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MATRICIAL STRUCTURE AND HOMOTOPY TYPE OF SIMPLE C^*-ALGEBRAS WITH REAL RANK ZERO

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Dedicated to the memory of John W. Bunce

INTRODUCTION

It is well known that the group of unitaries of any unital C^*-algebra $\mathcal{A}$, denoted by $U(\mathcal{A})$, is a deformation retract of the group of invertible elements, denoted by $\mathrm{GL}(\mathcal{A})$. These groups and the space of nontrivial projections (self-adjoint idempotents $\neq 0, 1$) of $\mathcal{A}$, called the Grassmann space and denoted by $\mathcal{P}(\mathcal{A})$, carry important information about the internal structure of $\mathcal{A}$. The purpose of this article is to determine the homotopy groups $\pi_k(U(\mathcal{A}))$, $\pi_k(\mathcal{P}(\mathcal{A}))$ ($k \geqslant 0$), and the group $U(C(S^k, \tilde{\mathcal{A}}))/U_0(C(S^k, \tilde{\mathcal{A}}))$ in terms of $\mathrm{K}_0(\mathcal{A})$ and $\mathrm{K}_1(\mathcal{A})$ in case $\mathcal{A}$ is a non-elementary simple C^*-algebras with real rank zero and stable rank one.

The homotopy groups $\pi_k(U(\mathcal{M}))$ ($k \geqslant 0$), where $\mathcal{M}$ is a von Neumann algebra without discrete summands, have been completely determined:

$$\pi_k(U(\mathcal{M})) \cong \begin{cases} \mathrm{K}(\mathcal{M}) & k \text{ odd} \\ 0 & k \text{ even}, \end{cases}$$

where by discrete summands we mean the form $\mathcal{M}_d := \bigoplus_{\alpha} M_{n_\alpha}(C(X_\alpha))$ for some hyperstonean spaces X_α's. The determination of $\pi_k(U(\mathcal{M}_d))$ remains open, since it is reduced to determine $\pi_k(U_m)$, where U_m is the unitary group of $m \times m$ full matrix algebras over complex numbers. In turn, if $2m \leqslant k$, $\pi_k(U_m)$ remains unknown in homotopy theory, since the problem is closely related to the fibration $U_{m-1} \longrightarrow U_m \longrightarrow S^{m-1}$. The reader is referred to the combination of [1, 8, 9, 14, 34, 35] and the book [23] for the details.

If $U_\infty(\mathcal{A})$ denotes the unitary group of the C^*-algebra obtained by joining an identity to the stabilization $\mathcal{A} \otimes \mathcal{K}$, where $\mathcal{A}$ is any unital C^*-algebra and $\mathcal{K}$ is the algebra of all compact operators on a separable Hilbert space, then it follows from the Bott periodicity of K-theory (see [23, III, 1.11; 7.7], for example) that

$$\pi_k(U_\infty(\mathcal{A})) \cong \begin{cases} K_0(\mathcal{A}) & k \text{ odd} \\ K_1(\mathcal{A}) & k \text{ even.} \end{cases}$$

If the groups of unitaries of the $m \times m$ matrix algebra over $\mathcal{A}$ $(m \geqslant 1)$, denoted by $U_m(\mathcal{A})$, are concerned, the homotopy groups $\pi_k(U_m(\mathcal{A}))$ change with respect to the matrix size m in general, even in case $\mathcal{A}$ is a commutative C^*-algebra. A well known example is the algebra $C(\mathbb{S}^3)$ consisting of all complex-valued continuous functions on the 3-sphere ([5, 8.1.2(c)]). However, there are some, actually many, classes of nonstable C^*-algebras for which $\pi_k(U(\mathcal{A}))$ $(k \geqslant 0)$ are stable invariants; i.e., $\pi_k(U(\mathcal{A})) = \pi_k(U_m(\mathcal{A}))$ for all $m \geqslant 1$ and $k \geqslant 0$. M. A. Rieffel proved [31] that if $\mathcal{A}_\theta$ is a non-commutative irrational torus, then for all $m \geqslant 1$

$$\pi_k(U_m(\mathcal{A}_\theta)) \cong \begin{cases} K_0(\mathcal{A}_\theta) & k \text{ odd} \\ K_1(\mathcal{A}_\theta) & k \text{ even.} \end{cases}$$

Among his other relevant results in [39, 2.12; 40, 3.7], K. Thomsen proved, roughly speaking, that if $\mathcal{A}$ is any C^*-algebra, and if $\mathcal{B}$ is either an infinite dimensional simple AF-algebra or a Cuntz algebra $\mathcal{O}_n$, then for $m \geqslant 0$

$$\pi_{2m+1}(\mathcal{U}(\mathcal{A} \otimes \mathcal{B})) \cong K_0(\mathcal{A} \otimes \mathcal{B}) \text{ and } \pi_{2m}(\mathcal{U}(\mathcal{A} \otimes \mathcal{B})) \cong K_1(\mathcal{A} \otimes \mathcal{B}),$$

where $\mathcal{U}(\cdot)$ denotes the group of quasi-unitary elements of a C^*-algebra (unital or not). The author proved [46] that if $\mathcal{A}$ is a purely infinite simple C^*-algebra (may not be unital) and p is any nonzero projection of $\mathcal{A}$, then for all $m \geqslant 1$

$$\pi_k(U_m(\tilde{\mathcal{A}})) \cong \pi_k(U_m(p\mathcal{A}p)) \cong \begin{cases} K_0(\mathcal{A}) & k \text{ odd} \\ K_1(\mathcal{A}) & k \text{ even,} \end{cases}$$

$$\pi_k(\mathcal{P}(\mathcal{A})) \cong \pi_k(\mathcal{P}(p\mathcal{A}p)) \cong \begin{cases} K_1(\mathcal{A}) & k \text{ odd} \\ K_0(\mathcal{A}) & k \text{ even.} \end{cases}$$

where $\tilde{\mathcal{A}}$ denotes the unital C^*-algebra obtained by joining an identity to $\mathcal{A}$ in case $\mathcal{A}$ is non-unital, and denotes $\mathcal{A}$ itself in case $\mathcal{A}$ is unital. To determine $\pi_k(U(\tilde{\mathcal{A}}))$, one of the key points was our previous result that any purely infinite simple C^*-algebra has real rank zero (i.e., the set of self-adjoint invertible elements is norm dense in the set of all self-adjoint elements; or equivalently, the set of self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements [45; 44, part I; 13]). Then a weak (Serre) fibration

$$\varphi_p : U(\tilde{\mathcal{A}}) \longrightarrow \mathcal{G}_p \text{ defined by } \varphi_p(u) = upu^*$$

induces a long exact sequence of homotopy groups, from which $\pi_k(\mathcal{P}(\mathcal{A}))$ were determined.

In this article, by different techniques we prove (Theorem II in Section 2) that if $\mathcal{A}$ is a non-elementary simple C^*-algebras with real rank zero and stable rank one (or equivalently, $\mathcal{A}$ has cancellation), and if p is any non-zero projection of $\mathcal{A}$, then for all $m \geqslant 1$

$$\pi_k(U_m(\tilde{\mathcal{A}})) \cong \pi_k(U_m(p\mathcal{A}p)) \cong \begin{cases} \mathrm{K}_0(\mathcal{A}) & k \text{ odd} \\ \mathrm{K}_1(\mathcal{A}) & k \text{ even}, \end{cases}$$

$$\pi_k(\mathcal{P}(\mathcal{A})) \cong \pi_k(\mathcal{P}(p\mathcal{A}p)) \cong \begin{cases} \mathrm{K}_1(\mathcal{A}) & k \text{ odd} \\ \mathrm{K}_0(\mathcal{A}) & k \text{ nonzero even}, \end{cases}$$

$$\pi_0(\mathcal{P}(\mathcal{A})) \cong D(\mathcal{A}) \setminus \{[0],[1]\}, \quad \pi_0(\mathcal{P}(p\mathcal{A}p)) \cong D(p\mathcal{A}p) \setminus \{[0],[p]\},$$

where $D(\mathcal{A})$ denotes the local semigroup consisting of Murray-von Neumann equivalence classes of projections in $\mathcal{A}$. This result also completely determines the homotopy type of the space of nontrivial symmetries (self-adjoint unitaries except $+1$ and -1) of $\mathcal{A}$, denoted by $\mathcal{S}(\mathcal{A})$, since $\mathcal{S}(\mathcal{A})$ is homeomorphic to $\mathcal{P}(\mathcal{A})$ via the map $p \mapsto 2p-1$. Combined with the classic theorem of Whitehead in homotopy theory, Theorem II leads the following isomorphism

$$U(C(X,\tilde{\mathcal{A}}))/U_0(C(X,\tilde{\mathcal{A}})) \cong \mathrm{K}_1(C(X,\mathcal{A})).$$

It then follows from Theorem II and the Bott periodicity that

$$U(C(\mathbf{S}^k,\tilde{\mathcal{A}}))/U_0(C(\mathbf{S}^k,\tilde{\mathcal{A}})) \cong \begin{cases} \mathrm{K}_0(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}) & k \text{ odd} \\ \mathrm{K}_1(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}) & k \text{ nonzero even}. \end{cases}$$

The same conclusion holds also in cases $\mathcal{A}$ is a purely infinite simple C^*-algebra or an irrational non-commutative torus considered in [31]. Here, we remind the reader that there is no known example of simple C^*-algebras of real rank zero which neither is purely infinite nor has stable rank one. The combination of the results in this article and in [46] may have given a complete account of information about $\pi_k(U(\tilde{\mathcal{A}}))$ and $\pi_k(\mathcal{P}(\mathcal{A}))$ for all simple C^*-algebras of real rank zero.

To prove our result, we will first consider *halving projections* in any simple C^*--algebras of real rank zero (Theorem I in Section 1), which is analogous to the well known fact that each projection in a factor can be halved (so can a projection in many other von Neumann algebras [24, Chapter 6]). The arguments involved in halving a projection in certain von Neumann algebras heavily rely on some special features of von Neumann algebras, for example, the operations '$\vee$, $\wedge$' and the powerful comparison theorem as well as traces. These, unfortunately, are not available in most

C^*-algebras. Despite the technical difficulties, we prove that each projection in any simple C^*-algebra $\mathcal{A}$ of real rank zero can be 'approximately halved', and a projection of the multiplier algebra $M(\mathcal{A})$ not in $\mathcal{A}$ can be exactly halved in case $\mathcal{A}$ is non-unital. Consequently, a non-unital simple C^*-algebra $\mathcal{A}$ of real rank zero is *divisible* in the terminology of M. Rieffel [31]; more precisely,

$$\mathcal{A} \cong \mathsf{M}_{2^n}(\mathcal{A}_n) \quad \text{for any } n \geqslant 1.$$

On the other hand, a simple unital C^*-algebra of real rank zero is *approximately divisible* in the sense that $\mathcal{A}$ is *-isomorphic to a C^*-algebra $\mathcal{D}_n$ such that

$$\mathsf{M}_{2^n}(\mathcal{A}_n) \subseteq \mathcal{D}_n \subseteq \mathsf{M}_{2^n+1}(\mathcal{A}_n).$$

The above matrix algebras are over a hereditary C^*-subalgebra $\mathcal{A}_n$ of $\mathcal{A}$. This discovery exposes a matricial structure of simple C^*-algebras of real rank zero, which combined with the technique in [34] and the Bott periodicity leads us to a complete determination of the homotopy groups $\pi_n(U(\tilde{\mathcal{A}}))$ and $\pi_n(\mathcal{P}(\mathcal{A}))$ $(n \geqslant 0)$ when $\mathcal{A}$ has cancellation.

A few words left for the class of simple C^*-algebras of real rank zero. This class of C^*-algebras has been intensively studied from various angles recently. All purely infinite simple C^*-algebras have real rank zero. Besides all type III factors and the Calkin algebra, the Cuntz algebras $\mathcal{O}_n$ $(2 \leqslant n \leqslant \infty)$ and certain Cuntz--Krieger algebras $\mathcal{O}_A$'s are purely infinite simple, so are all generalized Calkin algebras associated with purely infinite simple C^*-algebras [45; 44; 43, 3.3]. In addition to all type II_1 factors, many interesting simple C^*-algebras have real rank zero and topological stable rank one. The reader is referred to Section 2, (2.8) of this article for a list of examples of such C^*-algebras, and to [13; 15; 20; 3; 4; 18; 22; 28; 41–46] for more information.

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1. HALVING PROJECTION

The notation '$p_1 \sim p_2$' is reserved, as usual, for the Murray-von Neumann equivalence of projections p_1 and p_2 in a C^*-algebra $\mathcal{A}$; i.e, $p_1 \sim p_2$ iff there exists a partial isometry $v \in \mathcal{A}$ such that $vv^* = p_1$ and $v^*v = p_2$. $[p]$ stands for the equivalence class of projections of $\mathcal{A}$ containing p, and $D(\mathcal{A})$ denotes the set of equivalence classes of projections of $\mathcal{A}$, which is a local semi-group as mentioned earlier. Two elements $[p] < [q]$ iff p is equivalent to a proper subprojection of q, while $[p] = m[q]$ iff p is the sum of m mutually orthogonal subprojections equivalent to q.

A C^*-algebra $\mathcal{A}$ is said to be non-elementary if $\mathcal{A}$ is neither a matrix algebra over complex numbers nor the algebra $\mathcal{K}$ of all compact operators on an infinite dimensional separable Hilbert space. In case $\mathcal{A}$ is a σ-unital simple C^*-algebra of real rank zero, it follows immediately from [12, 2.8] that $\mathcal{A}$ is non-elementary iff $\mathcal{A}$ has no minimal nonzero projection.

1.1. THEOREM I. *Suppose that $\mathcal{A}$ is a non-elementary simple C^*-algebra of real rank zero.*

(i) (Approximately Halving Projections) *If p is a projection of $\mathcal{A}$, then for any integer $n \geqslant 1$ and any nonzero projection r of $\mathcal{A}$ there exist two subprojections p_n, q_n of p such that*

$$[p] = 2^n[p_n] + [q_n] \text{ with } [q_n] < [r] \text{ and } [q_n] < [p_n].$$

In other words, $p = (p'_1 \oplus p'_2 \oplus \cdots \oplus p'_{2^n}) \oplus q_n$, where $p'_1, \ldots, p'_{2^n}$ are mutually orthogonal, equivalent subprojections of p and q_n is a projection equivalent to both a subprojection of r and a subprojection of p'_1.

(ii) (Exactly Halving Projections) *If, in addition, $\mathcal{A}$ is σ-unital but non-unital, and if p is a projection of $M(\mathcal{A})$ not in $\mathcal{A}$, then for any integer $n \geqslant 1$ there exists a subprojection p_n of p such that*

$$[p] = 2^n[p_n] \quad \text{where } p_n \in M(\mathcal{A}) \setminus \mathcal{A}.$$

In other words, there are mutually orthogonal subprojections $p'_1, \ldots, p'_{2^n}$ of p in $M(\mathcal{A})$ not in $\mathcal{A}$ such that $p_i \sim p_n$ ($\forall\, 1 \leqslant i \leqslant 2^n$) and $p = p'_1 \oplus p'_2 \oplus \cdots \oplus p'_{2^n}$.

1.2. Before giving the proof, we first take a look at the structure of simple C^*-algebras of real rank zero and their hereditary C^*-subalgebras. If $\mathcal{B}$ is any σ-unital hereditary C^*-subalgebra of $\mathcal{A}$, then $\mathcal{B} = (x\mathcal{A}x)^-$ for some positive element x of $\mathcal{A}$. The unit of $\mathcal{B}$ is a projection r in the Banach space double dual $\mathcal{A}^{**}$ of $\mathcal{A}$ (actually, r is the range projection of x, called the open projection supporting $\mathcal{B}$). Then r is the identity of $M(\mathcal{B})$. It is clear that $\mathcal{B}$ is also simple, since $\mathcal{A}$ is simple. In case $\mathcal{B}$ is not unital (i.e., $r \notin \mathcal{A}$), applying Theorem I (ii) to the identity of $M(\mathcal{B})$, we can find a projection e_n of $\mathcal{M}(\mathcal{B})$ not in $\mathcal{B}$ such that

$$[r] = 2^n[e_n].$$

Immediately, we conclude that $\mathcal{B}$ is '*divisible*' in the sense indicated in the following corollary, which exposes a matricial structure of non-unital simple C^*-algebras of real rank zero:

1.3. COROLLARY. *Suppose that $\mathcal{A}$ is a non-elementary simple C^*-algebra of real rank zero.*

(i) *If $\mathcal{B}$ is a unital hereditary C^*-subalgebra of $\mathcal{A}$ (i.e., $\mathcal{B} = p\mathcal{A}p$ for some projection $p \in \mathcal{A}$), then for any integer $n \geqslant 1$ there exists a projection p_n in $\mathcal{B}$ such that $\mathcal{B}$ is *-isomorphic to a C^*-algebra $\mathcal{D}_n$ with*

$$\mathsf{M}_{2^n}(p_n\mathcal{A}p_n) \subseteq \mathcal{D}_n \subseteq \mathsf{M}_{2^n+1}(p_n\mathcal{A}p_n).$$

(ii) *If $\mathcal{B} = (x\mathcal{A}x)^-$ for some nonzero positive element x of $\mathcal{A}$, and $\mathcal{B}$ is non-unital, then for each $n \geqslant 1$ there exists a hereditary C^*-subalgebra $\mathcal{B}_n$ of $\mathcal{A}$ such that*

$$\mathcal{B} \cong \mathsf{M}_{2^n}(\mathcal{B}_n).$$

(iii) *If, in addition, $\mathcal{A}$ is σ-unital and p is a projection of $M(\mathcal{A})$ not in $\mathcal{A}$ (in particular, p is the identity of $M(\mathcal{A})$), then for any $n \geqslant 1$ there exists a projection q_n of $M(\mathcal{A})$ not in $\mathcal{A}$ such that*

$$p\mathcal{A}p \cong \mathsf{M}_{2^n}(q_n\mathcal{A}q_n), \quad \text{and hence } pM(\mathcal{A})p \cong \mathsf{M}_{2^n}(M(q_n\mathcal{A}q_n)).$$

We now turn to the proof of Theorem I, which is based on a previous result of the author as follows. The Riesz decomposition property in [43 or 42] is the key ingredient behind the lemma.

1.4. LEMMA ([44, Part III, Lemma 1.1]). *If $\mathcal{A}$ is a C^*-algebra of real rank zero and if p is a full projection of $\mathcal{A}$ (i.e., the closed ideal generated by p is $\mathcal{A}$), then for any projection q of $\mathcal{A}$ there exist subprojections $p_1, p_2, \ldots, p_n$ of p such that*

$$p_ip_j = 0 \quad \text{if } i \neq j,$$

$$[p_1] \geqslant [p_2] \geqslant \cdots \geqslant [p_n],$$

and

$$[q] = [p_1] + [p_2] + \cdots + [p_n].$$

1.5. THE PROOF FOR THEOREM I(i). We first deal with the case $n = 1$.

It follows from Lemma 1.4 that there exist partial isometries $v_1, v_2, \ldots, v_n$ in $\mathcal{A}$ such that

$$r > v_1^*v_1 \geqslant v_2^*v_2 \geqslant \cdots \geqslant v_n^*v_n,$$

and

$$p = v_1v_1^* \oplus v_2v_2^* \oplus \cdots \oplus v_nv_n^*.$$

We can assume that $n = 2m$ for some $m \geqslant 1$. Otherwise, add $v_{n+1} = 0$. Clearly, $v_{m+1}v_m^*, v_{m+2}v_{m-1}^*, \ldots, v_{2m-1}v_2^*, v_{2m}v_1^*$ are partial isometries of $\mathcal{A}$ with mutually orthogonal initial projections and mutually orthogonal final projections, since

$$(v_{m+i}v_{m-i+1}^*)(v_{m+i}v_{m-i+1}^*)^* = v_{m+i}v_{m+i}^*$$

and

$$(v_{m+i}v_{m-i+1}^*)^*(v_{m+i}v_{m-i+1}^*) \leqslant v_{m-i+1}v_{m-i+1}^* \quad (1 \leqslant i \leqslant m).$$

Set $x_1 = v_{2m}v_1^* + v_{2m-1}v_2^* + \cdots + v_{m+1}v_m^*$. Then it is easy to check that x_1 is a partial isometry of $\mathcal{A}$ such that

$$x_1x_1^* = v_{m+1}v_{m+1}^* \oplus v_{m+2}v_{m+2}^* \oplus \cdots \oplus v_{2m-1}v_{2m-1}^* \oplus v_{2m}v_{2m}^*,$$
$$x_1^*x_1 = v_1(v_{2m}^*v_{2m})v_1^* \oplus v_2(v_{2m-1}^*v_{2m-1})v_2^* \oplus \cdots \oplus v_m(v_{m+1}^*v_{m+1})v_m^*,$$
$$(x_1x_1^*)(x_1^*x_1) = 0.$$

Set

$$w_i = v_i(v_i^*v_i - v_{2m-i+1}^*v_{2m-i+1}) \quad \text{for } 1 \leqslant i \leqslant m.$$

Then

$$w_iw_i^* = v_iv_i^* - v_i(v_{2m-i+1}^*v_{2m-i+1})v_i^*$$

and

$$w_i^*w_i = v_i^*v_i - v_{2m-i+1}^*v_{2m-i+1}.$$

It is clear that

$$r - v_{2m}^*v_{2m} \geqslant w_1^*w_1 \geqslant w_2^*w_2 \geqslant \cdots \geqslant w_m^*w_m,$$

and

$$p - (x_1x_1^* \oplus x_1^*x_1) = w_1w_1^* \oplus w_2w_2^* \oplus \cdots \oplus w_mw_m^*.$$

We can now repeat the same arguments finitely many times to get partial isometries $x_2, x_3, \ldots, x_l$ such that

$$x_ix_i^* \oplus x_i^*x_i \leqslant p - \sum_{j=1}^{i-1}(x_jx_j^* \oplus x_j^*x_j)$$

and

$$x_l^*x_l \leqslant r - \sum_{j=1}^{l-1}(v_1v_1^*)(x_j^*x_j)(v_1v_1^*).$$

Set

$$p_1 = \sum_{j=1}^{l} x_j^* x_j,$$

$$p_2 = \sum_{j=1}^{l} x_j x_j^*,$$

$$s = r - \sum_{j=1}^{l} (v_1 v_1^*)(x_j^* x_j)(v_1 v_1^*).$$

Then

$$p = p_1 \oplus p_2 \oplus s, \quad s < r \quad \text{and} \quad p_1 \sim p_2.$$

If $s = r$, by [43, 3.2] we can get two mutually orthogonal equivalent subprojections r_1 and r_2 of r, and then replace p_1 and p_2 by $p_1 \oplus r_1$ and $p_2 \oplus r_2$, respectively. Hence, we can always assume that $s < r$. Now repeating the whole process to the projections p_1 and s, we can write

$$s = s_1 \oplus s_2 \oplus s' \quad \text{where } s_1 \sim s_2, \ s' \sim s'' < p_1.$$

It follows that

$$p = (p_1 \oplus s_1) \oplus (p_2 \oplus s_2) \oplus s';$$

i.e., $[p] = 2[p_1 \oplus s_1] + [s'']$, as wanted.

We now deal with arbitrary positive integer n. Applying [43, 3.2], we get a nonzero subprojection r_n of r such that

$$(2^{n+1} - 1)[r_n] \leqslant [r].$$

By recursively repeating the arguments for $n = 1$, we write

$$[p] = 2[p_1] + [s_1] \quad \text{with } [s_1] < [r_n],$$

$$[p_1] = 2[p_2] + [s_2] \quad \text{with } [s_2] < [r_n],$$

$$\ldots\ldots,$$

$$[p_{n-1}] = 2[p_n] + [s_n] \quad \text{with } [s_n] < [r_n].$$

It follows that

$$[p] = 2^n [p_n] + \sum_{i=1}^{n} 2^{i-1}[s_i].$$

It is clear that

$$\sum_{i=1}^{n} 2^{i-1}[s_i] < (2^{n+1} - 1)[r_n] \leqslant [r].$$

Choose a subprojection s of r such that

$$[s] = \sum_{i=1}^{n} 2^{i-1}[s_i].$$

Repeating the above process to $[s]$ and $[p_n]$, we can write

$$[s] = 2^n[s'_n] + [q_n] \quad \text{where } [q_n] < [p_n].$$

Then

$$[p] = 2^n[p_n \oplus s'_n] + [q_n], \quad \text{where clearly } [q_n] < [r],\ [q_n] < [p_n].$$

The proof has been completed.

1.6. THE PROOF FOR THEOREM I (ii). Since $p\mathcal{A}p$ is also σ-unital but non-unital, we write [41, 1.2]

$$p = \sum_{i=1}^{\infty} \oplus p_i \quad \text{(the sum converges strictly)},$$

where $\{p_i\}$ is a sequence of mutually orthogonal, nonzero projections of $p\mathcal{A}p$. Applying Theorem I (i) to p_1 and p_2, we can write

$$p_1 = r_1 \oplus s_1 \oplus q_1, \quad \text{where } r_1 \sim s_1,\ q_1 \sim q'_1 < p_2.$$

Let v_1 be a partial isometry in $(r_1 \oplus s_1)\mathcal{A}(r_1 \oplus s_1)$ and w_1 be a partial isometry in $(q_1 \oplus q'_1)\mathcal{A}(q_1 \oplus q'_1)$ such that

$$v_1v_1^* = r_1,\ v_1^*v_1 = s_1,\ w_1w_1^* = q_1,\ w_1^*w_1 = q'_1.$$

Applying Theorem I (i) again to $p_2 - q'_1$ and p_3, we can write

$$p_2 - q'_1 = r_2 \oplus s_2 \oplus q_2, \quad \text{where } r_2 \sim s_2,\ q_2 \sim q'_2 < p_3.$$

Then there are a partial isometry v_2 in $(r_2 \oplus s_2)\mathcal{A}(r_2 \oplus s_2)$ and another partial isometry w_2 in $(q_2 \oplus q'_2)\mathcal{A}(q_2 \oplus q'_2)$ such that

$$v_2v_2^* = r_2,\ v_2^*v_2 = s_2,\ w_2w_2^* = q_2,\ w_2^*w_2 = q'_2.$$

Repeating in this way, we find four sequence of mutually orthogonal projections, $\{r_i\}$, $\{s_i\}$, $\{q_i\}$, $\{q'_i\}$, two sequences of partial isometries $\{v_i\}$ and $\{w_i\}$ of $\mathcal{A}$ such that

(i) $r_i \sim s_i,\ \ q_i \sim q'_i < p_{i+1}$;

(ii) $r_i \oplus s_i \oplus q_i = p_i - q'_{i-1}$, where $q'_0 := 0$;

(iii) $v_i \in (r_i \oplus s_i)\mathcal{A}(r_i \oplus s_i),\ \ w_i \in (q_i \oplus q'_i)\mathcal{A}(q_i \oplus q'_i)$;

(iv) $v_i v_i^* = r_i, \quad v_i^* v_i = s_i, \quad w_i w_i^* = q_i, \quad w_i^* w_i = q_i'$.

Set

$$p_1 = \sum_{i=1}^{\infty} (r_i \oplus q_i),$$
$$p_2 = \sum_{i=1}^{\infty} (s_i \oplus q_i'),$$
$$v = \sum_{i=1}^{\infty} (v_i + w_i).$$

It is a routine to show, by construction, that p_1 and p_2 are mutually orthogonal projections of $M(\mathcal{A})$ such that

$$p_1 \oplus p_2 = p.$$

It is again a routine to show that v is a partial isometry of $M(\mathcal{A})$ such that

$$vv^* = p_1 \quad \text{and} \quad v^* v = p_2.$$

Then we can proceed in this way as many times as we want to conclude $[p] = 2^n [p_n]$ for some subprojection p_n of p for $n \geqslant 1$.

2. HOMOTOPY GROUPS $\pi_k(U_m(\tilde{\mathcal{A}}))$ AND $\pi_k(\mathcal{P}(\mathcal{A}))$

A unital C^*-algebra $\mathcal{A}$ has cancellation iff $[p - p'] = [q - q']$ whenever two projections p, q of $\mathsf{M}_m(\mathcal{A})$ and subprojections $p' < p$ and $q' < q$ such that $[p] = [q]$ and $[p'] = [q']$ for all $m \geqslant 1$ [5, 6.4]. If $\mathcal{A}$ is non-unital, we say that $\mathcal{A}$ has cancellation if $p\mathcal{A}p$ does for every projection p of $\mathcal{A}$. We will denote the path component of $U(\tilde{\mathcal{A}})$ containing the identity by $U_0(\tilde{\mathcal{A}})$.

2.1. PROPOSITION ([cf. 42, 3.4]). *Assume that $\mathcal{A}$ has real rank zero and cancellation (neither necessarily simple, nor necessarily unital). Then* $\pi_0(\mathcal{P}(\mathcal{A})) = D(\mathcal{A}) \setminus \{[0], [1]\}$. *In other words, two projections p and q of $\mathcal{A}$ are Murray-von Neumann equivalent iff p and q are in the same path component of $\mathcal{P}(\mathcal{A})$ (equivalently, there exists a unitary $v \in U_0(\tilde{\mathcal{A}})$ such that $vpv^* = q$.*

Proof. If $\mathcal{A}$ is unital, the author proved [42, 3.4] that $p \sim q$ iff p and q are homotopic. The same proof also works for non-unital C^*-algebra with real rank zero and cancellation.

2.2. COROLLARY ([cf. 20, 3.2]). *If u is any unitary of $\tilde{\mathcal{A}}$ and p is any projection of $M(\mathcal{A})$, then there is a unitary $v \in U_0(\tilde{\mathcal{A}})$ such that with respect to the decomposition*

$p + (1 - p) = 1$

$$vu = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad \text{where } u_1 \in U(p\mathcal{A}p) \text{ and } u_2 \in U((1-p)\mathcal{A}(1-p)).$$

If, in addition, $\mathcal{A}$ has cancellation, this together with the Riesz decomposition of $K_0(\mathcal{A})$ *proved in* [43] *warrants the Riesz decomposition property of the ordered group* $K_*(\mathcal{A}) = K_0(\mathcal{A}) \oplus K_1(\mathcal{A})$ *defined in* [20, 3.2] *for unital case, there an alternative proof is given.*

Proof. This is actually an immediate consequence of Proposition (2.1), or [42, 3.4], more precisely. Since upu^* is homotopic to p, there exists a unitary $v \in U_0(\tilde{\mathcal{A}})$ such that $vupu^*v^* = p$. It then follows that vu has the above matrix form.

2.3. Corollary. *Suppose that $\mathcal{A}$ is a C^*-algebra with real rank zero and cancellation.*

(i) *If p is a projection of $\mathcal{A}$ with $[p] \leqslant [1-p]$, then every unitary $u \in U(\tilde{\mathcal{A}})$ is homotopic to a unitary with the form*

$$\begin{pmatrix} p & 0 \\ 0 & u_0 \end{pmatrix}, \quad \text{where } u_0 \in U((1-p)\tilde{\mathcal{A}}(1-p)).$$

(The reader is referred to the proof of [46, 2.7], *if this is not clear.)*

(ii) *If u is a unitary of $\tilde{\mathcal{A}}$ and $p_1, p_2, \ldots, p_n$ are projections of $M(\mathcal{A})$ such that*

$$p_1 \oplus p_2 \oplus \cdots \oplus p_n = 1,$$

then u is path connected to a diagonal unitary $u_1 \oplus u_2 \oplus \cdots \oplus u_n$ with respect to the above decomposition of the identity.

(iii) *If, in addition, there exists $1 \leqslant i \leqslant n$ (we may assume $i = n$) such that $[p_j] \leqslant [p_n]$ for all $1 \leqslant j < n$, then u is connected to a unitary of the form $p_1 \oplus p_2 \oplus \oplus \cdots \oplus p_{n-1} \oplus u_0$ for some $u_0 \in U(p_n\tilde{\mathcal{A}}p_n)$. (This is immediate by applying Proposition* (2.1) *or* [42, 3.4] *recursively $n-1$ times.)*

2.4. Theorem II. *Suppose that $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and cancellation. If p is any nonzero projection of $\mathcal{A}$, then for all $m \geqslant 0$*

$$\pi_{2m}(U(\tilde{\mathcal{A}})) \cong \pi_{2m}(U(p\mathcal{A}p)) \cong K_1(\mathcal{A}),$$
$$\pi_{2m+1}(U(\tilde{\mathcal{A}})) \cong \pi_{2m+1}(U(p\mathcal{A}p)) \cong K_0(\mathcal{A}),$$
$$\pi_{2m+1}(\mathcal{P}(\mathcal{A})) \cong \pi_{2m+1}(\mathcal{P}(p\mathcal{A}p)) \cong K_1(\mathcal{A}),$$
$$\pi_{2m+2}(\mathcal{P}(\mathcal{A})) \cong \pi_{2m+2}(\mathcal{P}(p\mathcal{A}p)) \cong K_0(\mathcal{A})$$

and

$$\pi_0(\mathcal{P}(\mathcal{A})) \cong D(\mathcal{A}) \setminus \{[0], [1]\}.$$

The conclusion is also true if $\tilde{A}$ is replaced by $\mathsf{M}_n(\tilde{A})$ and p is replaced by a projection in $\mathsf{M}_n(\mathcal{A})$ for any $n \geqslant 1$ (since K_*-groups are stable invariants).

We will spend the whole Section 3 to prove Theorem II. Here we have the following remarks, corollaries and some specific examples in order:

2.5. REMARKS. (i) Here $\pi_k(\mathcal{P}(\mathcal{A}))$, $\pi_k(\mathcal{P}(p\mathcal{A}p))$, $\pi_k(U(\tilde{\mathcal{A}}))$ and $\pi_k(U(p\mathcal{A}p))$ are independent of the choices of the base points, which are omitted in the statements and proofs to ease the notations. The reader may take the base point of $\mathcal{P}(\mathcal{A})$ to be any fixed non-trivial projection q, the base point of $\mathcal{P}(p\mathcal{A}p)$ to be any nonzero subprojection of p, the base points of $U(\tilde{\mathcal{A}})$ to be the identity of $\mathcal{A}$, and the base point of $U(p\mathcal{A}p)$ to be the (local) identity p of $p\mathcal{A}p$. In fact, since any two path components of $U(\tilde{\mathcal{A}})$ (or $U(p\mathcal{A}p)$) are homeomorphic topological spaces, $\pi_k(U(\tilde{\mathcal{A}}))$ (or $\pi_k(U(p\mathcal{A}p))$) are independent of the choice of the base point. Clearly, the orbits of $\mathcal{P}(\mathcal{A})$ under the action of $U(\tilde{\mathcal{A}})$ defined by $u \mapsto uqu^*$ are disjoint subspaces (more precisely, the path components). For any fixed non-trivial projection q in $\mathcal{P}(\mathcal{A})$, it follows from Proposition (2.1) that the orbit of q is the path component $\mathcal{G}_q$ of $\mathcal{P}(\mathcal{A})$ containing q. Hence, the homotopy groups $\pi_k(\mathcal{P}(\mathcal{A}), r)$ $(k \geqslant 1)$ are independent of the choice of the base point $r \in \mathcal{G}_q$. Whenever such a base point $r \in \mathcal{G}_p$ is fixed, we have

$$\pi_k(\mathcal{P}(\mathcal{A}), r) = \pi_k(\mathcal{G}_q, r) \quad \text{for all } k \geqslant 1.$$

Furthermore, it turns out that

$$\pi_k(\mathcal{P}(\mathcal{A}), p) \cong \pi_k(\mathcal{P}(\mathcal{A}), q)$$

for any two nontrivial projections p and q, no matter whether they are in the same path component of $\mathcal{P}(\mathcal{A})$ or not, as the reader will see later from the proof. The same explanation applies to the Grassmann space of the corner C^*-subalgebra $p\mathcal{A}p$.

(ii) If $\mathcal{A}$ is any unital C^*-algebra and $\mathcal{S}(\mathcal{A})$ is the space of non-trivial symmetries of $\mathcal{A}$ (i.e., self-adjoint unitaries neither $+1$ nor -1), then the map

$$p \mapsto 2p - 1$$

is clearly a homeomorphism between $\mathcal{P}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A})$. Clearly, $\mathcal{S}(\mathcal{A})$ is the disjoint union of subsets (its path components) of $U_0(\mathcal{A})$. We will see in (3.11) that

$$\sigma_p : U_0(\mathcal{A}) \longrightarrow \mathcal{S}(p) \quad \text{defined by } u \mapsto 2upu^* - 1$$

is a weak fibration, where $\mathcal{S}(p)$ is the path component of $\mathcal{S}(\mathcal{A})$ containing $2p - 1$.

In case $\mathcal{A}$ is simple with real rank zero, the combination of Theorem II and Theorem B of [46] implies the following corollary:

2.6. COROLLARY. *If $\mathcal{A}$ is either a purely infinite simple C^*-algebra or a non--elementary simple C^*-algebra with real rank zero and cancellation, then*

$$\pi_0(\mathcal{S}(\tilde{\mathcal{A}})) \cong D(\mathcal{A}) \setminus \{[0],\ [1]\},$$

$$\pi_{2m+1}(\mathcal{S}(\tilde{\mathcal{A}})) \cong \mathrm{K}_1(\mathcal{A}) \ \text{ and } \ \pi_{2m+2}(\mathcal{S}(\tilde{\mathcal{A}})) \cong \mathrm{K}_0(\mathcal{A}) \quad (\forall m \geqslant 0).$$

If the base points of both $\pi_k(U(\tilde{\mathcal{A}}))$ and $\pi_k(\mathcal{S}(\tilde{\mathcal{A}}))$ are taken to be the identity, then we have the following immediate comparison:

$$\pi_{2m+1}(U_0(\tilde{\mathcal{A}})) \cong \mathrm{K}_0(\mathcal{A}), \quad \pi_{2m+1}(\mathcal{S}(p)) \cong \mathrm{K}_1(\mathcal{A}),$$

$$\pi_{2m+2}(U_0(\tilde{\mathcal{A}})) \cong \mathrm{K}_1(\mathcal{A}), \quad \pi_{2m+2}(\mathcal{S}(p)) \cong \mathrm{K}_0(\mathcal{A}),$$

where $m \geqslant 0$ and p is any projection in $\mathcal{P}(\mathcal{A})$. This comparison provides a new interpretation for $\mathrm{K}_0(\mathcal{A})$ and $\mathrm{K}_1(\mathcal{A})$ as the homotopy groups of $U_0(\mathcal{A})$ and its subspace $\mathcal{S}(p)$.

As a particular case, if $\mathcal{A} = \mathcal{M}$ is a type II_1 factor, H. Schröder determined $\pi_k(U(\mathcal{M}))$ in [34]. Our Theorem II offers some new information about the homotopy type of the Grassmann space and hence the space of symmetries, which are written down as the following corollary:

2.7. COROLLARY. *If $\mathcal{M}$ is a type II_1 factor, then*

$$\pi_{2m+1}(\mathcal{P}(\mathcal{M})) \cong \pi_{2m+1}(\mathcal{S}(\mathcal{M})) = 0$$

and

$$\pi_{2m+2}(\mathcal{P}(\mathcal{M})) \cong \pi_{2m+2}(\mathcal{S}(\mathcal{M})) \cong \mathrm{K}_0(\mathcal{M}).$$

2.8. EXAMPLES. Using their K-groups, one can read the homotopy type of $U(\tilde{\mathcal{A}})$ and $\mathcal{P}(\mathcal{A})$ if $\mathcal{A}$ is one of the following interesting C^*-algebras with cancellation:

(1) all simple AF algebras (in particular UHF algebra) [19];

(2) all Bunce-Deddens algebras [10, 11];

(3) irrational rotation C^*-algebras [15, 20];

(4) certain inductive limits of matrix algebras over algebras of continuous functions on a circle or on an interval, among many others ([3, 22, 4]).

3. A PROOF OF THEOREM II

We will spend this whole section to prove Theorem II. Main ingredients in determining $\pi_n(U(\tilde{\mathcal{A}}))$ are the *almost divisibility* in Theorem I, the general homotopy exact sequence associated with a Serre fibration [36; Chapter 7, 2.10] or [37; Chapter 4, 4.7], Bott periodicity [23; III. 1.11 and 7.7] and the idea of H. Schröder in [34] for determining the homotopy groups of type II_1 factors [34], while the ideas of [9] and [46, Section 3] apply to determine $\pi_k(\mathcal{P}(\mathcal{A}))$. For the convenience of the reader, we will spend a few lines to outline the main frame of [34] as follows.

3.1. Several known facts. If $\mathcal{A}$ is any unital C^*-algebra (not necessarily has real rank zero), we denote by $\mathrm{GL}(\mathcal{A})$ the group of invertible elements. It is well known that $U(\mathcal{A})$ is a deformation retract of $\mathrm{GL}(\mathcal{A})$, and hence they have the same homotopy groups.

As usual, the unitary group $U_{m-1}(\mathcal{A})$ of $\mathbf{M}_{m-1}(\mathcal{A})$ is naturally embedded in the unitary group $U_m(\mathcal{A})$ of $\mathbf{M}_m(\mathcal{A})$ via the map

$$u \longrightarrow \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, all $U_m(\mathcal{A})$ is embedded in the unitary group $U_\infty(\mathcal{A})$ of the C^*-algebra obtained by joining an identity to $\mathcal{A} \otimes \mathcal{K}$.

Set

$$\mathcal{A}^m = \mathcal{A} \oplus \mathcal{A} \oplus \ldots \oplus \mathcal{A} \quad (m \text{ copies}),$$

$$\mathcal{R}_m(\mathcal{A}) = \left\{ x = (x_i)_{i=1}^m \in \mathcal{A}^m : \sum_{i=1}^m x_i^* x_i \in \mathrm{GL}(\mathcal{A}) \right\},$$

$$\mathcal{S}_m(\mathcal{A}) = \left\{ x = (x_i)_{i=1}^m \in \mathcal{A}^m : \sum_{i=1}^m x_i^* x_i = 1 \right\}.$$

Then several known facts are in order, whose proofs are in [34] (also, a parallel treatment can be found in [30]).

(a) $\mathcal{R}_m(\mathcal{A})$ is an open subset of $\mathcal{A}^m$.

(b) $\mathcal{S}_m(\mathcal{A})$ is a deformation retract of $\mathcal{R}_m(\mathcal{A})$.

(c) $\mathcal{R}_m(\mathcal{A}) = \{(x_i)_{i=1}^m \in \mathcal{A}_m : x_1, x_2, \ldots, x_m$ generate $\mathcal{A}$ as a left ideal $\}$; $\mathcal{R}_m(\mathcal{A})$ is denoted by $\mathrm{Lg}_m(\mathcal{A})$ in [31].

(d) If $\mathcal{R}_m(\mathcal{A})$ is dense in $\mathcal{A}^m$ $(m \geqslant 1)$, then $\mathcal{R}_{m+k+1}(C(\mathbf{S}^k, \mathcal{A}))$ is dense in $C(\mathbf{S}^k, \mathcal{A})^{m+k+1}$, where $C(\mathbf{S}^k, \mathcal{A})$ is the C^*-algebra consisting of all norm-continuous maps from the k-sphere $\mathbf{S}^k$ to $\mathcal{A}$.

(e) If $\mathcal{R}_m(C(\mathbf{S}^k, \mathcal{A}))$ is dense in $C(\mathbf{S}^k, \mathcal{A})^m$, then the homotopy groups $\pi_k(\mathcal{S}_{m+n}(\mathcal{A})) = 0$ for $n \geqslant 1$.

3.2. Proposition. *If $\mathcal{A}$ is a unital C^*-algebra with cancellation (not necessarily with real rank zero), then the mapping*

$$U_m(\mathcal{A}) \xrightarrow{\varphi} \mathcal{S}_m(\mathcal{A})$$

defined by

$$\varphi\left(\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mm} \end{bmatrix}\right) = \left(\begin{bmatrix} 0 & \dots & 0 & x_{1m} \\ \vdots & \ddots & \vdots & x_{2m} \\ \vdots & \vdots & 0 & \vdots \\ 0 & \dots & 0 & x_{mm} \end{bmatrix}\right)$$

induces a Serre fibration with fiber homeomorphic to $U_{m-1}(\mathcal{A})$*, and hence* $\mathcal{S}_m(\mathcal{A})$ *is homeomorphic to the homogeneous space* $U_m(\mathcal{A})/U_{m-1}(\mathcal{A})$ *(the space of right cosets with the quotient topology).*

Proof. H. Schröder's proof for type II_1 factors in [34, 4] works for C^*-algebras with cancellation. The cancellation property of $\mathcal{A}$ implies that each partial isometry v of $\mathsf{M}_m(\mathcal{A})$ $(m \geqslant 1)$ is implemented by a unitary [5, 6.4], i.e., there exists a partial isometry v' of $\mathsf{M}_m(\mathcal{A})$ such that $v + v'$ is a unitary of $\mathsf{M}_m(\mathcal{A})$.

3.3. Homotopy exact sequence. Combining Corollary (2.2) or Corollary (2.3) and the general homotopy theory in [36] or [37], we have the following long exact sequence:

$$\cdots \to \pi_{k+1}(\mathcal{S}_m(\mathcal{A})) \to \pi_k(U_{m-1}(\mathcal{A})) \to \pi_k(U_m(\mathcal{A})) \to \pi_k(\mathcal{S}_m(\mathcal{A})) \to \cdots$$

$$\cdots \to \pi_1(\mathcal{S}_m(\mathcal{A})) \to \pi_0(U_{m-1}(\mathcal{A})) \to \pi_0(U_m(\mathcal{A})) \to 0.$$

The following proposition follows immediately from the above fact 3.1(d), (e), Proposition (3.2) and the above long exact sequence.

3.4. Proposition ([cf. 34]). *If* $\mathcal{A}$ *is a unital* C^**-algebra such that* $\mathrm{tsr}(\mathcal{A}) = 1$ *(again* $\mathcal{A}$ *is not necessarily of real rank zero), then*

$$\pi_k(\mathcal{S}_{k+n}(\mathcal{A})) = 0 \quad \text{for all } n \geqslant 3,$$

and hence

$$\pi_k(U_m(\mathcal{A})) \cong \pi_m(U_{m+1}(\mathcal{A})), \quad m \geqslant k+3.$$

Consequently,

$$\pi_k(U_{k+3}(\mathcal{A})) \cong \pi_k(U_{k+m}(\mathcal{A})) \cong \pi_k(U_\infty(\mathcal{A})) \quad (k \geqslant 0,\ m \geqslant 3).$$

We point out that the above also follow from another parallel approach of M. Rieffel in [31, Section 3] where he considered $\mathrm{GL}_m(\mathcal{A})$ instead.

3.5. Bott periodicity. If $\mathcal{A}$ is any unital C^*-algebra, applying the Bott periodicity of K-theory [23, III, 1.11;7.7], then

$$\pi_{2m}(U_\infty(\mathcal{A})) \cong \pi_0(U_\infty(\mathcal{A})) \cong \mathrm{K}_1(\mathcal{A}),$$

and

$$\pi_{2m+1}(U_\infty(\mathcal{A})) = \pi_1(U_\infty(\mathcal{A})) \cong \mathrm{K}_0(\mathcal{A}).$$

3.6. $K_*(\mathcal{A})$ AND $\mathrm{K}_*(p\mathcal{A}p)$. We now reduce our attention to the particular case that $\mathcal{A}$ has real rank zero and cancellation. We start with proving that for $m \geqslant 1$ and any nonzero projection of $\mathcal{A}$

$$\pi_{2m}(U(\tilde{\mathcal{A}})) \cong \pi_{2m}(U(p\mathcal{A}p)) \cong \mathrm{K}_1(\mathcal{A}),$$

and

$$\pi_{2m+1}(U(\tilde{\mathcal{A}})) \cong \pi_{2m+1}(U(p\mathcal{A}p)) \cong \mathrm{K}_0(\mathcal{A}).$$

To do so, we first notice that $\mathrm{K}_1(\mathcal{A}) = \mathrm{K}_1(p\mathcal{A}p)$ which is true for any simple C^*--algebra; and $\mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_{00}(\mathcal{A})$, since $\mathcal{A}$ has an approximate identity of projections ([5, 5.5.5]). Since $\mathcal{A}$ is simple, it is a routine to check that $\mathrm{K}_{00}(\mathcal{A}) \cong \mathrm{K}_0(p\mathcal{A}p)$, since for every projection q of $\mathcal{A} \otimes \mathcal{K}$ there exists $n \geqslant 1$ such that $[q] \leqslant n[p]$. Hence, $\mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_0(p\mathcal{A}p)$, although $\mathcal{A}$ may not be σ-unital. Applying these facts together with the Bott periodicity to $p\mathcal{A}p$, one has

$$\pi_{2m}(U_\infty(p\mathcal{A}p)) \cong \pi_0(U_\infty(p\mathcal{A}p)) \cong \mathrm{K}_1(p\mathcal{A}p) \cong \mathrm{K}_1(\mathcal{A}),$$

and

$$\pi_{2m+1}(U_\infty(p\mathcal{A}p)) = \pi_1(U_\infty(p\mathcal{A}p)) \cong \mathrm{K}_0(p\mathcal{A}p) \cong \mathrm{K}_0(\mathcal{A}).$$

Clearly, $p\mathcal{A}p \otimes \mathcal{K}$ is a hereditary C^*-subalgebra (actually a corner) of $\mathcal{A} \otimes \mathcal{K}$. To confirm our claim about $\pi_k(U(\tilde{\mathcal{A}}))$ we need to show that for any $k \geqslant 1$ and any non-zero projection p of $\mathcal{A}$

$$\pi_k(U(p\mathcal{A}p)) \cong \pi_k(U_\infty(p\mathcal{A}p),$$

and

$$\pi_k(U(\tilde{\mathcal{A}})) \cong \pi_k(U_\infty(\mathcal{A})) \cong \pi_k(U_\infty(p\mathcal{A}p)).$$

Consequently, $\pi_k(U(\tilde{\mathcal{A}}))$ would be a stable invariant for each $k \geqslant 0$. The key point in our proof is the almost divisibility of $\mathcal{A}$ in Theorem I. The reader will see a comparison between our work and the *divisibility* defined in [31, Section 4] and his result [31, 4.13].

Recall that $\mathcal{A}$ has real rank zero iff $\mathcal{A}$ has the (FS) property [13], while $\mathcal{A}$ has (FS) iff $\mathcal{A}$ has the (HP) property [6, Section 2]. Hence, under the assumption that $RR(\mathcal{A}) = 0$, $\mathcal{A}$ has cancellation iff $\mathrm{tsr}(\mathcal{A}) = 1$, no matter whether $\mathcal{A}$ is unital or not. Here we understand that $\mathrm{tsr}(\mathcal{A}) := \mathrm{tsr}(\tilde{\mathcal{A}})$ in case $\mathcal{A}$ is non-unital.

3.7. PROPOSITION. *Suppose that $\mathcal{A}$ is a C^*-algebra of real rank zero (not necessarily unital). Then $\mathrm{tsr}(\mathcal{A}) = 1$ iff $\mathcal{A}$ has cancellation.*

Proof. Since $\mathcal{A}$ has real rank zero, $\mathcal{A}$ has an approximate identity of projections. Then it is readily checked by definition that $\mathcal{A}$ has cancellation iff $p\mathcal{A}p$ has cancellation for every projection p of $\mathcal{A}$. If $\mathrm{tsr}(\mathcal{A}) = 1$, then $\mathcal{A}$ has cancellation by the same proof as in [5, 6.4]. Conversely, assume that $\mathcal{A}$ has cancellation. If x is any element of $\tilde{\mathcal{A}}$, there exists a projection p such that x is close in norm to an element with the form $pyp + t(1-p)$ for some scalar t. Now the proof of [5, 6.4] applies to $p\mathcal{A}p$. We can find an invertible element with the form $pzp + t(1-p)$ to approximate $pyp + t(1-p)$, and hence approximate x in norm.

3.8. LEMMA. *Suppose that $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and cancellation. If p is any nonzero projection of $\mathcal{A} \otimes \mathcal{K}$, then*

$$\pi_k(U_\infty(\mathcal{A})) \cong \pi_k(U_\infty(p\mathcal{A}p)) \cong \pi_k(U(p\mathcal{A}p)) \quad \forall\, k \geqslant 0.$$

Hence, $\pi_k(U(p\mathcal{A}p))$ is independent of the choice of p.

Proof. Since $p\mathcal{A}p \otimes \mathcal{K}$ is a corner of $\mathcal{A} \otimes \mathcal{K}$, we need only to show the following two statement:

(i) For any $k \geqslant 0$, each norm-continuous map from the k-sphere $\mathbf{S}^k$ to $U_\infty(\mathcal{A})$ whose image contains the identity is homotopic to a norm-continuous map of the form

$$\begin{pmatrix} v(\cdot) & 0 \\ 0 & 1-p \end{pmatrix},$$

where $v(\cdot)$ is a norm-continuous map from $\mathbf{S}^k$ to the unitary group of $p(\mathcal{A}\otimes\mathcal{K})p$, while the identity is contained in the image along the homotopy.

(ii) If a norm-continuous map from $\mathbf{S}^k$ to the subgroup of $U_\infty(\mathcal{A})$ consisting of elements of the form

$$\begin{pmatrix} v & 0 \\ 0 & 1-p \end{pmatrix}$$

is homotopic to the identity in $U_\infty(\mathcal{A})$, then it is homotopic to the identity via a path with the form

$$\begin{pmatrix} v(\cdot,t) & 0 \\ 0 & 1-p \end{pmatrix} \quad (0 \leqslant t \leqslant 1).$$

Of course $p\mathcal{A}p$ also has cancellation and real rank zero. It follows [31, 3.1] that $\mathcal{R}_1(p\mathcal{A}p) = \mathrm{GL}(p\mathcal{A}p)$, and $\mathrm{GL}(p\mathcal{A}p)$ is dense in $p\mathcal{A}p$ for any non-zero projection p. We carry out the proof of (i) and (ii) by the following three steps.

Step 1. We prove that for each $k \geqslant 0$ there is a nonzero subprojection p_k of p such that

$$\pi_k(U(p\mathcal{A}p)) \cong \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k)) \quad \text{for all } m \geqslant 0,$$

where n_k can be any integer greater than $k+3$.

For each $k \geqslant 0$ choose $n_k > 0$ to be any integer such that $2^{n_k} > k+3$. It follows from Corollary (1.3) that pAp is *-isomorphic to a C^*-algebra, say $\mathcal{D}_k$, such that

$$\mathsf{M}_{2^{n_k}}(p_k\mathcal{A}p_k) \subseteq \mathcal{D}_k \subseteq \mathsf{M}_{2^{n_k}+1}(p_k\mathcal{A}p_k) \quad (\hookrightarrow \mathsf{M}_{2^{n_k}+m}(p_k\mathcal{A}p_k) \text{ for all } m \geqslant 1).$$

for some nonzero subprojection p_k of p. Of course $\pi_k(U(\mathcal{D}_k)) \cong \pi_k(U(p\mathcal{A}p))$ for any $k \geqslant 0$.

We claim that

$$\pi_k(U(\mathcal{D}_k)) \cong \pi_k(U_{2^{n_k}}(p_k\mathcal{A}p_k)) \cong \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k)) \quad \text{for all } m \geqslant 1.$$

In fact, first, it follows from Proposition (3.4) that for all $m \geqslant 1$ the map

$$\pi_k(U_{2^{n_k}}(p_k\mathcal{A}p_k)) \xrightarrow{\omega_m} \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k))$$

defined by

$$\omega_m([u(\cdot)]_h) = \left[\begin{pmatrix} u(\cdot) & 0 \\ 0 & \sum_{i=2^{n_k}+1}^{2^{n_k}+m} p_k \otimes e_{ii} \end{pmatrix}\right]_h$$

is an isomorphism, where the braket $[\cdot]_h$ denotes a homotopy class in $\pi_k(\cdot)$. Secondly, the following maps

$$\pi_k(U_{2^{n_k}}(p_k\mathcal{A}p_k)) \xrightarrow{\varphi} \pi_k(U(\mathcal{D}_k)) \xrightarrow{\psi_m} \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k)) \quad (m \geqslant 1)$$

defined by

$$\varphi([u(\cdot)]_h) = \left[\begin{pmatrix} u(\cdot) & 0 \\ 0 & e_k \end{pmatrix}\right]_h$$

and

$$\psi_m([v(\cdot)]_h) = \left[\begin{pmatrix} v(\cdot) & 0 \\ 0 & f_k \oplus \sum_{i=2^{n_k}+2}^{2^{n_k}+m} p_k \otimes e_{ii} \end{pmatrix}\right]_h.$$

are bijections, where e_k equals to the difference between the identity of $\mathsf{M}_{2^{n_k}+1}(p_k\mathcal{A}p_k)$ and the identity of $\mathcal{D}_k$, and f_k equals to the difference between the identity of $\mathcal{D}_k$ and $\mathsf{M}_{2^{n_k}+1}(p_k\mathcal{A}p_k)$. Taking all three maps ω, ψ_m and φ into consideration and applying Proposition (3.4), one easily sees that for each $k \geqslant 0$

$$\pi_k(U_{2^{n_k}}(p_k\mathcal{A}p_k)) \cong \pi_k(U(\mathcal{D}_k)) \cong \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k)) \quad \text{for all } m \geqslant 0.$$

Therefore, for each $k \geqslant 0$

$$\pi_k(U(p\mathcal{A}p)) \cong \pi_k(U_{2^{n_k}+m}(p_k\mathcal{A}p_k)) \quad \text{for all } m \geqslant 0.$$

Step 2. We prove that every norm-continuous map $u(\cdot)$ from $\mathbf{S}^k$ to $U_\infty(\mathcal{A})$ is homotopic to a norm-continuous map from $\mathbf{S}^k$ to $U_\infty(\mathcal{A})$ with the following form

$$\begin{pmatrix} u_0(\cdot) & 0 \\ 0 & 1-\sum_{i=1}^{m} p_k \otimes e_{ii} \end{pmatrix} \quad \text{for some } m > 2^{n_k},$$

where $u_0(\cdot)$ is a norm-continuous map from $\mathbf{S}^k$ to the unitary group of

$$\left(\sum_{i=1}^{m} p_k \otimes e_{ii}\right)(\mathcal{A}\otimes\mathcal{K})\left(\sum_{i=1}^{m} p_k \otimes e_{ii}\right).$$

For any compact Hausdorff space X and any C^*-algebra $\mathcal{B}$ it is well known that

$$C(X,\mathcal{B}) \cong C(X)\otimes\mathcal{B} \quad ([38]).$$

Consider the case $X = \mathbf{S}^k$ and $\mathcal{B}$ is a hereditary C^*-subalgebra of $\mathcal{A}\otimes\mathcal{K}$. Let $\{e_\lambda\}$ be an approximate identity of $\mathcal{A}\otimes\mathcal{K}$ consisting of projections. It follows that $\{1\otimes e_\lambda\}$ is an approximate identity of $C(\mathbf{S}^k, \mathcal{A}\otimes\mathcal{K})$ consisting of projections. It then follows from [46, 2.6] that $u(\cdot)$ is homotopic to a norm-continuous map from $\mathbf{S}^k$ to $U_\infty(\mathcal{A})$ with the form

$$\begin{pmatrix} w(\cdot) & 0 \\ 0 & 1-q \end{pmatrix} \quad (\text{with respect to } q + (1-q) = 1),$$

where q is a projection of $\mathcal{A}\otimes\mathcal{K}$ such that $\sum_{i=1}^{m'} p_k \otimes e_{ii} < q$ for some $m' > 2^{n_k}$. Since $\mathcal{A}\otimes\mathcal{K}$ is simple, it follows from the Riesz decomposition property in [43]

$$n\left[\sum_{i=1}^{m'} p_k \otimes e_{ii}\right] > [q] \quad \text{for some } n \geqslant 1.$$

Set $m = nm'$. Then $\left[\sum_{i=1}^{m} p_k \otimes e_{ii}\right] > [q]$. Since two equivalent projections in a stable C^*-algebra are homotopic (or rather one can use Proposition (2.1)), there is a path

$$\{u_0(t) : 0 \leqslant t \leqslant 1\} \subset U_\infty(\mathcal{A})$$

such that

$$u_0(0) = 1, \quad u_0(1) q u_0(1)^* \leqslant \sum_{i=1}^{m} p_k \otimes e_{ii}.$$

It follows that

$$\begin{pmatrix} w(\cdot) & 0 \\ 0 & 1-q \end{pmatrix}$$

is homotopic to

$$u_0(1)\begin{pmatrix} w(\cdot) & 0 \\ 0 & 1-q \end{pmatrix}u_0(1)^*,$$

which has the desired form.

Step 3. We prove that if a norm-continuous map $u(\cdot)$ from $\mathbf{S}^k$ to a subgroup of $U_\infty(\mathcal{A})$ consisting of unitaries with the form

$$\begin{pmatrix} w & 0 \\ 0 & 1-\sum_{i=1}^{m} p_k \otimes e_{ii} \end{pmatrix} \quad \text{for some } m \geqslant 2^{n_k}$$

is homotopic to the identity as a map from $\mathbf{S}^k$ to $U_\infty(\mathcal{A})$, then $u(\cdot)$ is homotopic to the identity in the subgroup of $U_\infty(\mathcal{A})$ consisting of unitaries with the form

$$\begin{pmatrix} v(\cdot) & 0 \\ 0 & 1-\sum_{i=1}^{n} p_k \otimes e_{ii} \end{pmatrix} \quad \text{for some } n > m.$$

First, $C(\mathbf{S}^k \times [0,1], \mathcal{A}\otimes\mathcal{K})$ has an approximate identity, denoted by $\{1 \otimes e_\lambda\}$, consisting of projections, where the '1' means the identity of $C(\mathbf{S}^k \times [0,1])$. If $(\mathcal{A}\otimes\mathcal{K})^\dagger$ denotes the C^*-algebra obtained by joining an identity to $\mathcal{A}\otimes\mathcal{K}$, of course each norm-continuous path of the unitaries of $C(\mathbf{S}^k, (\mathcal{A}\otimes\mathcal{K})^\dagger)$ can be identified with a unitary in

$$C(\mathbf{S}^k \times [0,1], (\mathcal{A}\otimes\mathcal{K})^\dagger).$$

Using the same arguments as in Step 2, we can prove that $u(\cdot)$ is homotopic to the identity via a path to the subgroup of unitaries of $C(\mathbf{S}^k, (\mathcal{A}\otimes\mathcal{K})^\dagger)$ with the form

$$\begin{pmatrix} w_0(\cdot,t) & 0 \\ 0 & 1-q \end{pmatrix} \quad (0 \leqslant t \leqslant 1),$$

where q is a projection of $\mathcal{A}\otimes\mathcal{K}$ such that $q > \sum_{i=1}^{m} p_k \otimes e_{ii}$ for some $m > 2^{n_k}$ and

$$\{w_0(\cdot,t) : 0 \leqslant t \leqslant 1\}$$

is a path of unitaries of

$$C(\mathbf{S}^k, q(\mathcal{A}\otimes\mathcal{K})q).$$

Furthermore, there is a path of unitaries

$$\{v_0(t) : 0 \leqslant t \leqslant 1\} \subset U_\infty(\mathcal{A})$$

such that

$$v_0(0) = 1, \quad v_0(1)qv_0(1)^* \leqslant \sum_{i=1}^{n} p_k \otimes e_{ii} \text{ for some } n > m.$$

The following path of unitaries leads $u(\cdot)$ to the identity with the desired form:

$$v_0(1)\begin{pmatrix} w_0(\cdot,t) & 0 \\ 0 & 1-q \end{pmatrix} v_0(1)^*.$$

The three steps together effect the proof of Lemma 3.8.

3.9. LEMMA. *If $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and cancellation, and if p is any nonzero projection of $\mathcal{A}$, then*

$$\pi_k(U(p\mathcal{A}p)) \cong \pi_k(U(\tilde{\mathcal{A}})) \quad \textit{for } k \geqslant 0.$$

Proof. Let p_k be a subprojection of p defined in the proof of Lemma (3.8). If $\{e_\lambda\}$ is an approximate identity of $\mathcal{A}$ consisting of projections, then $\{1 \otimes e_\lambda\}$ is an approximate identity of $C(\mathbf{S}^k, \mathcal{A})$ consisting of projections. If $u(\cdot)$ is a unitary of $C(\mathbf{S}^k, \tilde{\mathcal{A}})$, applying [46, Lemma 2.6], there exists a projection $e_\lambda > p$ of $\mathcal{A}$ such that $u(\cdot)$ is close in norm (and hence homotopic) to a unitary with the form

$$\begin{pmatrix} u_1(\cdot) & 0 \\ 0 & 1-e_\lambda \end{pmatrix}$$

(with respect to the decomposition $e_\lambda + (1 - e_\lambda) = 1$), where $u_1(\cdot)$ is a unitary of $C(\mathbf{S}^k, e_\lambda \mathcal{A} e_\lambda)$. On the other hand, if a unitary $v(\cdot)$ with the matrix form

$$\begin{pmatrix} v_0(\cdot) & 0 \\ 0 & 1-e_\lambda \end{pmatrix}$$

homotopic to the identity of $C(\mathbf{S}^k, \tilde{\mathcal{A}})$, then applying [46, Lemma 2.8] to $C(\mathbf{S}^k \times \times [0,1], \tilde{\mathcal{A}})$ we can further assume, by properly choosing a projection $e_\mu \geqslant e_\lambda$ of $\mathcal{A}$, that $v(\cdot)$ is homotopic to the identity via a path of unitaries with matrix forms

$$\begin{pmatrix} v_1(\cdot,t) & 0 \\ 0 & 1-e_\mu \end{pmatrix},$$

where

$$\{v_1(\cdot,t) : (0 \leqslant t \leqslant 1)\}$$

is a path of unitaries of $C(\mathbf{S}^k, e_\mu \mathcal{A} e_\mu)$. Since $\mathcal{A}$ is simple, there exists $m > 2^{n_k}$, where $2^{n_k} > k+3$ such that

$$m[p_k \otimes e_{11}] > [e_\lambda].$$

Now we can use a similar argument as in the proof of Lemma (3.8) (Step 2 and Step 3) to conclude that

$$\pi_k(U(p\mathcal{A}p)) \cong \pi_k(U_\infty(e_\mu \mathcal{A} e_\mu) \cong \pi_k(U(\tilde{\mathcal{A}})) \quad \text{for } k \geqslant 0.$$

3.10. Combining Lemma (3.8), Lemma (3.9) and the Bott periodicity in (3.5), we conclude that

$$\pi_{2m}(U(\tilde{\mathcal{A}})) \cong \pi_{2m}(U(p\mathcal{A}p)) \cong \mathrm{K}_1(\mathcal{A}),$$

$$\pi_{2m+1}(U(\tilde{\mathcal{A}})) \cong \pi_{2m+1}(U(p\mathcal{A}p)) \cong \mathrm{K}_0(\mathcal{A}) \quad \text{for all } m \geqslant 0.$$

It is clear from the proof that $\pi_k(U_n(\tilde{\mathcal{A}})) \cong \pi_k(U(p\mathcal{A}p))$ for all $n \geqslant 1$ and $k \geqslant 0$.

3.11. ANOTHER HOMOTOPY SEQUENCE. Now we turn to the determination of $\pi_k(\mathcal{P}(\mathcal{A}))$ by using the same ideas as in [46, Section 3]. For each fixed projection $p \in \mathcal{P}(\mathcal{A})$, consider the following action $\varphi_p : U(\tilde{\mathcal{A}}) \longrightarrow \mathcal{P}(\mathcal{A})$ defined by $\varphi_p(u) = upu^*$ on $\mathcal{P}(\mathcal{A})$. Then φ_p is an open map. Since two equivalent projections are homotopic by Proposition (2.1), the orbit of p under this action is the path component $\mathcal{G}_p$ of $\mathcal{P}(\mathcal{A})$ containing p. We can take any projection in $\mathcal{G}_p$ as a base point of $\pi_k(\mathcal{P}(\mathcal{A}))$. In particular, we can take p as the base point. Clearly, $\pi_k(\mathcal{P}(\mathcal{A}), p) \cong \pi_k(\mathcal{G}_p, p)$ for $k \geqslant 1$. Furthermore, it follows from the same proof as in [9, 5.1] that local cross sections from a path component of $\mathcal{P}(\mathcal{A})$ to $U(\tilde{\mathcal{A}})$ exists.

Set

$$U(p) = \{u \in U(\tilde{\mathcal{A}}) : \ up = pu\}.$$

Then $U(p)$ is a closed subgroup of $U(\tilde{\mathcal{A}})$ whose elements have the form:

$$u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \quad \text{where } u_1 \in U(p\mathcal{A}p) \text{ and } u_2 \in U((1-p)\tilde{\mathcal{A}}(1-p)).$$

The same proof as in [9, 5.2] yields that $\mathcal{G}_p$ with the relative norm topology is homeomorphic to $U(\tilde{\mathcal{A}})/U(p)$, the space of all right cosets of $U(p)$ equipped with the quotient topology. It follows from [37; Chapter 4, 4.10–4.13] that

$$\varphi_p : U(\tilde{\mathcal{A}}) \longrightarrow \mathcal{G}_p$$

is a weak fibration with fiber $U(p)$. If one composes the action φ_p with the map

$$q \mapsto 2q - 1$$

from $\mathcal{G}_p$ to the corresponding path component $\mathcal{S}(p)$ of symmetries $\mathcal{S}(\tilde{\mathcal{A}})$, one has another weak fibration described in Remarks (2.5)(ii).

It then follows from the general homotopy theory [36] or [37] that the following long sequence is exact:

$$\cdots \to \pi_{k+1}(U(\tilde{\mathcal{A}})) \to \pi_{k+1}(\mathcal{G}_p) \to \pi_k(U(p)) \to \pi_k(U(\tilde{\mathcal{A}})) \to \cdots$$
$$\cdots \to \pi_2(\mathcal{G}_p) \to \pi_1(U(p)) \to \pi_1(U(\tilde{\mathcal{A}})) \to \pi_1(\mathcal{G}_p) \to \pi_0(U(p)) \to$$
$$\to \pi_0(U(\tilde{\mathcal{A}})) \to \pi_0(\mathcal{G}_p) = 0.$$

3.12. Proposition. *Assume that $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and cancellation. Then the following short sequences are exact:*

$$0 \to \pi_{k+1}(\mathcal{G}_p) \xrightarrow{\partial} \pi_k(U(p)) \xrightarrow{i_*} \pi_k(U(\tilde{\mathcal{A}})) \to 0 \quad (k \geqslant 0).$$

Proof. It follows from (3.10) that

$$\pi_{2m+1}(U(\tilde{\mathcal{A}})) \cong \mathrm{K}_0(\mathcal{A}) \quad \text{and} \quad \pi_{2m}(U(\tilde{\mathcal{A}})) \cong \mathrm{K}_1(\mathcal{A}) \quad (m \geqslant 0).$$

Since $U(p) \cong U(p\mathcal{A}p) \times U((1-p)\tilde{\mathcal{A}}(1-p))$, it is clear that

$$\pi_{2m+1}(U(p)) \cong \mathrm{K}_0(\mathcal{A}) \oplus \mathrm{K}_0(\mathcal{A}) \text{ and } \pi_{2m}(U(p)) \cong \mathrm{K}_1(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}) \quad (m \geqslant 0).$$

It again follows from (3.10) that the inclusion map $U(p) \xrightarrow{i} U(\tilde{\mathcal{A}})$ induces a surjective map

$$\pi_k(U(p)) \xrightarrow{i_*} \pi_k(U(\tilde{\mathcal{A}})).$$

Hence, the long exact sequence breaks into short exact sequences:

$$0 \to \pi_{k+1}(\mathcal{G}_p) \to \pi_k(U(p)) \to \pi_k(U(\tilde{\mathcal{A}})) \to 0 \quad (k \geqslant 0).$$

3.13. Proposition. *Suppose that $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and cancellation. If p is any fixed non-trivial projection of $\mathcal{A}$, then*

$$\pi_{2m-1}(\mathcal{G}_p) \cong \mathrm{K}_1(\mathcal{A}) \quad \text{and} \quad \pi_{2m}(\mathcal{G}_p) \cong \mathrm{K}_0(\mathcal{A}) \quad \text{for all } m \geqslant 1.$$

Proof. The same proof as in [46, 3.4] works here.

3.14. It was pointed out in (3.6) that

$$\mathrm{K}_0(\mathcal{A}) \cong \mathrm{K}_0(q(\mathcal{A} \otimes \mathcal{K})q) \quad \text{and} \quad \mathrm{K}_1(\mathcal{A}) \cong \mathrm{K}_1(q(\mathcal{A} \otimes \mathcal{K})q)$$

for any non-zero projection q of $\mathcal{A}\otimes\mathcal{K}$. It follows that

$$\pi_k(\mathcal{P}(q(\mathcal{A}\otimes\mathcal{K})q))\cong\pi_k(\mathcal{P}(\mathcal{A}))\quad\forall\, k\geqslant 1.$$

The combination of 3.8–3.14 effects the proof of Theorem II.

3.15. REMARK. We would like to point out that the conclusion for $\pi_k(U(\tilde{\mathcal{A}}))$ in Theorem II actually holds for C^*-algebras satisfying the following conditions:

(i) $\mathcal{A}$ is a simple C^*-algebra with the property (SP) (see [6]);

(ii) $\mathcal{A}$ has cancellation but does not have minimal projections; and

(iii) $\sup\limits_{p\in\mathcal{P}(\mathcal{A}\otimes\mathcal{K})}\ \operatorname{tsr}(p(\mathcal{A}\otimes\mathcal{K})p)=1.$

If, in addition, any two Murray-von Neumann equivalent projections are homotopic, then the conclusion for $\pi_k(\mathcal{P}(\mathcal{A}))$ in Theorem II holds also. Of course a simple non-elementary C^*-algebra with real rank zero and cancellation satisfies these conditions. However, we do not know any example of C^*-algebras with non-zero real rank and satisfying the above conditions. Hence, we choose not to state our result in a more general form.

4. THE UNITARY GROUP OF $C(X,\tilde{\mathcal{A}})$

4.0. If X is any compact Hausdorff space and $\mathcal{A}$ is a C^*-algebra, as usual let $C(X,\mathcal{A})$ stand for the C^*-algebra of all norm-continuous maps from X to $\mathcal{A}$ equipped with a norm $\|f(\cdot)\|=\sup\limits_{t\in X}\|f(t)\|$. Then it is well known that $C(X,\mathcal{A})$ is *-isomorphic to $C(X)\otimes\mathcal{A}$ ([38]). Consider the unitary group $U(C(X,\tilde{\mathcal{A}}))$ with the pointwise multiplication. It is clear that each unitary $u=u(\cdot)$ of $C(X,\tilde{\mathcal{A}})$ can be regarded as a norm-continuous map from X to $U(\tilde{\mathcal{A}})$; i.e., $U(C(X,\tilde{\mathcal{A}}))\cong C(X,U(\tilde{\mathcal{A}}))$. Obviously, the identity component $U_0(C(X,\tilde{\mathcal{A}}))$ of $U(C(X,\tilde{\mathcal{A}}))$ is a closed normal subgroup of both $C(X,U_0(\tilde{\mathcal{A}}))$ and $C(X,U(\tilde{\mathcal{A}}))$. Since $U_0(\tilde{\mathcal{A}})$ is a closed normal subgroup of $U(\tilde{\mathcal{A}})$, $C(X,U_0(\tilde{\mathcal{A}}))$ is a normal and closed subgroup of $C(X,U(\tilde{\mathcal{A}}))$.

The K-theory of $C(X,\mathcal{A})$ is under the control of the *Künneth formula* [5], but the group $U(C(X,\tilde{\mathcal{A}}))/U_0(C(X,\tilde{\mathcal{A}}))$ is not, where the group operation is induced by the pointwise multiplication. Of course, the structure of this group depends on both X and $\mathcal{A}$, and can be very complicated; even in case $X=\mathbf{S}^k$ and $\mathcal{A}$ is a full matrix algebra over the complex numbers, the group remains unknown in general.

For the special cases when $\mathcal{A}$ is a non-elementary simple C^*-algebra with real rank zero and $\mathcal{A}$ either has cancellation or is purely infinite, then Theorem II and the main result in [46] enable us to determine the group $U(C(X,\tilde{\mathcal{A}}))/U_0(C(X,\tilde{\mathcal{A}}))$.

4.1. LEMMA. *Suppose that $\mathcal{A}$ is a non-elementary, simple C^*-algebra of real rank zero, and $\mathcal{A}$ either is purely infinite or has cancellation. If p is any nonzero projection in $\tilde{\mathcal{A}} \otimes \mathcal{K}$, then the natural map from $U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)$ to $U_\infty(\tilde{\mathcal{A}})$ defined by $u \mapsto u \oplus (1-p)$ is a homotopy equivalence, where 1 is the identity of the unitization of $\mathcal{A} \otimes \mathcal{K}$.*

Proof. It is well known that the unitary group of a C^*-algebra is a homotopy retract of the group of all invertibles which is an open subset. Thus, $U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)$ and $U_\infty(\tilde{\mathcal{A}})$ are homotopy equivalent to CW-complexes. Theorem II asserts that the natural map $u \mapsto u \oplus (1-p)$ induces isomorphisms between the homotopy groups of $U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)$ and of $U_\infty(\tilde{\mathcal{A}})$. The classical theorem of Whitehead asserts that a continuous map between connected CW-complexes which induces isomorphisms of homotopy groups is automatically a homotopy equivalence [33]. Especially, this theorem applies to the identity path components of $U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)$ and of $U_\infty(\tilde{\mathcal{A}})$. The conclusion of this lemma is then confirmed; here we notice that all path components of the unitary group of a C^*-algebra are homeomorphic topological spaces.

4.2. THEOREM III. *Assume that $\mathcal{A}$ is as in Lemma 4.1, and X is a compact Hausdorff space. Then for each projection p in $\tilde{\mathcal{A}} \otimes \mathcal{K}$*

$$U(C(X, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(X, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \mathrm{K}_1(C(X, \mathcal{A})).$$

As a consequence,

$$U(C(\mathbb{S}^{2m+1}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(\mathbb{S}^{2m+1}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \mathrm{K}_0(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}),$$

$$U(C(\mathbb{S}^{2m+2}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(\mathbb{S}^{2m+2}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \mathrm{K}_1(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}), \quad (m \geqslant 0).$$

REMARK. The same argument as for Theorem III shows that

$$U(C(X, \tilde{\mathcal{A}}))/U_0(C(X, \tilde{\mathcal{A}})) \cong \mathrm{K}_1(C(X, \mathcal{A}))$$

as long as $\pi_k(U(\mathcal{A})) = \pi_k(U_\infty(\mathcal{A}))$ for $k \geqslant 0$. In particular, it holds if $\mathcal{A}$ is one of the irrational non-commutative tori considered in [31].

Proof. By Lemma (4.1) the natural map between two spaces $U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)$ and $U_\infty(\tilde{\mathcal{A}})$ is a homotopy equivalence. Then for any compact Hausdorff space X one has

$$[X, U(p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)] \cong [X, U_\infty(\tilde{\mathcal{A}})];$$

where $[X, U(\cdot)]$ denotes the set of homotopy classes of $C(X, U(\cdot))$ (without base points). It follows that

$$U(C(X, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(X, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \pi_0(C(X, U_\infty(\tilde{\mathcal{A}}))) \cong \mathrm{K}_1(C(X, \mathcal{A})).$$

In particular, if $X = \mathbb{S}^k$, then it follows from the *Künneth Formula* of K-theory that

$$U(C(\mathbb{S}^{2m+1}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(\mathbb{S}^{2m+1}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \mathrm{K}_0(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}),$$

$$U(C(\mathbb{S}^{2m+2}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p))/U_0(C(\mathbb{S}^{2m+2}, p(\tilde{\mathcal{A}} \otimes \mathcal{K})p)) \cong \mathrm{K}_1(\mathcal{A}) \oplus \mathrm{K}_1(\mathcal{A}), \quad (m \geqslant 0).$$

The two isomorphisms just mentioned can also be proved directly as follows. First, two unitaries $u_1(\cdot)$ and $u_2(\cdot)$ are in the same path component of $U(C(\mathbb{S}^k, \tilde{\mathcal{A}}))$ iff $u_1(\cdot)$ is homotopic to $u_2(\cdot)$ as maps in $C(\mathbb{S}^k, U(\tilde{\mathcal{A}}))$. Secondly, the image of each map in $C(\mathbb{S}^k, U(\tilde{\mathcal{A}}))$ is contained in one of the path components of $U(\tilde{\mathcal{A}})$. Using the fact asserted in Theorem II that

$$\mathrm{K}_1(\mathcal{A}) \cong U(\tilde{\mathcal{A}})/U_0(\tilde{\mathcal{A}}),$$

one sees that if the images of two elements $u_1(\cdot)$ and $u_2(\cdot)$ of $C(X, U(\tilde{\mathcal{A}}))$ are contained in the path components of $U(\tilde{\mathcal{A}})$ corresponding to $g_1, g_2 \in \mathrm{K}_1(\mathcal{A})$, respectively, then the image of the product $u_1(\cdot)u_2(\cdot)$ is contained in the path component of $U(\tilde{\mathcal{A}})$ corresponding to $g_1g_2 \in \mathrm{K}_1(\mathcal{A})$. These together yield

$$U(C(\mathbb{S}^k, \tilde{\mathcal{A}}))/U_0(C(\mathbb{S}^k, \tilde{\mathcal{A}})) \cong [\mathbb{S}^k, U_0(\tilde{\mathcal{A}})] \oplus \mathrm{K}_1(\mathcal{A}).$$

The conclusion follows from the fact (Theorem II and [46]) that

$$\pi_k(U(\tilde{\mathcal{A}})) \cong \pi_k(U_\infty(\mathcal{A})) \quad (k \geqslant 0)$$

and the following general lemma:

4.3. LEMMA. *If B is any unital C^*-algebra, then*

$$[\mathbb{S}^{2m+1}, U_0(B)] \cong \pi_{2m+1}(U_0(B))$$

and

$$[\mathbb{S}^{2m+2}, U_0(B)] \cong \pi_{2m+2}(U_0(B)) \quad (m \geqslant 0).$$

Proof. This is a particular case of [36, Theorem 5, page 382]. It is an exercise to give a direct proof.

4.4. REMARKS. (i) The relation between K-theory and the homotopy type of the group $\mathcal{U}(\mathcal{A})$ of quasi-unitaries of a C^*-algebra $\mathcal{A}$, denoted by $\mathcal{U}\mathcal{A}$, was intensively studied in [39, Section 2; 40, Section 3]. We recall [39, 1.1] that

$$\mathcal{U}\mathcal{A} = \{a \in \mathcal{A} : a^*a = a + a^* = aa^*\}.$$

If $\mathcal{A}$ is non-unital, it is easy to check that each unitary $u \in U(\tilde{\mathcal{A}})$ can be uniquely written as the form $\lambda \cdot 1 - a$ for some complex number λ with $|\lambda| = 1$ and an element $a \in \mathcal{A}$ (by considering the natural quotient map from $\tilde{\mathcal{A}}$ to $\tilde{\mathcal{A}}/\mathcal{A}$). One can easily check that $a \in \mathcal{U}(\mathcal{A})$ and that $\mathcal{U}(\mathcal{A})$ is isomorphic to the kernel of the canonical quotient map:

$$U(\tilde{\mathcal{A}}) \longrightarrow U(\tilde{\mathcal{A}}/\mathcal{A}).$$

Thus, $\mathcal{U}(\mathcal{A})$ can be identified with a closed normal subgroup of $U(\tilde{\mathcal{A}})$.

K. Thomsen proved [39, 40] that if $\mathcal{A}$ is any C^*-algebra and $\mathcal{B}$ is either an infinite dimensional simple AF-algebra or a Cuntz algebra $\mathcal{O}_n$ $(2 \leqslant n \leqslant \infty)$, then for $m \geqslant 0$

$$\pi_{2m+1}(\mathcal{U}(\mathcal{A} \otimes \mathcal{B})) \cong \mathrm{K}_0(\mathcal{A} \otimes \mathcal{B})$$

and

$$\pi_{2m}(\mathcal{U}(\mathcal{A} \otimes \mathcal{B})) \cong \mathrm{K}_1(\mathcal{A} \otimes \mathcal{B}).$$

Both the results in [39, 40] and ours in [46] and this article have the same target on non-stable K-theory and homotopy type, but are in different setting and end up with different results. If some results of [39, 40, 46] and this article are reduced to certain specific cases, we have some common corollaries. For example, the conclusions are the same about the homotopy type of the unitary groups of the Cuntz algebras $\mathcal{O}_n$ $(2 \leqslant n \leqslant +\infty)$ (see [46] and [39]).

(ii) Here we would like to point out that the homotopy classes of projections in $C(\mathbf{S}^k, \mathcal{A})$ can also be described in terms of K-theory for certain C^*-algebras by using techniques similar to the ones in this article. Since it is further technical and beyond the objective of this article, we will pursue the investigation elsewhere.

(iii) The author is indebted to K. R. Goodearl for calling his attention via an -email message to an inaccuracy in the statement of [43, 3.5]. We take this opportunity to make a correction to [43, 3.5] which should have been stated as follows:

Assume that A is a non-elementary, sigma-unital, simple C^*-algebra with $RR(A) = 0$ and cancellation. Then

(i) $\mathcal{J}_0 = \bigcap_{\tau \in \mathbf{S}([p]_0)} \mathcal{J}_t$ is a proper closed ideal of $M(\mathcal{A})$ strictly containing $\mathcal{A}$. If, in addition, each $P \in \mathcal{J}_0$ is continuous on $\mathbf{S}([p]_0)$, then $\mathcal{J}_0$ is the intersection of all closed ideals of $M(\mathcal{A})$ strictly containing $\mathcal{A}$.

(ii) $M(\mathcal{A})/\mathcal{A}$ is simple if and only if each projection $P \in M(\mathcal{A})$ is continuous on $\mathbf{S}([p]_0)$ (this is equivalent to that $\mathcal{A}$ has a continuous scale [26]).

(iii) is the same as before.

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