*{s-lim}_{r\to 1-0}(1-r)(I-rT)^{-1} = 0.$$

To prove this relation, note that for every $h \in \mathfrak{H}$

$$\left\|(I-rT)^{-1}D_T h - (I-T)^{-1}D_T h\right\| \leqslant$$
$$\leqslant \left\|(1-r)T(I-rT)^{-1}(I-T)^{-1}D_T h\right\| \leqslant \left\|(I-T)^{-1}D_T h\right\|$$

and consequently

$$\lim_{r\to 1-0}(1-r)(I-rT)^{-1}D_T h = 0.$$

Thus

$$\lim_{r\to 1-0}(1-r)(I-rT)^{-1} f = 0$$

for any vector f of the form

$$f = \sum_{k=-n}^{n} T^k D_T h_k \quad (n \geqslant 0,\ \{h_k\}_{-n}^{n} \subset \mathfrak{H}).$$

But $\|(1-r)(I-rT)^{-1}\| \leqslant 1$ $(0<r<1)$, and so (2.5) is proved for completely non-unitary operator T [20, Chapter 1]. The relation (2.5) is also true in the case when T has a unitary component U. Indeed, 1 is not an eigenvalue for U, as it is not an eigenvalue for T. Thus, it follows from the spectral decomposition of U that

$$\operatorname*{s-lim}_{r\to 1-0}(1-r)(I-rU)^{-1}=0.$$

2.2. *Proof of Theorem 2.1.* To prove the implication 1)⇒2), we shall write out the following equalities for characteristic operator-functions [20, 2]:

$$\begin{aligned} &W_T(\eta)W_T^*(\xi)=I-(1-\eta\bar{\xi})D_T(I-\eta T)^{-1}(I-\bar{\xi}T^*)^{-1}D_T,\\ (2.6)\quad &\\ &W_T^*(\xi)W_T(\eta)=I-(1-\eta\bar{\xi})W_T^{-1}(0)D_TT^*(I-\bar{\xi}T^*)^{-1}(I-\eta T)^{-1}TD_TW_T^{-1}(0).\end{aligned}$$

Hence we conclude

$$I-W_T(r)W_T^*(r)=(1-r^2)D_T(I-rT)^{-1}(I-rT^*)^{-1}D_T \quad (0\leqslant r<1)$$

and therefore

$$\frac{\|I-W_T(r)W_T^*(r)\|_p}{1-r^2}=\|D_T(I-rT)^{-1}(I-rT^*)^{-1}D_T\|_p=\|(I-rT^*)^{-1}D_T\|_{2p}^2.$$

On the other hand,

$$\begin{aligned}&\|(I-rT^*)^{-1}D_T-(I-T^*)^{-1}D_T\|_{2p}\leqslant\\ &\leqslant(1-r)\|(I-rT^*)^{-1}\|\cdot\|(I-T^*)^{-1}D_T\|_{2p}\leqslant\|(I-T^*)^{-1}D_T\|_{2p},\end{aligned}$$

so $\|(I-rT^*)^{-1}D_T\|_{2p}<C<+\infty$ $(0\leqslant r<1)$ and 2) follows.

2)⇒3). By Lemma 2.1

$$\begin{aligned}1-\mathcal{D}_T(r)\leqslant|\log\mathcal{D}_T(r)|&\leqslant M_p\|I-W_T(r)W_T^*(r)\|_p^p\leqslant\\ &\leqslant M_p'(1-r)^p \quad (r_0<r<1).\end{aligned}$$

3)⇒2). If $\{\lambda_k(r)\}$ is the sequence of eigenvalues of the nonnegative operator $W_T(r)W_T^*(r)$, then

$$\begin{aligned}\frac{1}{p}\sum_k[1-\lambda_k(r)]^p<\left|\sum_k\sum_{j=p}^{\infty}\frac{1}{j}[1-\lambda_k(r)]^j\right|=|\log\mathcal{D}_T(r)|\leqslant\\ \leqslant M[1-\mathcal{D}_T(r)]\leqslant M_1(1-r)^p \quad (0\leqslant r<1).\end{aligned}$$

Now it suffices to observe that

$$\|I - W_T(r)W_T^*(r)\|_p^p = \sum_k [1 - \lambda_k(r)]^p.$$

2)⇒1). It follows from 2), that

$$\frac{\|I - W_T(r)W_T^*(r)\|_p^p}{(1-r)^p} \leqslant K, \quad \frac{\|I - W_T^*(r)W_T(r)\|_p^p}{(1-r)^p} \leqslant K \quad (r_0 \leqslant r < 1).$$

Thus we obtain using the formulas (2.6)

$$\text{(2.7)} \qquad \|(I - rT^*)^{-1}D_T\|_{2p} \leqslant K_1, \quad \|T(I - rT)^{-1}D_T W_T^{-1}(0)\|_{2p} \leqslant K_1$$

and moreover, for any vector $h \in \mathfrak{D}_T$ we have

$$\sup_{0 \leqslant r<1} \|(I - rT^*)^{-1}D_T h\| < +\infty, \qquad \sup_{0 \leqslant r<1} \|(I - rT)^{-1}D_T h\| < +\infty.$$

Since 1 is not an eigenvalue for T, the well-known theorem of B. Sz.-Nagy and Foiaş [20, Chapter IV] gives that

$$\text{(2.8)} \qquad D_T\mathfrak{H} \subset (I - T^*)\mathfrak{H}, \quad D_T\mathfrak{H} \subset (I - T)\mathfrak{H}.$$

Moreover, by the same theorem the strong limits exist

$$\underset{r\to 1-0}{\text{s-lim}}(I - rT^*)^{-1}D_T = (I - T^*)^{-1}D_T, \quad \underset{r\to 1-0}{\text{s-lim}}(I - rT)^{-1}D_T = (I - T)^{-1}D_T$$

and we deduce using (2.7), that [12, Chapter III]

$$\text{(2.9)} \qquad (I - T^*)^{-1}D_T \in \mathfrak{S}_{2p}, \quad (I - T)^{-1}D_T \in \mathfrak{S}_{2p},$$

i.e. 1) is satisfied.

To prove the equivalence 3)⇔4), we write out (1.10) in the form

$$\text{(2.10)} \qquad \mathcal{D}_T(z) = \mathcal{D}_0(z)G(z),$$

where $\mathcal{D}_0$ and G are the Blaschke-type product and the exponential factor correspondingly. Theorem 1.2 indicates that the condition 3) is equivalent to the following pair of conditions

$$\text{(2.11)} \qquad \sup_{0 \leqslant r<1} (1-r)^{-p}[1 - \mathcal{D}_0(r)] < +\infty, \qquad \sup_{0 \leqslant r<1} (1-r)^{-p}[1 - G(r)] < +\infty.$$

But it is clear that the second condition of (2.11) is equivalent to the second condition of (2.1). Thus Lemma 2.2 completes the proof of the equivalence 3)⇔4).

1)⇒5). By Lemma 2.3

$$\underset{r\to 1-0}{\text{s-lim}}(I-T^*)(I-rT^*)^{-1}(I-T^*)^{-1}D_T=(I-T^*)^{-1}D_T,$$
$$\underset{r\to 1-0}{\text{s-lim}}(I-T)(I-rT)^{-1}(I-T)^{-1}D_T=(I-T)^{-1}D_T.$$

Consequently [12, Chapter III]

$$\lim_{r\to 1-0}\|(I-rT)^{-1}D_T-(I-T)^{-1}D_T\|_{2p}=$$
$$=\lim_{r\to 1-0}\|(I-rT^*)^{-1}D_T-(I-T^*)^{-1}D_T\|_{2p}=0,$$

which gives with the equalities (2.6)

$$\lim_{r\to 1-0}\|(I-rT^*)^{-1}D_T\|_{2p}^2=\lim_{r\to 1-0}(1-r^2)^{-1}\|I-W_T(r)W_T^*(r)\|_p= \\ =\|(I-T^*)^{-1}D_T\|_{2p}^2. \tag{2.12}$$

Further, if $\{\lambda_k(r)\}$ is the set of eigenvalues of the operator $W_T(r)W_T^*(r)$, then

$$\sum_k\sum_{j=p+1}^{\infty}\frac{1}{j}[1-\lambda_k(r)]^j\leqslant\frac{1}{p+1}\sum_k\sum_{j=p+1}^{\infty}[1-\lambda_k(r)]^j=$$
$$=\frac{1}{p+1}\sum_k[\lambda_k(r)]^{-1}[1-\lambda_k(r)]^{p+1}\leqslant C\cdot\max_k[1-\lambda_k(r)]\sum_k[1-\lambda_k(r)]^p=$$
$$=C'\|I-W_T(r)W_T^*(r)\|\cdot\|I-W_T(r)W_T^*(r)\|_p^p.$$

Here we used the equivalence 1)⇔2) which we have already proved. Thus when $r\to 1-0$ the condition 2) gives

$$(1-r)^{-p}\,|\log\mathcal{D}_T(r)|=\left|\sum_k(1-r)^{-p}\left\{\log\lambda_k(r)+\sum_{j=1}^{p-1}\frac{1}{j}[1-\lambda_k(r)]^j\right\}\right|=$$
$$=\sum_k\sum_{j=p}^{\infty}\frac{1}{j}(1-r)^{-p}[1-\lambda_k(r)]^j=\sum_k\frac{1}{p}(1-r)^{-p}[1-\lambda_k(r)]^p+$$
$$+\sum_k\sum_{j=p+1}^{\infty}\frac{1}{j}(1-r)^{-p}[1-\lambda_k(r)]^j=\frac{1}{p}(1-r)^{-p}\|I-W_T(r)W_T^*(r)\|_p^p+o(1).$$

Now (2.12) implies the existence of the limits

$$p\lim_{r\to 1-0}\frac{1-\mathcal{D}_T(r)}{(1-r)^p}=p\lim_{r\to 1-0}\frac{|\log\mathcal{D}_T(r)|}{(1-r)^p}=\lim_{r\to 1-0}\frac{\|I-W_T(r)W_T^*(r)\|_p^p}{(1-r)^p}. \tag{2.13}$$

If we take $B_k(r)$ instead of $\lambda_k(r)$ and do the same, then the existence of the following limits and their equality will appear as a strict analogue of (2.13):

$$p\lim_{r\to 1-0}(1-r)^{-p}[1-\mathcal{D}_0(r)]=\lim_{r\to 1-0}\sum_k(1-r)^{-p}[1-B_k(r)]^p.$$

Now we use the factorization (1.10) and its properties given in Theorem 1.2 and obtain

$$p\lim_{r\to 1-0}\frac{1-\mathcal{D}_T(r)}{(1-r)^p}=p\lim_{r\to 1-0}\frac{1-\mathcal{D}_0(r)}{(1-r)^p}+p\lim_{r\to 1-0}\frac{1-G(r)}{(1-r)^p}.$$

But since the inequality (2.4) is true,

$$\lim_{r\to 1-0}\sum_k (1-r)^{-p}[1-B_k(r)]^p=2^p\sum_k\left(\frac{1-|z_k|^2}{|1-z_k|^2}\right)^p.$$

Thus

$$\begin{aligned}p\lim_{r\to 1-0}(1-r)^{-p}[1-\mathcal{D}_T(r)]&=2^p\sum_k\left(\frac{1-|z_k|^2}{|1-z_k|^2}\right)^p+\\ &+\lim_{r\to 1-0}\frac{p}{2\pi}\int_{-\pi}^{\pi}(1-r)^{-p}S_{p-1+\varepsilon}(e^{-i\theta}r)\,d\psi_\varepsilon(\theta)\end{aligned}\tag{2.14}$$

and the implication 1)⇒5) is proved. The converse implication is an obvious consequence of previous results. Futher, the formula (2.2) follows from (2.12)–(2.14) and the proof of Theorem 2.1 is complete.

We close this section with two remarks concerning Theorem 2.1.

REMARK 2.1. Theorem 2.1 remains valid if we add to 1)–5), for example, the following conditions

6) $\sup_{z\in\Gamma}(1-|z|)^{-1}\|I-W_T(z)W_T^*(z)\|_p<\infty$,

7) $\sup_{z\in\Gamma}(1-|z|)^{-p}\,|1-\mathcal{D}_T(z)|<\infty$,

where Γ is an angular opening $<\pi$ in $|z|<1$, symmetric with respect to the real axis and with vertex at $z=1$. Moreover, all the limits of Theorem 2.1 exist when z tends to 1 by any non-tangential way.

The proof of such extension of Theorem 2.1 do not need any new idea and therefore we do not give it.

REMARK 2.2. We shall discuss the formula (2.2) again in Section 3. But we note now, that the formula (2.2) may be observed as a regularized trace formula for the operators of classes C_p ($p\geqslant 2$). The previous considerations may be used to obtain essentially more general relations of (2.2)-type, which are called trace formulas in the case when $p=1$ [15]. The existence of ε in (2.2) brings to some dissatisfaction. The way in which ε appears was explained in Section 1. We may get rid of it by letting $\varepsilon\to 0$. But it is unknown what properties has the limit function ψ_0 (which may be even a distribution). At last, note that Theorem 2.1 was proved earlier in [13] for the case $p=1$.

3. $\mathfrak{S}_p$-PERTURBATIONS OF SELF-ADJOINT OPERATORS

We will show in this section that the condition 1) of Theorem 2.1 is more clear for dissipative unbounded operators A whose Cayley transforms belong to C_p. It appears that the condition 1) is equivalent to representability of A as a sum of a self-adjoint operator and an operator from $\mathfrak{S}_p$.

3.1. We denote by Q_p the set of those operators A, whose Cayley transforms $T = (A - iI)(A + iI)^{-1}$ belong to C_p. It is easy to see that $A \in Q_p$ if and only if:

1) $\pm i \notin \sigma(A)$,

2) $\operatorname{Im}(Af, f) \geqslant 0$ for all $f \in D(A)$,

3) The operator $iR_{-i} - iR^*_{-i} - 2R^*_{-i}R_{-i}$ (where $R_\lambda = (A - \lambda I)^{-1}$) belongs to $\mathfrak{S}_p$.

It is clear that an operator $T \in C_p$ is the Cayley transform of an operator from Q_p if and only if 1 is not an eigenvalue for T. It is true

PROPOSITION 3.1. *Let $A \in Q_p$ be an arbitrary operator and let T be its Cayley transform. Then the operator T satisfies the condition* 1) *of Theorem* 2.1 *if and only if A is representable as*

$$A = A_R + iA_I, \tag{3.1}$$

where $A_R = A_R^*$ and A_I is nonnegative and belongs to $\mathfrak{S}_p$.

Proof. Let T satisfies the condition 1) of Theorem 2.1. Then it will follow from (2.8) that

$$(I - T^*T)\mathfrak{H} \subset (I - T^*)\mathfrak{H}, \quad (I - T^*T)\mathfrak{H} \subset (I - T)\mathfrak{H}. \tag{3.2}$$

Since $I - T^*T = 2(iR_{-i} - iR^*_{-i} - 2R^*_{-i}R_{-i})$, $D(A) = (I - T)\mathfrak{H}$ and $D(A^*) = (I - T^*)\mathfrak{H}$, the inclusions (3.2) mean that $(iR_{-i} - iR^*_{-i} - 2R^*_{-i}R_{-i})\mathfrak{H} \subset D(A)$ and $(iR_{-i} - iR^*_{-i} - 2R^*_{-i}R_{-i})\mathfrak{H} \subset D(A^*)$. It follows from the first of these relations that $R^*_{-i}Th \in D(A)$ for any $h \in \mathfrak{H}$. And since T is invertible, $D(A^*) \subset D(A)$. Similarly, the second relation gives $D(A) \subset D(A^*)$ and so $D(A) = D(A^*)$. Now we can write

$$i(A^* - iI)R_{-i} - iI - 2R_{-i} = 2^{-1}(A^* - iI)(I - T^*T) = 2^{-1}(A^* - iI)D_T^2.$$

Thus, if we take $h = (A + iI)f$ ($f \in D(A)$), then

$$2^{-1}(A^* - iI)D_T^2(A + iI)f = i(A^* - iI)f - i(A + iI)f - 2f = i(A^* - A)f$$

and so

$$A_I = \frac{A - A^*}{2i} = (I - T^*)^{-1}D_T^2(I - T)^{-1}. \tag{3.3}$$

It follows from the left relation of (2.9) that $A_I \in \mathfrak{S}_p$. In other words, the closure of A_I (which was initially defined on $D(A)$) is a nonnegative operator from $\mathfrak{S}_p$. We shall show now that the operator $A_R = (A + A^*)/2$ is self-adjoint. Since A is a closed symmetric operator, it suffices to show that its defect index is (0,0). If we suppose that its defect index is (n, m) $(n + m > 0)$, then we introduce the operator

$$\mathbb{A} = A \oplus (-A) = \mathbb{A}_R + \mathrm{i}\mathbb{A}_I, \quad \mathbb{A}_R = A_R \oplus (-A_R), \quad \mathbb{A}_I = A_I \oplus (-A_I)$$

acting in the space $\mathbb{H} = \mathfrak{H} \oplus \mathfrak{H}$ and find that the defect index of $\mathbb{A}_R$ is $(n+m, n+m)$. Thus $\mathbb{A}_R$ has an extension in $\mathbb{H}$. Further, since $\mathbb{A}_I$ is bounded, $\mathbb{A}$ also has an extension. But it is impossible, since A and hence $\mathbb{A}$ have a pair of regular points $\pm\mathrm{i}$.

Suppose now that A is representable in the form (3.1). Then by the same way we shall obtain $A_I f = (I - T^*)^{-1} D_T^2 (I - T)^{-1} f$ for any vector $f \in D(A)$ $(= D(A^*))$. Further, since A_I is bounded, we have for any $f \in D(A)$

$$\|D_T(I - T)^{-1} f\|^2 = ((I - T^*)^{-1} D_T^2 (I - T)^{-1} f, f) = (A_I f, f) \leqslant M\|f\|^2.$$

Consequently, the operator $S = D_T(I - T)^{-1}$ may be extended by taking its closure to a continuous operator in $\mathfrak{H}$. Now, since $A_I \in \mathfrak{S}_p$ and $A_I = S^* S$, it is easy to deduce that $D_T \mathfrak{H} \subset (I - T^*)\mathfrak{H}$ and $S^* = (I - T^*)^{-1} D_T \in \mathfrak{S}_{2p}$.

3.2. Here will be given a re-formulation of Theorem 2.1 for operators $A \in Q_p$. It will be needed to connect such operators with the functions

$$\mathcal{D}_A(w) = \mathcal{D}_T\left(\frac{\mathrm{i} + w}{\mathrm{i} - w}\right); \quad T = (A - \mathrm{i}I)(A + \mathrm{i}I)^{-1}, \ \operatorname{Im} w < 0 \tag{3.4}$$

analytic in the lower half-plane. The factorization (1.10) implies the representation

$$\mathcal{D}_A(w) = \mathcal{D}_0(w) g(w), \tag{3.5}$$

where the function g does not vanish in the lower half-plane and corresponds to the exponential factor of (1.10), and $\mathcal{D}_0$ is the product

$$\mathcal{D}_0(w) = \prod_k B_k(w) \exp\left\{\sum_{j=1}^{p-1} \frac{1}{j}[1 - B_k(w)]^j\right\}. \tag{3.6}$$

Here $B_k(w)$ are constructed by the discrete spectrum $\{\lambda_k\}$ of A:

$$B_k(w) = \frac{1 - \lambda_k w}{1 - \overline{\lambda}_k w} \cdot \frac{1 + \overline{\lambda}_k w}{1 + \lambda_k w}; \quad \operatorname{Im} \lambda_k > 0. \tag{3.7}$$

Note that the formal representation of the function g may be obtained from formulas

$$(3.8) \qquad g(w) = G\left(\frac{\mathrm{i}+w}{\mathrm{i}-w}\right), \quad G(z) = \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{p-1+\varepsilon}(\mathrm{e}^{-\mathrm{i}\theta}z)\mathrm{d}\psi_\varepsilon(\theta)\right\}.$$

But in contrast with the case of weak contractions, we do not observe such a representation as a natural one for problems connected with unbounded operators. Our nearest aim is to find a representation of g as an exponent of an integral with a kernel specified for the half-plane, i.e. a kernel which distinguishes the boundary point $w = \infty$. So, we shall use some methods of [8, 9] and additionally assume that the behaviour of the function $\mathcal{D}_A$ near the infinity is sufficiently regular. More exactly, we shall suppose that the operator $A \in Q_p$ is continuously invertible or, which is the same, $-1 \notin \sigma(T)$. By the way, this is the reason why in (3.4) we have used a conformal mapping which moves the point $w = \infty$ into $z = -1$.

The analogue of Theorem 1.2 for non-selfadjoint operators is

THEOREM 3.1. *If $p \geqslant 2$ is an integer and $A \in Q_p$ is an arbitrary continuously invertible operator, then for any $\varepsilon \in (0, 1/2)$*

$$(3.9) \qquad \mathcal{D}_A(w) = \mathcal{D}_0(w)\exp\left\{-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{[\mathrm{i}(w-t)]^{p+\varepsilon}}\right\}; \quad \operatorname{Im} w < 0.$$

Here $\mathcal{D}_0$ is the product (3.6)–(3.7) constructed by the set $\{\lambda_k\}$ of all non-real eigenvalues of A and μ_ε is a real-valued function of bounded variation in any finite interval from $(-\infty, +\infty)$, which satisfies the condition

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{(1+|t|)^{2\varepsilon}} < +\infty.$$

3.3. To prove Theorem 3.1 we need the following purely technical result on exchange of variable in an integral. We shall formulate it without proof.

LEMMA 3.1. *Let the function $G(z) \neq 0$ be analytic in $|z| < 1$ and also in a neighbourhood of the point $z = -1$, where it may be written as*

$$(3.10) \qquad G(z) = 1 + \sum_{k=p}^{\infty} a_k(z+1)^k \quad (p \geqslant 2, \ |z+1| < r_0 < 1).$$

Further, for any $\alpha \in (p-1, p)$ let

$$(3.11) \qquad I = \iint_{|z|<1} (1-|z|^2)^{\alpha-1}\,|\log|g(z)||\,\mathrm{d}\sigma(z) < +\infty,$$

where $d\sigma(z)$ *is the area element.*

Then the function g analytic in the lower half-plane, connected with G by the first of equalities (3.8), satisfies for every $\gamma \in (\alpha - p + 1, 1)$ the condition

$$J = \iint\limits_{\operatorname{Im} w<0} \frac{|\operatorname{Im} w|^{\alpha-1}}{1+|\operatorname{Re} w|^{\gamma}} |\log|g(w)|| \, d\sigma(w) < +\infty. \tag{3.12}$$

Proof of Theorem 3.1. Since the operator A is continuously invertible, $-1 \notin \sigma(T)$ (where $T = (A - iI)(A + iI)^{-1}$). If now we use for T the analogue of Theorem 2.1 for the point $z = -1$, we will have

$$\lim_{x \to -1+0} (1+x)^{-p}[1 - \mathcal{D}_T(x)] = b_p \quad (0 < b_p < +\infty)$$

and moreover

$$\lim_{x \to -1+0} (1+x)^{-p}[1 - G(x)] = a_p \quad (0 < a_p < +\infty),$$

where G is the exponential factor (1.10). But in our case G is analytic in a neighbourhood of the point $z = -1$. So it allows the representation (3.10). Further, for any $\alpha > p - 1$ and $r \in (0, 1)$ we have

$$\frac{1}{\Gamma(\alpha)} \int_0^r (r-t)^{\alpha-1} \left(\int_{-\pi}^{\pi} \log^+ |G(te^{i\theta})| d\theta \right) t dt \leqslant$$

$$\leqslant \int_{\pi}^{\pi} D^{-\alpha} \log^+ |G(re^{i\theta})| d\theta < M < +\infty$$

where M does not depend on r, as it was seen in the proof of Theorem 1.1. Thus

$$\sup_{0<r<1} \int_0^r \int_{-\pi}^{\pi} (r-t)^{\alpha-1} \log^+ |G(te^{i\theta})| t dt d\theta \leqslant M.$$

Further, since $G(z) \neq 0$ in $|z| < 1$, we use the equilibrium relation for Nevanlinna characteristics of G [19, Chapter VI, Section 2] and come to (3.11). Thus the function G satisfies the conditions of Lemma 3.1.

It follows from (3.10), that the inequality $\left| \log |g(w)| \right| \leqslant C |w|^{-p}$ with some constant C is true in a neighbourhood of the point $w = \infty$. Suppose now $\alpha = p - 1 + \varepsilon$ $(0 < \varepsilon < 1/2)$. Then from Lemma 1.9 of [8] it follows that for any w $(\operatorname{Im} w < 0)$ with sufficiently large modulus

$$\left| W^{-\alpha} \log |g(w)| \right| \leqslant C_1 |w|^{-(1-\varepsilon)}. \tag{3.13}$$

Here C_1 is a constant and by $W^{-\alpha}$ is denoted the operator of H. Weyl

$$W^{-\alpha}U(u+\mathrm{i}v)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha-1}U(u+\mathrm{i}t)\mathrm{d}t.$$

It is easy to deduce from (3.13) and (3.12) (with $\gamma=2\varepsilon$) that for any $v<0$

$$\int_{-\infty}^{+\infty}\left|W^{-\alpha}\log|g(u+\mathrm{i}v)|\right|\frac{\mathrm{d}u}{1+|u|^{2\varepsilon}}\leqslant$$
$$\leqslant\sup_{v<0}\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{v}(v-t)^{\alpha-1}\mathrm{d}t\int_{-\infty}^{+\infty}\left|\log|g(u+\mathrm{i}t)|\right|\frac{\mathrm{d}u}{1+|u|^{2\varepsilon}}=$$
$$=\Gamma^{-1}(\alpha)J<+\infty.$$

On the other hand, (3.12) gives

$$\lim_{R\to+\infty}\frac{1}{R}\int_{0}^{\pi}\left|W^{-\alpha}\log|g(Re^{-\mathrm{i}\theta})|\right|\sin\theta\mathrm{d}\theta=0.$$

But the function $W^{-\alpha}\log|g|$ is harmonic in the lower half-plane [8, Lemmas 1.5 and 1.8]. Therefore the results of [9] bring to the representation

$$W^{-\alpha}\log|g(w)|=\Gamma^{-1}(p+\varepsilon)\frac{v}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{(u-t)^2+v^2}\quad(w=u+\mathrm{i}v,\ v>0), \tag{3.14}$$

where the measure μ_ε has the needed properties. It is not difficult to check that [9]

$$W^{-(p-\alpha)}\frac{\partial^p}{\partial v^p}\Gamma^{-1}(p+\varepsilon)\frac{v}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{(u-t)^2+v^2}=-\mathrm{Re}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{[\mathrm{i}(w-t)]^{p+\varepsilon}}. \tag{3.15}$$

On the other hand, the properties of G imply that the analytic function $\log(g)$ may be represented in a neighbourhood of $w=\infty$ by the series

$$\log g(w)=\sum_{k=p}^{\infty}h_k(\mathrm{i}w)^{-k}. \tag{3.16}$$

But

$$W^{-(p-\alpha)}\frac{\partial^p}{\partial v^p}W^{-\alpha}(\mathrm{i}w)^{-k}=(\mathrm{i}w)^{-k}\quad(k\geqslant p)$$

and so, if we apply subsequently the operators $W^{-\alpha}$, $\partial^p/\partial v^p$ and $W^{-(p-\alpha)}$ to both left and right parts of (3.16) (the uniform convergence will not fail) and take the real parts, we shall obtain

$$W^{-(p-\alpha)}\frac{\partial^p}{\partial v^p}W^{-\alpha}\log|g(w)| = \log|g(w)|; \quad \operatorname{Im} w = v < 0.$$

Thus it follows from (3.13) that

$$g(w) = \exp\left\{-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu_\varepsilon(t)}{[i(w-t)]^{p+\varepsilon}} + iC\right\}; \quad \operatorname{Im} w < 0,$$

where C is a real number. But, since $g(w) \to 1$ as $w \to \infty$ and also

$$\lim_{v\to-\infty}\int_{-\infty}^{+\infty}\frac{d\mu_\varepsilon(t)}{(|v|-it)^{p+\varepsilon}} = 0,$$

we have $C = 0$ and the theorem follows from (3.5).

3.4. Previous results of this section infer

THEOREM 3.2. *If $p \geqslant 2$ is an integer, then the following conditions are equivalent for any operator $A \in Q_p$:*

1) *A is representable as*

$$A = A_R + iA_I,$$

where $A_R = A_R^*$, $A_I \geqslant 0$ and belongs to $\mathfrak{S}_p$;

2) $\sup\limits_{-1<v<0} |v|^{-p}[1 - \mathcal{D}_A(iv)] < +\infty$;

3) *The following limit exists*

$$\lim_{v\to-0}|v|^{-p}[1-\mathcal{D}_A(iv)].$$

If, additionally, A is continuously invertible, then the conditions 1)–3) are equivalent to

4) $\sum\limits_k (\operatorname{Im}\lambda_k)^p < +\infty$ *and the following limit exists*

$$\lim_{v\to-0}\frac{p}{4^p\pi}\int_{-\infty}^{+\infty}\frac{d\mu_\varepsilon(t)}{|v|^p(|v|-it)^{p+\varepsilon}},$$

where $\{\lambda_k\}$ and μ_ε are the parameters of the factorization (3.9).

Further, if A is invertible and any of conditions 1)–4) *is satisfied, then the following equalities are true*

$$\|A_I\|_p^p = \tfrac{p}{4^p}\lim_{v\to -0} |v|^{-p}[1-\mathcal{D}_A(\mathrm{i}v)] =$$
$$= \sum_k (\operatorname{Im}\lambda_k)^p + \lim_{v\to -0}\frac{p}{4^p\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\mu_\varepsilon(t)}{|v|^p(|v|-\mathrm{i}t)^{p+\varepsilon}}. \tag{3.17}$$

The proof follows immediatly from Theorem 2.1, Proposition 3.1 and Theorem 3.1, if we use the formulas

$$(1-x)^{-p}[1-\mathcal{D}_T(x)] = (1-v)^p(2|v|)^{-p}[1-\mathcal{D}_A(\mathrm{i}v)]; \quad v<0,$$
$$z_k = \frac{\lambda_k - \mathrm{i}}{\lambda_k+\mathrm{i}}, \quad \frac{1-|z_k|^2}{|1-z_k|^2} = \operatorname{Im}\lambda_k$$

and the equality (3.3) which implies

$$\|A_I\|_p^p = \|(I-T^*)^{-1}D_T\|_{2p}^{2p}.$$

Note that the equality (3.17) improves the well-known (at least for bounded dissipative operators) inequality

$$\sum_k (\operatorname{Im}\lambda_k)^p \leqslant \|A_I\|_p^p.$$

In the case $p=1$, $\operatorname{Sp}(A_I)$ was calculated by the parameters of the factorization of a bounded analytic function in the half-plane in [13].

4. COMPLETENESS OF SOME OPERATORS FROM C_p

In this section we give an application of previous notions in a completeness problem for non-weak contractions. We shall not try to formulate maximally general results. The aim will be simpler: to show how may be applied the properties of the regularized determinant $\mathcal{D}_T$ in the questions of completeness. Remind that an operator is called complete if the closed linear hull of its root subspaces corresponding to its eigenvalues coincides with the whole space.

4.1. The following theorem strenghthens in the case of dissipative operators a well-known result of M. V. Keldych [14, or 12, Chapter V].

THEOREM 4.1. *Let* $p\geqslant 1$ *be an integer and let* $T\in C_p$ *be an arbitrary operator for which* 1 *is not an eigenvalue. Further, let the spectrum of* T *be a sequence* $\{z_k\}$ *with the only limit point* 1 *and let*

$$\liminf_{r\to 1-0}\log\frac{\mathcal{D}_T(r)}{\mathcal{D}_0(r)} = 0 \tag{4.1}$$

for the functions $\mathcal{D}_T$ and $\mathcal{D}_0$ defined by formulas (1.2) *and* (1.8). *Then the operator T is complete.*

Before we prove this theorem, we shall show that it is really a generalization of the mentioned result of M. V. Keldych. Indeed, (4.1) is satisfied "with a considerable reserve" if

$$\sup_{0<r<1} (1-r)^{-p}[1-G(r)] < +\infty; \quad G(z) = \mathcal{D}_T(z)/\mathcal{D}_0(z).$$

And if we demand in addition

$$\sup_{0<r<1} (1-r)^{-p}[1-\mathcal{D}_0(r)] < +\infty,$$

then

$$\sup_{0<r<1} (1-r)^{-p}[1-\mathcal{D}_T(r)] < +\infty$$

and the conditions of Theorem 2.1 will be satisfied. Thus, by Theorem 3.2, the corresponding operator A (the inverse Cayley transform of T) is representable as $A = A_R + iA_I$ where $A_R = A_R^*$, $A_I \geqslant 0$ and $A_I \in \mathfrak{S}_p$. Further, since compact perturbations do not change the continuous spectrum, the continuous spectrums of the operators A and A_R are the same. Thus, by a condition of Theorem 4.1, the continuous spectrum of the operator A_R is located in ∞, i.e. A_R has only discrete spectrum. So, in this particular case Theorem 4.1 states: an $\mathfrak{S}_p$-perturbation of a self-adjoint operator with discrete spectrum is a complete operator. Namely this was stated first by M. V. Keldych [14].

If we put $p = 1$ in the formulas for $\mathcal{D}_T$ and $\mathcal{D}_0$, then $\mathcal{D}_T(z)$ will become the usual determinant of $W_T(z)W_T^*(\bar{z})$ and $\mathcal{D}_0(z)$ will become a Blaschke product. Thus the condition (4.1) is also necessary for completeness of the operator T, if it is a weak contraction. The last fact is an immediate consequence of the well-known completeness criterion for weak contractions [12, Chapter V].

4.2. *Proof of Theorem 4.1.* Suppose the statement is not true. Then it is clear that $\mathfrak{H}_1 =$Clos span$\{\mathcal{L}_k : |z_k| < 1\} \neq \mathfrak{H}$ and if we denote $\mathfrak{H}_2 = \mathfrak{H} \ominus \mathfrak{H}_1$, then it will be true the following triangulation given by the invariant subspace $\mathfrak{H}_1$:

$$T = \begin{pmatrix} T_1 & \Gamma \\ 0 & T_2 \end{pmatrix}; \quad T_1 = T\Big|\mathfrak{H}_1,\ T_2^* = T^*\Big|\mathfrak{H}_2.$$

Here T_2 is an operator whose spectrum is only the point $z = 1$, which is not an eigenvalue of T_2. Thus the characteristic function $W_T(z)$ allows the factorization [2]

$$W_T(z) = W_1(z)W_2(z) \tag{4.2}$$

the factors of which are defined by the formulas

$$W_1(z)W_1^*(0) = I - D_T P_1 (I - zT_1)^{-1} P_1 D_T,$$
$$W_2(z)W_2^*(0) = I - W_1^{-1}(0) D_T P_2 (I - zT)^{-1} P_2 D_T W_T^{*-1}(0),$$
$$W_2(0) = W_1^{-1}(0) W_T(0),$$

where $W_1(0)$ is any invertible solution of the equation [2]

$$W_1(0)W_1^*(0) = I - D_T P_1 D_T.$$

In these formulas P_1 and P_2 are the orthogonal projections on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ correspondingly. Since the operator T_1 is complete, its characteristic function W_1 may be represented by the product

$$W_1(z) = W_{z_1}(z)W_{z_2}(z)\cdots W_{z_n}(z)\cdots = \overrightarrow{\prod_k} W_{z_k}(z),$$

where $W_{z_k}(z)$ are operatorial Blaschke factors. This may be checked by standard methods introduced into operator theory by M. S. Livšic [17]. Note that this infinite product is convergent in the sence that the operator-function

$$\Lambda_n(z) = \left(\prod_{k=1}^{n} W_{z_k}(z)\right)^{-1} W_1(z)$$

uniformly converges to I on any compact from $|z| < 1$ in $\mathfrak{S}_p$-norm:

$$\|I - \Lambda_n(z)\|_p \to 0; \quad n \to \infty.$$

Thus from (4.2) it follows that when $n \to \infty$

$$W_T^{(n)}(z) \overset{\text{def}}{=} \left(\prod_{k=1}^{n} W_{z_k}(z)\right)^{-1} W_T(z) = \overrightarrow{\prod_{k \geqslant n+1}} W_{z_k}(z) W_2(z) \to W_2(z)$$

in $\mathfrak{S}_p$-norm, uniformly in $|z| < 1$. Consequently, the properties of regularized determinants [12, Chapter IV] give that uniformly in $|z| < 1$

$$\text{(4.3)} \qquad \det_p W_T^{(n)}(z) W_T^{(n)*}(z) \to \det_p W_2(z) W_2^*(z).$$

We shall show now that

$$\text{(4.4)} \qquad \mathcal{D}_T(x)/\mathcal{D}_0(x) \leqslant \det_p W_2(x) W_2^*(x); \quad -1 < x < 1.$$

Indeed, Lemma 1.2 and the relations from the end of the proof of Theorem 1.2 give that

$$\mathcal{D}_T(x) = \det_p W_T(x)W_T^*(x) = \det_p W_{z_1}(x)W_T^{(1)}(x)W_T^{(1)*}(x)W_{z_1}^*(x) =$$
$$= \det_p W_{z_1}^*(x)W_{z_1}(x)W_T^{(1)}(x)W_T^{(1)*}(x) \leqslant \Phi_1 \det_p W_T^{(1)}(x)W_T^{(1)*}(x),$$

where, as before, Φ_1 is the elementary factor of the product $\mathcal{D}_0$, corresponding to the eigenvalue z_1. It is clear that we can do the same also with the term $\det_p W_T^{(1)}(x) \cdot W_T^{(1)*}(x)$ and so on. Thus for each $n \geqslant 1$

$$\mathcal{D}_T(x) \leqslant \prod_{k=1}^{n} \Phi_k(x) \det_p W_T^{(n)}(x)W_T^{(n)*}(x).$$

Letting here $n \to \infty$ and taking into account (4.3), we come to (4.4). Now we introduce the notation

$$\mathcal{D}_2(z) = \det_p W_2(z)W_2^*(\bar{z}).$$

If $\{\lambda_k(x)\}$ is the sequence of eigenvalues of the operator $W_2(x)W_2^*(x)$, then it is easy to see that

(4.5)
$$|\log \mathcal{D}_2(x)| = \sum_{k=1}^{\infty}\sum_{j=p}^{\infty} \frac{1}{j}[1 - \lambda_k(x)]^j =$$
$$= \sum_{j=p}^{\infty} \frac{1}{j} \|(I - W_2(x)W_2^*(x))^j\|_1 \geqslant \sum_{j=p}^{\infty} \frac{1}{j} \|I - W_2(x)W_2^*(x)\|^j.$$

Further, it follows from inequality (4.4) and the second condition of our theorem that

$$\lim_{r \to 1-0} |\log \mathcal{D}_2(r)| = \lim_{r \to 1-0} \sum_{j=p}^{\infty} \frac{1}{j} \|I - W_2(r)W_2^*(r)\|^j = 0$$

and consequently

(4.6)
$$\lim_{r \to 1-0} \|I - W_2(r)W_2^*(r)\| = 0.$$

We shall denote by Γ the angle with vertex $z = 1$, defined by the inequality

$$\frac{|1 - z|}{1 - |z|} < \alpha \quad (|z| < 1,\ \alpha > 1).$$

Suppose $z \in \Gamma$ and observe that by formulas (2.6)

$$((I - W_2(x)W_2^*(x))h, h) = (1 - |z|^2)\|(I - \bar{z}T_2^*)^{-1}Q_2h\|^2; \quad h \in \mathfrak{D}_T$$

where $Q_2 = P_2 D_T W_1^{*-1}(0)$. Further, since

$$(I - \bar{z}T_2^*)^{-1}Q_2h - (I - rT^*)^{-1}Q_2h = (\bar{z} - r)T_2^*(I - \bar{z}T_2^*)^{-1}(I - rT_2^*)^{-1}Q_2h \quad (r = |z|)$$

and

$$\frac{|z-r|}{1-r} \leqslant \frac{1-r}{1-r} + \frac{|1-z|}{1-r} < 1+\alpha = K \quad (z \in \Gamma),$$

it will follow that

$$\|(I-\bar{z}T_2^*)^{-1}Q_2h\| \leqslant \|(I-rT_2^*)^{-1}Q_2h\| +$$
$$+(1-r)^{-1}|z-r| \cdot \|(I-rT_2^*)^{-1}Q_2h\| \leqslant K\|(I-rT_2^*)^{-1}Q_2h.$$

Thus for any $h \in \mathfrak{D}_T$

$$((I-W_2(z)W_2^*(z))h,h) \leqslant K((I-W_2(r)W_2^*(r))h,h)$$

and so, by (4.6)

$$\lim_{z\to 1} \|I-W_2(z)W_2^*(z)\| = 0; \quad z \in \Gamma.$$

Therefore we have in any angle Γ the inequality

$$\|(W_2(z)W_2^*(z))^{-1}\| \leqslant \sum_{k=0}^{\infty} \|I-W_2(z)W_2^*(z)\|^k < C_\alpha < +\infty,$$

where the constant C_α depends only on α. We shall suppose further that the opening of Γ is $> \pi(1-2/p)$. Now we may state that for any $h_1, h_2 \in \mathfrak{D}_T$ the function $f(z) = (W_2^{-1}(z)h_1, h_2)$ is analytic in the whole closed complex plane, except of the point $z = 1$, and bounded in Γ. Further, the entire function

$$F(w) = f(z); \quad z = (w-\mathrm{i})/(w+\mathrm{i})$$

is bounded in upper half-plane, except of two angles $< \pi/p$ adjoining the real axis (because $W_2(z)$ is an unitary operator when $|z| = 1$ [2]). As the aim is to apply Phragmén-Lindelöf principle in mentioned angles, we shall show that F is of order p. First we note that for any z ($|z| < 1$)

$$\|W_2^{-1}(z)\| = \|W_2^*(\bar{z}^{-1})\| \leqslant C_1 + C_2\|(I-\bar{z}^{-1}T_2^*)^{-1}\| =$$
$$= C_1 + C_2|z| \cdot \|(T_2 - zI)^{-1}\|.$$

Now we introduce the operator $A_2 \in Q_p$ which is connected with T_2 by the equation $T_2 = (A_2 - \mathrm{i}I)(A_2 + \mathrm{i}I)^{-1}$. If we put $z = (w-\mathrm{i})/(w+\mathrm{i})$, then an elementary calculation will bring us to the equation

$$(T_2 - zI)^{-1} = (2\mathrm{i})^{-1}(w+\mathrm{i})I + (2\mathrm{i})^{-1}(w+\mathrm{i})(\mathrm{i}-w)(A_2+wI)^{-1}.$$

On the other hand, if $B_2 = A_2^{-1}$, then $B_2 = -\mathrm{i}(I-T_2)(I+T_2)^{-1}$ and $B_2 - B_2^* \in \mathfrak{S}_p$, since $B_2 - B_2^* = -2\mathrm{i}(I+T_2)^{-1}(I-T_2T_2^*)(I+T_2^*)^{-1}$. Since the operator A_2 has an empty spectrum, the operator B_2 is quasinilpotent and thus it is compact [12, Chapter I]. But then, according to a theorem of V. I. Macaev, $B_2 \in \mathfrak{S}_p$ and, as it is well known, the order of the resolvent $(A_2 - wI)^{-1}$ is equal to p [12, Chapter IV]. Hence the function F is of order p. Now we apply Phragmén-Lindelöf principle and conclude that the function F is bounded in the upper half-plane. It follows from elementary properties of characteristic operator-functions [2], that F is bounded also in the lower half-plane. Consequently the operator-function W_2 is a constant in the unit disk, i.e. T_2 is a unitary operator. Since $\sigma(T_2) = 1$, we come to the equality $T_2 = I$ which contradicts the conditions of theorem.

4.3. If we carefully examine the proof of Theorem 4.1, we shall see that a more general result is true. Namely, instead of the condition (4.1) we may ask the function $|\log(\mathcal{D}_T(x)/\mathcal{D}_0(x))|$ to be bounded by a constant depending on p (which may be calculated). Of course, stronger criterions of completeness formulated in terms of the regularized determinant $\mathcal{D}_T$ are possible. They may be obtained using more delicate Phragmén-Lindelöf type theorems. For example, the following result is true for the case $p = 2$ presenting a considerable interest.

THEOREM 4.2. *Let T be a contraction from C_2 for which 1 is not an eigenvalue. Further, let the spectrum of T be a sequence $\{z_k\}$ with the only limit point 1 and let*

$$\liminf_{r\to 1-0}(1-r)\log\frac{\mathcal{D}_T(r)}{\mathcal{D}_0(r)} = 0. \tag{4.7}$$

Then T is complete. Moreover, there exist non-complete operators from C_2, for which the lower limit (4.7) is equal to any given number $a < 0$.

Proof. It follows from (4.4) that

$$\liminf_{r\to 1-0}(1-r)\log\mathcal{D}_2(r) = 0$$

where $\mathcal{D}_2$ is the function introduced in the proof of Theorem 4.1. We suppose $\mathcal{D}_2(r) = \exp\{-\varphi(r)\}$ ($\varphi(r) > 0$). Then, according to Theorem 1.2, $(1-r)\varphi(r) \to 0$ as $r \to 1-0$. Now note that (4.5) may be written in the form

$$\varphi(r) > -\log(1 - \|I - W_2(r)W_2^*(r)\|) - \sum_{j=1}^{p-1}\frac{1}{j}\|I - W_2(r)W_2^*(r)\|^j.$$

We use it and come to the inequality $(1 - \|I - W_2(r)W_2^*(r)\|)^{-1} \leqslant C\exp\{\varphi(r)\}$ which implies

$$\|(W_2(r)W_2^*(r))^{-1}\| \leqslant \sum_{j=0}^{\infty}\|I - W_2(r)W_2^*(r)\|^j \leqslant C\exp\{\varphi(r)\}$$

and if we turn as earlier to the entire function F, then

$$\limsup_{v \to +\infty} v^{-1} \log |F(iv)| \leqslant 0 \tag{4.8}$$

and F is of second order and minimal type [12, Chapter V]. Now if Phragmén-Lindelöf principle will be applied to the function $F_1(w) = F(w)\exp\{iw\}$ in both quadrants of the upper half-plane, then the result will be that F is of first order and, consequently, it belongs to M. Kartwright class [16, Chapter V]. Thus (4.8) gives that the indicator of F is non-positive and so $F(w) \equiv$ const. The end of the proof of completeness is quite the same as that of Theorem 4.1 and we omit it. Now we introduce an obviously non-complete operator $T \in C_2$ as an orthogonal sum $T = T_1 \oplus T_2$, where T_1 is a complete operator from C_2, for which (4.7) is true, and T_2 is a contraction with one-dimensional defect, the characteristic function of which is of form

$$W_2(z) = \exp\left\{\frac{a}{2}\frac{1+z}{1-z}\right\}; \quad a < 0.$$

Then $\mathcal{D}_T(z) = \mathcal{D}_{T_1}(z)W_2(z)\exp\{1 - W_2(z)\}$ and consequently

$$\limsup_{r \to 1-0} \log \frac{\mathcal{D}_T(r)}{\mathcal{D}_0(r)} = \lim_{r \to 1-0}(1-r)\log W_2(r) = a < 0.$$

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