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THE SOFT TORUS III: THE FLIP

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1. INTRODUCTION

As a loose definition of what is a softening of a compact Hausdorff space X, we mean a non-commutative C^*-algebra A such that the abelianization of A is isomorphic to $C(X)$. In case of the torus $\mathbf{T}^2$, the second author defined, for each ε in [0, 2], a soft-torus A_ε. This was universally generated by two unitaries u and v subject to the relation

$$\|uv - vu\| \leqslant \varepsilon$$

In this paper, we examine the prospect of softening the sphere $\mathbf{S}^2$.

The soft-tori were tractable because they were generated by unitaries. As $C(\mathbf{S}^2)$ is not, in any natural way, generated by unitaries, the direct approach has not been successful. We proceed indirectly via the flip $\sigma : u \mapsto u^{-1}, v \mapsto v^{-1}$ on the soft torus. The connection with the sphere is that $A_0 = C(\mathbf{T}^2)$,

$$C(\mathbf{T}^2)^\sigma \cong C(\mathbf{S}^2)$$

and

$$C(\mathbf{T}^2) \rtimes \mathbf{Z}/2 \subseteq C(\mathbf{S}^2, M_2)$$

See [1,2,3,6] for details.

Accepting $C(\mathbf{T}^2) \rtimes_\sigma \mathbf{Z}/2$ as a resonable replacement for $C(\mathbf{S}^2)$, we set about softening it. Using generators and relations, we have a more general notion of softening. Giving a C^*-algebra A generated by elements, $x_1, \ldots, x_n$, universal for some relations $p_j(x_1, \ldots, x_n) = 0$, we replace some or all of these relations by $\|p_j(x_1, \ldots, x_n)\| \leqslant \varepsilon$. See [7] for a general discussion of such softenings.

We consider two softenings of $C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}/2$. The first is just $A_\varepsilon \rtimes_\sigma \mathbb{Z}/2$. The second is obtained by replacing the flip by a "soft-flip". That is, rather than adjoining to A_ε an order-two unitary w such that

$$wuw^* = u^{-1} \quad \text{and} \quad wvw^* = v^{-1},$$

we require only that

$$\|wuw^* - u^{-1}\| \leqslant \varepsilon \quad \text{and} \quad \|wvw^* - v^{-1}\| \leqslant \varepsilon.$$

Soft crossed products are defined in a more general setting, but we concentrate on actions of $\mathbb{Z}$ and the group

$$\mathbb{Z} \rtimes \mathbb{Z}/2 \cong \mathbb{Z}/2 * \mathbb{Z}/2.$$

For $0 \leqslant \theta \leqslant 1$, let A_θ^{rot} denote the associated rotation algebra [9]. These are not softenings of $C(\mathbb{T}^2)$ as their abelianizations are zero, except when $\theta = 0$. Rather, these are quantizations (in the sense of [10]) of $C(\mathbb{T}^2)$. Notice that these are natural surjections $A_\varepsilon \longrightarrow A_\theta^{\text{rot}}$ when $\theta \approx 0$. The non-commutative spheres $A_\theta^{\text{rot}} \rtimes \mathbb{Z}/2$ studied in [1], [2] are seen as quantized spheres, not softened spheres. The remarkable discovery in [3] that $A_\theta^{\text{rot}} \rtimes \mathbb{Z}/2$ is AF, for θ irrational, does not seem to have any counterpart regarding $A_\varepsilon \rtimes_\sigma \mathbb{Z}/2$.

We calculate the K-theory of $A_\varepsilon \rtimes_\sigma \mathbb{Z}/2$ and of the soft crossed product. In each case, the map onto $C(\mathbb{T}^2) \rtimes \mathbb{Z}/2$ induces an isomorphism on K-theory. We also show that the regular and soft crossed products form continuous fields of C^*-algebras. Our approach to these problems is to realize a soft crossed product of A by G as a regular crossed product $B \rtimes G$, where B contains many copies of A. Exact sequences in K-theory reduce the problem to finding special retractions of B onto A.

These softenings are not, we hope, the end of the story. A truly soft torus would be a unital C^*-algebra A'_ε, generated by two elements subject to the relations:

$$\|ab - ba\| \leqslant \varepsilon$$

$$\|a^*a - 1\| \leqslant \varepsilon, \quad \|aa^* - 1\| \leqslant \varepsilon,$$

$$\|b^*b - 1\| \leqslant \varepsilon \quad \text{and} \quad \|bb^* - 1\| \leqslant \varepsilon.$$

An open question is whether the natural surjection $A'_\varepsilon \longrightarrow C(\mathbb{T}^2)$ is an isomorphism on K-theory.

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2. THE FLIP ON THE SOFT TORUS

We begin by recalling the definition of the soft torus from [4].

DEFINITION 2.1. For every ε in the interval [0, 2] we let A_ε be the universal unital C^*-algebra generated by unitary elements u_ε and v_ε subject to the relation

$$\|u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon\| \leqslant \varepsilon.$$

By the flip on A_ε we mean the automorphism σ defined by $\sigma(u_\varepsilon) = u_\varepsilon^{-1}$ and $\sigma(v_\varepsilon) = v_\varepsilon^{-1}$. We shall also denote by σ the obvious action of $\mathbf{Z}_2$ on A_ε.

Proposition 2.3 in [4] states that for all ε in [0, 2] A_ε is isomorphic to the crossed product

$$A_\varepsilon \cong B_\varepsilon \rtimes_\tau \mathbf{Z},$$

where B_ε is the universal unital C^*-algebra generated by a sequence $\{u_n : n \in \mathbf{Z}\}$ of unitary elements subject to the relations

$$\|u_{n+1} - u_n\| \leqslant \varepsilon, \quad n \in \mathbf{Z},$$

and τ is the automorphism of B_ε specified by

$$\tau(u_n) = u_{n+1}, \quad n \in \mathbf{Z}.$$

We shall be interested also in the automorphisms α_0 and α_1 of B_ε defined by

$$\alpha_0(u_n) = u_{-n}^{-1}, \quad n \in \mathbf{Z},$$

and

$$\alpha_1(u_n) = u_{1-n}^{-1}, \quad n \in \mathbf{Z}.$$

Clearly, both α_0 and α_1 are involutions so together they define an action of the free product $\mathbf{Z}_2 * \mathbf{Z}_2$ on B_ε, which we shall denote by $\alpha_1 * \alpha_2$.

PROPOSITION 2.2. *For all ε in* [0, 2] *there is an isomorphism*

$$A_\varepsilon \rtimes_\sigma \mathbf{Z}_2 \cong B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2).$$

Proof. It is easy to check, using the universal property of crossed product algebras, that $A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$ can be characterized as being the universal unital C^*-algebra generated by a set $G \doteq \{u, v, y\}$ of unitary elements such that

(i) $\|uv - vu\| \leqslant \varepsilon$

(ii) $yuy^{-1} = u^{-1}$

(iii) $yvy^{-1} = v^{-1}$

(iv) $y^2 = 1$.

On the other hand, $B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2)$ is the universal unital C^*-algebra generated by a set

$$H = \{u_n : n \in \mathbf{Z}\} \cup \{z_1, z_2\}$$

of unitary elements under the relations

(i) $\|u_{n+1} - u_n\| \leqslant \varepsilon$

(ii) $z_1 u_n z_1^{-1} = u_{-n}^{-1}$

(iii) $z_2 u_n z_2^{-1} = u_{1-n}^{-1}$

(iv) $z_1^2 = z_2^2 = 1$.

Consider the functions

$$\varphi : G \longrightarrow B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2)$$

and

$$\psi : H \longrightarrow A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$$

where φ is defined by $\varphi(u) = u_0$, $\varphi(v) = z_1 z_2$ and $\varphi(y) = z_1$ while ψ is given by $\psi(u_n) = v^{-n} u v^n$, $\psi(z_1) = y$ and $\psi(z_2) = yv$.

The reader may easily verify that both φ and ψ extend to $*$-homomorphisms and that they are each other's inverse. ■

DEFINITION 2.3. We shall say that two C^*-dynamical systems (A, α, Γ) and (A', α', Γ') are *homotopically equivalent* if there are homomorphisms $\varphi : A \longrightarrow A'$ and $\psi : A' \longrightarrow A$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are both homotopic to the respective identity maps in such a way that the homotopies involved commute with the corresponding group actions.

Clearly, homotopically equivalent dynamical systems give rise to homotopically equivalent crossed product algebras.

Let us now consider the following C^*-dynamical systems:

(a) $(C(\mathbf{S}^1), \alpha, \mathbf{Z}_2)$

(b) $(B_\varepsilon, \alpha_0, \mathbf{Z}_2)$

(c) $(B_\varepsilon, \alpha_1, \mathbf{Z}_2)$

where the action α of $\mathbf{Z}_2$ on $C(\mathbf{S}^1)$ given by $\alpha(z) = z^{-1}$ (by z we mean the standard generator of $C(\mathbf{S}^1)$).

Consider the map $\beta : B_\varepsilon \longrightarrow C(\mathbf{S}^1)$ given by $\beta(u_n) = z$ for all n. It is clear that β is an equivariant map regardless of which of the above two actions we choose for B_ε.

PROPOSITION 2.4. *If* $\varepsilon < 2$ *then* β *is a homotopy equivalence from* $(B_\varepsilon, \alpha_i, \mathbb{Z}_2)$ *to* $(C(\mathbb{S}^1), \alpha, \mathbb{Z}_2)$ *for* $i = 0, 1$.

Proof. The proof for $i = 0$ is essentially contained in [4]. In fact, one just needs to observe that the homotopy given in Theorem 2.2 of [4] is equivariant.

Let us now consider the case $i = 1$. Put $m = (u_0 + u_1)/2$. Since $\varepsilon < 2$ one can easily check that m is an invertible element of B_ε. Let therefore $\varphi : C(\mathbb{S}^1) \longrightarrow B_\varepsilon$ be given by $\varphi(z) = \hat{m}$ where $\hat{m}$ stands for the unitary part of the polar decomposition of m, i.e.,

$$\hat{m} = m|m|^{-1} = m(m^*m)^{-1/2}.$$

We have that $\alpha_1(m) = m^*$, and so

$$\alpha_1(\varphi(z)) = \alpha_1(m(m^*m)^{-1/2}) = m^*(mm^*)^{-1/2} =$$

$$= (m^*m)^{-1/2}m^* = (\hat{m})^* = \varphi(\alpha(z)),$$

from which it follows that φ is equivariant.

Clearly, $\beta \circ \varphi$ is the identity on $C(\mathbb{S}^1)$ and so the proof will be complete once we check that $\varphi \circ \beta$ is equivariantly homotopic to the identity map on B_ε. As a first step we claim that $\varphi \circ \beta$ is equivariantly homotopic to the map

$$\psi : B_\varepsilon \longrightarrow B_\varepsilon$$

given by

$$\psi(u_n) = \begin{cases} u_0 & \text{if } n \leqslant 0 \\ u_1 & \text{if } n \geqslant 1 \end{cases}.$$

Let, for all $t \in [0, 1]$,

$$a_t = (1-t)u_0 + tu_1$$

and put $v_t = \hat{a}_t$. Define $\psi_t : B_\varepsilon \longrightarrow B_\varepsilon$, for all $t \in \left[0, \frac{1}{2}\right]$, by

$$\psi_t(u_n) = \begin{cases} v_t & \text{if } n \leqslant 0 \\ v_{1-t} & \text{if } n \geqslant 1 \end{cases}.$$

In order to verify that ψ_t is a well-defined endomorphism of B_ε for each t one needs to check that $\|\psi_t(u_{n+1}) - \psi_t(u_n)\| \leqslant \varepsilon$ for all n. This follows from inequality

$$\|v_t - v_s\| \leqslant \varepsilon, \quad t, s \in [0, 1],$$

which we prove next. We have

$$\|v_t - v_s\| = \|a_t|a_t|^{-1} - a_s|a_s|^{-1}\| = \|u_0^*a_t|u_0^*a_t|^{-1} - u_0^*a_s|u_0^*a_s|^{-1}\|.$$

If we now let $b_t = u_0^* a_t = 1 - t + t u_0^* u_1$ then

$$\|v_t - v_s\| = \|\hat{b}_t - \hat{b}_s\|.$$

The relevant fact to observe here is that b_t is in the commutative algebra A generated by $u_0^* u_1$. Therefore,

$$\|\hat{b}_t - \hat{b}_s\| = \sup|\chi(\hat{b}_t) - \chi(\hat{b}_s)|$$

where the supremum is taken over all complex homomorphisms χ of A. Now obseve that for all such χ the path $\chi(b_z)$ is just the segment joining

$$\chi(b_0) = 1$$

and

$$\chi(b_1) = \chi(u_0^* u_1)$$

which are points in the unit circle within ε of each other. Now, $\chi(\hat{b}_t)$ is the radial projection of $\chi(b_t)$ onto the unit circle and hence it lies in the arc from $\chi(b_0)$ to $\chi(b_1)$. One now needs to verify the elementary fact that any two points in the arc are within ε of each other. This shows ψ_t to be well defined for all t.

The proof that ψ_t is equivariant is a straightforward consequence of the fact that $\alpha_1(a_t) = a_{1-t}^*$. Our assertion is thus proved since $\psi_0 = \psi$ and $\psi_{1/2} = \varphi \circ \beta$.

A similar argument to the one used in [4] shows ψ to be equivariantly homotopic to the identity. This completes the proof. ∎

THEOREM 2.5. *If $\varepsilon < 2$ then the natural homomorphism*

$$\eta : A_\varepsilon \rtimes_\sigma \mathbf{Z}_2 \longrightarrow A_0 \rtimes_\sigma \mathbf{Z}_2$$

induces isomorphisms at the level of K*-theory groups. Thus $A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$ has $\mathbf{Z}^6$ as its* K_0 *group, and trivial* K_1 *group.*

Proof. Using the isomorphisms of Proposition 1.2 it is enough to prove the corresponding result for the natural map

$$\varphi : B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2) \longrightarrow B_0 \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2).$$

Recall that by [8] there is a cyclic exact sequence

$$\begin{array}{ccc} \mathrm{K}_0(B_\varepsilon) \to (B_\varepsilon \rtimes_{\alpha_0} \mathbf{Z}_2) \oplus \mathrm{K}_0(B_\varepsilon \rtimes_{\alpha_1} \mathbf{Z}_2) & \to & \mathrm{K}_0(B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2)) \\ \uparrow & & \downarrow \\ \mathrm{K}_1(B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2)) \leftarrow \mathrm{K}_1(B_\varepsilon \rtimes_{\alpha_0} \mathbf{Z}_2) \oplus \mathrm{K}_1(B_\varepsilon \rtimes_{\alpha_1} \mathbf{Z}_2) & \leftarrow & \mathrm{K}_1(B_\varepsilon). \end{array}$$

If we also write down the corresponding sequence for $\varepsilon = 0$ as well as the natural maps between the two sequences we see that the result follows from Proposition 2.4 and the five lemma. See [2,6] for the K-groups. ■

LEMMA 2.6. *Let Γ be a discrete amenable group and let, for each ε in $[0, 2]$, α_ε be an action of Γ on B_ε such that the canonical map $\varphi_\varepsilon : B_2 \longrightarrow B_\varepsilon$ is Γ-equivariant. Then there exists a continuous field of C^*-algebra over the interval $[0, 2]$ such that $B_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is the fiber over ε and such that for every $a \in B_2 \rtimes_{\alpha_2} \Gamma$ the map*

$$\varepsilon \in [0, 2] \mapsto \bar{\varphi}_\varepsilon(a) \in B_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$$

is a continuous section, where $\bar{\varphi}_\varepsilon : B_2 \rtimes_{\alpha_2} \Gamma \longrightarrow B_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is the natural extension of φ_ε.

Proof. Let L_ε be the kernel of φ_ε. We claim that

$$L_\varepsilon = \overline{\bigcup_{\varepsilon' > \varepsilon} L_{\varepsilon'}} \quad \text{for } \varepsilon \in [0, 2)$$

and that

$$L_\varepsilon = \bigcap_{\varepsilon' < \varepsilon} L_{\varepsilon'} \quad \text{for } \varepsilon \in (0, 2].$$

The first assertion can be proved based on the universal properties of B_ε and of full crossed products as it was done in [5], Proposition 1.2.

As for our second claim, we first recall that, if B_ε is the kernel of φ_ε, then we know from [5] that for $\varepsilon \in (0, 2)$,

$$\bigcap_{\varepsilon' < \varepsilon} K'_\varepsilon = K_\varepsilon.$$

The same fact also holds for $\varepsilon = 2$ since, with the notations of [5], we have $K_\varepsilon = J_\varepsilon \cap B_2$ and hence

$$\bigcap_{\varepsilon' < 2} K_{\varepsilon'} \subseteq \bigcap_{\varepsilon' < 2} J_{\varepsilon'} = (0) = K_2.$$

We next need to check that Lemma 2.5 of [5] extends to crosed products by discrete amenable groups. They key point for this is the fact that an element x in $B_\varepsilon \rtimes_{\alpha_\varepsilon} \Gamma$ is zero if and only if $E(x\delta_{t^{-1}}) = 0$ for all $t \in \Gamma$, which is a consequence of the fact that Γ is amenable.

Now the proof of [4], Theorem 2.6 extends to our case and our assertion follows. To conclude our proof we may proceed as in Theorem 3.4 of [4]. ■

THEOREM 2.7. *There exists a continuous field of C^*-algebras over the interval $[0, 2]$ such that $A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$ is the fiber over ε and such that for all $a \in A_2 \rtimes_\sigma \mathbf{Z}_2$ the function*

$$\varepsilon \in [0, 2] \mapsto \varphi_\varepsilon(a) \in A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$$

is a continuous section, where $\varphi_\varepsilon : A_2 \rtimes \mathbf{Z}_2 \longrightarrow A_\varepsilon \rtimes \mathbf{Z}_2$ is the natural map.

Proof. This follows from the previous lemma on noting that $A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$ is isomorphic to $B_\varepsilon \rtimes_{\alpha_0 * \alpha_1} (\mathbf{Z}_2 * \mathbf{Z}_2)$ and that $\mathbf{Z}_2 * \mathbf{Z}_2$ is amenable. ∎

3. SOFT CROSSED PRODUCTS

The object of study in [4], namely, the soft torus, is the result of a construction which we shall call a "soft crossed product".

In order to precisely define this concept let (A, α, Γ) be a C^*-dynamical system where Γ is a discrete group and A is unital.

Assume that A is generated as a C^*-algebra by a set $\{a_i\}_{i \in I}$ and that Γ is generated as a group by a set $\{g_j\}_{j \in J}$.

DEFINITION 3.1. For every $\varepsilon \geqslant 0$ the *soft crossed product* C^*-algebra associated to the C^*-dynamical system (A, α, Γ) and the generating sets $\{a_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ is the universal unital C^*-algebra generated by a copy of A and one unitary element u_g for each g in Γ subject to the relations:

(i) $\|u_{g_j} a_i u_{g_j}^{-1} - \alpha_{g_j}(a_i)\| \leqslant \varepsilon, \quad i \in I, j \in J,$

(ii) $u_g u_h = u_{gh}, \quad g, h \in \Gamma.$

The resulting algebra is denoted $A \rtimes_\alpha^\varepsilon \Gamma$.

Note that when $\varepsilon = 0$ we recover the "hard", i.e., the usual, crossed product algebra $A \rtimes_\alpha \Gamma$ regardless of the choice of generators.

The soft torus clearly fits into this picture with $C(\mathbf{S}^1)$ in the role of A, $\mathbf{Z}$ in place of Γ and the action being the trivial action. Also we should choose z as the generator of A and 1 as the generator of Γ.

On the other hand, the algebra

$$A_\varepsilon \rtimes_\sigma \mathbf{Z}_2 = (C(\mathbf{S}^1) \rtimes^\varepsilon \mathbf{Z}) \rtimes_\sigma \mathbf{Z}_2$$

can be considered a "semi-soft" crossed product since the action of $\mathbf{Z}$ is "soft" and the action of $\mathbf{Z}_2$ is "hard".

It seems therefore natural to investigate the corresponding "truly soft" crossed product. That is, let Γ be the semi-direct product $\Gamma = \mathbf{Z} \rtimes \mathbf{Z}_2$ where the action of $\mathbf{Z}_2$ on $\mathbf{Z}$ is given by the involution $n \mapsto -n$.

Denote the positive generator of $\mathbf{Z}$ by v and the generator of $\mathbf{Z}_2$ by y so that $\{v, y\}$ generates Γ. In fact it is well known that Γ admits the presentation

$$\Gamma = \langle v, y : yvy^{-1} = v^{-1}, y^2 = 1 \rangle.$$

Consider the action γ of Γ on $C(\mathbf{S}^1)$ given by

$$\gamma_v(z) = z$$

$$\gamma_y(z) = z^{-1}.$$

The "semi-soft crossed product" mentioned above is thus obtained by a "soft" action of v and a "hard" action of y as is clearly seen by the description of $A_\varepsilon \rtimes_\sigma \mathbf{Z}_2$ given in the proof of Proposition 2.2.

We now propose to compute the K-theory groups of $C(\mathbf{S}^1) \rtimes_\gamma^\varepsilon \Gamma$ for $\varepsilon < 2$. In order to do this we must first choose our generators. It is well-known that Γ is isomorphic to the free product group $\mathbf{Z}_2 * \mathbf{Z}_2$ under the isomorphism taking the canonical generators of the latter group into y and vy respectively. We shall therefore choose y and vy as our generators for Γ. Also, let us take z to be our choice of generators for $C(\mathbf{S}^1)$.

PROPOSITION 3.2. *With the above choice of generators one has*

$$C(\mathbf{S}^1) \rtimes_\gamma^\varepsilon \Gamma \cong B_\varepsilon \rtimes_{\alpha_1 * \alpha_3} (\mathbf{Z}_2 * \mathbf{Z}_2)$$

where α_3 is the involution of B_ε defined by $\alpha_3(u_n) = u_{3-n}^*$.

Proof. In order to simplify our notation we shall describe B_ε as the universal unital C^*-algebra generated by the set

$$\{a_n : n \in \mathbf{Z}\} \cup \{b_n : n \in \mathbf{Z}\}$$

of unitary elements subject to the relations

(i) $\|a_n - b_n\| \leqslant \varepsilon, \ n \in \mathbf{Z}$
(ii) $\|b_n - a_{n+1}\| \leqslant \varepsilon, \ n \in \mathbf{Z}$.

In other words, we are relabeling the old generating set by $a_n = u_{2n}$ and $b_n = u_{2n+1}$.

The automorphisms α_1 and α_3 are then given by

(i) $\alpha_1(a_n) = b_{-n}^{-1}$
(ii) $\alpha_1(b_n) = a_{-n}^{-1}$

and

(iii) $\alpha_3(a_n) = b_{1-n}^{-1}$
(iv) $\alpha_3(b_n) = a_{1-n}^{-1}$.

We may therefore describe $B_\varepsilon \rtimes_{\alpha_1 * \alpha_3} (\mathbb{Z}_2 * \mathbb{Z}_2)$ as being the universal unital C^*-algebra generated by the set of unitary elements

$$\{a_n : n \in \mathbb{Z}\} \cup \{b_n : n \in \mathbb{Z}\} \cup \{w_1, w_3\}$$

subject to the relations

(i) $\|a_n - b_n\| \leqslant \varepsilon$
(ii) $\|b_n - a_{n+1}\| \leqslant \varepsilon$
(iii) $w_1 a_n w_1^{-1} = b_{-n}^{-1}$
(iv) $w_1 b_n w_1^{-1} = a_{-n}^{-1}$
(v) $w_3 a_n w_3^{-1} = b_{1-n}^{-1}$
(vi) $w_3 b_n w_3^{-1} = a_{1-n}^{-1}$
(vii) $w_1^2 = w_3^2 = 1$.

On the other hand, $C(\mathbb{S}^1) \rtimes_\gamma^\varepsilon \Gamma$ is, given our choice of generators, the universal unital C^*-algebra on unitaries z, x_1 and x_2 with

(i) $\|x_1 z x_1^{-1} - z^{-1}\| \leqslant \varepsilon$
(ii) $\|x_2 z x_2^{-1} - z^{-1}\| \leqslant \varepsilon$
(iii) $x_1^2 = x_2^2 = 1$.

The maps

$$B_\varepsilon \rtimes_{\alpha_1 * \alpha_3} (\mathbb{Z}_2 * \mathbb{Z}_2) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} C(\mathbb{S}^1) \rtimes_\gamma^\varepsilon \Gamma$$

given by

(i) $\varphi(b_n) = (x_1x_2)^n z (x_1x_2)^{-n}$
(ii) $\varphi(a_n) = x_2 (x_1x_2)^{-n} z^* (x_1x_2)^n x_2$
(iii) $\varphi(w_3) = x_1$
(iv) $\varphi(w_1) = x_2$

and

(i) $\psi(x_1) = w_3$
(ii) $\psi(x_2) = w_1$
(iii) $\psi(z) = b_0$

show these algebras to be isomorphic. ∎

THEOREM 3.3. *The canonical map*

$$C(\mathbb{S}^1) \rtimes_\gamma^\varepsilon \Gamma \longrightarrow C(\mathbb{S}^1) \rtimes_\gamma \Gamma$$

is an isomorphism at the level of K*-theory groups as long as* $\varepsilon < 2$.

Proof. Follows easily by [8] as in the proof of Theorem 2.5. ∎

THEOREM 3.4. *There exists a continuous field of C^*-algebras over the interval* [0, 2] *such that $C(\mathbf{S}^1) \rtimes_\gamma^\varepsilon \Gamma$ is the fiber ε and such that for all $a \in C(\mathbf{S}^1) \rtimes_\gamma^2 \Gamma$ the function*

$$\varepsilon \in [0, 2] \longrightarrow \varphi_\varepsilon(a) \in C(\mathbf{S}^1) \rtimes_\gamma^\varepsilon \Gamma$$

is a continuous section.

Proof. The proof is similar to the proof of Theorem 2.7. ■

REMARK 3.5. Our definition of soft crossed product relies heavily on a choice of generators. A very natural choice of a generating set for Γ is clearly the set $\{v, y\}$. We can prove the soft crossed product defined under that choice to be isomorphic to a hard crossed product of Γ by the algebra C_ε defined to be the universal unital C^*-algebra generated by the set

$$\{u_{n,m} : n \in \mathbf{Z}, m \in \mathbf{Z}_2\}$$

subject to

(i) $\|u_{n,0} - u_{n,1}\| \leqslant \varepsilon, \quad n \in \mathbf{Z}$

(ii) $\|u_{n,m} - u_{n+1,m}\| \leqslant \varepsilon, \quad n \in \mathbf{Z}, m \in \mathbf{Z}_2$.

Nevertheless, we are unable to identify the homotopy class of C_ε.

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REFERENCES

1. BRATTELI, O.; ELLIOTT, G. A.; EVANS, D. E.; KISHIMOTO, A., Noncommutative spheres. I, *International J. Math.*, **2**(1991), 139–166.
2. BRATTELI, O.; ELLIOTT, G. A.; EVANS, D. E.; KISHIMOTO, A., Noncommutative spheres. II: Rational rotations, *J. Operator Theory*, to appear.
3. BRATTELI, O.; KISHIMOTO, A., Noncommutative spheres. III: Irrational rotations, *Comm. Math. Phys.*, to appear.
4. EXEL, R., The soft torus and applications to almost commuting matrices, *Pacific J. Math.*, to appear.
5. EXEL, R., The soft torus. II: A variational analysis of commutator norms, preprint.
6. KUMJIAN, A., Non-commutative spherical orbifolds, *C. R. Math. Rep. Acad. Sci. Canada*, **12**(1990), 87–89.
7. LORING, T. A., C^*-algebras generated by stable relations, *J. Functional Analysis*, to appear.
8. NATSUME, T., On $K_*(C^*(SL_2(\mathbf{Z})))$ (appendix to K-theory for certain group C^*-algebras, by E. C. Lance), *J. Operator Theory*, **13**(1985), 119–129.

9. RIEFFEL, M. A., C^*-algebras associated with irrational rotations, *Pacific J. Math.*, **93**(1981), 415–429.
10. RIEFFEL, M. A., Deformation quantization and operator algebras, in *Proc. Sympos. Pure Math.*, Vol. **51**, Amer Math. Soc., Providence, 1990, pp. 411–423.

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