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COORDINATES, NUCLEARITY AND SPECTRAL SUBSPACES IN OPERATOR ALGEBRAS

PAUL S. MUHLY, CHAOXIN QIU and BARUCH SOLEL

Dedicated to the memory of John Bunce

1. INTRODUCTION

In recent years, coordinate representations for operator algebras have played an important role in their analysis. When they are used, it is often useful to know whether they are essential for the problem at hand or whether their intervention is an artifact of poorly conceived proofs. The purpose of this paper is to prove two theorems that will help assist in this determination.

The first theorem asserts that if B is a C^*-algebra admitting a diagonal D in the sense of Kumjian [6], then B is nuclear if and only if the so-called $\mathbf{T}$-groupoid associated with B (i.e. the coordinate representation of B) is measurewise amenable in the sense of Renault [16]. This assertion is analogous to the well known fact from the theory of group C^*-algebras that a group is amenable if and only if its group C^*-algebra is nuclear. One direction of the implication is contained, essentially, in [16], and the other will be seen to be a corollary of the famous theorem of Connes, Feldman and Weiss [2]. It clears up, in the case of r-discrete principal groupoids, some questions raised in Section II.3 of [16].

The second theorem is a generalization to $\mathbf{T}$-groupoids of the so-called Spectral Theorem for Bimodules proved in [9]. This theorem asserts that if $\mathfrak{M}$ is a norm--closed subspace of B that is a bimodule over D in the sense that $d \cdot \mathfrak{M}$ and $\mathfrak{M} \cdot d$ are contained in $\mathfrak{M}$ for all d in D, then $\mathfrak{M}$ may be parametrized in terms of the same coordinates that represent B *provided* they are measurewise amenable. Thus, taken together, the two theorems assure us that when we wish to study (not necessarily self-adjoint) subalgebras of B containing D, the coordinates used to describe B work equally well to describe the subalgebra, assuming that B is nuclear. As will be shown elsewhere (see [10] and [8]), this leads to a number of theorems that may be stated in a coordinate free fashion but whose proofs seem to require coordinates in essential ways.

2. PRELIMINARIES

In order to avoid any confusion in the sequel, we want to emphasize here that unless specified to the contrary, all of our C^*-algebras and Hilbert spaces are assumed to be separable and all of our topological spaces are assumed to be Hausdorff and second countable. We follow the notation and terminology of Renault [16,17] and Kumjian [6], but with some minor modifications which we now set forth.

First of all, we recall that to say that a unital C^*-algebra B has a *diagonal* D in the sense of Kumjian means that D is an abelian subalgebra of B containing the unit of B and that there is a faithful expectation $\mathbf{P}$ from B onto D with the property that the kernel of $\mathbf{P}$ is the (closed) span of the so-called *free normalizers* of D in B. A *normalizer* of D in B is an element a such that both aDa^* and a^*Da are contained in D. These are denoted $\mathrm{N}(D)$. A normalizer of D in B is called *free* if its square is zero. These are denoted $\mathrm{N}_f(D)$. The example one should keep in mind is the setting where B is the algebra of $n \times n$ matrices over $\mathbf{C}$, and D is the algebra of diagonal matrices. Then $\mathbf{P}$ is the map that replaces all the off-diagonal entries of a matrix with zeros, typical elements of $\mathrm{N}(D)$ are (scalar multiples of) matrix units, and typical elements of $\mathrm{N}_f(D)$ are (scalar multiples of) matrix units that are not in D.

If B is not unital and D is an abelian subalgebra of B, then D will be called a diagonal in B if after a unit is adjoined to B and D, D (with the unit) is a diagonal in B (with the unit) in the sense just defined. In this paper, units play no special role, and we will not assume that they exist.

We note in passing that in Proposition 1.4 of [6], it is shown that a diagonal in a C^*-algebra is *maximal* abelian.

We let X denote the maximal ideal space of D. By theorem 3.1 of [6], there is an r-discrete principal groupoid G with the unit space $G^{(0)}$ homeomorphic to X and a $\mathbf{T}$-groupoid E over G such that B is isomorphic to $C^*_{\mathrm{red}}(G; E)$ with D being sent to $C_0(G^{(0)})$. To describe these notions in a bit greater detail, we recall that to say that G is an r-discrete principal groupoid with unit space X is to say that G may be viewed as an equivalence relation contained in $X \times X$ with the property that the equivalence class (or orbit) of each point in X is countable. Furthermore, G carries its own, locally compact Hausdorff topology, *which may be different from the relative* *topology inherited from* $X \times X$, making G a topological groupoid in the sense of [16] and such that the map $r: G \to X$, defined by $r(x,y) = x$, is open. It follows from this assumption that counting measure, λ^x, on each of the sets $G^x := r^{-1}(x), x \in X$, is a Haar system, $\{\lambda^x\}_{x \in X}$, in the sense of [16]. To say that E is a $\mathbf{T}$-groupoid over G may be described in a number of equivalent ways. Perhaps the most congenial is to say that E is a locally compact groupoid with unit space (homeomorphic to) X having the structure of a principal $\mathbf{T}$-bundle over G such that for t_1 and t_2 in $\mathbf{T}$ and γ_1 and γ_2 in E, with $(\gamma_1, \gamma_2) \in E^{(2)}$, $(t_1\gamma_1, t_2\gamma_2)$ also lies in $E^{(2)}$ with $(t_1\gamma_1)(t_2\gamma_2) = (t_1t_2)(\gamma_1\gamma_2)$ and such that every γ, with $r(\gamma) = s(\gamma) = u \in E^{(0)} (\approx X)$, can be written as $t \cdot u$ for some t in $\mathbf{T}$. That is, the isotropy group bundle of E is naturally isomorphic to $\mathbf{T} \times X$ and G is the principal groupoid associated with E. We write j for the quotient map from E onto G, i.e., $j(\gamma) = (r(\gamma), s(\gamma))$. It will often be convenient to abbreviate $j(\gamma)$ by $\dot{\gamma}$.

Let $C_C(G;E)$ be the space of compactly supported continuous functions f on E with the property that $f(t\gamma) = tf(\gamma)$ for all γ in E. Then with the inductive limit topology and operations defined by the formulae

$$f * g(\alpha) = \int f(\beta) g(\beta^{-1}\alpha)\, \mathrm{d}\lambda^{r(\alpha)}(\beta)$$

and

$$f^*(\gamma) = \overline{f(\gamma^{-1})},$$

$C_C(G;E)$ becomes a topological $*$-algebra. By a *representation* of $C_C(G;E)$ we mean a non-degenerate $*$-homomorphism from $C_C(G;E)$ into the algebra $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on the (complex) Hilbert space $\mathcal{H}$ that is continuous when $\mathcal{L}(\mathcal{H})$ is given the weak operator topology. It is proved in [17] that the quantity $||f||$ defined by the formula

$$||f|| = \sup\{||\pi(f)|| \mid \pi \text{ — a representation}\}$$

is finite and defines a C^*-norm on $C_C(G;E)$. The completion of $C_C(G;E)$ in this norm, then, is a C^*-algebra denoted $C^*(G;E)$.

Whenever it is convenient, we will identify X with $G^{(0)}$, which is the diagonal $\Delta \subseteq X \times X$, or with $E^{(0)}$, so that $C_0(X)$ sits inside $C^*(G;E)$ in a natural way which, however, is *not* as $C_0(E^{(0)})$. Recall that each γ in $j^{-1}(G^{(0)})$ can be written uniquely as $t \cdot u$ for some $u \in E^{(0)}$. So given $f \in C_0(X)$ we identify f with the function $\tilde{f} \in C_0(j^{-1}(G^{(0)}))$ given by the formula $\tilde{f}(t \cdot u) = t \cdot f(u)$, where in the right hand product, of course, the product is the product of the numbers t and $f(u)$ and we are thinking of u as an element of X. Extending $\tilde{f}$ to be zero off the clopen set $j^{-1}(G^{(0)})$ yields a function, also denoted $\tilde{f}$, on E that represents an element in

$C^*(G;E)$. Actually, the fact that $\tilde{f}$ represents an element in $C^*(G;E)$ requires a brief check. If $f \in C_C(X)$, then $\tilde{f} \in C_C(G;E)$ and from Renault's Disintegration Theorem, it follows that the C^*-norm of $\tilde{f}$ is the sup-norm of f. Since any f in $C_0(X)$ is the sup-norm limit of functions in $C_C(X)$, our assertions follows.

In general, the C^*-algebra B is not isomorphic to $C^*(G;E)$, but only to a quotient, $C^*_{\text{red}}(G;E)$, of $C^*(G;E)$ determined by the common kernel of all the representations $\{\operatorname{Ind}\varepsilon_x\}_{x\in X}$, where $\operatorname{Ind}\varepsilon_x$ will be defined in a moment. This is Kumjian's representation theorem, Theorem 3.1 of [6]. We will show as a consequence of Theorem I that if B is nuclear then, in fact, $C^*_{\text{red}}(G;E) = C^*(G;E)$.

It will be helpfull to have at our disposal some facts about representations of groupoids and groupoid C^*-algebras. The key theorem is Renault's Disintegration Theorem [17, Théorème 4.1] which asserts that every representation of $C_C(G;E)$ can be written as the integrated form of a representation of (G,E). To explain this, we follow the notation of [16,17] specialized to our special setting. We let $X * \mathcal{H}$ denote a (*Borel*) Hilbert bundle over X and we let $\int_X^{\oplus} \mathcal{H}(x)\,\mathrm{d}\mu(x)$ denote the direct integral, or cross sectional L^2-space of $X*\mathcal{H}$ associated with a measure μ on X. (Throughout, our measures will always be Radon measures.) Thus elements of $\int_X^{\oplus} \mathcal{H}(x)\,\mathrm{d}\mu(x)$ are Borel cross sections $\xi: X \to X * \mathcal{H}$ such that the function $x \to \|\xi(x)\|_{\mathcal{H}(x)}$ is in $L^2(\mu)$ and the norm of ξ in $\int_X^{\oplus} \mathcal{H}(x)\,\mathrm{d}\mu(x)$ is the L^2-norm of $x \to \|\xi(x)\|_{\mathcal{H}(x)}$. The *isomorphism groupoid* of $X * \mathcal{H}$, $\operatorname{Iso}(X * \mathcal{H})$, is the collection

$$\{U_{x,y}: \mathcal{H}(y) \to \mathcal{H}(x) \mid U_{x,y} \text{ is a Hilbert space isomorphism}\}$$

endowed with its natural Borel structure. A *representation* of (G,E) on $X*\mathcal{H}$ consists of a pair (μ, L) where μ is a quasi-invariant measure on X in the sense of Renault [16] and where L is a Borel homomorphism from E into $\operatorname{Iso}(X * \mathcal{H})$ such that

(1) $L(\gamma): \mathcal{H}(s(\gamma)) \to \mathcal{H}(r(\gamma))$, for all γ in E;

(2) $L(x) = I_{\mathcal{H}(x)}$, for all x in $X = E^0$;

(3) $L(t\gamma) = \bar{t} \cdot L(\gamma)$, for all γ in E and t in $\mathbf{T}$.

(Thanks to Theorem 3.2 in [15], one does not need to use the weaker definition given in [16,17], where certain null sets intrude.) Given a representation (μ, L) of (G,E), form $\nu = \int \lambda^x\,\mathrm{d}\mu(x)$, let ν^{-1} be the image of ν under the map $\gamma \to \gamma^{-1}$ on G, and let $\Delta = \mathrm{d}\nu/\mathrm{d}\nu^{-1}$, chosen to be a homomorphism from G into $\mathbf{R}^*$. This may be done by the assumption that μ is quasi-invariant and Ramsay's theorem just cited. Then for f in $C_C(G;E)$ and $\xi,\mu \in \int_X^{\oplus} \mathcal{H}(x)\,d\mu(x)$, the formula

$$(2.1)\qquad (\pi(f)\xi,\mu) = \int_G f(\gamma)(L(\gamma)\xi(s(\gamma)),\eta(r(\gamma))\,d\nu_0(\dot\gamma),$$

where $\nu_0 := \Delta^{-1/2}\nu$, defines a representation, π, of $C_C(G;E)$ on $\int_X^{\oplus} \mathcal{H}(x)\,d\mu(x)$. We call π the *integrated form* of (μ, L). (Note that the transformation properties of f and L imply that the integrated in equation (2.1) is invariant under the action of $\mathbf{T}$ and so is, really, a function on G.) Renault's Disintegration Thorem, Théorème 4.1 of [17] says, conversely, that every representation of $C_C(G;E)$ is the integrand form of some representation of $(G;E)$.

It is evident from Renault's Disintegration Theorem that a representation π of $C_C(G;E)$ may be extended to the space, $B_C(G;E)$, consisting of all bounded Borel functions f with compact support on E such that $f(t\gamma) = tf(\gamma)$ for all t in $\mathbf{T}$ and γ in E. It is easy to see that $B_C(G;E)$ is a $*$-algebra under the same operations used to define $C_C(G;E)$ and that the extension of π is a $*$-homomorphism that is continuous with respect to the weak operator topology on the image of π and the "topology" on $B_C(G;E)$ defined by declaring that a sequence $(f_n)_{n\in\mathbf{N}}$ in $B_C(G;E)$ converges to f in $B_C(G;E)$ if and only if $f_n \to f$ pointwise boundedly and the supports of f and all of the f_n are contained in some fixed compact set in E. Of course the weak closure of $\pi(C_C(G;E))$ coincides with the weak closure of $\pi(B_C(G;E))$.

The reason for bringing $B_C(G;E)$ into the discussion is that it is isomorphic in an explicit useful way to the algebra $B_C(G,\sigma)$ of bounded Borel functions with compact support on G *twisted by a 2-cocycle* σ. To develop this isomorphism, recall that because our groupoids are locally compact and second countable, they are σ--compact. Consequently, while there may not be a continuous cross section to the quotient map $j\colon E \to G$, there always exists a *Borel* cross section λ with the property that λ maps compact sets in G to pre-compact sets in E. In fact, as is shown in [13], one can arrange for $\lambda(x) = x$ for x in X and for $\lambda(\gamma^{-1}) = \lambda(\gamma)^{-1}$. We say that such a cross section is *normalized*. It follows that the map $\varphi\colon \mathbf{T}\times G \to E$ defined by $\varphi(t,\dot\gamma) = t\lambda(\dot\gamma)$ is a Borel isomorphism between E and $\mathbf{T}\times G$ when $\mathbf{T}\times G$ is endowed with the product $(t_1,\dot\gamma_1)\cdot(t_2,\dot\gamma_2) = (t_1t_2\cdot\sigma(\dot\gamma_1\dot\gamma_2),\dot\gamma_1\dot\gamma_2)$ where σ is the (necessarily Borel) function on $G^{(2)}$ defined by the formula

$$\sigma(\dot\gamma_1,\dot\gamma_2)\cdot\lambda(\dot\gamma_1\dot\gamma_2) = \lambda(\dot\gamma_1)\lambda(\dot\gamma_2).$$

The function σ depends on the choice of λ clearly, and is a Borel 2-cocycle in the sense defined in [16]. (See [13] for a proof of the fact that σ is a 2-cocycle.) The normalization condition implies that $\sigma(\dot\gamma, \dot\gamma^{-1}) = 1$ for all $\dot\gamma \in G$. We fix λ once and for all. The space $B_C(G,\sigma)$ of compactly supported bounded Borel functions on G becomes a $*$-algebra under the operations

$$f * g(\dot\gamma) = \int f(\dot\gamma\dot\alpha) g(\dot\alpha^{-1}) \sigma(\dot\gamma\dot\alpha, \dot\alpha^{-1}) \,\mathrm{d}\lambda^{s(\dot\gamma)}(\dot\alpha),$$

$$f^*(\dot\gamma) = \overline{f(\dot\gamma^{-1})},$$

where f and g are in $B_C(G,\sigma)$. (Note that the factor of $\overline{\sigma(\dot\gamma, \dot\gamma^{-1})}$ that the normally appears in the definition of f^* is missing because of our normalization assumptions.) The map $\Phi\colon B_C(G;E) \to B_C(G,\sigma)$ defined by the formula $\Phi(f)(\dot\gamma) = f(\lambda(\dot\gamma)), \dot\gamma \in \in G$, is easily seen to be an algebra $*$-isomorphism with inverse given by the formula $\Phi^{-1}(f)(\gamma) = t(\gamma) \cdot f(\dot\gamma)$, where γ is in E and $t(\gamma) = \gamma \cdot \lambda(\dot\gamma)^{-1}$, viewed as an element of $\mathbf{T}$.

Given a measure μ on X, we can form a representation, which we denote by σ-Ind μ, of $B_C(G,\sigma)$ on the Hilbert space $L^2(G, \nu^{-1})$ by the formula

$$\sigma\text{-Ind}\,\mu(f)\xi(\dot\gamma) = \int f(\dot\gamma\dot\alpha)\xi(\dot\alpha^{-1})\sigma(\dot\gamma\dot\alpha, \dot\alpha^{-1}) \,\mathrm{d}\lambda^{s(\dot\gamma)}(\dot\alpha),$$

$f \in B_C(G,\sigma), \xi \in L^2(G,\nu^{-1})$. It is an easy matter to check that the weak closure of the image of $B_C(G;\sigma)$ under σ-Ind μ is the von Neumann algebra $M(G,\sigma)$ of Felman and Moore [3,4] when μ is quasi-invariant. This follows essentialy from Proposition II.1.10 of [16]. The map Φ extends to a Hilbert space isomorphism $U\colon L^2(G, E, \nu^{-1}) \to \to L^2(G, \nu^{-1})$ such that

$$U \operatorname{Ind}\mu(f) U^{-1} = \sigma\text{-Ind}\,\mu(\Phi(f)),$$

for all f in $B_C(G;E)$. Since the weak closure of Ind $\mu(B_C(G;E))$, coincides with the weak closure of Ind $\mu(B_C(G;E))$, we may conclude that the following proposition has been proved.

PROPOSITION 2.1. *With the notation just established, if μ is a quasi-invariant measure on X then the Hilbert space isomorphism $U\colon L^2(G,E,\nu^{-1}) \to L^2(G,\nu^{-1})$ effects a spatial isomorphism between the weak closure of* Ind $\mu(C^*(G;E))$ *and the Feldman-Moore von Neumann algebra $M(G,\sigma)$ determined by μ.*

It will be useful to note that the analysis that went into the proof of Proposition 2.1 applies more generally than simply to induced representations.

DEFINITION 2.2. Let G and σ be as above. A *σ-representation* of G is a pair (μ, L) where μ is a quasi-invariant measure on X and where L is a Borel map from G

to Iso$(X * \mathcal{H})$ for a suitable Hilbert bundle $X * \mathcal{H}$ such that $L(\gamma)$ is a Hilbert space isomorphism from $\mathcal{H}(s(\gamma))$ onto $\mathcal{H}(r(\gamma))$ for all γ in G, satisfying $L(x) = I_{\mathcal{H}(x)}$ for $x \in X$, and such that

$$L(\alpha)L(\beta) = \overline{\sigma(\alpha,\beta)} \cdot L(\alpha\beta), \quad (\alpha\beta) \in G^{(2)}.$$

We note in pasing that the formulas in Definition 2.2 look slightly different from those in [16], first because σ is conjugated, and second because no null sets appear. However, one should keep in mind that $\overline{\sigma}$ is just as much a 2-cocycle as is σ and the conjugate appears in our analysis as an artifact of the way we defined $C_C(G; E)$, as can be seen by reflecting on Proposition II.1.22 in [16]. Note, too, that it is not necessary to specify the condition $L(\gamma)^{-1} = \sigma(\gamma^{-1}, \gamma) \cdot L(\gamma^{-1})$, as is done in [16] because of our normalization assumption on λ. The null sets disappear for σ-representations for the same reason, essentially, they disappear for representations; namely, Theorem 3.2 of [15]. This will be clear in a moment.

Given a representation L of (G, E) on the Hilbert bundle $X * \mathcal{H}$, define $\tilde{L}: G \to$ $\to$ Iso$(X * \mathcal{H})$ by the formula $\tilde{L}(\dot{\gamma}) = L(\lambda(\dot{\gamma}))$. Then it is an easy calculation to see that $\tilde{L}$ is a σ-representation of G and conversely, given a σ-representation $\tilde{L}$ of G, setting $L(\gamma) = t(\gamma) \cdot \tilde{L}(\lambda(\dot{\gamma}))$, where $t(\gamma)$ is the unique element $\mathbf{T}$ such that $\gamma = t(\gamma) \cdot \lambda(\dot{\gamma})$, yields a representation of (G, E). Moreover, if π is the integrated form of L and if $\tilde{\pi}$ is the integrated form of $\tilde{L}$, then $\pi(f) = \tilde{\pi}(\Phi(f))$, for $f \in B_C(G; E)$, as the following calculation indicates

$$\pi(f) = \int f(\gamma)L(\gamma)\,\mathrm{d}\nu_0(\dot{\gamma}) =$$

$$= \int f(t(\gamma)\lambda(\dot{\gamma}))L(t(\gamma)\lambda(\dot{\gamma}))\,\mathrm{d}\nu_0(\dot{\gamma}) =$$

$$= \int f(\lambda(\dot{\gamma}))L(\lambda(\dot{\gamma}))\,\mathrm{d}\nu_0(\dot{\gamma}) =$$

$$= \int \Phi(f)(\dot{\gamma})\tilde{L}(\dot{\gamma})\,\mathrm{d}\nu_0(\dot{\gamma}) =$$

$$= \tilde{\pi}(\Phi(f)).$$

This will prove useful in the next section.

3. NUCLEARITY vs. AMENABILTY

Of the many equivalent definitions of "nuclearity" (see [7]), the following is the one that will be of use here: A C^*-algebra A is *nuclear* if and only if $\pi(A)''$ is an injective von Neumann algebra for every representation π of A. There are essentialy two different notions of amenability for an r-discrete principal groupoid G that we will want to compare, measurewise amenability in the sense of Renault [16] and amenability in the sense of Zimmer [19]. We will show that the two notions are equivalent and that are equivalent to the assertion that $C^*(G;E)$ is nuclear. It follows that $C^*(G;E) = C_{\text{red}}(G;E)$ if either algebra is nuclear. Thus expectations based on what happens in the context of discrete groups are borne out. However, in the context of groupoids, whether the equality of $C^*(G;E)$ and $C_{\text{red}}(G;E)$ implies amenability of G remains unsolved.

DEFINITION 3.1. 1) We say that a quasi-invariant measure μ on $G^{(0)}$ is *amenable in the sense of Renault* in case there is a sequence of functions in $B_C(G)$, $\{f_n\}_{n\in N}$, such that

(i) the functions $x \to \int |f_n|^2 \,\mathrm{d}\lambda^x$ converge to 1 in the weak $*$-topology on $L^\infty(G^{(0)},\mu)$, and

(ii) the functions $\gamma \to \int f_n(\gamma\beta)\overline{f_n(\beta)}\,\mathrm{d}\lambda^{s(\gamma)}(\beta)$ converge to 1 in the weak $*$-topology on $L^\infty(G,\nu)$, where $\nu = \int \lambda^x \,\mathrm{d}\mu(x)$. If every quasi-invariant measure on $G^{(0)}$ is amenable in the sense of Renault, we say that G is *measurewise amenable in the sense of Renault.*

2) We say that a quasi-invariant measure μ on $G^{(0)}$ is *amenable in the sense of Zimmer* in case there is a positive, unital, linear map $\mathcal{M}\colon L^\infty(G,\nu) \to L^\infty(G^{(0)},\mu)$, called an *invariant mean* on $L^\infty(G,\nu)$, such that

(i) $\mathcal{M}(h\cdot\varphi) = h\cdot\mathcal{M}(\varphi)$, for $\varphi\in L^\infty(G,\nu)$ and $h\in B_C(G^{(0)})$, where $h\cdot\varphi(\gamma) = h(r(\gamma))\varphi(\gamma)$, and

(ii) $\mathcal{M}(f\cdot\varphi) = f\cdot\mathcal{M}(\varphi)$, for $\varphi\in L^\infty(G,\mu)$ and $f\in B_C(G)$, where

$$f\cdot\varphi(\gamma) = \int f(\beta)\varphi(\beta^{-1}\gamma)\,\mathrm{d}\lambda^{r(\gamma)}(\beta),$$

and where, for $\psi\in L^\infty(G^{(0)},\mu)$,

$$f\cdot\psi(x) = \int f(\beta)\psi(s(\beta))\cdot\mathrm{d}\lambda^x(\beta).$$

If every quasi-invariant measure on $G^{(0)}$ is amenable in the sense of Zimmer, we say that G is *measurewise amenable in the sense of Zimmer.*

Several remarks need to be made here. First, in [16], Renault restricts his attention to functions in $C_C(G)$ in the definition of amenability. However, this is not a material restriction since every function in $B_C(G)$ is the bounded pointwise limit on its support of a sequence in $C_C(G)$ with supports contained in a suitable prescribed compact set containing the support of the limit function. Second, since our groupoids are second countable, we may use sequences in the definitions instead of nets, as used by Renault. Third, as Renault points out in Remark II.3.5. of [16], if a quasi-invariant measure is amenable in the sense of Renault, then it is amenable in the sense of Zimmer and he asks about the reverse implication.

PROPOSITION 3.2. *If μ is a quasi-invariant measure on $G^{(0)}$ that is amenable in the sense of Zimmer, then μ is amenable in the sense of Renault.*

Proof. Since G is second countable, so is $X = G^{(0)}$. Hence in the Borel category, X is a standard Borel space and G is a Borel equivalence relation in $X \times X$ with countable equivalence classes, which, together with μ becomes a measured equivalence relation in the sense of Feldman and Moore [3,4]. Our hypothesis on μ together with the famous theorem of Connes, Feldman and Weiss [9] implies that there is a standard Borel measure space $(\tilde{X}, \tilde{\mu})$ carrying a non-singular measurable transformation τ such that G is Borel isomorphic to the orbit equivalence relation $\tilde{R}$ determined by τ. A bit more precisely, there is a co-null Borel set F in X for μ and a Borel isomorphism $\varphi: F \to \tilde{X}$ carrying μ to $\tilde{\mu}$ such that $\varphi^{(2)}$ mapping $F \times F$ to $\tilde{X} \times \tilde{X}$ carries $G|_F :=$ $= G \cap F \times F$ to $\tilde{R}$, where $\varphi^{(2)}(x, y) = (\varphi(x), \varphi(y))$. But as is well known, we may assume that $\tilde{X}$ actually is locally compact and second countable and that τ is a homeomorphism. (This assertion is quite old and "lost in antiquity". Ramsay's paper [15] contains it in the discussion of universal G spaces on page 321. However, in the setting of a single non-singular measurable transformation τ, all one needs to do is to choose a *separable* τ-invariant C^*-subalgebra of $L^\infty(\tilde{\mu})$ which separates the points of $\tilde{X}$. Then one can replace $\tilde{X}$ by the maximal ideal space of this subalgebra and one can replace τ by the induced homeomorphism.) Thus $\tilde{R}$ is an r-discrete principle groupoid of the type we are discussing. Since $\tilde{\mu}$ is easily seen to be amenable in the sense of Renault, either directly or by Proposition II.3.9 in [16], it follows that so is μ. ■

By virtue of Proposition 3.2, we will drop the qualifying phrases "in the sense of Renault" and "in the sense of Zimmer" when referring to amenable quasi-invariant measures and measurewise amenable groupoids.

THEOREM 3.3. *Suppose that E is a $\mathbf{T}$-groupoid over an r-discrete principal groupoid G. Then the following assertions are equivalent.*

1) $C^*(G; E)$ is nuclear;

2) $C^*_{\text{red}}(G; E)$ is nuclear; and

3) G is measurewise amenable.

Proof. Suppose G is measurewise amenable. To show that $C^*(G; E)$ is nuclear, we need to show that if π is a representation of $C^*(G; E)$, then $\pi(C^*(G; E))''$ is an injective von Neumann algebra. However, the analysis that follows Definition 2.2 shows that if π is the integrated form of the representation L of (G, H), then $\pi(C^*(G;E))'' = \tilde{\pi}(B_C(G,\sigma))''$ where $\tilde{\pi}$ is the integrated form of the σ-representation $\tilde{L}$ associated with L. By Proposition II.3.5 of [16], $\tilde{\pi}(B_C(G,\sigma))''$ is injective. Thus, as it is well known (see [7]), $C^*(G;E)$ is nuclear. (We note in passing that Renault uses continuous 2-cocycles and works in the context of $C_C(G,\sigma)$, but inspection of the proof of Proposition II.3.5 in [16] reveals that this assumption is not at all necessary and the argument works in the more general context under consideration.)

Since $C^*_{\mathrm{red}}(G;E)$ is a quotient of $C^*(G;E)$, the nuclearity of $C^*(G;E)$ implies that for $C^*_{\mathrm{red}}(G;E)$. So to complete the proof, we need to show that if $C^*_{\mathrm{red}}(G;E)$ is nuclear and if μ is a quasi-invariant meaure on X, then μ is amenable. It is easy to see that $\mathrm{Ind}\,\mu$ is the direct integral of $\mathrm{Ind}\,\varepsilon_x$, $\mathrm{Ind}\,\mu = \int_X^{\oplus} \mathrm{Ind}\,\varepsilon_x \, d\mu(x)$. Consequently, $\mathrm{Ind}\,\mu(C^*(G;E))'' = \mathrm{Ind}\mu(C^*_{\mathrm{red}}(G;E))''$. By Proposition 2.1, this is the Feldman-Moore von Neumann algebra, $M(G,\sigma)$ associated with μ. Since $C^*_{\mathrm{red}}(G;E)$ is assumed to be nuclear, $M(G,\sigma)$ is injective and is therefore the range of projection of norm one on $\mathcal{L}(L^2(G,\nu^{-1}))$. By Proposition 7 of [2], μ is amenable. ∎

The following proposition is proved as Theorem 3.6 in [18] in much greater generality. Our restricted setting allows for a simpler proof, which we present.

PROPOSITION 3.4. *Let L be a representation of (G,E) and let μ be an amenable quasi-invariant measure on $G^{(0)}$. If π is the integrated form of L determined by μ, then π is weakly contained in $\mathrm{Ind}\,\mu$.*

Proof. One can imitate the proof of Proposition II.3.2 of [16], or one can tap into that proof by noting that $\pi(B_C(G;E)) = \tilde{\pi}(B_C(G,\sigma))$, where $\tilde{\pi}$ is the integrated form of the σ-representation $\tilde{L}$ associated with L as in the discussion after Definition 2.2. The argument in [16] shows that any vector state on $\tilde{\pi}(B_C(G,\sigma))$ is the weak limit of vector states on σ-$\mathrm{Ind}\mu(B_C(G,\sigma))''$ which is unitarily equivalent to $\mathrm{Ind}\,\mu(C_C(G;E))''$ by Proposition 2.1. This shows that any vector state on $\pi(C^*(G;E))$ is the weak limit of vector states on $\mathrm{Ind}\,\mu(C_C(G;E))''$, because any vector state on $\pi(C^*(G;E))$ extends to a vector state on $\pi(B_C(G;E))$ since π can be expressed in integrated form. Thus π is weakly contained in $\mathrm{Ind}\,\mu$. ∎

The following corollary is now immediate.

COROLLARY 3.5. *If E is a $\mathbf{T}$-groupoid over G such that $C^*(G;E)$ is nuclear, $C^*(G;E) = C^*_{\mathrm{red}}(G;E)$.*

One would expect a converse, as was first suggested by Renault [16], but still at this stage, nothing is known.

In concluding section, we remark that if G is amenable and if μ is a quasi-invariant measure on $G^{(0)}$ with induced measure $\nu = \int \lambda \, d\mu(x)$ then it is possible to find an increasing sequence $G_1 \subseteq G_2 \subseteq G_3 \subseteq \dots$ of *Borel* groupoids on X such that when viewed as an equivalence relation on X, each G_n has only finitely many equivalence classes and such that $G = \cup G_n$ a.e. ν. It would be interesting to know if it is possible to find such a sequence that is independent of ν, so that there are no exceptional null sets.

4. THE SPECTRAL THEOREM FOR BIMODULES

Our primary objective in this section is to prove the following theorem.

THEOREM 4.1 (The Spectral Theorem For Bimodules). *Suppose that G is measurewise amenable and that $\mathfrak{M}$ is a norm closed linear subspace of $C^*(G;E)$ that is a bimodule over $C_0(X)$. Then there is an open subset Q in G such that $\mathfrak{M} = \mathfrak{M}(Q)$, where $\mathfrak{M}(Q)$ is defined to be the closure of $\{f \in C_C(G;E) | \operatorname{supp}(f) \subseteq j^{-1}(Q)\}$ in $C^*(G;E)$.*

The proof follows the lines of the special case of the theorem proved in [9]. However, there are important technical differences. Many of the ingredients in [9], such as the assumption that G has a cover by compact open G-sets, are not available here. We will not assume that G is measurewise amenable until we have to. Thus we maintain the distinction between $C^*(G;E)$ and $C^*_{\text{red}}(G;E)$.

Recall that a *G-set* is a subset U of G such that r and s are one-to-one on U. Such a U, then, is a graph of a partially defined transformation whose domain is $r(U)$ and whose range is $s(U)$. Conversely, such a graph is evidently a G-set. We write $\Omega(G)$ for the collection of all *open G-sets*. It is shown in [6] that $\Omega(G)$ covers G. We write N_C for the collection of f in $C_C(G;E)$ such that the image of the support of f under j is contained in some U in $\Omega(G)$. Such an f evidently has the property that $f * d * f^*$ and $f^* * d * f$ are contained in $C_0(X)$ for all d in $C_0(X)$. That is, N_C consists of normalizers of the diagonal of $C^*_{\text{red}}(G;E)$ that are compactly supported. These will be used repeatedly in the proof of Proposition 4.4 below, which is a key ingredient for the proof of Theorem 4.1.

We want to show next that elements in $C^*_{\text{red}}(G;E)$ may be viewed as C_0-functions on E. This is an extension of Proposition II.4.2 in [16]. To this end, let J denote the identity map from $C_C(G;E)$ to $C_C(G;E)$. Since

$$\|f\|_\infty \leqslant \|\mathbf{P}(f^* * f)\|_\infty^{1/2} \leqslant \|f\|_{\text{red}},$$

it is clear that J may be extended to a contractive linear map from $(C^*_{\text{red}}(G; E), \|\cdot\|_{\text{red}})$ to $(C_0(E), \|\cdot\|_\infty)$. We need to show that J is one-to-one. To this end, given x in X, consider $\operatorname{Ind}\varepsilon_x$. It acts on $L^2(G, E, k_x)$ where, recall, $\{k^x\}_{x\in X}$ is the Haar system for E and k_x is the image of k^x under inversion. It is easy to see that the characteristic function of $j^{-1}(G^0)(\approx \mathbf{T}\times X), \varphi_0$, is a unit cyclic and separating vector for the image of $C^*_{\text{red}}(G; E)$ under $\operatorname{Ind}\varepsilon_x$. Consequently, the map $\underline{J}: C_C(G; E) \to$ $\to L^2(G, E, k_x)$ defined by $\underline{J}(f) = \operatorname{Ind}\varepsilon_x(f)\varphi_0$ satisfies $\|\underline{J}(f)\|_2 \leqslant \|f\|_{\text{red}}$. Furthermore, as functions, $\underline{J}(f) = J(f)$ a.e. (k_x). Thus, $\underline{J}$ may be extended to a contractive linear map from $C^*_{\text{red}}(G; E)$ to $L^2(G, E, k_x)$ such that $\underline{J}(f) = J(f)$ a.e. (k_x) for all f in $C^*_{\text{red}}(G; E)$. Consequently, if $J(f) = 0$, then $J(f) = 0$ as a vector in $L^2(G, E, k_x)$. Since $\underline{J}(f) = \operatorname{Ind}\varepsilon_x(f)\varphi_0$ and φ_0 is cyclic and separating for $\operatorname{Ind}\varepsilon_x(C^*_{\text{red}}(G; E))$, we conclude that $\operatorname{Ind}\varepsilon_x(f) = 0$. Since x is arbitrary, it follows that f is 0 in $C^*_{\text{red}}(G; E)$. For the sake of reference, we summarize our discussion as

PROPOSITION 4.2. *Each f in $C^*_{\text{red}}(G; E)$ may be viewed as a function in $C_0(E)$ with $\|f\|_\infty \leqslant \|f\|_{\text{red}}$.*

DEFINITION 4.3. If $\mathfrak{M}$ is a norm closed subspace of $C^*_{\text{red}}(G; E)$ that is a bimodule over $C_0(X)$, then we let $Q(\mathfrak{M})$ be $\{\dot\gamma \in G \mid f(\gamma) \neq 0, \text{ for some } f \in \mathfrak{M}\}$ and we call $Q(\mathfrak{M})$ the *spectrum* of $\mathfrak{M}$. Given an open subset Q of G, we set $\mathfrak{M}_C(Q) = \{f \in$ $\in C_C(G; E) | \operatorname{supp}(f) \subseteq \mathrm{j}^{-1}(Q)\}$ and we set $\mathfrak{M}_0(Q) = \{f \in C^*_{\text{red}}(G; E) | \operatorname{supp}(f) \in$ $\in \mathrm{j}^{-1}(Q)\}$. We call $\mathfrak{M}_C(Q)$ the *minimal spectral subspace determined by Q* and we call $\mathfrak{M}_0(Q)$ the *maximal spectral subspace determined by Q*.

By definition and Proposition 4.2, spectral subsets are open. Also, given an open subset Q of G, $\mathfrak{M}_C(Q) \subseteq \mathfrak{M}_0(Q)$ and these spaces are non-zero bimodules over $C_0(X)$. The use of the adjectives "minimal" and "maximal" is justified by

PROPOSITION 4.4. *If $\mathfrak{M}$ is a norm closed linear subspace of $C^*_{\text{red}}(G; E)$ that is a bimodule over $C_0(X)$, then*

$$\mathfrak{M}_C(Q(\mathfrak{M})) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_0(Q(\mathfrak{M})).$$

Proof. The right inclusion is a consequence of the definitions. What needs proof is the left hand inclusion. To this end, given an element g of N_C, define a map $\Psi_g: C^*_{\text{red}}(G; E) \to C^*_{\text{red}}(G; E)$ by the formula $\Psi_g(f) = \mathbf{P}(f * g^*)g$. A calculation shows that Ψ_g is continuous with norm $\|g\|^2$ and that Ψ_g maps $C^*_{\text{red}}(G; E)$ into N_C. Given g in N_C and $f \in C_C(G; E)$, the proof of Lemma 2.10 in [6] shows that there is an n-tuple $d = \{d_1, \ldots, d_n\}$ in $C_0(X)$ (*depending on f and g*) such that $\sum d_i^* d_i \leqslant 1$ and $\Psi_g(f) = \sum d_i^* * f * (g^* * d_i * g)$. Let us use this sum to define a map, denoted

Ψ_{dg}, so that for the particular f in question, $\Psi_g(f) = \Psi_{dg}(f)$. Since $\Psi_{dg}(f)$ may be viewed as the 1,1-entry of the matrix product

$$\begin{bmatrix} d_1^* & d_2^* & \dots & d_n^* \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} f & 0 & \dots & 0 \\ 0 & f & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & f \end{bmatrix} \cdot \begin{bmatrix} g^*d_1g & 0 & \dots & 0 \\ g^*d_2g & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ g^*d_ng & 0 & \dots & 0 \end{bmatrix}$$

it is clear that the norm of Ψ_{dg} is less or equal to $\|g\|^2$.

For any n-tuple d and any g in N_C, it is evident that Ψ_{dg} maps $\mathfrak{M}$ into $\mathfrak{M}$ because $\mathfrak{M}$ is a $C_0(X)$ bimodule. But this is enough to show that Ψ_g maps $\mathfrak{M}$ into $\mathfrak{M}$ for any g in N_C. Indeed, with g in N_C fixed (and nonzero of course), choose f in $\mathfrak{M}$ and choose h in $C_C(G;E)$ with $\|f-h\| < \varepsilon/(2\|g\|)^2$, where ε is any prescribed positive number. Then choose an n-tuple d as above so that $\Psi_g(h) = \Psi_{dg}(h)$. We have

$$\|\Psi_g(f) - \Psi_{dg}(f)\| \leqslant \|\Psi_g(f) - \Psi_g(h)\| + \|\Psi_g(h) - \Psi_{dg}(f)\| =$$
$$= \|\Psi_g(f) - \Psi_g(h)\| + \|\Psi_{dg}(h) - \Psi_{dg}(f)\| \leqslant \varepsilon.$$

Since $\Psi_{dg}(f)$ lies in $\mathfrak{M}$ for all d and g, so does $\Psi_g(f)$. This shows that $\mathfrak{M} \cap \mathfrak{M}_c(Q(\mathfrak{M}))$ is non-empty.

Next we prove that every f in $\mathfrak{M}_C(Q(\mathfrak{M}))$ is a finite sum $f = \sum g_i$, with g_i in $N_C \cap \mathfrak{M}_0(Q(\mathfrak{M}))$. Since $\Omega(G)$ covers G and since $j(\mathrm{supp}\,(f))$ is compact, there is a finite collection $V_1, \dots, V_n$ in $\Omega(G)$ that covers $j(\mathrm{supp}\,(f))$. Choose $h_1, \dots, h_n$ in $C_C(G)$ so that $\{h_1, \dots, h_n\}$ forms a partition of unity subordinate to $\{V_1, \dots, V_n\}$; i.e., so that $\sum h_i(\dot\gamma) = 1$ for all γ in $\mathrm{supp}\,(f)$ and so that $\mathrm{supp}\,(h_i) \subseteq V_i$. If $g_i(\gamma)$ is defined to be $f(\gamma) \cdot h_i(\dot\gamma)$, then g_i is in $N_C \cap \mathfrak{M}_0(Q(\mathfrak{M}))$ and $f = \sum g_i$.

Thus, to complete the proof, we need only show that every g in $N_C \cap \mathfrak{M}_0(Q(\mathfrak{M}))$ lies in $\mathfrak{M}$. So fix g in $N_C \cap \mathfrak{M}_0(Q(\mathfrak{M}))$. Given any f in $C^*_{\mathrm{red}}(G;E)$, it is easy to expand $f * g^*$ and to see that

$$\Psi_g(f)(\gamma) = \mathsf{P}(f * g^*)(r(\gamma))g(\gamma) = f(\gamma)|g(\gamma)|^2.$$

For each γ such that $g(\gamma) \neq 0$, there is an f in $\mathfrak{M}$ such that $f(\gamma) \neq 0$ (because $g(\gamma) \neq 0$ implies that $j(\gamma) \in Q(\mathfrak{M})$). By premultiplying f by a suitable d in $C_0(X)$ we may assume that $f(\gamma) > 0$. Now the interior of the support of g (i.e., $\{\gamma \in E \mid g(\gamma) \neq 0\}$) is mapped by r onto an open subset W in X. Since W is a second countable, locally compact Hausdorff space, W is paracompact. Thus we can find a locally finite open cover $\{U_i\}$ of W, a partition of unity $\{k_i\}$ subordinate to $\{U_i\}$, and elements f_i in $\mathfrak{M}$ such that $\mathsf{P}(f_i * g^*)(r(\gamma))$ is bounded away from zero on U_i. If we set $h_i(x) = k_i(x)/\mathsf{P}(f_i * g^*)(x)$, for $x \in U_i$, and set $h_i(x) = 0$ when $x \notin U_i$, then the

h_i's are compactly supported continuous functions on X and the series $\sum h_i \cdot \Psi_g(f_i)$ converges in $\mathfrak{M}$ to g. ∎

To complete the proof of Theorem 4.1, we need only show that under the hypothesis that G is measurewise amenable, the closure of $\mathfrak{M}_C(Q)$ in $C^*(G;E)$ is $\mathfrak{M}_0(Q)$, for any open subset Q of G. That is, we need to prove under the hypothesis of measurewise amenability that every open subset of G is a set of *"spectral synthesis"*. This, in turn, is to be expected on the basis of the analogy that exists between r--discrete principal groupoids and discrete groups.

The proof is really quite similar to the proof of Theorem 3.10 in [9]. Alternatively, one can appeal to that proof using the same technique used to prove Proposition 3.4 on the basis of Proposition II.3.2 of [16]. However, since the details are somewhat involved, we outline them for the sake of completeness.

We begin by supposing that the closure of $\mathfrak{M}(Q)$ in $C^*(G;E)$ is different from $\mathfrak{M}_0(Q)$ and appealing to Lemma 3.9 of [9] to assert that there is a representation π of $C^*(G;E)$ on a Hilbert space $\mathcal{H}$ and vectors ξ and μ in $\mathcal{H}$ such that the functional φ on $C^*(G;E)$ defined by the formula $\varphi(f) = (\pi(f)\xi, \mu)$, $f \in C^*(G;E)$, vanishes on $\mathfrak{M}_C(Q)$ but not on $\mathfrak{M}_0(Q)$. We assume, too, as we may, that π is written in integrated form as in equation (2.1). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $B_C(G)$ satisfying the conditions of Definition 3.1.1 and set

$$h_n(\gamma) = \int f_n(\gamma\alpha) \cdot \overline{f_n(\alpha)}\, \mathrm{d}\lambda^{s(\gamma)}(\alpha), \qquad n \in \mathbb{N};$$

that is, $h_n = f_n * f_n^*$. Then each h_n lies in $B_C(G)$. We define φ_n on $C^*(G;E)$ by the formula

$$\varphi_n(f) = \int h_n(\dot\gamma) \cdot f(\gamma)(L(\gamma)\xi(s(\gamma)),\, \eta(r(\gamma)))\, \mathrm{d}\nu_0(\dot\gamma),$$

$f \in C^*(G;E)$, and note that by condition (ii) in Definition 3.1.1, $\varphi_n(f) \to \varphi(f)$, for every $f \in C^*(G;E)$.

Observe that if g is a bounded Borel function with compact support contained in Q, then g is the pointwise bounded limit of a sequence of continuous functions, all of whose supports are conatined in some prescribed compact subset of Q. It follows that for such a g,

$$\int g(\gamma)(L(\gamma)\xi(s(\gamma)), \eta(r(\gamma)))\, \mathrm{d}\nu_0(\dot\gamma) = 0.$$

But $h_n(\dot\gamma) \cdot f(\gamma)$ is such a function for any f in $\mathfrak{M}_C(Q)$, and so we see that each φ_n vanishes on $\mathfrak{M}_C(Q)$. Since the φ_n converge pointwise to φ, we conclude that for some f_0 in $\mathfrak{M}_0(Q)$ and some n, $\varphi_n(f_0) \neq 0$.

Following the proof of Proposition II.3.2 in [16], we may write

$$\varphi_n(f) = (\mathrm{Ind}\,(M)(f)\xi_n, \eta_n), \tag{4.1}$$

where $f \in C^*(G;E), \mathrm{Ind}(M)$ is the representation of $C^*(G;E)$ induced by the restriction of π to $C_0(X)$, and ξ_n and η_n lie in the Hilbert space of $\mathrm{Ind}(M)$. In greater detail, which will be important for us, the Hilbert bundle for the disintegration of $\mathrm{Ind}(M)$ is given by

$$\mathcal{K}(x) = L^2(G, E, k^x) \otimes \mathcal{H}(x) = L^2(G, E, \mathcal{H}(x), k^x),$$

and the associated representation of (G, E), which we denote $(\mu, \tilde{L})$, is simply given by left translation. That is, for $\xi \in \mathcal{K}(x)$, and γ satisfying $s(\gamma) = x$ and $r(\gamma) = y$, $\tilde{L}(\gamma)\xi(\alpha) = \xi(\gamma^{-1}\alpha)$, if $r(\alpha) = y$. The vectors ξ_n and η_n in (4.1) lie in the direct integral $\int_X^{\oplus} \mathcal{K}(x)\,\mathrm{d}\eta(x)$ and so may be viewed as functions of two variables, $\xi_n(x,\gamma)$ and $\eta_n(x,\gamma)$, each supported on $\{(x,\gamma) \mid r(\gamma) = x\}$, and given by the formulae:

$$\xi_n(x,\gamma) = \Delta^{1/2}(\dot{\gamma}) \cdot \overline{f_n(\dot{\gamma})} \cdot L(\gamma^{-1})\xi(x),$$

and

$$\eta_n(x,\gamma) = \Delta^{1/2}(\dot{\gamma}) \cdot \overline{f_n((\dot{\gamma})} \cdot L(\gamma^{-1})\eta(x),$$

where ξ and η define φ. The calculation that leads to (4.1), now, may be read from the proof of Proposition II.3.2 of [16] in a straightforward fashion.

Since n will be fixed from now on, we drop it and write

$$\varphi(f) = (\mathrm{Ind}(M)(f)\xi, \eta).$$

Expanding this expression, we arrive at

$$\int_G f(\alpha\beta) \cdot (\xi(s(\beta), \beta^{-1}), \eta(s(\alpha), \alpha))\,\mathrm{d}k^x(\beta)\,\mathrm{d}k_x(\alpha)\,\mathrm{d}\mu(x) =$$

$$= \int_G f(\alpha\beta) \cdot (\xi(s(\beta), \beta^{-1}), \eta(x, \alpha))\,\mathrm{d}k^x(\beta)\,\mathrm{d}k_x(\alpha)\,\mathrm{d}\mu(x).$$

If we assume, as we may that $\dim(\mathcal{H}(x))$ is constant almost everywhere with value, say, n, than we may assume that our original Hibert bundle is the constant bundle with n-dimensional fiber $\mathcal{H}_0$. Choose an orthonormal basis $\{e_i\}$ for $\mathcal{H}_0$ and expand the inner products in the above integrals in terms of it, to obtain the representation $\varphi(f) = \sum \psi_i(f)$, where

$$\psi_i(f) = \int_G f(\alpha\beta) \cdot (\xi(s(\beta), \beta^{-1}), e_i) \cdot \overline{(\eta(x,\alpha), e_i)}\,\mathrm{d}k^x(\beta)\,\mathrm{d}k_x(\alpha)\,\mathrm{d}\mu(x).$$

If we write $\xi_i(x,\gamma)$ for $(\xi(x,\gamma),e_i)$ and $\eta_i(x,\gamma)$ for $(\eta(x,\gamma),e_i)$, we see that ξ_i and η_i are cross sections for the Hilbert bundle associated with $\operatorname{Ind}\mu$. (See pages 51 and 52 of [9].) That is,

$$\psi_i(f) = \int_G f(\alpha\beta)\cdot\xi_i(s(\beta),\beta^{-1})\cdot\overline{\eta_i(x,\alpha)}\,dk^x(\beta)\,dk_x(\alpha)\,d\mu(x) = (\operatorname{Ind}\mu(f)\xi_i,\eta_i).$$

Thus, in particular, we see that $\|\psi_i\| \leqslant \|\xi_i\|\cdot\|\eta_i\|$ for every i. Since $\|\xi\| = \left(\sum\|\xi_i\|^2\right)^{\frac{1}{2}}$ and $\|\eta\| = \left(\sum\|\eta_i\|^2\right)^{\frac{1}{2}}$, it follows that the series $\sum\|\psi_i\|$ converges and, so, $\varphi = \sum\psi_i$, with the series converging in norm.

We assume, without loss of generality, that π is faithfull. This can be arranged by adding a faithful direct summand, if necessary. It results that the closed support of μ is all of X. (Otherwise, the complement, which is open and invariant, gives rise to an ideal in $C^*(G;E)$ which is killed by π.) Consequently, we may assume that $\operatorname{Ind}\mu$ is faithful, too.

By Proposition 2.1, the weak closure of $\operatorname{Ind}\mu(C^*(G;E))$ is (isomorphic to) the Feldman-Moore von Neumann algebra, $M(G,\sigma)$, determined by μ. The functionals ψ_i may be viewed as vector functionals on $M(G,\sigma)$, and so are σ-weakly continuous. Since $\varphi = \sum\psi_i$, with the series converging in norm, φ is a σ-weakly continuous functional on $M(G,\sigma)$. (Actually, since $M(G,\sigma)$ is in standard form, φ is itself a vector functional.) It follows that the functional φ separates the *σ-weak closure* of $\mathfrak{M}_c(Q)$ from the *σ-weak closure* of $\mathfrak{M}_0(Q)$. But both of these σ-weak closures are bimodules over $L^\infty(\mu)$ regarded as multiplication operators on the diagonal of G, i.e., both of these are bimodules over the Cartan subalgebra of $M(G,\sigma)$. On the other hand, it is clear that the closure in $L^2(G,\nu^{-1})$ of the intersection of each of these spaces with $L^2(G,\nu^{-1})$ coincides with $L^2(Q)$, the set of functions in $L^2(G,\gamma^{-1})$ supported on Q. Thus by Theorem 2.5 in [12], the two σ-weak closures coincide. This contradiction completes the proof. ∎

We remark in passing that at first glance, one might be inclined to think that the last part of the argument for the proof of Theorem 4.1 is an immediate consequence of Proposition 3.4. But all that Proposition 3.4 shows is that φ is the weak limit of vector functionals affiliated with $\operatorname{Ind}\mu$. It does *not* show that φ can be realized in the Hilbert space of $\operatorname{Ind}\mu$, namely $L^2(G,E,\nu^{-1})$. That appears to require special argument.

We conclude by briefly indicating extensions to T-groupoids of three key results of [9]. Recall that an *open* set P in G is called a *preorder* (on X or in G) in case $X\subseteq P$ and $P\cdot P\subseteq P$, where $P\cdot P = \{\alpha\beta \mid (\alpha,\beta)\in(P\times P)\cap G^{(2)}\}$. A preorder P satisfying

$P \cap P^{-1} = X$ is called a *partial* order. A partial order P satisfying $P \cup P^{-1} = G$ is called a *total order*. The term "total order" is a little imprecise, unless $G = X \times X$. But note that a total order in the sense we are using the term totally orders each equivalence class. Given a preorder P on G, we write $\mathfrak{A}(P)$ for what we were just writing as $\mathfrak{M}(P)$. The change in notation is justified by the following result that generalizes the Theorem 4.1 in [9].

THEOREM 4.5. *Suppose G is measurewise amenable. For each preorder P in G, $\mathfrak{A}(P)$ is a norm closed subalgebra of $C^*(G; E)$ containing $C_0(X)$ Conversely, each subalgebra $\mathfrak{A}$ of $C^*(G; E)$ containing $C_0(X)$ is of the form $\mathfrak{A}(P)$ for a unique preorder P. The correspondence $P \longrightarrow \mathfrak{A}(P)$ is an inclusion preserving bijection between the collection of preorders in G and norm closed subalgebras of $C^*(G; E)$ containing $C_0(X)$.*

Proof. All that needs to be checked, really, is the assertion that $\mathfrak{A}(P)$ is a subalgebra of $C^*(G; E)$ if and only if P is a preorder. Since $X \subseteq P$ if and only if $C_0(X) \subseteq \mathfrak{A}(P)$, all that needs checking is the assertion that $\mathfrak{A}(P) \cdot \mathfrak{A}(P) \subseteq \mathfrak{A}(P)$ if and only if $P \cdot P \subseteq P$. For this it suffices to note first that for f and g in N_C, $j(\text{supp}(f * g)) = j(\text{supp}(f)) \cdot j(\text{supp}(g))$, an easy calculation, and second that in the proof of Proposition 4.4 it is shown that every element in $C_C(P)$ is the sum of elements in N_C supported in P. ∎

The following proposition is a generalization of Lemma 4.3 of [MS1] and describes ideals for $\mathfrak{A}(P)$ in terms of subsets of G. Its proof is a straightforward application of the key observations in the proof of Theorem 4.5 and so will be omitted.

PROPOSITION 4.6. *Assume that G is measurewise amenable and that P is a preorder in G. If Q is an open subset of P satisfying $P \cdot Q \cdot P \subseteq Q$, then $\mathfrak{M}(Q)$ is a (norm closed, two sided) ideal in $\mathfrak{A}(P)$. The map $Q \longrightarrow \mathfrak{M}(Q)$, from the collection of open sets Q satisfying $P \cdot Q \cdot P \subseteq Q$ to such ideals for $\mathfrak{A}(P)$, is an order preserving bijection between the colllection of such sets and the collection of ideals.*

Suppose that B is a C^*-algebra and that D is simply an abelian C^*-subalgebra. A subalgebra $\mathfrak{T}$ of B is called *triangular* (with respect to D or with diagonal D) in case $\mathfrak{T} \cap \mathfrak{T}^* = D$, where $\mathfrak{T}^* = \{a^* | a \in \mathfrak{T}\}$. A triangular algebra $\mathfrak{T}$ in B is called *maximal triangular* in case it is not contained in any larger triangular algebra with the same diagonal. If, in addition, $\mathfrak{T} + \mathfrak{T}^*$ is norm dense in B, then we say that $\mathfrak{T}$ is *strongly maximal*. Peters, Poon and Wagner first gave examples of maximal triangular algebras that are not strongly maximal in [14]. Their finding was rather surprising in view of work that had been done in the von Neumann algebra setting. See [12] and [10].

A notion that is related to triangularity is "subdiagonality". In the von Neumann algebra setting, the notion is due to Arveson [1]; in the C^*-setting, it is due to Kawamura and Tomiyama [5]. To define it, suppose that B is a C^*-algebra and suppose first that B is unital with identity 1. A closed subalgebra A of B is called a *(C^*-)subdiagonal subalgebra* of B (with diagonal $A \cap A^*$) in case $1 \in A$, $A + A^*$ is norm dense in B, and there is a faithful conditional expectation, say Φ, from B onto $A \cap A^*$ that is multiplicative on A; i.e., $\Phi(ab) = \Phi(a) \cdot \Phi(b)$, for all a and b in A. If B is not unital and A is a subalgebra of B that becomes a subdiagonal subalgebra of B after a unit is adjoined to A and B, then we continue to say that A is a (C^*-) subdiagonal subalgebra of B. If A is a subdiagonal subalgebra of B with respect to the conditional expectation $\Phi: B \longrightarrow A \cap A^*$ and if A is not contained in any larger subalgebra of B with the same diagonal on which Φ is multiplicative, then A is called a *maximal* subdiagonal subalgebra of B (with respect to Φ). In [5] it is shown that every subdiagonal subalgebra A of a C^*-algebra B is contained in a maximal subdiagonal subalgebra, namely, $A_m := \{a \in B \mid b, c \in A, \text{ with } b \in \ker(\Phi),\ \Phi(bac) = \Phi(cab) = 0\}$. Note that in [5] it is assumed that B and A are unital, but this is not a necessary restriction.

The following theorem, which generalizes Theorem 4.2 of [9], relates the notions of strongly maximal triangular subalgebras and maximal subdiagonal subalgebras when the containing C^*-algebra is nuclear and has a diagonal in the sense of Kumjian (which serves as a diagonal for the triangular or subdiagonal subalgebra). It exhibits very nicely coordinate free statements that seem to require coordinates for their proofs.

THEOREM 4.7. *Let B be a nuclear C^*-algebra with digaonal D and conditional expectation $\mathbf{P}$. Suppose $\mathfrak{T}$ is a triangular subalgebra of B with diagonal D. Then $\mathbf{P}$ is multipicative on $\mathfrak{T}$. Furthermore, $\mathfrak{T}$ is subdiagonal (with respect to D and $\mathbf{P}$) if and only if $\mathfrak{T}$ is strongly maximal triangular. In this case, $\mathfrak{T}$ is, in fact, maximal subdiagonal. Finally, if B is realized as $C^*(G;E)$, with $\mathfrak{T}$ realized as $\mathfrak{A}(P)$ for a suitable preorder P, then P is a partial order; P is a total order if and only if $\mathfrak{T}$ is strongly maximal triangular.*

Proof. Realize B as $C^*(G;E)$ and $\mathfrak{T}$ as $\mathfrak{A}(P)$ for a suitable preorder P. By Theorem 4.5, it is clear that $\mathfrak{T} \cap \mathfrak{T} = \mathfrak{A}(P \cap P^{-1}) = \mathfrak{A}(G^{(0)})$, if the diagonal of $\mathfrak{T}$ is assumed to be D. Thus it is clear that P must be a partial order if $\mathfrak{T}$ is triangular. As is shown in [5], $\mathbf{P}$ is given by the formula

$$\mathbf{P}(f)(\gamma) = \begin{cases} f(\gamma), \ \gamma \in j^{-1}(G^{(0)}) \\ 0, \ \text{otherwise} \end{cases},$$

$f \in C^*(G;E)$. Thus $\ker(\mathbf{P}) \cap \mathfrak{A}(P) = \mathfrak{M}(P \backslash G^{(0)})$. Since $P \backslash G^{(0)}$ is an open subset of P that clearly satisfies the conditions of Proposition 4.6, it follows that $\ker(\mathbf{P}) \cap \mathfrak{A}(P)$

is an ideal in $\mathfrak{A}$. Since $\mathbf{P}$ is idempotent, it follows that $\mathbf{P}$ is multiplicative on $\mathfrak{T}$. It is now evident that $\mathfrak{T}$ is strongly maximal triangular if and only if $\mathfrak{T}$ is subdiagonal. It is also evident that if $P \cup P^{-1} = G$, then $\mathfrak{T}$ is strongly maximal (and hence subdiagonal). But this condition clearly implies, too, that $\mathfrak{T}$ is maximal subdiagonal because if $\mathfrak{T}'$ is any larger subdiagonal algebra with the same diagonal D, and if $\mathfrak{T}'$ is written as $\mathfrak{A}(P')$ for an open partial order P', then the conditions that $P \subset P'$ and $P \cup P^{-1} = G$ are easily seen to imply that $P = P'$. All that remains, then, is to show that if $\mathfrak{T}$ is strongly maximal triangular, then $P \cup P^{-1} = G$. Suppose to the contrary that there is a $\dot{\gamma} \in G \backslash (P \cup P^{-1})$ and choose a function $f \in C_C(G;E)$ such that f does not vanish on $j^{-1}(\dot{\gamma})$. By assumption and Theorem 4.1, there are sequences $\{f_n\}$ and $\{g_n\}$ in $C_C^*(G;E) \cap \mathfrak{T}$ such that $f_n + g_n^* \longrightarrow f$ in $C^*(G;E)$. By Proposition 4.2, $f_n + g_n^* \longrightarrow f$ pointwise. However, f_n is supported on $j^{-1}(P)$ and so vanishes on $j^{-1}(G \backslash P)$. Likewise, g_n^* vanishes on $j^{-1}((G \backslash P)^{-1}) = j^{-1}(G \backslash P^{-1})$. Consequently, $f_n + g_n^*$ vanishes on $j^{-1}((G \backslash P) \cap (G \backslash (P^{-1}))) = j^{-1}(G \backslash (P \cup P^{-1}))$. Therefore

$$0 \equiv f_n(\alpha) + g_n^*(\alpha) \longrightarrow f(\alpha) \neq 0,$$

for any $\alpha \in j^{-1}(\dot{\gamma})$. This contradiction shows that $P \cup P^{-1} = G$ and completes the proof.

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PAUL S. MUHLY
Department of Mathematics,
University of Iowa,
Iowa City, Iowa 52242,
U.S.A.

CHAOXIN QIU
Department of Mathematics,
University of Nevada,
Reno, Nevada 89557,
U.S.A.

BARUCH SOLEL
Department of Mathematics,
Technion,
Haifa 35000,
Israel

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