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DISCRETIZED CCR ALGEBRAS

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To the memory of John Bunce

1. INTRODUCTION

We consider the problem of discretizing the Hamiltonian of a one-dimensional quantum system in a form that is appropriate for carrying out numerical studies. Specifically, we start with a formal Schrödinger operator

$$H = \frac{1}{2}P^2 + v(Q)$$

acting on the Hilbert space $L^2(\mathbf{R})$, where P and Q are the canonical operators

$$P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

$$Q = \text{multiplication by } x,$$

and v is a real-valued continuous function of a real variable. The problem of discretizing H is that of finding an approximation to H which satisfies two requirements: (a) the basic principles of numerical analysis are satisfied, and (b) the uncertainty principle is preserved.

In [1, Sections 1–2], we argued that in order to satisfy these two conditions one must first replace P, Q with the pair

$$P_\tau = \frac{1}{\tau}\sin(\tau P)$$

$$Q_\tau = \frac{1}{\tau}\sin(\tau P).$$

Here, τ is a fixed positive real number, the numerical step size. The discretized Hamiltonian is then defined as the following bounded self-adjoint operator on $L^2(\mathbf{R})$:

$$H_\tau = \frac{1}{2}P_\tau^2 + v(Q_\tau).$$

Obviously, H_τ belongs to the unital C^*-algebra $C^*(P_\tau, Q_\tau)$ generated by P_τ and Q_τ. We show that when τ^2/π is irrational (e.g., when τ is a rational number), $C^*(P_\tau, Q_\tau)$ is isomorphic to the non-commutative sphere $\mathcal{B}_{\tau^2}$ of Bratteli, Evans, Elliott and Kishimoto [5], [6]; hence it is a simple C^*-algebra with a unique trace. We also describe the way in which the canonical commutation relations muts be "discretized" in order to accomodate pairs of operators (P_τ, Q_τ) of this type. Together, these observations serve to make a more philosophical point, namely *non-commutative spheres will arise in any serious attempt to model quantum systems on a computer.*

In the "linear" case where v has the form $v(x) = cx^2/2$, c being a positive constant, the operator H_τ turns out to be unitarily equivalent to an operator of the form $\lambda M + \mu I$, where λ and μ are real constants and M is the almost Mathieu Hamiltonian

$$M = U + U^* + c(V + V^*),$$

associated with a pair of unitary operators U, V satisfying

$$VU = e^{i4\tau^2}UV.$$

An extensive amount of work has been done to compute the spectra of such operators. Here, we mention only [2], [3], [4], [9], [16] and refer the reader to the monograph [8] for further references.

Finally, I would like to thank Larry Schweitzer for pointing out the references [11] and [13] (as well as the relevances of his own work [17]) in connection with the spectral invariance property of the Banach $*$-algebra $\ell^1(\mathbf{Z} \oplus \mathbf{Z}, w)$.

2. DISCRETIZED CCR ALGEBRAS

Let θ be a real number such that θ/π is irrational, and let w be the bicharacter of the discrete abelian group $G = \mathbf{Z} \oplus \mathbf{Z}$ defined by

$$w((m,n),(p,q)) = e^{i(np-mq)\theta/2}. \tag{2.1}$$

A uniformly bounded family $\{D_x : x \in G\}$ of self-adjoint operators on a Hilbert space H is said to satisfy the *discretized canonical commutation relations* if

$$D_x D_y = w(x,y)D_{x+y} + w(y,x)D_{x-y}, \quad x, y \in G. \tag{2.2}$$

REMARKS. Notice that the formula (2.2) is a generalization of the elementary trigonometric identity

$$2\cos A \cos B = \cos(A+B) + \cos(A-B),$$

in which phase shifts have been added by way of the cocycle w. Indeed, for any pair of real numbers α, β, the function $D: G \longrightarrow \mathbf{R}$ defined by

$$D(m,n) = 2\cos(\alpha m + \beta n)$$

satisfies (2.2) for the trivial cocycle $w = 1$. It is related to formula (2.2) of [5], except that our operators are self-adjoint and the phase factor is associated with a *nondegenerate* bicharacter w.

The purpose of this section is to associate a C^*-algebra with the relations (2.2), and to point out some of its basic properties. Let $\{D_x : x \in G\}$ satisfy (2.2). It is clear that the norm closed linear span

$$\mathcal{D} = \overline{\operatorname{span}}\{D_x : x \in G\}$$

is a separable C^*-algebra. Thus by passing form H to the subspace $[\mathcal{D}H]$ if necessary, we can assume that $\mathcal{D}$ is nondegenerate.

PROPOSITION 2.3.
(i) $D_0 = 2I$.
(ii) $D_{-x} = D_x$.
(iii) $\|D_x\| \leqslant 2$, for every $x \in G$.

Proof. Setting $y = 0$ in (2.2) we obtain $D_x D_0 = 2D_x$ for all $x \in G$, from which (i) is evident. Setting $x = 0$ in (2.2) now leads to $2D_y = D_0 D_y = D_y + D_{-y}$, hence (ii). For (iii), let

$$M = \sup_{x \in G} \|D_x\|.$$

By hypothesis, $M < \infty$. Moreover, setting $y = x$ in (2.2) gives

$$D_x^2 = D_{2x} + D_0 = D_{2x} + 2I$$

and thus $M^2 \leqslant M + 2$. This inequality implies that $-1 \leqslant M \leqslant 2$, hence (iii). ∎

We now construct a Banach $*$-algebra whose representations are associated with operator realizations of (2.2). Let $\ell^1(G, w)$ denote the Banach space of all absolutely summable complex functions on G, endowed with the multiplication and involution

$$f * g(x) = \sum_y w(y,x) f(y) g(x-y)$$

$$f^*(x) = \overline{f(-x)}.$$

It is easily checked that the linear subspace

$$D_\theta = \{f \in \ell^1(G, w) : f(-x) = f(x),\ x \in G\}$$

is in fact a $*$-subalgebra of $\ell^1(G, w)$. Of course, the adjoint operation in D_θ simplifies to $f^*(x) = \overline{f(x)}$. Moreover, D_θ is linearly spanned by the elements

$$d_x = \delta_x + \delta_{-x},$$

δ_x denoting the unit function supported at x, and one has

$$d_x d_y = w(x, y) d_{x+y} + w(y, x) d_{x-y}$$

$$\|d_x\| = 2$$

$$d_x = d_{-x} = d_x^*.$$

PROPOSITION 2.4. *Let $\{D_x : x \in G\}$ be a uniformly bounded family of self-adjoint operators on a Hilbert H satisfying (2.2). Then there is a unique representation $\pi : D_\theta \longrightarrow \mathcal{B}(H)$ such that*

$$\pi(d_x) = D_x,\ x \in G.$$

Proof. By Proposition 2.3, we know that $\|D_x\| \leqslant 2$; hence

$$\pi(f) = \frac{1}{2} \sum_{x \in G} f(x) D_x$$

defines a contractive self-adjoint linear mapping of D_θ into $\mathcal{B}(H)$. Moreover, using (2.2) we have

$$\pi(f)\pi(g) = \frac{1}{4} \sum_{x,y} f(x) g(y) (w(x, y) D_{x+y} + w(y, x) D_{x-y}) =$$

$$= \frac{1}{4} \sum_z \left(\sum_x f(x) g(z - x) w(x, z) \right) D_z + \frac{1}{4} \sum_z \left(\sum_x f(x) g(x - z) w(-z, x) \right) D_z.$$

Using the fact that $g(x - z) = g(z - x)$ and $w(-z, x) = w(x, z)$, the right side becomes

$$\frac{1}{2} \sum_x \left(\sum_x f(x) g(z - x) w(x, z) \right) D_z = \pi(f * g),$$

as required.

Finally, taking $f = \delta_x + \delta_{-x} = d_x$ and using (ii) of (2.3), we find that

$$\pi(d_x) = \frac{1}{2}\left(f(x)D_x + f(-x)D_x\right) = D_x,$$

as required. ■

REMARKS. It follows that the enveloping C^*-algebra $C^*(D_\theta)$ is the universal C^*--algebra generated by the commutation relations (2.2).

Let α be an automorphism of the discrete abelian group $\mathbf{Z} \oplus \mathbf{Z}$. Then α is given by a 2×2 integer matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by way of $\alpha(m, n) = (am + bn, cm + dn)$, where $a, b, c, d \in \mathbf{Z}$ satisfy the condition

$$\det \alpha = ad - bc = \pm 1.$$

It folows that

$$w(\alpha x, \alpha y) = \det \alpha \cdot w(x, y).$$

Hence the group $\mathrm{SL}(2, \mathbf{Z})$ of determinant 1 automorphisms acts naturally on D_θ (resp. $C^*(D_\theta)$) as a group of $*$-automorphisms. Any $\alpha \in \mathrm{Aut}\,(\mathbf{Z} \oplus \mathbf{Z})$ satisfying $\det \alpha = -1$ gives rise to a $*$-anti-automorphism of D_θ (resp. $C^*(D_\theta)$).

Finally, notice that there is a natural $*$-homomorphism which carries D_θ into the irrational rotation C^*-algebra $\mathcal{A}_\theta$. Indeed, D_θ is obviously contained in the larger Banach $*$-algebra $\ell^1(\mathbf{Z} \oplus \mathbf{Z}, w)$ obtained by simply dropping the requirement that $f(-x) = f(x)$. It is clear that $\ell^1(\mathbf{Z} \oplus \mathbf{Z}, w)$ is the universal Banach $*$-algebra generated by unitary operators $\{W_x : x \in \mathbf{Z} \oplus \mathbf{Z}\}$ satisfying

$$W_x W_y = w(x, y) W_{x+y}, \qquad x, y \in \mathbf{Z} \oplus \mathbf{Z}.$$

Because of the formula (2.1) giving w in terms of θ, the unitary elements U, V defined by $U = W_{(1,0)}, V = W_{(0,1)}$ satisfy $VU = e^{i\theta} UV$, and of course they generate $\ell^1(\mathbf{Z} \oplus \oplus \mathbf{Z}, w)$ as a Banach $*$-algebra. It follows that the enveloping C^*-algebra of $\ell^1(\mathbf{Z} \oplus \oplus \mathbf{Z}, w)$ is $\mathcal{A}_\theta$. Thus we obtain a morphism of D_θ into $\mathcal{A}_\theta$ by simply restricting the completion map

$$\gamma : \ell^1(\mathbf{Z} \oplus \mathbf{Z}, w) \longrightarrow \mathcal{A}_\theta$$

to D_θ. By the universal property of enveloping C^*-algebras there is correspondingly a unique morphism of C^*-algebras

$$\gamma_B : C^*(D_\theta) \longrightarrow \mathcal{A}_\theta.$$

In the next section it will be shown that γ_B is injective and we will identify its range.

3. SPECTRAL INVARIANCE AND EXTENSIONS OF STATES

Let A be a Banach $*$-algebra with unit, and let

$$A^+ = \overline{\{a_1^*a_1 + a_2^*a_2 + \ldots + a_n^*a_n \,:\, a_k \in A,\ n \geqslant 1\}}$$

denote the closed positive cone in A. For simplicity, we assume throughout this section that the completion map γ of A into its enveloping C^*-algebra is *injective*.

Let B be a unital self-adjoint Banach subalgebra of A. We are interested in determining whether or not the C^*-algebra obtained by closing $\gamma(B)$ in the norm of $C^*(A)$ is the enveloping C^*-algebra of B. More precisely, we seek conditions under which the $*$-homomorphism $\gamma_B : C^*(B) \longrightarrow C^*(A)$ defined by the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\text{incl}} & A \\ \downarrow & & \downarrow \\ C^*(B) & \xrightarrow{\gamma_B} & C^*(A) \end{array} \tag{3.1}$$

should be *injective*. Elementary considerations show that the following three conditions are equivalent:

(1) γ_B is injective.

(2) Every positive linear functional on B can be extended to a positive linear functional on A.

(3) $A^+ \cap B \subseteq B^+$.

Note, for example, that the implication (3)⇒(2) is the extension theorem of M. G. Krein [15, p. 227], whereas (2)⇒(3) follows from a standard separation theorem. It is not hard to find examples showing that these conditions are not always satisfied (see Appendix).

A is said to have the *spectral invariance property* if for every element $a \in A$ which is invertible in $C^*(A)$, we have $a^{-1} \in A$. This is equivalent to the assertion that the spectrum of any element of A is the same whether it is computed in A or in $C^*(A)$, or that A is closed under the holomorphic functional calculus of $C^*(A)$ (see [10, p. 52] for further significant consequences of spectral invariance in more general Frèchet algebras).

A familiar Tauberian theorem of Wiener asserts that if a continuous function on the unit circle never vanishes and has an absolutely convergent Fourier series

$$f\left(e^{i\theta}\right) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta},$$

$\sum |a_n| < \infty$, then $1/f$ has an absolutely convergent Fourier series. Of course, this is precisely the assertion that the group algebra $\ell^1(\mathbf{Z})$ has the spectral invariance property. While this theorem has a simple proof using the Gelfand theory, it is certainly not a triviality.

The significance of spectral invariance for our purposes derives from the following.

PROPOSITION 3.2. *Let A be a unital Banach $*$-algebra which admits spectral invariance. Then for every self-adjoint unital Banach subalgebra B of A, the natural $*$--homomorphism*

$$\theta_B : C^*(B) \longrightarrow C^*(A)$$

is injective.

Proof. We will verify property (3) above by showing that $A^+ \cap B \subseteq B^+$. We may clearly assume that $A \subseteq C^*(A)$, as a self-adjoint subalgebra which is a Banach algebra relative to a larger norm than that of $C^*(A)$.

Choose $x \in A^+ \cap B$; without loss of generality we may assume that the B-norm of x is less than 1. Since x belongs to the positive cone of $C^*(A)$ its spectrum in $C^*(A)$ is nonnegative. By spectral invariance we have $\sigma_A(x) \subseteq [0,1)$. Moreover, since $\sigma_A(x)$ cannot separate the complex plane, we see from the spectral permanence theorem that $\sigma_B(x) = \sigma_A(x) \subseteq [0,1)$. Hence for sufficiently small ε we have $\sigma_B(x+ +\varepsilon 1) \subseteq (\varepsilon, 1)$. Thus we may apply the power series

$$\sqrt{t} = \sum_{n=0}^{\infty} a_n (1-t)^n, \quad |1-t| < 1$$

to the element $x + \varepsilon 1$ to obtain a square root in B, i.e., a self-adjoint element $h \in B$ satisfying $x + \varepsilon 1 = h^2$. This shows that $x + \varepsilon 1 \in B^+$, and we obtain the desired conclusion by allowing ε to tend to zero. ■

We now apply this to show that the enveloping C^*-algebra $C^*(D_\theta)$ is isomorphic to the non-commutative sphere $\mathcal{B}_\theta$ of [6]. If we realize the irrational rotation C^*-algebra $\mathcal{A}_\theta$ as the C^*-algebra generated by a pair of unitary operators U, V satisfying $VU = e^{i\theta} UV$, then there is a unique automorphism σ of $\mathcal{A}_\theta$ satisfying $\sigma(U) = U^{-1}$, $\sigma(V) = V^{-1}$. In case θ/π is irrational, $\mathcal{B}_\theta$ is defined to be the fixed subalgebra

$$\mathcal{B}_\theta = \{a \in \mathcal{A}_\theta : \sigma(a) = a\}.$$

Let $\{W_x : x \in \mathbf{Z} \oplus \mathbf{Z}\}$ be the family of unitary operators in $\mathcal{A}_\theta$ defined by

$$W_{(m,n)} = e^{imn\theta/2} U^m V^n, \quad m, n \in \mathbf{Z}.$$

One verifies that

$$W_x W_y = w(x,y) W_{x+y},$$

where w is the bicharacter on $\mathbf{Z} \oplus \mathbf{Z}$ defined in (2.1), and moreover the action of σ is given by

$$\sigma(W_x) = W_{-x}, \quad x \in \mathbf{Z} \oplus \mathbf{Z}.$$

Since $\mathcal{A}_\theta$ is spanned by $\{W_x : x \in \mathbf{Z} \oplus \mathbf{Z}\}$, we conclude that $\mathcal{B}_\theta$ is spanned by $\{W_x + +W_{-x} : x \in \mathbf{Z} \oplus \mathbf{Z}\}$.

COROLLARY. *Supppose θ is not a rational multiple of π, and let $\alpha : D_\theta \longrightarrow \mathcal{A}_\theta$ be the morphism defined by*

$$\alpha(d_x) = W_x + W_{-x}, \quad x \in \mathbf{Z} \times \mathbf{Z}.$$

Then the natural extension $\tilde{\alpha} : C^*(D_\theta) \longrightarrow \mathcal{A}_\theta$ gives an isomorphism of C^*-algebras

$$C^*(D_\theta) \cong \mathcal{B}_\theta.$$

Proof. Let $G = \mathbf{Z} \oplus \mathbf{Z}$ and let $w : G \times G \longrightarrow \mathbf{T}$ be the bicharacter of (2.1). Consider the Banach $*$-algebra $\ell^1(G, w)$, where multiplication and involution are defined by

$$f * g(x) = \sum_y w(y,x) f(y) g(y-x)$$

$$f^*(x) = \overline{f(-x)}.$$

Notice first that $C^*(G, w)$ is naturally identified with the irrational rotation C^*--algebra $A_\theta = C^*(U, V)$, where U and V are unitary operators satisfying the above relation $VU = e^{i\theta} UV$. Indeed, letting $\{W_x : x \in G\}$ be the operators of $\mathcal{A}_\theta$ defined in the preceding remarks, it is clear that we can define a morphism of $\ell^1(G, w)$ into $\mathcal{A}_\theta$ by

$$\gamma(\delta_x) = W_x, \quad x \in G,$$

δ_x denoting the unit function at x. The range of γ is dense in $\mathcal{A}_\theta$, and the natural extension of γ to $C^*(\ell^1(G, w))$ is injective because of the familiar universal property of such pairs U, V.

D_θ is identified (via an isometric isomorphism of Banach $*$-algebras) with a subalgebra of $\ell^1(G, w)$,

$$D_\theta = \{f \in \ell^1(G, w) : \sigma_0(f) = f\},$$

where σ_0 is the $*$-automorphism of $\ell^1(G, \omega)$ given by

$$\sigma_0(f)(x) = f(-x), \quad x \in G.$$

It is clear that the restriction of γ to D_θ carries $d_x = \delta_x + \delta_{-x}$ to $W_x + W_{-x}$; hence by the preceding remarks $\gamma(D_\theta)$ is a dense $*$-subalgebra of $\mathcal{B}_\theta$. Thus γ extends naturally to a surjective $*$-homomorphism of $C^*(\ell^1(G,w))$ onto $\mathcal{B}_\theta$, and it remains only to show that the latter morphism is injective.

Now it is known that $\ell^1(G,w)$ admits spectral invariance (see [13, Satz 5] for example, or apply Theorem 1.1.3 of [17] together with the results of [11] on the symmetry of the group algebra of the rank 3 discrete Heisenberg group); hence Proposition 3.2 implies that $\gamma \restriction D_\theta$ extends uniquely to a $*$-isomorphism of $C^*(D_\theta)$ onto $\overline{\gamma(D_\theta)} = \mathcal{D}_\theta$. ■

4. REPRESENTATIONS

In this section we make some general comments about the representations theory of the discretized CCRs (2.2) and we make the connection between (2.2) and the operators P_τ, Q_τ described in Section 1. We assume throughout that θ is a real number such that θ/π is irrational.

REMARK 4.1. *Finite representations*

The unique trace on the irrational rotation algebra $\mathcal{A}_\theta$ gives rise to a representation of $\mathcal{A}_\theta$ which generates the hyperfinite II_1 factor R. The closure of $\mathcal{B}_\theta$ in this representation is a sub von Neumann algebra of R. Since $\mathcal{B}_\theta$ has a unique tracial state [5], it follows that the closure of $\mathcal{B}_\theta$ is a subfactor of R, and hence is also isomorphic to R. Moreover, since $\mathcal{B}_\theta$ is also simple [5], any finite representation of $\mathcal{B}_\theta$ is quasi-equivalent to this one.

It is not hard to show that the subfactor of R generated by $\mathcal{B}_\theta$ in the above representation has Jones index 2. Since any two subfactors of R of index 2 are known to be isomorphic [12], we have here a very stable invariant for the embedding of the discretized CCR algebra in the irrational rotation algebra $\mathcal{A}_\theta$.

In particular, by the corollary of 3.2 we may conclude from these remarks that *there is a representation of the discretized* CCRs (2.2) *which generates R as a von Neumann algebra; moreover any finite representation of the discretized* CCRs *is quasi--equivalent to this one.*

Now let τ be a positive real number such that τ^2/π is irrational, and let P_τ and Q_τ be the discretized canonical operators on $L^2(\mathbf{R})$ associated with the step size τ as in Section 1. We want to make explicit the relation that exists between the pair (P_τ, Q_τ) and the C^*-algebra $C^*(\mathcal{B}_{\tau^2})$ discussed in Section 2.

THEOREM 4.2. *There is a unique representation π of D_{τ^2} on $L^2(\mathbf{R})$ satisfying*

$$\pi(d_{(1,0)}) = 2\tau Q_\tau,$$

$$\pi(d_{(0,1)}) = 2\tau P_\tau.$$

$\pi(D_{\tau^2})$ and $\{P_\tau, Q_\tau\}$ generate the same unital C^*-algebra. Thus, the three C^*-algebras

$$C^*(D_{\tau^2}),\ \ \mathcal{B}_{\tau^2},\ \ C^*(P_\tau, Q_\tau)$$

are mutually isomorphic.

Proof. Let U, V be the one-parameter groups

$$U_t f(x) = e^{itx} f(x),$$

$$V_t f(x) = f(x+t) \qquad f \in L^2(\mathbf{R}).$$

As in Section 1 we have

$$\begin{aligned} Q_\tau &= \frac{1}{2i\tau}(U_\tau - U_{-\tau}) = \frac{1}{\tau}\sin(\tau Q), \\ P_\tau &= \frac{1}{2i\tau}(V_\tau - V_{-\tau}) = \frac{1}{\tau}\sin(\tau P). \end{aligned} \tag{4.3}$$

We claim first that the *sines* in (4.3) can be replaced by *cosines* in the sense that the pair (P_τ, Q_τ) is unitarily equivalent to the pair $(\tilde{P}_\tau, \tilde{Q}_\tau)$ given by

$$\begin{aligned} \tilde{Q}_\tau &= \frac{1}{2\tau}(U_\tau + U_{-\tau}) \\ \tilde{P}_\tau &= \frac{1}{2\tau}(V_\tau + V_{-\tau}). \end{aligned} \tag{4.4}$$

To see this, put $\lambda = \pi/2\tau$ and let R denote the reflection on $L^2(\mathbf{R})$ given by $Rf(x) = f(-x)$. Consider the unitary operator

$$W = RU_{-\lambda}V_\lambda.$$

Using the commutation relations $V_tU_s = e^{ist}U_sV_t$ together with $RU_sR^* = U_{-s}$ and $RV_tR^* = V_{-t}$, one finds that

$$WU_sW^* = e^{i\lambda s}U_{-s}$$

$$WV_tW^* = e^{i\lambda s}V_{-t}.$$

Noting that $e^{i\lambda\tau} = \sqrt{-1}$, we obtain (4.4) by applying $\mathrm{ad}W$ to (4.3), i.e.,

$$WQ_\tau W^* = \frac{1}{2\tau}(U_\tau + U_{-\tau}) = \frac{1}{2\tau}\cos(\tau Q)$$

$$WP_\tau W^* = \frac{1}{2\tau}(V_\tau + V_{-\tau}) = \frac{1}{2\tau}\cos(\tau P).$$

We may therefore assume that the pair (Q_τ, P_τ) is defined by (4.4).

For each $x = (m, n) \in \mathbf{Z} \oplus \mathbf{Z}$, define a unitary operator W_x by

$$W_{(m,n)} = e^{imn\tau^2/2} U_{m\tau} V_{n\tau}.$$

A straightforward computation shows that the family of unitaries $\{W_x : x \in \mathbf{Z} \oplus \mathbf{Z}\}$ satisfies

$$W_x W_y = w(x, y) W_{x+y}$$

w being the cocycle of (2.1) for the value $\theta = \tau^2$, and hence there is a representation π of $\ell^1(\mathbf{Z} \oplus \mathbf{Z}, \tau^2)$ on $L^2(\mathbf{R})$ such that

$$\pi(w_x) = W_x, \quad x \in \mathbf{Z} \oplus \mathbf{Z}.$$

It is clear that π carries $d_{(1,0)}$ (resp. $d_{(0,1)}$) to $U_\tau + U_{-\tau} = 2\tau Q_\tau$ (resp. $2\tau P_\tau$).

It remains to show that the restriction of π to $C^*(D_{\tau^2})$ is uniquely defined by its values on the two elements $\{d_{(1,0)}, d_{(0,1)}\}$, and that Q_τ and P_τ generate $\pi(C^*(D_\tau))$ as a unital C^*-algebra. We will prove both by showing that the two elements $\{d_{(1,0)}, d_{(0,1)}\}$ and the identity generate the Banach $*$-algebra D_{τ^2}. It is not hard to adapt the results of [5] to prove that these three elements generate D_{τ^2}. Instead, we present the following argument since it gives somewhat more structural information.

Actually, we will give a fairly explicit method for calculating each element $d_x = = \delta_x + \delta_{-x}$ in terms of the self-adjoint elements $p = d_{(1,0)}$ and $q = d_{(0,1)}$, using a "generating function" for the family $\{d_x : x \in \mathbf{Z} \oplus \mathbf{Z}\}$. Indeed, it suffices to establish the following lemma.

LEMMMA 4.5. *Let θ be a real number such that θ/π is irrational, and consider the real analytic function $F\colon (-1,1) \times (-1,1) \longrightarrow D_\theta$ defined by*

$$F(s,t) = \sum_{m,n=-\infty}^{+\infty} s^{|m|} t^{|n|} e^{-imn\theta/2} d_{(m,n)}. \tag{4.6}$$

(i) *For $-1 < u < 1$, $-2 \leqslant x \leqslant 2$, let*

$$\varphi(u, x) = \frac{1 - u^2}{1 + u^2 - ux}.$$

Noting that φ is separately analytic in each variable, we have

$$F(s,t) = 2\varphi(s, q)\varphi(t, p), \quad |s|, |t| < 1,$$

where q, p are the elements of D_θ defined by

$$q = d_{(1,0)}, \quad p = d_{(0,1)}.$$

(ii) *The Banach $*$-algebra D_θ is spanned by the set $\mathcal{F} \cup \mathcal{F}^*$, where*

$$\mathcal{F} = \{F(s,t) : |s|, |t| < 1\}.$$

Proof of (i). Let w be the bicharacter of $\mathbf{Z} \oplus \mathbf{Z}$ defined by

$$w((p,q),(m,n)) = e^{i(qm-pn)\theta/2}$$

and let u, v be the following elements of $\ell^1(\mathbf{Z} \oplus \mathbf{Z}, w)$:

$$u = \delta_{(1,0)}, \quad v = \delta_{(0,1)}.$$

Then $w_{(m,n)} = e^{imn\theta/2} u^m v^n$, hence

$$d_{(m,n)} = e^{imn\theta/2}\left(u^m v^n + u^{-m} v^{-n}\right).$$

It follows that

$$F(s,t) = \sum_{m,n=-\infty}^{\infty} s^{|m|} t^{|n|}\left(u^m v^n + u^{-m} v^{-n}\right) = 2 \sum_{m,n=-\infty}^{\infty} s^{|m|} t^{|n|} u^m v^n =$$

$$= 2 \sum_{m=-\infty}^{\infty} s^{|m|} u^m \sum_{n=-\infty}^{\infty} t^{|n|} v^n.$$

An elementary calculation shows that if z is any complex number having absolute value 1 and $-1 < s < 1$, then

$$\sum_{m=-\infty}^{\infty} s^{|m|} z^m = \frac{1-s^2}{1+s^2-s(z+\bar{z})} = \varphi(s, z+\bar{z}).$$

Since $q = d_{(1,0)} = u + u^*$ and $p = d_{(0,1)} = v + v^*$, the assertion (i) follows from the analytic functional calculus.

To prove (ii), let A_{pq} be the coefficients in the power series expansion of F,

$$F(s,t) = \sum_{p,q=0}^{\infty} A_{pq} s^p t^q.$$

Obviously, $\{F(s,t): s,t \in (-1,1)\}$ and $\{A_{pq}: p,q \geqslant 0\}$ have the same closed linear span. Using the fact that $d_{(-m,-n)} = d_{(m,n)}$, a straightforward computation shows that

$$A_{pq} = 2e^{-ipq\theta/2} d_{(p,q)} + 2e^{ipq\theta/2} d_{(-p,q)}.$$

Thus,

$$A_{0q} = 2\left(d_{(0,q)} + d_{(0,q)}\right) = 4d_{(0,q)},$$

and

$$A_{p0} = 2\left(d_{(p,0)} + d_{(-p,0)}\right) = 4d_{(p,0)}.$$

In the remaining cases where $pq \neq 0$, the determinant of the coefficients of the 2×2 system of operator equations

$$\begin{aligned} A_{pq} &= 2e^{-ipq\theta/2} d_{(p,q)} + 2e^{ipq\theta/2} d_{(-p,q)} \\ A_{pq}^{*} &= 2e^{ipq\theta/2} d_{(p,q)} + 2e^{-ipq\theta/2} d_{(-p,q)} \end{aligned} \tag{4.7}$$

is $4\left(e^{-ipq\theta} - e^{ipq\theta}\right) \neq 0$, and in particular we can solve (4.7) for $d_{(p,q)}$ as a complex linear combination of A_{pq} and A_{pq}^{*}. This argument shows that the closed linear span of $\mathcal{F} \cup \mathcal{F}^{*}$ contains $\{d_{(p,q)}: p,q \in \mathbb{Z}\}$, and (ii) follows. That completes the proof of Theorem 4.2. ∎

REMARK. In some very recent work [7], Bratteli and Kishimoto have established the striking result that $\mathcal{B}_\theta$ is an AF-algebra.

APPENDIX. *Failure of extensions*

We present a simple example of a pair of commutative unital Banach $*$-algebras $B \subseteq A$ such that A is a subalgebra of its enveloping C^*-algebra, but such that the natural morphism $\gamma_B: C^*(B) \longrightarrow C^*(A)$ is not injective. Let A be the algebra of all complex-valued continuous functions defined on the annulus $\{1 \leqslant |z| \leqslant 2\}$ which are analytic in its interior. With norm and involution defined by

$$\|f\| = \sup_{1 \leqslant |z| \leqslant 2} |f(z)|, \quad f^*(z) = \overline{f}(\overline{z}),$$

$\overline{f}$ denoting the complex conjugate of f, A is a unital Banach $*$-algebra. $C^*(A)$ is the commutative C^*-algebra $C(X)$,

$$X = [-2,-1] \cup [+1,+2]$$

denoting the intersection of the annulus $\{1 \leqslant |z| \leqslant 2\}$ with the real axis, and the completion map $\gamma: A \longrightarrow C(X)$ is defined by restriction to X. Let B be the norm closure of all holomorphic polynomials in A. Then B is a self-adjoint subalgebra whose enveloping C^*-algebra is $C(Y)$, Y being the intersection of the polynomially convex hull of the annulus with the real axis, namely

$$Y = [-2, +2].$$

The morphism $\gamma_B : C(Y) \longrightarrow C(X)$ is given by restriction to X, and hence there is a nontrivial kernel. But differently, for every real $\lambda \in (-1, +1)$, the complex homomorphism of B defined by

$$w_\lambda(f) = f(\lambda), \quad f \in B$$

is a bounded positive linear functional on B which cannot be extended to a positive linear functional on A.

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