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IDEAL PERTURBATIONS OF NESTS

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Dedicated to the memory of John Bunce

Voiculescu [13, 14] has shown that it is interesting to consider perturbations relative to ideals properly contained in the compact operators. He introduced a quantitative measure of the obstruction to the existence of a quasi-central approximate unit relative to such an ideal. We carry out this program for nests. This yields a version of Andersen's Theorem relative to *every* ideal properly containing the trace class operators. On the other hand, approximate unitary equivalence relative to the trace class operators implies unitary equivalence.

The prototype for this type of theorem has its roots in Weyl's Theorem that every self-adjoint operator is the sum of a diagonal operator and a small compact one. Kuroda [11] generalized this to show that the compact operator may be chosen to have small norm in any ideal properly containing the trace class. On the other hand, the Kato-Rosenblum Theorem [10,12] shows that the part of the spectrum absolutely continuous to Lebesgue measure is an obstruction to a trace class perturbation which is diagonal. Voiculescu, in a celebrated paper [13], applied his ideas to n-tuples of commuting self-adjoint operators. By constructing an appropriate quasi-central approximate unit, he was able to simultaneously diagonalize this n-tuple modulo the ideal $\mathcal{C}_n$. At the same time, he identified an ideal $\mathcal{C}_{n-}$. The (joint) absolutely continuous part of the spectral measure yielded a *finite* obstruction to a $\mathcal{C}_{n-}$ quasi--central approximate unit, and hence to the desired ideal perturbation.

The theory of nests in many ways parallels the theory of a single self-adjoint operator. The unitary invariants for a nest [6] are exactly analogous to the multiplicity theory of a self-adjoint operator. On the other hand, Andersen [1] showed that any two continuous nests are approximately unitarily equivalent. In [2], I show that there is a quasi-central approximate unit for any nest, and use this to give a much easier proof in the spirit of Voiculescu and Arveson. In this paper, we show that every (*separable*) nest has a quasi-central approximate unit relative to any ideal properly containing the trace class. This is used to obtain a version of Andersen's Theorem for all such ideals. On the other hand, the obstruction to a quasi-central approximate unit relative to $\mathcal{C}_1$ is exactly proportional to the maximum multiplicity of the nest's non-atomic part. This in turn leads to the fact that nests which are approximately unitary equivalent relative to the trace class are in fact unitarily equivalent. The reason that we obtain a much stronger result here than in the self-adjoint theory is that the parametrization of a nest is arbitrary. In particular, it can be reparametrized to make any portion of the continuous part of the nest absolutely continuous to Lebesgue measure.

For background on nests, see [3]. In particular, we will make frequent reference to the material on approximate unitary equivalence and similarity of nests in chapters 12 and 13.

1. QUASI-CENTRAL APPROXIMATE UNITS

Let Φ be an ideal of compact operators with norm $\|\cdot\|_\Phi$ satisfying

$$\|AKB\|_\Phi \leqslant \|A\|\,\|K\|_\Phi\,\|B\| \qquad \text{for all } A, B \in \mathcal{B}(\mathcal{H}),\ K \in \Phi.$$

It is well-known that $\|K\|_\Phi$ is a function of the singular values of K (cf. [7]). If Φ properly contains $\mathcal{C}_1$, the norm of projections P_n of rank n satisfy $\|P\|_\Phi = \varepsilon_n n = o(n)$. Hence for finite rank operators,

$$\|F\|_\Phi \leqslant \|F\|\varepsilon_{\mathrm{rank}F}\mathrm{rank}F.$$

Voiculescu [13] introduced the following measure of the obstruction to a quasi--central approximate unit relative to the ideal Φ. Let $\mathcal{F}_+^1$ denote the set of positive finite rank contractations endowed with the usual order on positive operators. If $\mathcal{S}$ is a set of operators, define

$$\kappa_\Phi(\mathcal{S}) = \lim_{R\in\mathcal{F}_+^1} \inf_{R\uparrow I} \max_{S\in\mathcal{S}} \|[R,S]\|_\Phi.$$

It is clear that if $\kappa_\Phi(\mathcal{S}) = 0$, then a quasi-central approximate unit relative to Φ exists for $\mathcal{S}$. For most ideals, this quantity is either 0 or ∞. An ideal for which this quantity is finite and non-zero can be thought of as a dimension for the set in the spirit of Hausdorff dimension for Euclidian sets. We will write κ_1 instead of $\kappa_{\mathcal{C}_1}$.

In [2], we construct a quasi-central approximate unit for a continuous nest $\mathcal{N}$ with cyclic vector. We will refer to the treatment in [3, Lemma 13.5]. This quasi-central approximate unit F_n has the following properties:

(i) $F_n \in \mathcal{F}^1_+$
(ii) $F_n F_{n+1} = F_n$ for all $n \geqslant 1$
(iii) $F_n \uparrow I$
(iv) $[P(N), F_n]$ has rank at most n for all $N \in \mathcal{N}$
(v) $\|[P(N), F_n]\| \leqslant 2/n$ for all $N \in \mathcal{N}$.

Hence for any ideal Φ properly containing $\mathcal{C}_1$, we obtain

$$\|[P(N), F_n]\| \leqslant \varepsilon_n \|[P(N), F_n]\| \operatorname{rank}[P(N), F_n] \leqslant 2\varepsilon_n = o(1).$$

Hence this sequence is a quasi-central approximate unit relative to every ideal Φ properly containing $\mathcal{C}_1$. From this, it is easy to deduce the following lemma:

LEMMA 1.1. *Let $\mathcal{M}$ be a nest on a separable Hilbert space, and let Φ be an ideal properly containing the trace class. There is a quasi-central approximate unit for $\mathcal{M}$ relative to Φ. It may be chosen to satisfy property* (ii) *above.*

Proof. It is trivial to construct a central approximate unit for the atomic part of $\mathcal{M}$, so without loss of generality, we may assume that $\mathcal{M}$ is continuous. By [3, Lemma 13.3], $\mathcal{M}$ is unitarily equivalent to the direct sum of cyclic continuous nests $\mathcal{M} \cong \sum \oplus \mathcal{M}_j$. Let $F^{(j)}_n$ be a quasi-central approximate unit for $\mathcal{M}_j$ as constructed above. Choose a strictly increasing sequence n_k so that $\varepsilon_{n_k} < k^{-2}$. Then

$$F_k = \sum_{j=1}^{k} \oplus F^{(j)}_{n_k}$$

is the desired quasi-central approximate unit. ∎

Let M_x be the operator of multiplication by x on $L^2(0,1)$. Voiculescu [13] quantifies the Kato-Rosenblum Theorem in the statement $\kappa_1(M_x) = 1/\pi$. Let $\mathcal{N}_0$ denote the Volterra nest on $L^2(0,1)$ consisting of the subspaces

$$N_t = \{\, f \in L^2(0,1) \mid \operatorname{supp}(f) \subset [0,t] \,\}.$$

Set $\gamma = \kappa_1(\mathcal{N}_0)$. We will show that this lies in the interval $[1/\pi, 1]$. From the unitary invariants of a nest, we have the multiplicity function. For a nest $\mathcal{M}$, let $\operatorname{mcm}(\mathcal{M})$ denote the maximum multiplicity of the continuous part of $\mathcal{M}$. This is always an element of $\{0, 1, 2, \ldots, \infty\}$.

THEOREM 1.2. *For any nest $\mathcal{M}$, $\kappa_1(\mathcal{M}) = \gamma \operatorname{mcm}(\mathcal{M})$.*

Proof. First consider the Volterra nest. Let F_n be the quasi-central approximate unit constructed above. Following the notation in [3, Lemma 13.5], we obtain

$$\|[P(N), F_n]\|_1 \leqslant \frac{1}{n} \sum_{j=1}^{n} \|y_j \otimes z_j^* - z_j \otimes y_j^*\|_1 \leqslant \frac{1}{n} \sum_{j=1}^{n} 2\|y_j\|\|z_j\|$$

where y_j is orthogonal to z_j and $\|y_j + z_j\| = 1$. Thus $2\|y_j\|\|z_j\| \leqslant 1$. Consequently, $\gamma = \kappa_1(\mathcal{N}_0) \leqslant 1$.

On the other hand,

$$M_x = \int_0^1 P(N_t)^{\perp} \mathrm{d}t.$$

Thus,

$$\|[M_x, F]\|_1 = \|\int_0^1 [F, P(N_t)]\mathrm{d}t\|_1 \leqslant \max \|[P(N_t), f]\|_1.$$

Thus, $1/\pi = \kappa_1(M_x) \leqslant \kappa_1(\mathcal{N}_0) = \gamma$.

It is an easy exercise (c.f. [13, Prop. 1.5]) to see that

$$\kappa_1(\mathcal{N} \otimes \mathbf{C}^m) = m\kappa_1(\mathcal{N}_0) = m\gamma.$$

Now, let us consider a general nest $\mathcal{M}$. As before, the atomic part of $\mathcal{M}$ has a central approximate unit. So we may assume that $\mathcal{M}$ is continuous. First, suppose that $\mathcal{M}$ is cyclic (i.e. $\mathrm{mcm}(\mathcal{M}) = 1$). Then by a theorem of Kadison and Singer [9](c.f. [3, Prop. 7.17]), $\mathcal{M}$ is unitarily equivalent to $\mathcal{N}_0$. Hence, $\kappa_1(\mathcal{M}) = \gamma$.

Next suppose that $\mathcal{M}$ has multiplicity m on a set Ω of positive spectral measure. The restriction $\mathcal{M}_1$ of $\mathcal{M}$ to the spectral subspace $E_{\mathcal{M}}(\Omega)\mathcal{H}$ has uniform multiplicity m, and thus is unitarily equivalent to $\mathcal{N} \otimes \mathbf{C}^m$. Thus

$$\kappa_1(\mathcal{M}) \geqslant \kappa_1(\mathcal{M}_1) = m\gamma.$$

Hence, $\kappa_1(\mathcal{M}) \geqslant \mathrm{mcm}(\mathcal{M})\gamma$.

Conversely, suppose that $\mathcal{M}$ has finite $\mathrm{mcm}(\mathcal{M}) = m$. Then $\mathcal{M}$ can be written as the direct sum of m cyclic nests, say $\mathcal{M}_j$ for $1 \leqslant j \leqslant m$. Each is unitarily equivalent to the Volterra nest (with different reparametizations), and so has a quasi-central approximate unit $F_n^{(j)}$ with

$$\lim_{n\to\infty} \sup_{M \in \mathcal{M}_j} \|[F_n^{(j)}, P(M)]\| = \gamma.$$

Clearly, $F_n = \sum_{j=1}^{n} \oplus F_n^{(j)}$ is a quasi-central approximate unit such that

$$\kappa_1(\mathcal{M}) \leqslant \lim_{n\to\infty} \|[F_n, P(M)]\|_1 = \mathrm{mcm}(\mathcal{M})\gamma.$$

Hence equality is assured. ∎

I have been unable to determine the exact value of γ, but my gut feeling is that that $\gamma = 1$.

2. CONTINUOUS NESTS MOD Φ

The proof of Andersen's Theorem given in [2], (also [3, Theorem 13.9]) can be modified to work for any ideal with a quasi-central approximate unit. In this section, let Φ be a fixed ideal properly containing the trace class.

LEMMA 2.1. *Let $\mathcal{N}$ and $\mathcal{M}$ be cyclic continuous nests with cyclic vectors x and y respectively. For each positive integer k, let P_k be the projection onto the subspace* $\operatorname{span}\{ E_{\mathcal{N}}((2^{-k}(j-1), 2^{-k}j])x \mid 1 \leqslant j \leqslant 2^k \}$. *For $\varepsilon > 0$, there is a unitary operator U satisfying*

$$\sup_{t\in[0,1]} \|(P(M_t)U - UP(N_t))P_k\|_1 < \varepsilon.$$

Proof. The proof of [3, Lemma 13.6] produces a unitary U with the required property. Indeed, it is shown that the difference is a *rank one* operator with norm at most ε. Hence the trace norm is the same. ∎

In the operator norm version, we used a lemma [3, Lemma 13.7] stating that for a positive operator E, one has

$$\|[E^{1/2}, T]\| \leqslant \sqrt{2}\|[E,T]\|^{1/2}\|T\|^{1/2}.$$

This enables us to get a reasonable estimate of the quantity

$$\|[(F_m - F_n)^{1/2}, P(N)]\|.$$

However, we do not expect that such an estimate is valid for all ideal norms. Helton and Howe [8] show that an appropiate estimate is valid for C^∞ functions, but the square root does not qualify. Instead, we use a device that we introduced in [4].

Recall that our quasi-central approximate unit $\{F_n\}$ has the special property that

$$F_n F_m = F_{\min(n,m)}.$$

We will see that $G_n = \sin^2(\frac{\pi}{2}F_n)$ is a quasi-central approximate unit with the same properties plus, for $n > m$,

$$(G_n - G_m)^{1/2} = (G_n(I - G_m))^{1/2} = \sin\left(\frac{\pi}{2}F_n\right)\cos\left(\frac{\pi}{2}F_m\right)$$

LEMMA 2.2. *Let F_n and G_n be given as above. Then for all operators T,*

$$\|[G_n, T]\|_\Phi \leqslant 2\cosh\left(\frac{\pi}{2}\right)\|[F_n, T]\|_\Phi$$

and for $n > m$,

$$\|[(G_N - G_m)^{1/2}, T]\|_\Phi \leqslant \cosh\left(\frac{\pi}{2}\right)\|[F_n, T]\|_\Phi + \sinh\left(\frac{\pi}{2}\right)\|[F_m, T]\|_\Phi.$$

Proof. As in [4], this is a routine power series calculation. We have

$$\sin\left(\frac{\pi}{2}F\right) = \sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}F\right)^{2n+1}$$

and

$$\cos\left(\frac{\pi}{2}F\right) = \sum_{n\geqslant 0}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}F\right)^{2n}.$$

The well known estimate $|||[A^k,T]|||_\Phi \leqslant k||A||^{k-1}|||[A,T]|||_\Phi$ is valid for all ideal norms. So it is easy to obtain the estimates

$$\left\|\left[\sin\left(\frac{\pi}{2}F\right),T\right]\right\|_\Phi \leqslant \left(\sum_{n\geqslant 0}\frac{1}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}\right)|||[F,T]|||_\Phi = \cosh\left(\frac{\pi}{2}\right)|||[F,T]|||_\Phi$$

and

$$\left\|\left[\cos\left(\frac{\pi}{2}F\right),T\right]\right\|_\Phi \leqslant \left(\sum_{n\geqslant 0}\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\right)|||[F,T]|||_\Phi = \sinh\left(\frac{\pi}{2}\right)|||[F,T]|||_\Phi.$$

Thus we obtain the estimates

$$|||[G_n,T]|||_\Phi \leqslant 2\left\|\sin\left(\frac{\pi}{2}F_n\right)\right\|\left\|\left[\sin\left(\frac{\pi}{2}F\right),T\right]\right\|_\Phi \leqslant 2\cosh\left(\frac{\pi}{2}\right)|||[F_n,T]|||_\Phi$$

and for $n > m$, $|||[(G_n - G_m)^{1/2},T]|||_\Phi$ is dominated by

$$\left\|\left[\sin\left(\frac{\pi}{2}F_n\right),T\right]\right\|_\Phi\left\|\cos\left(\frac{\pi}{2}F_m\right)\right\| + \left\|\sin\left(\frac{\pi}{2}F_n\right)\right\|\left\|\left[\cos\left(\frac{\pi}{2}F\right),T\right]\right\|_\Phi \leqslant$$

$$\leqslant \cosh\left(\frac{\pi}{2}\right)|||[F_n,T]|||_\Phi + \sinh\left(\frac{\pi}{2}\right)|||[F_m,T]|||_\Phi. \quad \blacksquare$$

Now the intertwining lemma replacing [3, Lemma 13.8] follows by standard technique.

LEMMA 2.3. *Let $\mathcal{N}$ be a continuous nest with cyclic vector, and let $\mathcal{M}$ be another continuous nest. Given an $\varepsilon > 0$, there is an isometry U so that $f(t) = P(M_t)U - UP(N_t)$ is a continuous function from $[0,1]$ into Φ with norm*

$$||f|| = \sup_{1\leqslant t\leqslant 1}||f(t)||_\Phi < \varepsilon.$$

Sketch. As nothing new is introduced here, the ideas will only be briefly outlined. Split $\mathcal{M}$ as a direct sum of countably many cyclic continuous nests $\mathcal{M}_j$. Choose a sequence n_j growing sufficiently fast, and use the lemma above to construct unitaries U_j so that

$$\|[F_{n_j}, P(N_t)]\|_{\Phi} < 2^{-j}\varepsilon/(4\cosh(\frac{\pi}{2}))$$

and

$$\|\big(P(M_t^{(j)})U_j - U_jP(N_t)\big)P_{n_j}\|_{\Phi} < 2^{-j}\varepsilon/2.$$

Then

$$U = \sum_{j\geqslant 1} U_j(G_{n_j} - G_{n_{j-1}})^{1/2}$$

is an isometry satisfying $\|P(M_t)U - UP(N_t)\|_{\Phi} \leqslant \varepsilon$. ∎

We are ready to prove the main theorem of this section, the analogoue of Andersen's Theorem for the ideal Φ.

THEOREM 2.4. *Let $\mathcal{N}$ and $\mathcal{M}$ be continuous nests, and let Φ be an ideal properly containing the trace class. Let θ be any order isomorphism of $\mathcal{N}$ onto $\mathcal{M}$. Given $\varepsilon > 0$, there is a unitary operator W so that the function $f(N) = P(\theta(N)) - WP(N)W^*$ is a continuous function from $\mathcal{N}$ into Φ with norm at most ε.*

Proof. Parametrize $\mathcal{N}$ and $\mathcal{M}$ as $\{N_t\}$ and $\{M_t\}$ so that $\theta(N_t) = M_t$. Following the proof of [3, Theorem 13.9] verbatim yields a isometry U from $\mathcal{H}^{\infty}$ into $\mathcal{H}$ so that

$$g(t) = P(M_t)U - UP(N_t)^{(\infty)}$$

is a continuous function into Φ with norm at most ε. From this it follows that $h(t) = [P(M_t), UU^*]$ is a continuous function into Φ with the norm at most 2ε. The formalism is that $\mathcal{M}$ has been approximately split as

$$\mathcal{M} \sim \mathcal{L} \oplus \mathcal{N}^{(\infty)} \cong (\mathcal{L} \oplus \mathcal{N}^{(\infty)}) \oplus \mathcal{N} \sim \mathcal{M} \oplus \mathcal{N}.$$

Chasing through the calculations exactly as in [3, Theorem 13.9] yields the theorem. ∎

3. TRACE CLASS PERTURBATIONS

The sharp dichotomy between the trace class and larger ideals observed for quasi--central approximate units persist for approximate unitary equivalence. Although we cannot get the whole picture from the κ_1 invariant, we will extract some of the information from it.

LEMMA 3.1. *Suppose that θ is an dimension preserving order isomorphism of a nest $\mathcal{N}$ onto another nest $\mathcal{M}$, and that W is a unitary operator such that $f(N) = P(\theta(N)) - WP(N)W^*$ is a continuous function of $\mathcal{N}$ into C_1. Then there is an approximate unitary equivalence of $\mathcal{N}$ and $\mathcal{M}$ modulo C_1 implementing θ.*

Proof. We may assume that $W = I$. Fix ε with $0 < \varepsilon < .5$. By the uniform continuity of f, the nest $\mathcal{N}$ may be partitioned into intervals by a finite subset

$$0 = N_0 < N_1 < \ldots < N_k = \mathcal{H}$$

so that either $\|f(N) - f(N_{j-1})\|_1 < \varepsilon$ for all N in the interval $[N_{j-1}, N_n]$, or N_{j-1} is the immediate predeccessor of N_j in the nest. Let $E_j = P(N_j) - P(N_{j-1})$ and $F_j = P(\theta(N_j)) - P(\theta(N_{j-1}))$. Whenever E_j is an atom of $\mathcal{N}$, the fact that θ preserves dimension guarantees that $\operatorname{rank}(E_j) = \operatorname{rank}(F_j)$ and hence there is a partial isometry U_j with initial space E_j and final space F_j. Otherwise, let U_j be the partial isometry in the polar decomposition of F_jE_j. Since $F_j - E_j$ is trace class of norm less than one, U_j intertwines E_j and F_j and $U_j - E_j$ is trace class. Define $U = \sum_{j=1}^{n} U_j$. This is unitary and a trace class perturbation of the identity. It is clear that $UN_j = \theta(N_j)$ for $0 \leqslant j \leqslant n$. It remains to show that the function $g(N) = P(\theta(N)) - UP(N)U^*$ has the desired properties. Clearly, it is a continuous trace class valued function. Furthermore, $g(N_j) = 0$ for each j. Suppose that N is an element of $\mathcal{N}$ such that $N_{j-1} < N < N_j$. Let $E = P(N) - P(N_{j-1})$ and $F = P(\theta(N)) - P(\theta(N_{j-1}))$, and $R = (E_j^\perp + E_jF_jE_j)^{-1/2}$. As in [3, Theorem 12.16], compute

$$\begin{aligned} g(N)U &= FU_j - U_jE = (FF_jE_j - F_jE_jE)(R) = \\ &= \big(F(F-E)E_jE^\perp - F^\perp F_j(F-E)E\big)R. \end{aligned}$$

It follows that $\|g(N)\|_1 \leqslant 2\|E - F\|_1\|R\| < 2\varepsilon(1-\varepsilon)^{-1/2}$. As ε may be chosen arbitrarily small, $\mathcal{N}$ and $\mathcal{M}$ are approximately unitarily equivalent modulo the trace class. ∎

The following corollary is an immediate consequence.

COROLLARY 3.2. *If $f(N) = P(\theta(N)) - WP(N)W^*$ is a continuous function of $\mathcal{N}$ into C_1, then $\kappa_1(\mathcal{N}) = \kappa_1(\mathcal{M})$, and hence $\mathrm{mcm}(\mathcal{N}) = \mathrm{mcm}(\mathcal{M})$.*

COROLLARY 3.3. *Suppose that $\mathrm{mcm}(\mathcal{N})$ is finite. If $f(N) = P(\theta(N)) - WP(N)W^*$ is a continuous function of $\mathcal{N}$ into C_1, then $\mathcal{N}$ and $\mathcal{M}$ are unitarily equivalent.*

Proof. Let μ be a scalar measure so that both spectral measures of $\mathcal{N}$ and $\mathcal{M}$ are absolutely continuous with respect to μ. Suppose that there is a set Ω of positive μ measure on which $\mathcal{N}$ has uniform multiplicity n and $\mathcal{M}$ has uniform multiplicity

m. Let $\mathcal{L}$ be a cyclic nest supported on Ω such that the restrictions of $\mathcal{N}$ and $\mathcal{M}$ to Ω are unitarily equivalent to $\mathcal{L} \otimes \mathbf{C}^n$ and $\mathcal{L} \otimes \mathbf{C}^m$ respectively. Pick an integer $p > \mathrm{mcm}(\mathcal{N})$, and consider the nests $\mathcal{N}_1 = \mathcal{N} \oplus \mathcal{L}^{(p)}$ and $\mathcal{M}_1 = \mathcal{M} \oplus \mathcal{L}^{(p)}$. It is clear that these nests are approximately unitary equivalent modulo $\mathcal{C}_1$. Moreover, since $\mathcal{L}^{(p)}$ has multiplicity 0 off of Ω, and uniform multiplicity p on Ω,

$$n + p = \mathrm{mcm}(\mathcal{N}_1) = \mathrm{mcm}(\mathcal{M}_1) = m + p.$$

Consequently, $n = m$. This holds μ almost everywhere, and hence $\mathcal{N}$ and $\mathcal{M}$ are unitarily equivalent. ∎

In order to deal with nests of infinite multiplicity, a way is needed to isolate parts of the nests.

LEMMA 3.4. *Suppose that $f(N) = P(\theta(N)) - WP(N)W^*$ is a continuous function of $\mathcal{N}$ into $\mathcal{C}_1$. For any monotone scalar function $h(N)$,*

$$\int h(N)\mathrm{d}P(\theta(N)) - W \int h(N)\mathrm{d}P(N)W^*$$

is in the trace class.

Proof. The quantity in question is easily seen to be $\int h(N)\mathrm{d}f(N)$, which after integration by parts becomes $-\int f(N)\mathrm{d}h(N)$. Standard estimates for the Riemann--Stieltes integral show that this converges uniformly in the trace norm. ∎

THEOREM 3.5. *Suppose that θ is a dimension preserving order isomorphism of a nest $\mathcal{N}$ onto another nest $\mathcal{M}$, and that U is a unitary operator such that $f(N) = P(\theta(N)) - UP(N)U^*$ is a continuous function of $\mathcal{N}$ into $\mathcal{C}_1$. Then there is a unitary operator implementing the isomorphism θ.*

Proof. Let μ be a scalar measure dominating the spectral measures of both $\mathcal{N}$ and $\mathcal{M}$. Suppose that there is a set Ω of positive μ measure on which $\mathcal{N}$ and $\mathcal{M}$ have uniform multiplicities n and m respectively. Let $h(N) = \mu(\Omega \cap [0, N])$. Consider the operators $A = \int h(N)\mathrm{d}P(N)$ and $B = \int h(N)\mathrm{d}P(\theta(N))$. By the lemma, these two self-adjoint operators are unitarily equivalent modulo the trace class. Thus by Kato-Rosenblum Theorem [10, 12], the absolutely continuous parts of A and B are unitarily equivalent. By construction, the spectral measures of A and B are equivalent to Lebesgue measure on $(0, \mu(\Omega)]$ plus a large point mass at 0. However, they have multiplicities n and m respectively. Consequently, $n = m$. It follows that the multiplicity functions of $\mathcal{N}$ and $\mathcal{M}$ agree μ almost everywhere. Hence $\mathcal{N}$ and $\mathcal{M}$ are unitarily equivalent. ∎

4. ARBITRARILY NESTS MOD Φ

In this section, the results for continuous nests of section 2 will be extended to arbitrary nests. This turns out to be mostly a technical exercise. Let us fix an ideal Φ properly containing the trace class. Let us also fix a pair of nests $\mathcal{N}$ and $\mathcal{M}$ and a dimension preserving order isomorphism θ from $\mathcal{N}$ onto $\mathcal{M}$. By the author's Similarity Theorem [5], there is a similarity and also an approximate unitary equivalence between $\mathcal{N}$ and $\mathcal{M}$ implementing θ. One of the points of the proof is that the order type of $\mathcal{N}$, say ω, can be split into a countable subset c and the maximal perfect subset ω_0. The portions of $\mathcal{N}$ and $\mathcal{M}$ corresponding to c are unitarily equivalent. So we can restrict our attention to ω_0. So without loss of generality, we may assume that ω is perfect. We may assume that $\mathcal{N}$ and $\mathcal{M}$ are parametrized as $\{N_t\}$ and $\{M_t\}$ for $t \in \omega$.

The problem would appear to be that $\mathcal{N}$ might be atomic while $\mathcal{M}$ may have non-atomic part. For then the non-atomic parts cannot be matched up. In the proof of the Similarity Theorem, this problem was overcome by showing that there existed some (essentially non-constructible) similarity which introduced a non-atomic part everywhere. By a Theorem of Gohberg and Krein [7], the similarity cannot have the form *scalar plus Macaev ideal.* This makes it hard to imagine a way to force the difference to lie in the trace class. We proceed in a different way.

First we note that Lemma 2.1 does not require continuous nests. Some extra care is needed if the embedded nest $\mathcal{N}$ has atoms. We do not need this, so $\mathcal{N}$ will continue to be a non-atomic. However, the case of real interest here occurs when $\mathcal{M}$ is totally atomic with a dense set of atoms. The prototype is the Cantor nest on $\ell^2(\mathbf{Q})$ with usual order (see [3, sections 2.14, 13.15]). In this case, the order type is the Cantor set. The Cantor set is similar to the analogous nest defined on $\ell^2(\mathbf{Q}) \oplus L^2(\mathbf{R})$, which has a large continuous part. Our problem is to embed this continuous part into the Cantor nest with a small Φ-norm perturbation.

LEMMA 4.1. *Let ω be a perfect compact subset of $\mathbf{R}$. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are cyclic nests with support ω, and that $\mathcal{N}$ is non-atomic. Given $\varepsilon > 0$ and a finite rank projection P, there is a unitary operator U so that*

$$\sup_{t\in\omega} \|(P(M_t)U - UP(N_t))P\|_\Phi < \varepsilon.$$

Proof. It is not really necessary to consider arbitrary finite projections P. It is enough to demonstrate this for a sequence of such projections increasing strongly to the identity as Lemma 2.1. For such a sequence will eventually dominate any P within an arbitrary ε in the trace norm. Fix cyclic vectors x and y for $\mathcal{N}$ and $\mathcal{M}$ respectively. Let μ be a (non-atomic) probability measure on ω mutually absolutely continuous to the spectral measure of $\mathcal{N}$. Then map ω onto $[0,1]$ by $h(t) = \mu((-\infty, t])$.

For each positive integer n, let $E_{k,n}$ denote the spectral projection $E_{\mathcal{N}}((h^{-1}((k-1)/n), h^{-1}(k/n)])$ for $1 \leqslant k \leqslant 2^n$. Let $x_{k,n} = E_{k,n}x\|E_{k,n}x\|^{-1}$, and set P_n to be the projection onto $\operatorname{span}\{x_{k,n} \mid 1 \leqslant k \leqslant 2^n\}$. Because x is a cyclic vector, the projections P_n increase strongly to the identity. Similarly, let $F_{k,n}$ and $y_{k,n}$ be the corresponding projections and vectors corresponding to $\mathcal{M}$. Note that we are following the notation of [3, Lemmas 13.5-6]. Even though $\mathcal{M}$ may be totally atomic, it has the important feature that $F_{k,n}$ are always non-zero, and hence the vectors $y_{k,n}$ are always well defined.

The arguments of [3, Lemmas 13.5-6] and Lemma 2.1 go through verbatim. It is worth noting that the proof of Lemma 13.5 uses the continuity of $\mathcal{N}$ but not of $\mathcal{M}$. This is crucial here. ∎

THEOREM 4.2. *Let $\mathcal{N}$ and $\mathcal{M}$ be isomorphic nests, and let θ be a dimension preserving order isomorphism from $\mathcal{N}$ onto $\mathcal{M}$. Then for any ideal Φ properly containing the trace class, there is an approximate unitary equivalence of $\mathcal{N}$ and $\mathcal{M}$ modulo Φ implementing θ.*

Proof. The discussion at the beginning of this section reduces the problem to nests of perfect order type, say ω. It suffices to show that the nest $\mathcal{M}$ (and similarly $\mathcal{N}$) is approximately unitary equivalent to another with its non-atomic part supported on all of ω. For then Theorem 2.4 will allow us to match up the non-atomic parts. This follows the spirit of our proof of the Similarity Theorem. To this end, let $\mathcal{L}$ be a cyclic continuous nest parametrized by ω. It is easy to split $\mathcal{M}$ into an infinite direct sum of cyclic nests $\mathcal{M}_j$ each with support ω. This is analogous to [3, Lemma 13.3]. To deal with the atomic part, one need only split the atoms into countably many subsets, each dense in the largest possible set (namely, the maximal perfect subset of the closure of the atoms in ω).

Now, follow the details of Lemma 2.3 to intertwine $\mathcal{M}$ and $\mathcal{L}^\infty$ via an isometry U so that $f(t) = P(M_t)U - UP(L_t)^{(\infty)}$ is a continuous function from ω into Φ with norm at most ε. It follows as in the proof of Theorem 2.4 that $\mathcal{M}$ and $\mathcal{M} \oplus \mathcal{L}$ are approximately unitary equivalent modulo Φ. ∎

5. FINAL REMARKS

We are at a loss to explain the connection between these results and the Theorem of Gohberg and Krein. Their result states that if an operator T has the form unitary plus a compact in the Macaev ideal, then the nests $\mathcal{N}$ and $T\mathcal{N}$ are unitarily equivalent. The reason this is so is that such T factor as $T = UA$ where U is unitary and A is invertible in the nest algebra $\mathcal{T}(\mathcal{N})$. Their proof has two parts. The most important ingredient is that triangular truncation is bounded from the Macaev ideal into the compact operators. It is then possible to use this operator to explicitly compute the factorization when the compact perturbation is norm at most 1 in the Macaev ideal. The other step is to prove the result for finite rank operators. Gohberg and Krein's original proof of this is quite independent from the first part, but in [3, Lemma 14.8] we give an easier proof as a corollary of the small Macaev norm case. Now using the results of this section 3, it is easy to deduce that if T is unitary plus trace class, then $\mathcal{N}$ and $T\mathcal{N}$ are unitarily equivalent. Unfortunately, this is a far cry from the Macaev ideal.

The reason we cannot find a tight connection betweem these two results is that if T is unitary plus Macaev, there is no apparent reason for $\mathcal{N}$ and $T\mathcal{N}$ to be unitarily equivalent modulo trace class. They are unitarily equivalent modulo the Macaev ideal, but we have now seem that this (and much more!) is true of all similar nests. This means that similarity by compact operators (which are necessarily not in the Macaev ideal) can perturb a nest $\mathcal{N}$ to another which is a small $\mathcal{C}_2$ (or even $\mathcal{C}_n$) perturbation of $\mathcal{N}$, but is not unitarily equivalent. It would seem to be an important problem to draw some connection between these results. The hope is still to shed some light on the similarities of nests which change the unitary equivalence class.

REFERENCES

1. ANDERSEN, N. T., Compact perturbations of reflexive algebras, *J. Functional Analysis* **38**(1980),366–400.
2. DAVIDSON, K. R., Approximate unitary equivalences of continuous nests, *Proc. Amer. Math. Soc.* **97**(1986), 655–660.
3. DAVIDSON, K. R., Nest Algebras, *Pitman Research Notes in Mathematics Series*, vol. **191**, Longman Scientific and Technical Pub. Co., London—New York, 1988.
4. DAVIDSON, K. R., Almost commuting Hermitian matrices, *Math. Scand.* **56**(1985), 222–240.
5. DAVIDSON, K. R., Similarity and compact perturbations of nest algebras, *J. Reine Angew. Math.* **348**(1984), 286–294.
6. ERDÖS, J. A., Unitary invariants for nests, *Pacific J. Math.* **23**(1967), 229–256.
7. GOHBERG, I. C.; KREIN, M. G., *Theory and applications of Volterra operators in Hilbert space*, Nauka, Moscow, 1967; English transl., *Transl. Math. Monographs*, **24**, *Amer. Math. Soc.*, Providence, RI, 1970.
8. HELTON, J. W.; HOWE, R., Integral operators, commutator traces, index and homology, Springer *Lect. Notes Math.* **345**(1973), 141–209.
9. KADISON, R. V.; SINGER, I. M., Triangular operator algebras, *Amer. J. Math.* **82**(1960), 227–259.
10. KATO, T., Perturbation of the continuous spectrum by trace class operators, *Proc. Japan Acad.* **33**(1957), 260–264.
11. KURODA, S. T., On a theorem of Weyl-von Neumann, *Proc. Japan Acad.* **34** (1958), 11–15.

12. ROSENBLUM, M., Perturbation of the continuous spectrum and unitary equivalence, *Pacific J. Math.* **7**(1957), 997–1010.
13. VOICULESCU, D. V., Some results on norm ideal perturbations of Hilbert space operators, *J. Operator Theory.* **2**(1979), 3–37.
14. VOICULESCU, D. V., On the existence of quasi-central approximate unit relative to normed ideals, Part I, *J. Functional Analysis* **91**(1990), 1–36.

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