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THE λ-FUNCTION IN OPERATOR ALGEBRAS

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1. INTRODUCTION

Let $\mathfrak{A}$ be a normed space with closed unit ball $\mathfrak{B}$, and denote by $\mathfrak{E}$ the set of extreme points of the convex set $\mathfrak{B}$. In [2] Aron and Lohman investigate the λ--function, defined on elements T of $\mathfrak{B}$ to be the supremum, $\lambda(T)$, of numbers λ in [0,1] for which there exists a pair V, B in $\mathfrak{E} \times \mathfrak{B}$, such that

$$T = \lambda V + (1-\lambda)B.$$

Among other things they show that when $\mathfrak{A}$ has the *λ-property* ($\lambda(T) > 0\ \forall T \in \mathfrak{B}$), then every closed face of $\mathfrak{B}$ contains extreme points, so that any convex function on $\mathfrak{B}$ that attains its maximum must do so on $\mathfrak{E}$. Moreover, if $\mathfrak{A}$ has the *uniform λ-property* ($\lambda(T) \geqslant \varepsilon > 0\ \forall T \in \mathfrak{B}$), then $\mathfrak{A}$ has the Krein-Milman-like property that

$$(\mathrm{conv}(\mathfrak{E} \cap \mathfrak{F}))^{=} = \mathfrak{F},$$

for every closed face $\mathfrak{F}$ of $\mathfrak{B}$. Further results on the λ-function for various classes of normed spaces were obtained in [3], [16], [17], and [18].

Aron and Lohman ask in [2, Question 4.1]: "what spaces of operators have the λ-property, and what does the λ-function look like for these spaces?" This paper is written in an attempt to answer their question, in the case where $\mathfrak{A}$ is a normclosed $*$-algebra of operators on some Hilbert space $\mathfrak{H}$, i.e. when $\mathfrak{A}$ is a C^*-algebra. It turns out, namely, that a similar problem — originating in the classical Russo-Dye theorem — has received considerable interest lately in operator algebra theory, see [7], [9], [10], [13], [21], [24], [25], [28], and [31]. Thus we now have quite detailed knowledge of the geometry of unit balls in C^*-algebras. Indeed, we have — in principle — a complete characterization of the facial structure of such balls [1]. In order to make this information available to non-specialists (i.e. mathematicians who specialize in other areas), the author has chosen a somewhat expansive style, repeating now and then a wellknown definition, and indicating the proofs of wellknown results. Honi soit qui mal y pense.

It is a pleasure to thank Richard Aron for bringing the λ-function and its literature to my attention, when both of us were attending the X Escuela Latinoamericano Matematica in the hills of Cordoba last year. Thanks also (less pleasure, more embarrasment) to Larry Brown for some last-minute corrections.

2. NOTATIONS AND PRELIMINARIES

Throughtout this article $\mathfrak{A}$ will denote a $*$-invariant algebra of bounded operators on some Hilbert space $\mathfrak{H}$, closed in the norm topology on $B(\mathfrak{H})$. Abstractly this means that $\mathfrak{A}$ is a *C^*-algebra*, i.e. a Banach algebra with an involution that satisfies the condition $\|T^*T\| = \|T\|^2$ for every T in $\mathfrak{A}$, cf. [14] or [22]. Sometimes we will further assume that $\mathfrak{A}$ is closed in the weak operator topology on $B(\mathfrak{H})$, in which case $\mathfrak{A}$ is *von Neumann algebra*. Abstractly this means that $\mathfrak{A}$ is a dual space. In a von Neumann algebra the unit ball is (weakly) compact, so that the Krein-Milman theorem applies. Also $\mathfrak{A}$ is generated by its projections (in the strong sense that the spectral resolution of every normal operator in $\mathfrak{A}$ belongs to $\mathfrak{A}$), and the set of projections in $\mathfrak{A}$ forms a complete lattice (a sublattice of the set of closed subspaces of $\mathfrak{H}$). Since this fact is being used crucially a number of times, it is only fair to point out that a larger class of C^*-algebras (Kaplansky's *AW-algebras*) have the same property. (But now the lattice of projections is not necessarily a sublattice of subspaces of any Hilbert space).

The closed unit ball of our C^*-algebra $\mathfrak{A}$ will always be denoted by $\mathfrak{B}$, and the set of extreme points of $\mathfrak{B}$ is denoted by $\mathfrak{E}$. We shall assume throughout that $\mathfrak{A}$ is *unital*, i.e. $I \in \mathfrak{A}$, for the simple reason that otherwise $\mathfrak{E} = \emptyset$. For a unital C^*-algebra the elements in $\mathfrak{E}$ were characterized by Kadison in 1951 as follows ([12] or [22, 1.4.7]):

$$\mathfrak{E} = \{V \in \mathfrak{B} | (I - VV^*)\mathfrak{A}(I - V^*V) = 0\}.$$

Thus $V \in \mathfrak{E}$ if it is a partial isometry such that the two projections $I - V^*V$ and $I - VV^*$ (on the kernels of V and V^*, respectively) are *centrally orthogonal* (so that even the two-sided ideals they generate are orthogonal). If therefore $\mathfrak{A}$ is a *prime* C^*-algebra (like $B(\mathfrak{H})$), or, even better, *simple* (no non-trivial closed ideals), then elements V in $\mathfrak{E}$ are either *isometries* ($V^*V = I$) or *co-isometries* ($VV^* = I$).

An important class of extreme points is the set $\mathfrak{U}$ of *unitary* elements in $\mathfrak{A}$ (elements U such that $U^*U = UU^* = I$, i.e. $U^* = U^{-1}$). In contrast to general elements in $\mathfrak{E}$, the elements in $\mathfrak{U}$ are normal operators. Moreover, they form a group under multiplication, a subgroup of the group $\mathfrak{A}^{-1}$ of invertible elements in $\mathfrak{A}$. For these reasons there has been a natural tendency in operator algebra theory to concentrate attention on $\mathfrak{U}$, in favour of the much more elusive elements in $\mathfrak{E} \setminus \mathfrak{U}$.

Under quite general circumstances we can deduce that $\mathfrak{E} = \mathfrak{U}$, thus avoiding the problem above. We say that the C^*-algebra is *finite*, if $T^*T = I$ implies $TT^* = I$ for all T in $\mathfrak{A}$, i.e. if every isometry is unitary. In case $\mathfrak{A}$ is a von Neumann algebra (and this is where the definition was first coined by Murray and von Neumann), this implies that $\mathfrak{E} = \mathfrak{U}$. For if $V \in \mathfrak{E}$, there is a central projection Z in $\mathfrak{A}$ such that ZV is an isometry in $Z\mathfrak{A}$, and $(I - Z)V$ is a co-isometry in $(I - Z)\mathfrak{A}$ (see the proof of Theorem 4.2); and the finiteness of $\mathfrak{A}$ now implies that $ZV + (I - Z)V^*$ is unitary, whence also $V \in \mathfrak{U}$. The same argument will work if $\mathfrak{A}$ is a finite AW^*-algebra. For a general finite C^*-algebra $\mathfrak{A}$ we need not have $\mathfrak{E} = \mathfrak{U}$ (see Proposition 9.4), but in the important case where $\mathfrak{A}$ is simple (or just prime), so that elements in $\mathfrak{E}$ are isometries or co-isometries, finiteness will, of course, imply $\mathfrak{E} = \mathfrak{U}$.

Rieffel introduced and studied the notion of topological *stable rank* for C^*-algebras [29], later identified with Bass' stable rank from ring theory [11]. The lowest rank, one, is the more tractable, and $\mathrm{sr}(\mathfrak{A}) = 1$ simply means that $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$. Also in this case we can conclude that $\mathfrak{E} = \mathfrak{U}$ (and thus $\mathfrak{A}$ is finite, as well, see Corollary 3.3).

Even when non-unitary extreme points exists, the group $\mathfrak{U}$ is rich enough to ensure that $\mathrm{conv}\mathfrak{U}$ is dense in $\mathfrak{B}$. This fact is the Russo-Dye theorem [7], [32]. We now have much more precise information. As a crowning achievement, building on earlier results from [13] and [21], Rørdam proved in [31, 3.3] the following theorem.

THEOREM 2.1. *If T is a non-invertible element in $\mathfrak{B}$, such that $\alpha(T) = \mathrm{dist}(T, \mathfrak{A}^{-1}) < 1$, there is for every $\beta > 2(1 - \alpha(T))^{-1}$ unitaries $U_1, U_2, \ldots, U_n$ in $\mathfrak{U}$, where $n - 1 < \beta \leqslant n$, such that*

$$T = \beta^{-1}(U_1 + U_2 + \cdots + U_{n-1}) + \beta^{-1}(\beta + 1 - n)U_n.$$

When $\mathfrak{E} = \mathfrak{U}$, this allows us to determine the λ-function, see Theorem 5.1.

3. POLAR DECOMPOSITIONS

As von Neumann showed, every operator T in $\mathcal{B}(\mathfrak{H})$ has a polar decomposition $T = V|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and V is a (unique) partial isometry such that

$\ker V = \ker T$. The construction

$$V = \lim T\left(\frac{1}{n}I + |T|^{-1}\right),$$

where the limit is taken in the strong operator topology, see e.g. [22, 2.2.9], shows immediately that if $\mathfrak{A}$ is a von Neumann algebra and $T \in \mathfrak{A}$, then $|T| \in \mathfrak{A}$ and $V \in \mathfrak{A}$. If $\mathfrak{A}$ is only a C^*-algebra, we can not be certain that $V \in \mathfrak{A}$ (although, of course, $|T| \in \mathfrak{A}$; and since $V|T| \in \mathfrak{A}$, it follows from (Stone-)Weierstrass' theorem that $Vf(|T|) \in \mathfrak{A}$ if f is continuous and $f(0) = 0$). Rather, V belongs to the von Neumann algebra generated by $\mathfrak{A}$ — equal to the double commutant $\mathfrak{A}''$ of $\mathfrak{A}$. But if $T \in \mathfrak{A}^{-1}$, then $T = U|T|$ for the unitary $U = T|T|^{-1}$ in $\mathfrak{A}$. Also in other cases it is possible to write elements in $\mathfrak{A}$ with a unitary "sign". The strongest known result in this direction follows. It uses the spectral resolution of $|T|$ in $\mathfrak{A}''$. Thus for each $\delta > 0$ we denote by E_δ the spectral projection of $|T|$ corresponding to the open interval $]\delta, \infty[$.

THEOREM 3.1. *If T is an element in a C^*-algebra $\mathfrak{A}$ with polar decomposition $V|T|$, then for each $\delta > \operatorname{dist}(T, \mathfrak{A}^{-1})$ there is a unitary U in $\mathfrak{U}$, such that $UE_\delta = VE_\delta$. For $\delta < \operatorname{dist}(T, \mathfrak{A}^{-1})$ there is no unitary extension of VE_δ in $\mathfrak{A}$.*

Proof. See [24, Theorem 5] or [31, Theorem 2.2]. ∎

COROLLARY 3.2. *Each element of the form $Vf(|T|)$, where f is a continuous function on $\operatorname{sp}|T|$ such that $f(t) = 0$ for $t \leqslant \delta$, and where $\delta > \operatorname{dist}(T, \mathfrak{A}^{-1})$, has a unitary polar decomposition $Uf(|T|) = Vf(|T|)$ in $\mathfrak{A}$.*

PROPOSITION 3.3. *If V is an extreme point in $\mathfrak{B}$ with $\operatorname{dist}(V, \mathfrak{A}^{-1}) < 1$, then $V \in \mathfrak{U}$.*

Proof. Let $P = V^*V$ and $Q = VV^*$ be the projections on the support and the range of V, respectively. Then $V = VP$ is the polar decomposition of V, so that, in the notation above, $P = E_\delta$ for any δ in the interval $]0, 1[$. By Theorem 3.1 there is a unitary U in $\mathfrak{U}$, such that $UP = VP = V$. Consequently, $Q = VV^* = UPU^*$.

By assumption, $I - Q$ and $I - P$ are centrally orthogonal, so

$$0 = (I - Q)U(I - P) = U(I - P)U^*U(I - P) = U(I - P).$$

It follows that $U = UP = V$, so $V \in \mathfrak{U}$. ∎

COROLLARY 3.4. *If $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$ then $\mathfrak{E} = \mathfrak{U}$.*

Theorem 3.1 also gives a neat proof of the formula relating the distances of an element from $\mathfrak{A}^{-1}$ and from $\mathfrak{U}$. This formula was established by Olsen when $\mathfrak{A}$ is a von Neumann algebra [20, Theorem 3.8], and by Rørdam in the general case.

PROPOSITION 3.5. *If* $T \notin \mathfrak{A}^{-1}$ *then*

$$\operatorname{dist}(T, \mathfrak{U}) = \max\{\|T\| - 1, \operatorname{dist}(T, \mathfrak{A}^{-1}) + 1\}.$$

Proof. [24, Theorem 10] or [31, Theorem 2.7]. ∎

The next result is known to most experts in von Neumann algebra theory, but the author has been unable to locate a precise reference.

PROPOSITION 3.6. *If* $\mathfrak{A}$ *is a von Neumann algebra and* $T \in \mathfrak{A}$*, there is an extreme point* W *in* $\mathfrak{E}$ *such that* $T = W|T|$.

Proof. The set

$$\mathfrak{C} = \{W \in \mathfrak{B} \mid T = W|T|\}$$

is a non-empty, convex, weakly closed subset of the weakly compact unit ball in $B(\mathfrak{H})$. By Krein-Milman's theorem we can therefore find an extreme point W in $\mathfrak{C}$. Since $\|W\| = 1$ we have $W^*W|T| = |T|$, because

$$|T|(I - W^*W)|T| = T^*T - T^*T = 0.$$

Since $W|W|$ and $W(2 - |W|)$ both belongs to $\mathfrak{B}$, and

$$T = W|W|\,|T| = W(2 - |W|)|T|,$$

$$W = \frac{1}{2}(W|W| + W(2 - |W|)),$$

we conclude from the extremality of W that $W = W|W|$, i.e., $|W|$ is a projection and W is a partial isometry. If now

$$A \in (I - WW^*)\mathfrak{B}(I - W^*W),$$

then $A|T| = 0$, so $T = (W \pm A)|T|$. Since $\|W \pm A\| \leqslant 1$ it follows, again from the extremality of W, that $A = 0$. This holds for all such A, whence $W \in \mathfrak{E}$.

A more sophisticated proof is obtained by considering the classical polar decomposition $T = V|T|$, and then note that the set

$$\mathfrak{D} = V + (I - VV^*)\mathfrak{B}(I - V^*V),$$

is a weakly closed face of $\mathfrak{B}$. In fact, as shown by C. M. Edwards and G. T. Rüttimann, every weakly closed face in $\mathfrak{B}$ has this form, see [1, Theorem 4.4]. An extreme point W of $\mathfrak{D}$ therefore belongs to $\mathfrak{E}$, and writing

$$W = V + (I - VV^*)B(I - V^*V)$$

it follows that

$$W|T| = V|T| = T. \qquad \blacksquare$$

4. VON NEUMANN ALGEBRAS

For an operator T in $\mathcal{B}(\mathfrak{H})$ we define

$$m(T) = \inf\{\|Tx\| \mid x \in \mathfrak{H},\ \|x\| = 1\}.$$

With $|T| = (T^*T)^{\frac{1}{2}}$ we know that $\|Tx\| = \|\,|T|x\|$ for every x in $\mathfrak{H}$, and it follows easily that

$$m(T) = m(|T|) = \min\{\varepsilon > 0 \mid \varepsilon \in \mathrm{sp}(|T|)\} =$$

$$= \max\{\varepsilon \geqslant 0 \mid \varepsilon I \leqslant |T|\} = \|\,|T|^{-1}\|^{-1}$$

(with a suitable interpretation if $|T| \notin \mathfrak{A}^{-1}$). By the open mapping theorem the condition $m(T) > 0$ is equivalent to T being injective with closed range. So in this case $T = U|T|$ for a unique isometry $U = T|T|^{-1}$.

Now consider T as an element of some von Neumann algebra in $\mathcal{B}(\mathfrak{H})$ with center $\mathfrak{Z}$ $(= \mathfrak{A} \cap \mathfrak{A}')$, and denote by $\mathfrak{Z}_{\mathrm{p}}$ the set of projections in $\mathfrak{Z}$. To obtain a common lower bound for T and T^* and their central splittings, define

$$(*) \qquad m_{\mathrm{q}}(T) = \sup\{m(ZT + (I-Z)T^*) | Z \in \mathfrak{Z}_{\mathrm{p}}\}.$$

Note that if we decompose $T = H + \mathrm{i}K$ in real and imaginary parts, then

$$(**) \qquad m_{\mathrm{q}}(T) = \sup\{m(H + \mathrm{i}SK) | S \in \mathfrak{Z}_{\mathrm{s}}\},$$

where $\mathfrak{Z}_{\mathrm{s}}$ denotes the set of *symmetries* S in $\mathfrak{Z}$ (of the form $S = 2Z - I$ for some Z in $\mathfrak{Z}_{\mathrm{p}}$). Of course, $m_{\mathrm{q}}(T)$ depends on the algebra $\mathfrak{A}$ (as well as on T), and if $\mathfrak{Z}_{\mathrm{p}}$ is small, the definition of $m_{\mathrm{q}}(T)$ is easier. Thus, taking $\mathfrak{A} = \mathcal{B}(\mathfrak{H})$, we simply get

$$(***) \qquad m_{\mathrm{q}}(T) = m(T) \vee m(T^*).$$

LEMMA 4.1. *If $\mathfrak{A}$ is a von Neumann algebra and $T \in \mathfrak{A}$, there is a central projection Z in $\mathfrak{A}$ such that*

$$m_{\mathrm{q}}(T) = m(ZT + (I-Z)T^*).$$

Proof. If Y and Z both belong to $\mathfrak{Z}_{\mathrm{p}}$, and

$$\varepsilon I \leqslant Y|T| + (I-Y)|T^*|, \quad \varepsilon I \leqslant Z|T| + (I-Z)|T^*|,$$

then by spectral theory

$$\varepsilon X \leqslant X|T| \quad \text{and} \quad \varepsilon(I-X) \leqslant (I-X)|T^*|$$

for $X = Y \vee Z$ and also for $X = Y \wedge Z$, because these statements only involve the three commuting elements Y, Z and $|T|$, respectively Y, Z and $|T^*|$. Since

$$m(ZT + (I-Z)T^*) = \max\{\varepsilon \mid \varepsilon I \leqslant |ZT + (I-Z)T^*|\} =$$

$$= \max\{\varepsilon \mid \varepsilon I \leqslant Z|T| + (I-Z)|T^*|\},$$

this means that if (Z_n) is a sequence in $\mathfrak{Z}_{\mathrm{p}}$ such that the sequence (ε_n) of numbers $\varepsilon_n = m(Z_nT + (I-Z_n)T^*)$ increases to $m_{\mathrm{q}}(T)$, then with $Y_k = \bigvee_{n \geqslant k} Z_n$ we have

$$\varepsilon_m \leqslant m(Y_kT + (I-Y_k)T^*)$$

for every k. Arguing in the same way on the decreasing sequence (Y_k) in $\mathfrak{Z}_{\mathrm{p}}$, we see that if $Z = \bigwedge Y_k$, then

$$\varepsilon_k \leqslant m(ZT + (I-Z)T^*)$$

for every k; whence

$$m_{\mathrm{q}}(T) = m(ZT + (I-Z)T^*). \qquad \blacksquare$$

Recalling the definition of the λ-function from the introduction:

$$\lambda(T) = \sup\{\lambda \in [0,1] \mid T = \lambda V + (1-\lambda)B, \ V \in \mathfrak{E}, \ B \in \mathfrak{B}\},$$

we are ready for our first result.

THEOREM 4.2. *If $\mathfrak{A}$ is a von Neumann algebra with unit ball $\mathfrak{B}$, and $T \in \mathfrak{B}$, then*

$$\lambda(T) = \frac{1}{2}(1 + m_{\mathrm{q}}(T)).$$

Moreover, if $\frac{1}{2} \leqslant \lambda \leqslant \lambda(T)$, there are extreme points V and W in $\mathfrak{B}$, such that

$$T = \lambda V + (1-\lambda)W.$$

Proof. If $T = \lambda V + (1-\lambda)B$ for some V in $\mathfrak{E}$ and B in $\mathfrak{B}$, put $P = V^*V$ and $Q = VV^*$. We can find a central projection Z in $\mathfrak{Z}_{\mathrm{p}}$ such that

$$(*) \qquad I - Q \leqslant Z \leqslant P.$$

To see this, note that $I-P$ and $I-Q$ are centrally orthogonal (since $V \in \mathfrak{E}$), so for every unitary U in $\mathfrak{U}$

$$I - Q \perp U(I-P)U^* = I - UPU^*,$$

i.e., $I - Q \leqslant UPU^*$. Take $Z = \bigwedge UPU^*$, the infinum being taken over all U in $\mathfrak{U}$. Evidently $UZU^* = Z$ for every U in $\mathfrak{U}$, i.e. $UZ = ZU^*$; and since $\mathfrak{A} = \operatorname{span}(\mathfrak{U})$ (in fact, every element is a linear combination of 4 (even 3) unitaries), it follows that $Z \in \mathfrak{Z}_{\mathrm{p}}$. By (*) we have $Z \leqslant P$ and $1 - Z \leqslant Q$, so

$$(ZV)^*(ZV) = ZP = Z;$$

$$(I-Z)V((I-Z)V^*) = (I-Z)Q = I - Z.$$

It follows that $W = ZV + (I-Z)V^*$ is an isometry in $\mathfrak{A}$. With the notations $T_0 = ZT + (I-Z)T^*$ and $B_0 = ZB + (I-Z)B^*$ we can rewrite the equation $T = \lambda V + (1-\lambda)B$ as $T_0 = \lambda W + (1-\lambda)B_0$. Since $B_0 \in \mathfrak{B}$ we compute

$$\begin{aligned} m_{\mathrm{q}}(T) &\geqslant m(T_0) = m(\lambda W + (1-\lambda)B_0) = \\ &= \inf\{\|(\lambda W + (1-\lambda)B_0 x\| \mid \|x\| = 1\} \geqslant \\ &\geqslant \inf\{\lambda\|Wx\| - (1-\lambda)\|B_0\| \mid \|x\| = 1\} \geqslant 2\lambda - 1 \end{aligned}$$

This inequality holds for any decomposition $T = \lambda V + (1-\lambda)B$, and we conclude that

$$(**) \qquad m_{\mathrm{q}}(T) \geqslant 2\lambda(T) - 1.$$

To prove the reverse inequality we take by Lemma 4.1 a projection Z in $\mathfrak{Z}_{\mathrm{p}}$, such that with $\varepsilon = m_{\mathrm{q}}(T)$ we have

$$m(ZT + (1-Z)T^*) = \varepsilon.$$

Setting

$$A = |ZT + (I-Z)T^*| = Z|T| + (I-Z)|T^*|,$$

this means that $\varepsilon I \leqslant A$. As shown in [13, Lemma 6] this implies that for any λ in the interval $\left[\frac{1}{2}, \frac{1}{2}(1+\varepsilon)\right]$ we can find unitaries U_1, U_2 in $\mathfrak{U}$, such that $A = \lambda U_1 + (1-\lambda)U_2$. This fact is easily verified by writing $U_1 = B + \mathrm{i}(1-\lambda)D$ and $U_2 = C - \mathrm{i}\lambda D$, where B, C and D are the self-adjoint elements in $\mathfrak{A}$ given by

$$\begin{aligned} B &= \frac{1}{2}\lambda^{-1}(A + (2\lambda-1)A^{-1}), \\ C &= \frac{1}{2}(1-\lambda)^{-1}(A - (2\lambda-1)A^{-1}), \\ D &= (1-\lambda)^{-1}(I - B^2)^{\frac{1}{2}} = \lambda^{-1}(I - C^2)^{\frac{1}{2}}. \end{aligned}$$

Here $(2\lambda - 1)A^{-1}$ should be interpreted as 0 when $\lambda = \frac{1}{2}$ (if $m_q(T) = 0$ this may be the only choice), and if $\lambda = 1$ (so that $A = I$) the formulae for C and D should be interpreted as 0. Thus for $\frac{1}{2\leqslant\lambda\leqslant\frac{1}{2}(1+m_q(T))}$ we have

$$(***) \qquad Z|T| + (1 - Z)|T^*| = \lambda U_1 + (1 - \lambda)U_2.$$

By proposition 3.6 we can choose extreme points W_1 and W_2 in $\mathfrak{E}$ such that $T = W_1|T|$ and $T^* = W_2|T^*|$. But then by $(***)$

$$\begin{aligned} T = ZT + ((1 - Z)T^*)^* = W_1Z|T| + (I - Z)|T^*|W_2^* = \\ = W_1Z(\lambda U_1 + (1 - \lambda)U_2) + (I - Z)(\lambda U_1 + (1 - \lambda)U_2)W_2^* = \\ = \lambda(ZW_1U_1 + (I - Z)U_1W_2^*) + (1 - \lambda)(ZW_1U_2 + (I - Z)U_2W_2^*). \end{aligned}$$

Evidently the elements

$$V = ZW_1U_1 + (I - Z)U_1W_2^*, \qquad W = ZW_1U_2 + (I - Z)U_2W_2^*$$

are extreme points, and we have $T = \lambda V + (1 - \lambda)W$, as desired. Choosing $\lambda = \frac{1}{2}(1 + m_q(T))$, we get $\lambda(T) \geqslant \frac{1}{2}(1 + m_q(T))$, which in conjunction with $(**)$ implies equality, and the proof is complete. ■

5. C^*-ALGEBRAS AND THE λ_u-FUNCTION

As mentioned before, the non-unitary extreme points in the unit ball $\mathfrak{B}$ of a C^*-algebra $\mathfrak{A}$ are somewhat elusive. Our first result overcome this problem in a time-honoured fashion — by changing the definition. For each T in $\mathfrak{B}$ we define

$$\lambda_u(T) = \sup\{\lambda \in [0, 1] \mid T = \lambda U + (1 - \lambda)B,\ U \in \mathfrak{U},\ B \in \mathfrak{B}\}.$$

Clearly $\lambda_u(T) \leqslant \lambda(T)$ and, more importantly, the two functions agree whenever $\mathfrak{E} = \mathfrak{U}$.

If $T \in \mathfrak{A}$ and $\mathfrak{A}^{-1}$ denotes the group of invertible elements in $\mathfrak{A}$, we set

$$\alpha(T) = \mathrm{dist}(T, \mathfrak{A}^{-1}).$$

THEOREM 5.1. *If $\mathfrak{A}$ is a C^*-algebra with unit ball $\mathfrak{B}$, and T is a non-invertible element of $\mathfrak{B}$, then*

$$\lambda_u(T) = \frac{1}{2}(1 - \alpha(T)).$$

If T is invertible, then

$$\lambda_u(T) = \frac{1}{2}(1 + \|T^{-1}\|^{-1}).$$

Proof. Consider first the case where $T \notin \mathfrak{A}^{-1}$. If

$$T + \lambda U + (1-\lambda)B, \quad U \in \mathfrak{U},\ B \in \mathfrak{B},$$

then $\lambda \leqslant \frac{1}{2}$, since otherwise

$$T = \lambda U(I + \lambda^{-1}(1-\lambda)U^*B) \in \mathfrak{A}^{-1},$$

because $\|\lambda^{-1}(1-\lambda)U^*B\| < 1$. Now,

$$\|T - \lambda(U + B)\| = \|(1-2\lambda)B\| \leqslant 1 - 2\lambda.$$

Since $U + sB = U(I + sU^*B) \in \mathfrak{A}^{-1}$ for every $s < 1$, we see that $U + B \in (\mathfrak{A}^{-1})^=$, whence $\alpha(T) \leqslant 1 - 2\lambda$. Since this holds for all decompositions, we conclude that

$$\alpha(T) \leqslant 1 - 2\lambda_{\mathrm{u}}(T).$$

An argument, using Proposition 3.5, estimating

$$\|T - U\| = \|(1-\lambda)B - (1-\lambda)U\| \leqslant 2(1-\lambda),$$

is also available (and gives the same result!).

When $\alpha(T) = 1$ the result above shows that $\lambda_{\mathrm{u}}(T) = 0$, so in order to prove the reverse inequality we may assume that $\alpha(T) < 1$. But then, by Theorem 2.1, there is for every $\beta > 2(1-\alpha(T))^{-1}$ a convex combination

$$T = \beta^{-1}(U_1 + \cdots + U_{n-1}) + \beta^{-1}(\beta + 1 - n)U_n,$$

with the U_k's in $\mathfrak{U}$ and $n - 1 < \beta \leqslant n$. Taking

$$B = (\beta - 1)^{-1}(U_2 + \cdots + U_{n-1} + (\beta + 1 - n)U_n),$$

this reads: $T = \beta^{-1}U_1 + (1 - \beta^{-1})B$, with B in $\mathfrak{B}$, so that $\lambda_{\mathrm{u}}(T) \geqslant \beta^{-1}$. It follows that

$$\lambda_{\mathrm{u}}(T) \geqslant \frac{1}{2}(1 - \alpha(T)),$$

giving the desired equation.

If $T \in \mathfrak{A}^{-1}$ we have $T = U|T|$ with U in $\mathfrak{U}$. Thus $T^{-1} = |T|^{-1}U^*$ and $\|T^{-1}\| = \| |T|^{-1}\|$. With $m(T)$ as in section 4 we see that

$$(*) \qquad m(T) = \|T^{-1}\|^{-1}.$$

Since $|T| \geqslant m(T)I$ we can use [13, Lemma 6] as in the proof of Theorem 4.2 to find unitaries U_1, U_2 in $\mathfrak{U}$, such that with $\lambda_0 = \frac{1}{2}(1 + m(T))$ we have

$$|T| = \lambda_0 U_1 + (1 - \lambda_0) U_2.$$

Multiplying this equation with U we see that

$$(**) \qquad \lambda_u(T) \geqslant \lambda_0 = \frac{1}{2}(1 + m(T)).$$

Conversely, if $T = \lambda U + (1 - \lambda)B$ with U in $\mathfrak{U}$ and B in $\mathfrak{B}$ we get (as in the proof of Theorem 4.2)

$$m(T) = \inf\{\|\lambda U x + (1 - \lambda)Bx\| \mid \|x\| = 1\} \geqslant 2\lambda - 1.$$

This holds for any decomposition, so

$$m(T) \geqslant 2\lambda_u(T) - 1.$$

Combined with (**) (and inserting (*)) we get the desired equation. ∎

REMARK 5.2. It is amusing to note, that when T is invertible the number $m(T)$ $(= \|T^{-1}\|^{-1})$ in formulas serves as a measure of the "negative distance" from T to $\mathfrak{A}^{-1}$. (It *is* the distance to the boundary of $\mathfrak{A}^{-1}$.) This happens in Theorem 5.1 but also in Proposition 3.5. For if $T \in \mathfrak{A}^{-1}$, so that $T = U|T|$ with U in $\mathfrak{A}$, then U is an approximant to T in $\mathfrak{U}$, and

$$\operatorname{dist}(T, \mathfrak{U}) = \|T - U\| = \|\,|T| - I\| =$$

$$= \max\{\|T\| - 1,\ 1 - m(T)\},$$

cf. [19, Proposition 3.5].

It is also worthwhile to realize that when T is invertible the two functions λ and λ_u agree on T.

PROPOSITION 5.3. *If T is an invertible element in the unit ball of a C^*-algebra $\mathfrak{A}$, then*

$$\lambda_u(T) = \lambda(T).$$

Proof. We have $\lambda_u(T) = \frac{1}{2}(1 + \|T^{-1}\|^{-1}) > \frac{1}{2}$. If we had $\lambda_u(T) < \lambda(T)$, there would be a pair V, B in $\mathfrak{E} \times \mathfrak{B}$ such that

$$T = \lambda V + (1 - \lambda)B, \quad \lambda > \lambda_u(T) > \frac{1}{2}.$$

With $A = \lambda^{-1}T$ we then have $A \in \mathfrak{A}^{-1}$ and

$$\|V - A\| = \|(\lambda^{-1} - 1)B\| \leqslant \lambda^{-1} - 1 < 1.$$

But then $V \in \mathfrak{U}$ by proposition 3.3, so $\lambda \leqslant \lambda_u(T)$, a contradiction. Consequently $\lambda(T) \leqslant \lambda_u(T)$, whence $\lambda(T) = \lambda_u(T)$. ■

THEOREM 5.4 *For a C^*-algebra $\mathfrak{A}$ the following conditions are equivalent:*

(i) *For every T in $\mathfrak{B}$ and $0 < \varepsilon < \frac{1}{2}$ there are unitaries U_1, U_2, U_3 such that*

$$T = \frac{1}{2}(1-\varepsilon)U_1 + \frac{1}{2}(1-\varepsilon)U_2 + \varepsilon U_3.$$

(ii) $\lambda_u(T) \geqslant \frac{1}{2}$ *for every T in $\mathfrak{B}$.*

(iii) $\mathfrak{A}$ *has the uniform λ_u-property ($\lambda_u(T) \geqslant \varepsilon > 0\ \forall T \in \mathfrak{B}$).*

(iv) $\mathfrak{A}$ *has the λ_u-property ($\lambda_u(T) > 0\ \forall T \in \mathfrak{B}$).*

(v) $\operatorname{dist}(T, \mathfrak{A}^{-1}) < 1$ *for every T in $\mathfrak{B}$.*

(vi) $\operatorname{sr}(\mathfrak{A}) = 1$ *(i.e. $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$).*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are trivial; and (iv) ⇒ (v), because if $T = \lambda U + (1-\lambda)B$ with $\lambda > 0$, then $\lambda U \in \mathfrak{A}^{-1}$ so

$$\operatorname{dist}(T, \mathfrak{A}^{-1}) \leqslant \|T - \lambda U\| \leqslant 1 - \lambda < 1.$$

The implications (v) ⇒ (vi) and (vi) ⇒ (i) are due to Rørdam, [31, Theorem 2.6 and Corollary 3.6]. The first is proved by negation: if $T \in \mathfrak{B}$ such that $\alpha = \operatorname{dist}(T, \mathfrak{A}^{-1}) > 0$, define $S = Vf|T|$, where $T = V|T|$ is the polar decomposition of T (in $B(\mathfrak{H})$) and $f(t) = 1 \wedge \alpha^{-1}t$. Since $f(0) = 0$ and f is continuous it follows that $S \in \mathfrak{A}$ (so $S \in \mathfrak{B}$). But if $\operatorname{dist}(S, \mathfrak{A}^{-1}) < 1$, then with E_δ as the spectral projection of $|S|$ corresponding to the interval $]\delta, \infty[$ there is by Theorem 3.1 a unitary U in $\mathfrak{U}$, such that $VE_\delta = UE_\delta$ for some $\delta < 1$. Now, since $S = Vf(|T|)$ and $T = V|T|$, it follows that E_δ is also a spectral projection of $|T|$, but corresponding to the interval $]\alpha\delta, \infty[$. Since $\alpha\delta < \alpha$, this contradicts Theorem 3.1 applied to T. Thus $\operatorname{dist}(S, \mathfrak{A}^{-1}) = 1$, as desired. The other implication (vi) ⇒ (i) is an immediate corollary to Theorem 2.1. ■

COROLLARY 5.5. *A function algebra $C(X)$, where X is a compact Hausdorff space, has the λ-property if and only if the (covering) dimension of X is at most one, in which case $C(X)$ has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Since we are explicitly dealing with algebras over the *complex* numbers, the invertible functions on X are dense in $C(X)$ iff $\dim(X) \leqslant 1$. See [29, §1] for a general discussion. ■

Replacing stable rank with real rank we have a partial generalisation of Theorem 5.4 to infinite C^*-algebras, see Theorem 10.4.

6. ATTAINING THE λ-VALUES

The definition of the λ-function and of the λ_u-function involves a supremum, and it is natural to ask when this supremum is attained. Thus, if $\lambda(T) > 0$ we ask for the existence of a pair V, B in $\mathfrak{E} \times \mathfrak{B}$ such that

$$T = \lambda(T)V + (1 - \lambda(T))B.$$

For spaces of the form $C(X, \mathfrak{M})$, X compact Hausdorff and $\mathfrak{M}$ an infinite dimensional, strictly convex normed space, Aron and Lohman find this to be the case if the element (function) does not attain the value 0 [12, Theorems 1.6 and 1.9]. In our case the spaces involved are never strictly convex, but the same phenomenon persists.

PROPOSITION 6.1. *If T is an invertible element in the unit ball of a C^*-algebra $\mathfrak{A}$, then*

$$T = \lambda(T)V + (1 - \lambda(T))W$$

for some V, W in $\mathfrak{U} \times \mathfrak{U}$.

Proof. In the proof of Theorem 5.1, see (**), we constructed a decomposition

$$T = \lambda_0 U U_1 + (1 - \lambda_0) U U_2,$$

with U, U_1, U_2 in $\mathfrak{U}$ and $\lambda_0 = \frac{1}{2}(1 + m(T))$. We further showed that $\lambda_0 = \lambda_u(T)$, when $T \in \mathfrak{A}^{-1}$ with $T = U|T|$. Since $\lambda_u(T) = \lambda(T)$ by Proposition 5.3, we are done. ∎

When the elements are non-invertible, the λ-function values are only attained in the presense of severe conditions. Thus, for example, we see from Theorem 4.2 that if $\mathfrak{A}$ is a von Neumann algebra, then the λ-values are attained. Less will do, but only by resorting to highly non-separable spaces. We illustrate the problems in the commutative case.

PROPOSITION 6.2 *If X is a compact, metric space, such that for every f in $C(X)$ the number $\lambda(f)$ is attained in a decomposition for f, then X is finite.*

Proof. If X is infinite there is a convergent sequence (x_n) in X with $x_n \neq x_m$ for $n \neq m$. Passing if necessary to a subsequence we may assume that $\text{dist}(x_n, x_0) < 2(n\pi)^{-1}$ for all n, where $x_0 = \lim x_n$ and dist denotes the metric on X. Put $Y = \{x_n | n \geqslant 0\}$ and define h in $C(Y)$ by $h(x_0) = 0$, $h(x_n) = 2(n\pi)^{-1}$, $1 \leqslant n$. By

Tietze's extension theorem h extends to an element in $C(X)$ (again denoted by h) with $0 \leqslant h \leqslant 1$. We may assume that $h(x) > 0$ for all $x \neq x_0$, replacing otherwise h with the expression

$$(h(x) \vee \operatorname{dist}(x, x_0)) \wedge 1,$$

which does not change h on Y. Now define f in $C(X)$ by

$$f(x) = h(x)\exp(\mathrm{i}h(x)^{-1}), \quad x \in X \setminus \{x\},$$

and $f(x_0) = 0$. By construction, $f(x_n) = 2(n\pi)^{-1}\mathrm{i}^n$ for all n (with $\mathrm{i} = \sqrt{-1}$). We claim that f can be approximated by invertible functions. Indeed, with

$$f_n(x) = h(x)\exp(\mathrm{i}(h(x)^{-1} \wedge n))$$

we clearly have elements in $C(X)$ with distance zero to the set of invertible elements, because f_n is a product of a positive and an invertible function. But $f_n(x) = f(x)$ if $h(x) \geqslant n^{-1}$, and

$$|f_n(x) - f(x)| \leqslant 2h(x) \leqslant 2n^{-1}$$

otherwise. It follows from Theorem 5.1 that $\lambda(f) = \frac{1}{2}$.

If we had a decomposition

$$f = \frac{1}{2}(u + b)$$

with u unitary (i.e. a circle-valued function) and b of norm $\leqslant 1$ in $C(X)$, then

$$2\mathrm{Re}\, u^* f = \mathrm{Re}\,(1 + u^* b) \geqslant 0.$$

Consequently $\mathrm{Re}\,\overline{u}(x_n)\mathrm{i}^n \geqslant 0$ for all n, which is impossible since $(u(x_n))$ converges to $u(x_0)$. ∎

For the characterization of function algebras in which the λ-values are attained, we need the concept of a *sub-Stonean* space, meaning a (locally) compact Hausdorff space X such that any two disjoint, open, σ-compact subsets of X have disjoint closures. These spaces have amused (general) topologists since 1956, when they appeared in works by L. Gillman, M. Henriksen and M. Jerison under the name of F-spaces. They were rediscovered by G. Choquet, who coined the term sub-Stonean spaces. Some of their properties are discussed in [8].

If Y is an open subset of a compact Hausdorff space X, there is a continuous map $\Phi : \beta(Y) \to \overline{Y}$ from the Stone-Čech compactification $\beta(Y)$ of Y onto the closure if Y in X, extending the embedding map of Y into X.

LEMMA 6.3. *The map $\Phi : \beta(Y) \to \overline{Y}$ mentioned above is a homeomorphism for every open, σ-compact subset Y of X if and only if X is a sub-Stonean space.*

Proof. If X is a sub-Stonean, then each of the maps Φ is a homeomorphism by [8, Theorem 1.10]. Conversely, if all the Φ's are homeomorphisms, take open, σ-compact, disjoint sets Y and Z in X. Then $\beta(Y \cup Z) = \beta(Y) \oplus \beta(Z)$ (topological direct sum), and since

$$\Phi : \beta(Y) \oplus \beta(Z) \longrightarrow \overline{Y} \cup \overline{Z}$$

is a homeomorphism, it follows that $\overline{Y}$ and $\overline{Z}$ are disjoint. Therefore X is sub-Stonean, as claimed. ■

PROPOSITION 6.4. [30] *If X is a compact Hausdorff space, the following conditions are equivalent:*

(i) *X is sub-Stonean and* $\dim(X) \leqslant 1$.

(ii) *Every element f in $C(X)$ has a unitary polar decomposition $f = u|f|$, with u in $\mathfrak{U}(C(X))$.*

(iii) $\mathfrak{B}(C(X)) = \frac{1}{2}(\mathfrak{U}(C(X)) + \mathfrak{U}(C(X)))$.

Proof. (i) ⇒ (ii). Given f in $C(X)$, let

$$Y = \{x \in X \mid f(x) \neq 0\},$$

and define w on Y by $w(x) = f(x)|f(x)|^{-1}$. Since Y is open and σ-compact, $\overline{Y}$ is homeomorphic to $\beta(Y)$ (cf. Lemma 6.3), and as $w \in C_b(Y) = C(\beta(Y))$, it extends by continuity to a unitary function on $\overline{Y}$ (again denoted by w). Because $\dim(X) \leqslant 1$, every unitary function w on a closed subset extends to an element u in $\mathfrak{U}(C(X))$, and evidently $f = u|f|$.

(ii) ⇒ (iii). If $f \in C(X)$, with $\|f\| \leqslant 1$, we have $f = u|f|$ for some u in $\mathfrak{U}(C(X))$. Define

$$v = |f| + \mathrm{i}(1 - |f|^2)^{\frac{1}{2}}.$$

Then v is unitary and

$$\frac{1}{2}(uv + uv^*) = u|f| = f.$$

(iii) ⇒ (i). If $f = \frac{1}{2}(u + v)$, u and v unitaries, then

$$g = \frac{1}{2}(1 + \varepsilon)u + \frac{1}{2}(1 - \varepsilon)v$$

is invertible for every $\varepsilon > 0$, and $\|f - g\| \leqslant \varepsilon$. It follows that the invertible elements are dense in $C(X)$, whence $\dim(X) \leqslant 1$.

Now let Y and Z be the disjoint open, σ-compact subsets of X, and choose positive functions f and g in $C(X)$ of norm less than 1, with Y and Z as their co-zero sets. By assumption we have unitary functions u and v, such that

$$f - \mathrm{i}g = \frac{1}{2}(u + v).$$

If x is a point on the boundary of Z, it follows from plane geometry that $u(x)$ (and $v(x)$) equals ± 1. Similarly, if x is on boundary of Y we get $u(x) \in \{\pm i\}$. Consequently $\overline{Y} \cap \overline{Z} = \emptyset$, so that X is sub-Stonean. ■

We say that a compact Hausdorff space X is *2-sub-Stonean* if for each open, σ-compact subset Y of X the map $\Phi : \beta(Y) \to \overline{Y}$, mentioned above, is at most of order 2 at any point. This means that a point x on the boundary of Y can be reached as a limit of at most two distinct universal nets in Y.

LEMMA 6.5. *A compact Hausdorff space X is 2-sub-Stonean if and only if given any pairwise disjoint, open, σ-compact subsets Y_1, Y_2, Y_3 of X we have*

$$\overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3 = \emptyset.$$

Proof. If $x \in \overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3$ for some disjoint, open, σ-compact subsets of X, put $Y = Y_1 \cup Y_2 \cup Y_3$. Then the map $\Phi : \beta(Y) \to \overline{Y}$ will have order 3 at the point x (i.e. $\Phi^{-1}(x)$ consist at least 3 points), because

$$\beta(Y) = \beta(Y_1) \oplus \beta(Y_2) \oplus \beta(Y_3).$$

Conversely, assume that X is not 2-sub-Stonean. Thus for some open, σ-compact subset Y of X we have distinct points $\gamma_1, \gamma_2, \gamma_3$ in $\beta(Y)$, such that $\Phi(\gamma_i) = x$ for some x in $\overline{Y}$ and $i = 1, 2, 3$. Choose f in $C_b(Y)$ such that $f(\gamma_i)$, $i = 1, 2, 3$, are three distinct points in $\mathbb{C}$, and let A_i, $i = 1, 2, 3$, be the pairwise disjoint, open neighbourhoods of the $f(\gamma_i)$'s. Put $Y_i = f^{-1}(A_i)$ (as subsets of Y) to obtain pairwise disjoint, open, σ-compact subsets of X. Since $\beta(Y_i)$ is the closure of Y_i in $\beta(Y)$, it follows that $\gamma_i \in \beta(Y_i)$, whence $x \in \Phi(\beta(Y_i)) = \overline{Y_i}$ for all i. ■

THEOREM 6.6. *If $\mathfrak{A} = C(X)$ is a comutative C^*-algebra with the unit ball $\mathfrak{B}$ and unitary group $\mathfrak{U}$, then*

$$(*) \qquad \mathfrak{B} = \frac{1}{2}(\mathfrak{U} + \mathfrak{B})$$

if X is a sub-Stonean space with $\dim(X) \leqslant 1$. *Conversely, if $(*)$ is satisfied then X must be 2-sub-Stonean with* $\dim(X) \leqslant 1$.

Proof. The first statement is contained in Proposition 6.4. To prove the second, note first that if $u \in \mathfrak{U}$ and $b \in \mathfrak{B}$, then

$$(1 + \varepsilon)u + (1 - \varepsilon)b$$

is invertible for every $\varepsilon > 0$ and close to $u + b$. As in the proof of Proposition 6.4 it follows that $(*)$ implies that $\dim(X) \leqslant 1$.

To prove that X is 2-sub-Stonean, let Y_1, Y_2, Y_3 be pairwise disjoint, open, σ--compact subsets of X, and choose positive functions f_1, f_2, f_3 in $C(X)$ of norm less than 2 with co-zero sets Y_1, Y_2, Y_3, respectively. Let $\theta = \exp\left(\frac{2}{3}\pi i\right)$, and define

$$f = \theta f_1 + \theta^2 f_2 + f_3.$$

If $(*)$ is satisfied we have $f = u + b$ for some u in $\mathfrak{U}$ and b in $\mathfrak{B}$. For any x in Y_3 we have

$$u(x) + b(x) = f_3(x) > 0,$$

and it follows from plane geometry that $\operatorname{Re} u(x) \geqslant \frac{1}{2} f_3(x)$. If therefore x belongs to the boundary of Y_3 we see from the continuity of u that

$$\operatorname{Re} u(x) \geqslant 0.$$

Similar arguments show that if x belongs to the boundary of Y_2 or Y_1, then

$$\operatorname{Re} \theta u(x) \geqslant 0 \quad \text{and} \quad \operatorname{Re} \theta^2 u(x) \geqslant 0,$$

respectively. If now $x \in \overline{Y}_1 \cap \overline{Y}_2 \cap \overline{Y}_3$, then $u(x)$ should belong to 3 half-spaces in $\mathbf{C}$, whose intersections is $\{0\}$. But $|u(x)| = 1$, and we have reached a contradiction. Thus X is 2-sub-Stonean, as claimed. ∎

PROPOSITION 6.7. *There exists a compact Hausdorff space X, which is not sub-Stonean (but only 2-sub-Stonean), such that $C(X)$ satisfies the condition $(*)$ in Theorem 6.6.*

Proof. Take $X_1 = \beta(\mathbf{R}_+) \setminus \mathbf{R}_+$. Combining Theorem 3.2, Proposition 3.5 and Theorem 3.6 in [8] we see that X_1 is a connected, sub-Stonean space of dimension 1. Choose any non-trivial open, σ-compact subset Y of X_1, and let x_0 be a point in $\overline{Y} \setminus Y$. (If none existed, Y would be closed as well as open, contradicting the fact that X_1 is connected.) Let X_2 be another copy of X_1, and define X to be the topological union of X_1 and X_2, glued together at x_0. Thus X is a compact, connected Hausdorff space of dimension 1; but it is not sub-Stonean, because the two copies of Y (in X_1 and X_2) are disjoint open, σ-compact subsets of X with a common boundary point, viz. x_0.

We wish to prove that the λ-values are attained for any element in the unit ball of $C(X)$ (which is slightly more than promised by $(*)$). To do this, note that

$$C(X) = \{(f_1, f_2) \in C(X_1) \times C(X_2) \mid f_1(x_0) = f_2(x_0)\},$$

and take $f = (f_1, f_2)$ in the unit ball of $C(X)$. If f is invertible, its λ-value is attained by Proposition 6.1. We may therefore assume that f is not invertible, whence

$\lambda(f) \leqslant \frac{1}{2}$. However, since $\dim(X) = 1$, we know from Corollary 5.5 that $\lambda(f) \geqslant \frac{1}{2}$ for every f; so in our case $\lambda(f) = \frac{1}{2}$.

If $f(x_0) \neq 0$ we choose by Proposition 6.4 unitaries u_1, u_2 in $\mathfrak{U}(C(X_1))$ and $\mathfrak{U}(C(X_2))$, respectively, such that $f_i = u_i|f_i|$, $i = 1, 2$. Then

$$u_1(x_0) = f_1(x_0)|f_1(x_0)|^{-1} = f_2(x_0)|f_2(x_0)|^{-1} = u_2(x_0),$$

so that $u = (u_1, u_2)$ is a unitary in $C(X)$ with $f = u|f|$. With $v = |f| + i(1 - |f|^2)^{\frac{1}{2}}$ this, as on previous occasions, gives

$$f = \frac{1}{2}(uv + uv^*).$$

We are left with the case where $f(x_0) = 0$. To simplify matters, use Proposition 6.4 to find w_1 in $\mathfrak{U}(C(X_1))$ such that $f_1 = w_1|f_1|$, and extend it from the closed subset X_1 to a continuous, circle-valued function w on the one-dimensional space X. Replacing f with w^*f we see that it suffices to consider the case where $f = (f_1, f_2)$, $f_1(x_0) = f_2(x_0) = 0$ and $f_1 \geqslant 0$. Working in the sub-Stonean space X_2, we use Proposition 6.4 to find elements v_1, v_2 in $\mathfrak{U}(C(X_2))$, such that $f_2 = \frac{1}{2}(v_1 + v_2)$. The remaining task is to find suitable continuous extensions of these functions on X_1. We may assume, without loss of generality, that $\operatorname{Re}(v_1(x_0)) \geqslant 0$. (Otherwise we consider v_2; and since $v_1(x_0) + v_2(x_0) = 0$, one of them will work.) Furthermore we may assume that $\operatorname{Im}(v_1(x_0)) \geqslant 0$, since the argument for $\operatorname{Im}(v_1(x_0)) \leqslant 0$ is quite symmetric. Let

$$Z = \{x \in X_1 \mid f_1(x) \leqslant \operatorname{Re} v_1(x_0)\}.$$

This is a closed subset of X_1 containing x_0. For each x in Z we define

$$v(x) = v_1(x_0), \quad b(x) = 2f_1(x) - v_1(x_0).$$

It is easy to check that $|b(x)| \leqslant 1$, and clearly $f_1 = \frac{1}{2}(v + b)$ on Z. Moreover, v and b are continuous extensions of v_1 and v_2 from X_2 to $X_2 \bigcup_{x_0} Z$, because

$$b(x_0) = -v_1(x_0) = v_2(x_0).$$

For each x in $X_1 \setminus Z$ we define

$$v(x) = f_1(x) + i(1 - |f_1(x)|^2)^{\frac{1}{2}},$$

$$b(x) = f_1(x) - i(1 - |f_1(x)|^2)^{\frac{1}{2}}.$$

Again it is clear that these functions are unitary and continuous on $X_1 \setminus Z$, with $f_1 = \frac{1}{2}(v + b)$. To see that v and b are, in fact, continuous on X, consider any point

x on the boundary of Z. By definition of Z we must then have $f_1(x) = \mathrm{Re}(v_1(x_0))$, which implies that

$$v(x) = v_1(x_0) = f_1(x) + \mathrm{i}(1 - |f_1(x)|^2)^{\frac{1}{2}}.$$

Moreover,

$$b(x) = 2f_1(x) - v_1(x_0) = f_1(x) - \mathrm{i}(1 - |f_1(x)^2)^{\frac{1}{2}}.$$

We see that v and b on the boundary of Z agree with the definitions of v and b given on $X_1 \setminus Z$, and thus v and b are continuous on all of X_1. Put $w = (v, v_1)$ and $c = (b, v_2)$. Then w and c belong to $C(X)$, w is unitary and $\|c\| \leqslant 1$; and, most importantly,

$$f = \frac{1}{2}(w + c).$$ ■

Theorem 6.6 is not quite satisfactory, since we do not obtain a classification of those compact Hausdorff spaces X for which the λ-values are attained for all elements in $C(X)$. Moreover, the concept of 2-sub-Stonean spaces is somewhat artificial; and the author has been unable to produce examples of such spaces except, as in Proposition 6.7, by glueing together sub-Stonean spaces at a finite number of points. The author fears that the problem does not admit a clean solution.

Extending the results above to the general, non-comutative case seems nontrivial. For C^*-algebras that are *σ-finite* (in the sense that any family of non-zero, pairwise orthogonal elements is countable) Haagerup and Rørdam in [10] showed that the condition

$$\mathfrak{B} = \frac{1}{2}(\mathfrak{U} + \mathfrak{U})$$

implies that $\mathfrak{U}$ is a finite AW^*-algebra. It was known from [25, Proposition 2.7] that the condition implies that $\mathfrak{B} = \mathfrak{U}\mathfrak{B}_{sa}$, so the aim of their ingeneous argument was to provide a unitary polar decomposition $T = U|T|$ for every self-adjoint T in $\mathfrak{B}$. Replacing one copy of $\mathfrak{U}$ with $\mathfrak{B}$ as in Theorem 6.6, we no longer have $\mathfrak{B} = \mathfrak{U}\mathfrak{B}_{sa}$ for free, so the Haagerup-Rørdam argument is not directly applicable.

Looking for sufficient conditions that will give unitary polar decompositions we meet the SAW^*-algebras. A C^*-algebra $\mathfrak{A}$ is an *SAW^*-algebra* (sub-AW^*) if for any two elements S, T in $\mathfrak{A}$ with $ST = 0$, there is an E in $\mathfrak{A}$, with $0 \leqslant E \leqslant I$, such that

$$SE = 0 = (I - E)T.$$

As shown in [23] a commutative C^*-algebra $\mathfrak{A} = C(X)$ is an SAW^*-algebra if and only if X is a sub-Stonean space. Pertinent to our discussion is the fact from [25, Theorem 3.5], that if every element in a C^*-algebra $\mathfrak{A}$ has a unitary polar decomposition (in symbols: $\mathfrak{A} = \mathfrak{U}\mathfrak{A}_+$), then $\mathfrak{A}$ is an SAW^*-algebra with $\mathfrak{A}^{-1}$ dense in $\mathfrak{A}$. The converse holds under the additional hypothesis (which may be vacuously true) that also $\mathcal{M}_2(\mathfrak{A})$ is an SAW^*-algebra.

Thus from the Haagerup-Rørdam result, Proposition 6.4 and Theorem 6.6 we can make an educated guess of a general result.

CONJECTURE 6.8. *For a unital C^*-algebra $\mathfrak{A}$ the following conditions are equivalent:*

(i) $\mathfrak{B} = \frac{1}{2}(\mathfrak{U} + \mathfrak{U})$.

(ii) $\mathfrak{A} = \mathfrak{U}\mathfrak{A}_+$.

(iii) $\mathfrak{A}$ *is an SAW^*-algebra with* $\mathrm{sr}(\mathfrak{A}) = 1$ *(i.e. $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$).*

Moreover, if $\mathfrak{A} \subset \mathcal{B}(\mathfrak{H})$, with $\mathfrak{H}$ separable, then already the condition

(iv) $\mathfrak{B} = \frac{1}{2}(\mathfrak{U} + \mathfrak{B})$

will imply that $\mathfrak{A}$ is a von Neumann algebra.

7. LEFT INVERTIBLE ELEMENTS

The relative ease with which the λ_u-function can be calculated for elements in a C^*-algebra $\mathfrak{A}$ is partly due to the easily established series of inclusions:

$$(*) \qquad \mathfrak{B}^{-1} \subset \mathfrak{U}\mathfrak{B}_+ \subset \frac{1}{2}(\mathfrak{U} + \mathfrak{U}) \subset \frac{1}{2}(\mathfrak{U} + \mathfrak{B}) \subset (\mathfrak{B}^{-1})^=;$$

where $\mathfrak{B}^{-1} = \mathfrak{B} \cap \mathfrak{A}^{-1}$ by definition. The distance, $\alpha(T)$, of an element T in $\mathfrak{B}$ to (any of) the sets in $(*)$ determines $\lambda_u(T)$ (if $T \notin \mathfrak{A}^{-1}$), cf. Theorem 5.1.

To determine the λ-function we shall need the multiplicative semigroup of *left invertible* elements

$$\mathfrak{A}_\ell^{-1} = \{A \in \mathfrak{A} \mid \mathfrak{A}A = \mathfrak{A}\}.$$

Thus $A \in \mathfrak{A}_\ell^{-1}$ if $BA = I$ for some B in $\mathfrak{A}$. It follows from the open mapping theorem that $A \in \mathfrak{A}_\ell^{-1}$ if and only if A is injective with closed range, so that

$$\mathfrak{A}_\ell^{-1} = \{A \in \mathfrak{A} \mid m(A) > 0\},$$

with $m(A)$ as in section 4. From this, or using the equation $BA = I$, so that

$$I = A^*B^*BA \leqslant \|B\|^2 A^*A = \|B\|^2 |A|^2,$$

we see that $|A|$ is invertible. Moreover, $V = A|A|^{-1}$ is an isometry ($V^*V = I$). Consequently, $A \in \mathfrak{A}_\ell^{-1}$ if and only if it has a polar decomposition $A = V|A|$, with V an isometry and $|A|$ invertible. In symbols,

$$\mathfrak{A}_\ell^{-1} = \mathfrak{E}_i \mathfrak{A}_+^{-1},$$

where $\mathfrak{E}_i$ denotes the set of isometries in $\mathfrak{A}$ (so that $\mathfrak{E}_i \subset \mathfrak{E}$).

Setting $\mathfrak{B}_\ell^{-1} = \mathfrak{B} \cap \mathfrak{A}_\ell^{-1}$ we have a series of inclusions, corresponding to (*), given by

$$\mathfrak{B}_\ell^{-1} \subset \mathfrak{E}_i \mathfrak{B}_+ \subset \frac{1}{2}(\mathfrak{E}_i + \mathfrak{E}_i) \subset \frac{1}{2}(\mathfrak{E}_i + \mathfrak{B}) \subset (\mathfrak{B}_\ell^{-1})^=.$$

The last of these from the observations that if $V \in \mathfrak{E}_i$ and $B \in \mathfrak{B}$, then for $t < 1$,

$$V + tB = (I + tBV^*)V \in \mathfrak{A}^{-1}V \subset \mathfrak{A}_\ell^{-1},$$

or, equally effective,

$$m(V + tB) \geqslant 1 - \|tB\| \geqslant 1 - t > 0.$$

In section 8 we will consider $*$-invariant conditions, so we shall need also the set $\mathfrak{A}_r^{-1}$ of *right invertible elements*. Note that $\mathfrak{A}_r^{-1} = (\mathfrak{A}_\ell^{-1})^*$.

If $\mathfrak{A}$ is a finite C^*-algebra, then $\mathfrak{A}_\ell^{-1} = \mathfrak{A}^{-1}$. Indeed, if $A \in \mathfrak{A}_\ell^{-1}$, then $A = U|A|$ with $|A|$ in $\mathfrak{A}^{-1}$ and $U^*U = I$. By finiteness, $UU^* = I$, so U is unitary and $A \in \mathfrak{A}^{-1}$. On the other hand, if $\mathfrak{A}_\ell^{-1}$ is dense in $\mathfrak{A}$, then $\mathfrak{A}$ is finite. For if $A \in \mathfrak{A}_\ell^{-1}$ and $BA = I$, we can find C in $\mathfrak{A}_\ell^{-1}$ close to B such that $\|I - CA\| < 1$. But then $CA \in \mathfrak{A}^{-1}$, and if $DC = I$ we have

$$A = DCA \in D\mathfrak{A}^{-1} \in \mathfrak{A}_r^{-1};$$

whence $A \in \mathfrak{A}_\ell^{-1} \cap \mathfrak{A}_r^{-1} = \mathfrak{A}^{-1}$, and Corollary 3.4 applies.

By contrast, consider $\mathfrak{A} = \mathbf{B}(\mathfrak{H})$, and let S be the unilateral shift on $\ell^2 = \mathfrak{H}$, cf. section 9. Then, of course, $S \in \mathfrak{A}_\ell^{-1}$, since $S^*S = I$, but $\|S - A\| \geqslant 1$ for every A in $\mathfrak{A}_r^{-1}$. Yet $\mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}$ is dense in $\mathfrak{A}$ (for any factorial von Neumann algebra $\mathfrak{A}$) by Proposition 3.6.

For any element T in a C^*-algebra $\mathfrak{A}$ we define

$$\alpha_\ell(T) = \operatorname{dist}(T, \mathfrak{A}_\ell^{-1}).$$

Moreover, working on some Hilbert space $\mathfrak{H}$, we consider the polar decomposition $T = V|T|$ in $B(\mathfrak{H})$, and denote by E_δ the spectral projection of $|T|$ corresponding to the interval $]\delta, \infty[$, cf. section 3.

THEOREM 7.1. *For each $\delta > \alpha_\ell(T)$ there is an isometry U in $\mathfrak{E}_i$, such that $UE_\delta = VE_\delta$. For $\delta < \alpha_\ell(T)$ there is no isometric extension of VE_δ in $\mathfrak{A}$.*

Proof. If $\delta > \alpha_\ell(T)$ there is an A in $\mathfrak{A}_\ell^{-1}$ such that $\|T - A\| < \delta$. Write $A = W|A|$, where $W \in \mathfrak{E}_i$ and $|A| \in \mathfrak{A}^{-1}$. Then $\|TW^* - AW^*\| < \delta$, and since $AW^* \in (\mathfrak{A}^{-1})^=$ (because $W|A|W^* + \varepsilon I \in \mathfrak{A}^{-1}$ for every $\varepsilon > 0$) it follows that $\alpha(TW^*) < \delta$, with α as in section 5.

Note now that TW^* has the polar decomposition $VW^*(W|T|W^*)$. Moreover, if f is a polynomial without constant term, then $f(W|T|W^*) = Wf(|T|)W^*$. The relation therefore holds when f is a Borel function (with $f(0) = 0$), so if E_δ is the spectral projection for $|T|$, corresponding to the interval $]\delta, \infty[$, then $WE_\delta W^*$ is the corresponding spectral projection for $W|T|W^*$.

Applying Theorem 3.1 to TW^* we find a unitary U in $\mathfrak{U}$ such that

$$UWE_\delta W^* = VW^*(WE_\delta W^*) = VE_\delta W^*.$$

Therefore, $UW \in \mathfrak{C}_{\mathrm{i}}$ and $UWE_\delta = VE_\delta$, as desired.

Conversely, if $U \in \mathfrak{C}_{\mathrm{i}}$ such that $UE_\delta = VE_\delta$ for some $\delta > 0$, define $f(t) = (t - \delta) \vee 0$. Then $S = Vf(|T|) \in \mathfrak{A}$. In fact, since $f(t) = 0$ for $t \leqslant \delta$, we have

$$S = Vf(|T|)E_\delta = UE_\delta f(|T|) = Uf(|T|) \in (\mathfrak{A}_\ell^{-1})^=,$$

because $f(|T|) + \varepsilon I \in \mathfrak{A}^{-1}$ for every $\varepsilon > 0$. Since

$$\|T - S\| = \|\, |T| - f(|T|)\| \leqslant \delta,$$

it follows that $\alpha_\ell(T) \leqslant \delta$. ∎

Corollary 7.2. *Each element of the form $Vf(|T|)$, where f is a continuous function on* sp$|T|$ *such that $f(t) = 0$ for $t \leqslant \delta$, for some $\delta > \alpha_\ell(T)$, has a polar decomposition $Uf(|T|) = Vf(|T|)$, where U is an isometry in $\mathfrak{A}$.*

Proposition 7.3. *If a C^*-algebra $\mathfrak{A}$ contains an element T with $\alpha_\ell(T) > 0$, then there is an S in $\mathfrak{B}$ with $\alpha_\ell(S) = 1$. If, moreover, $\alpha_\ell(T^*) \geqslant \alpha_\ell(T)$ we may assume that also $\alpha_\ell(S^*) = 1$.*

Proof. As in the proof of Theorem 5.4 we regard $\mathfrak{A}$ as a C^*-subalgebra of some $\mathcal{B}(\mathfrak{H})$ and let $T = V|T|$ be the polar decomposition of $|T|$. Assuming, as we may, that $\|T\| = 1$ we let $S = Vf(|T|)$, where $f(t) = 1 \wedge \alpha_\ell(T)^{-1}t$ for $0 \leqslant t \leqslant 1$.

If $\alpha_\ell(S) < 1$, then with E_δ as the spectral projection of $|S|$ corresponding to the interval $]\delta, \infty[$, there is by Theorem 7.1 an isometry U in $\mathfrak{C}_{\mathrm{i}}$ such that $UE_\delta = VE_\delta$ for some $\delta < 1$. Since $S = Vf(|T|)$ and $T = V|T|$, it follows that E_δ is also a spectral projection for $|T|$, but corresponding to the interval $]\delta\alpha_\ell(T), \infty[$. Since $\delta\alpha_\ell(T) < \alpha_\ell(T)$, this contradicts Theorem 7.1, applied to T. Thus $\alpha_\ell(S) = 1$.

If $\alpha_\ell(T^*) \geqslant \alpha_\ell(T)$ and $\alpha_\ell(S^*) < 1$, then, since $S^* = V^*(V|S|V^*)$, we can apply the previous argument to find an isometry U in $\mathfrak{C}_{\mathrm{i}}$, such that

$$U(VE_\delta V^*) = V^*(VE_\delta V^*)$$

with $\delta > 1$. Since $VE_\delta V^*$ is the spectral projection of $|T^*|$ $(= V|T|V^*)$ corresponding to the interval $]\delta\alpha_\ell(T), \infty[$, we conclude from Theorem 7.1, applied to T^*, that $\alpha_\ell(T^*) \leqslant \delta\alpha_\ell(T)$. But $\alpha_\ell(T^*) \geqslant \alpha_\ell(T)$ by assumption, and again we have reached a contradiction. Thus also $\alpha_\ell(S^*) = 1$. ∎

LEMMA 7.4. (Cf. [20, Cor. 2.3]). *For every T in a C^*-algebra $\mathfrak{A}$ and any isometry U in $\mathfrak{E}_i$ the spectrum of TU^* contains a disc about the origin with radius $\alpha_\ell(T)$.*

Proof. If $\lambda \in \mathbf{C}$ with $|\lambda| < \alpha_\ell(T)$, but $\lambda \notin \mathrm{sp}(TU^*)$, then $TU^* - \lambda I = A \in \mathfrak{A}^{-1}$. But then

$$\|T - AU\| = \|\lambda U\| = |\lambda| < \alpha_\ell(T),$$

a contradiction, since $AU \in \mathfrak{A}_\ell^{-1}$. ∎

THEOREM 7.5. *Let $\mathfrak{E}_i$ be the set of isometries in a C^*-algebra $\mathfrak{A}$, and take T in $\mathfrak{A}$. If $T \notin \mathfrak{A}_\ell^{-1}$ then*

$$\mathrm{dist}(T, \mathfrak{E}_i) = \max\{\|T\| - 1,\ \alpha_\ell(T) + 1\}.$$

Otherwise we have an approximant V in $\mathfrak{E}_i$ with

$$\mathrm{dist}(T, \mathfrak{E}_i) = \|T - V\| = \max\{\|T\| - 1,\ 1 - m(T)\}.$$

Proof. If $T \notin \mathfrak{A}_\ell^{-1}$ then for any U in $\mathfrak{E}_i$

$$\|T - U\| \geqslant \|U^*T - I\| \geqslant r(U^*T - I) \geqslant 1 + \alpha_\ell(T),$$

because the spectral radius of $U^*T - I$ must be at least $1 + \alpha_\ell(T)$ by Lemma 7.4. Clearly we also have $\|T - U\| \geqslant \|T\| - 1$, so we have established inequality in the formula above. To prove the reverse inequality, consider $\delta > \alpha_\ell(T)$. As in the proof of Theorem 7.1 there is an isometry W in $\mathfrak{E}_i$ such that $\alpha(TW^*) < \delta$. By Proposition 3.5 there is for any $\varepsilon > 0$ a unitary U in $\mathfrak{U}$ such that

$$\|TW^* - U\| < \max\{\|TW^*\| - 1 + \varepsilon,\ \delta + 1\}.$$

But then $UW \in \mathfrak{E}_i$ with

$$\|T - UW\| \leqslant \|TW^* - U\| < \max\{\|T\| - 1 + \varepsilon,\ \delta + 1\}.$$

Since ε and δ are arbitrary we get

$$\mathrm{dist}(T, \mathfrak{E}_i) \leqslant (\|T\| - 1) \vee (\alpha_\ell(T) + 1),$$

and thus equality.

If $T \in \mathfrak{A}_\ell^{-1}$ we have a polar decomposition $T = V|T|$ with V in $\mathfrak{E}_i$ and $|T|$ in $\mathfrak{A}^{-1}$. Thus, as in the proof of Theorem 5.1 (see $(*)$), we have $m(T) = \| \, |T|^{-1} \|^{-1}$ and evidently

$$\|T - V\| = \| \, |T| - I\| = (\|T\| - 1) \vee (1 - m(T)).$$

Conversely, if $U \in \mathfrak{E}_i$ then, of course, $\|T - U\| \geqslant \|T\| - 1$, and moreover

$$\|U - T\| = \sup \|(U - T)x\| \geqslant$$

$$\geqslant \sup \|Ux\| - \|Tx\| = 1 - \inf \|Tx\| = 1 - m(T),$$

when x ranges over the set of unit vectors in $\mathfrak{H}$. Thus, in this case

$$\operatorname{dist}(T, \mathfrak{E}_i) \geqslant (\|T\| - 1) \vee (1 - m(T)),$$

which, in conjunction with the previous result, completes the proof. ■

8. PRIME C^*-ALGEBRAS

Using the result from section 7 we can extend the λ-theory to a large class of infinite C^*-algebras.

For any element T in a C^*-algebra $\mathfrak{A}$ we define

$$\alpha_q(T) = \operatorname{dist}(T, \mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}) =$$
$$= \alpha_\ell(T) \wedge \alpha_\ell(T^*).$$

As with the definition of $m_q(T)$ from $m(T)$ in section 4, we would have liked to define $\alpha_q(T)$ as an infinum of distances to $\mathfrak{A}_\ell^{-1}$ of elements $ZT + (I - Z)T^*$; instead of just taking $Z = 0$ and $Z = I$, cf. $(* * *)$ in section 4. The absense of spectral projections in C^*-algebra theory prevents this, but it also explains why our program will only work in C^*-algebras that are *prime*, i.e. where $S\mathfrak{A}T = 0$ implies $S = 0$ or $T = 0$ for any pair S, T in $\mathfrak{A}$. The λ-theory for general C^*-algebras, as well as the theory for extremal extensions, approximations and convex decompositions, will be carried out in [34]. In a prime C^*-algebra $\mathfrak{A}$, the extreme points are either isometries or co-isometries, so that

$$\mathfrak{E} = \mathfrak{E}_i \cup \mathfrak{E}_i^*;$$

and we see why $\alpha_q(T)$ can be used in this context. Alternatively, we could define a special λ-function on general C^*-algebras, related to $\mathfrak{E}_i \cup \mathfrak{E}_i^*$ in analogy with the λ_u-function in section 5. But this time we will play fair.

THEOREM 8.1. *If $\mathfrak{A}$ is prime C^*-algebra and $T \in \mathfrak{B}$, then*

$$\lambda(T) = \frac{1}{2}(1 - \alpha_q(T))$$

if $T \notin \mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}$. Otherwise

$$\lambda(T) = \frac{1}{2}(1 + m_q(T)),$$

where $m_q(T) = m(T) \vee m(T^*)$.

Proof. If $\alpha > \alpha_q(T)$ there is an element A in $\mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}$ with $\|T - A\| < \alpha$. We may assume that $A = V|A|$ with V in $\mathfrak{E}_i$ and $|A|$ in $\mathfrak{A}^{-1}$, passing otherwise to T^*. Consequently

$$\alpha(TV^*) = \operatorname{dist}(TV^*, \mathfrak{A}^{-1}) \leqslant \|TV^* - V|A|V^*\| < \alpha.$$

By Theorem 5.1 there is a pair U, B in $\mathfrak{U} \times \mathfrak{B}$ such that

$$TV^* = \frac{1}{2}(1 - \alpha)U + \frac{1}{2}(1 + \alpha)B.$$

Since $V^*V = I$ this means that

$$T = \frac{1}{2}(1 - \alpha)UV + \frac{1}{2}(1 + \alpha)BV;$$

and since $UV \in \mathfrak{E}$ and $\alpha > \alpha_q(T)$ was arbitrary we conclude that

$$(*) \qquad \lambda(T) \geqslant \frac{1}{2}(1 - \alpha_q(T)).$$

Conversely, if

$$T = \lambda V + (1 - \lambda)B$$

for some V in $\mathfrak{E}$ and B in $\mathfrak{B}$, we may assume that $V^*V = I$ (passing otherwise to T^*). Assume for the moment that $\lambda \leqslant \frac{1}{2}$ and $\|B\| < 1$. Then, as we pointed out in the beginning of section 7, cf (**),

$$V + B = (I + BV^*)V \in \mathfrak{A}_\ell^{-1},$$

because $I + BV^* \in \mathfrak{A}^{-1}$, so that

$$V^*(1 + BV^*)^{-1}(V + B) = I.$$

Since in our case

$$T - \lambda(V + B) = (1 - 2\lambda)B,$$

we conclude that $\alpha_\ell(T) \leqslant 1 - 2\lambda$. The condition $\|B\| < 1$ can be removed by continuity, and the conclusion is that

$$(**) \qquad \alpha_q(T) \leqslant 1 - 2\lambda(T)$$

provided that $\lambda(T) \leqslant \frac{1}{2}$. But if $\lambda > \frac{1}{2}$, the same arguments shows that

$$T = \lambda(V + \lambda^{-1}(1-\lambda)B) \in \mathfrak{A}_\ell^{-1}.$$

Thus if $T \notin \mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}$ it follows from $(*)$ and $(**)$ that

$$\lambda(T) = \frac{1}{2}(1 - \alpha_q(T)).$$

If $T \in \mathfrak{A}_\ell^{-1}$ we have $T = V|T|$ with V in $\mathfrak{E}_i$ and $|T|$ in $\mathfrak{A}^{-1}$. Thus

$$m(T) = \| \, |T|^{-1} \|^{-1}.$$

As in the proof of Theorem 5.1 this implies that

$$|T| = \lambda_0 U_1 + (1-\lambda_0)U_2,$$

where $\lambda_0 = \frac{1}{2}(1 + m(T))$ and $U_1, U_2 \in \mathfrak{U}$. It follows that

$$(***) \qquad T = \lambda_0 V U_1 + (1-\lambda_0) V U_2,$$

so that $\lambda(T) \geqslant \lambda_0$. Conversely, if $T = \lambda V + (1-\lambda)B$ with V in $\mathfrak{E}_i$ and B in $\mathfrak{B}$ then

$$m(T) = \inf\{\|Tx\| \mid \|x\| = 1\} \geqslant 2\lambda - 1,$$

so that $m(T) \geqslant 2\lambda(T) - 1$. Applying the same arguments to T^* we finally conclude that if T is left or right invertible, then

$$\lambda(T) = \frac{1}{2}(1 + m_q(T)). \qquad \blacksquare$$

COROLLARY 8.2. *If T is a left or right invertible element of a prime C^*-algebra $\mathfrak{A}$, then*

$$T = \lambda(T)V + (1 - \lambda(T))W$$

for some V, W in $\mathfrak{E}$.

Proof. See $(***)$ in the proof of Theorem 8.1. ■

THEOREM 8.3. *A prime C^*-algebra $\mathfrak{A}$ has the λ-property if and only if*

$$(\mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1})^= = \mathfrak{A},$$

in which case it has the uniform λ-property for $\lambda = \frac{1}{2}$.

Proof. By Theorem 8.1 we have $\lambda(T) \geqslant \frac{1}{2}$ if and only if $\alpha_q(T) = 0$ for all T in $\mathfrak{B}$, i.e. if and only if $\mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1}$ is dense in $\mathfrak{A}$. Moreover, $\mathfrak{A}$ has the λ-property if and only if $\alpha_q(S) < 1$ for every S in $\mathfrak{B}$. But if $\alpha_q(T) > 0$ for some T in $\mathfrak{B}$, we may assume that $\alpha_\ell(T^*) \geqslant \alpha_\ell(T) > 0$, whence $\alpha_q(S) = 1$ for some S in $\mathfrak{B}$ by Proposition 7.3. ∎

9. EXAMPLES OF INFINITE C^*-ALGEBRAS

For general (infinite, non-prime) C^*-algebras we have no explicit formula for the λ-function. Some interesting examples can, however, be computed. For this we need a few results from classical index theory, found in any number of textbooks, e.g. [26, 3.3]. We also wish to mention some results from C^*-algebraic K-theory (non-classical index theory), and refer the reader to [4], [19] or [33], in decreasing order of complexity.

On the separable Hilbert space $\mathfrak{H}$ $(= \ell^2)$ we let $\mathfrak{K}$ denote the algebra of compact operators, and we denote by $\mathfrak{F}$ the set of Fredholm operators in $\mathbf{B}(\mathfrak{H})$ — the operators whose images in the Calkin algebra $\mathbf{B}(\mathfrak{H})/\mathfrak{K}$ are invertible.

THEOREM 9.1. *Let* $\mathfrak{A}$ *be a* C^**-subalgebra of* $\mathbf{B}(\mathfrak{H})$ *containing* $\mathfrak{K}$, *such that* $\mathfrak{F} \cap \mathfrak{A}$ *is dense in* $\mathfrak{A}$. *Then* $\lambda(T) \geqslant \frac{1}{2}$ *for every* T *in* $\mathfrak{B}$.

Proof. Since $\mathfrak{K}$ is a minimal ideal, $\mathfrak{A}$ is prime, so by Theorem 8.3 it suffices to show that the left or right invertible elements in $\mathfrak{A}$ are dense.

Given T in $\mathfrak{B}$ and $\varepsilon > 0$ we can by assumption find F in $\mathfrak{F} \cap \mathfrak{A}$ such that $\|T - F\| < \varepsilon$. Since $\lambda(T) = \lambda(T^*)$ we may assume, without loss of generality, that the index n of F is $\leqslant 0$, considering otherwise T^* and F^*. Since $\mathfrak{K} \subset \mathfrak{A}$ we can choose a partial isometry A of finite rank from $\ker F$ into $\ker F^*$. As $\ker F^* = F(\mathfrak{H})^\perp$, the operator $F + A$ is an injection of $\mathfrak{H}$ onto a closed subspace $(= F(\mathfrak{K}) \oplus A(\mathfrak{H}))$ of co-dimension $-n$. By the open mapping theorem $(F + A)^*(F + A)$ is invertible, so that

$$F + A = V|F + A|$$

for some isometry V $(= (F + A)|F + A|^{-1})$ in $\mathfrak{A}$. Since $F^*A = FA^* = 0$, it follows that $F = V|F|$. Likewise, $F + \varepsilon A = V|F + \varepsilon A|$, where $|F + \varepsilon A| \in \mathfrak{A}^{-1}$. Thus $F + \varepsilon A \in \mathfrak{A}_\ell^{-1}$, and

$$\|T - (F + \varepsilon A)\| < 2\varepsilon.$$

Since ε is arbitrary it follows that $\alpha_q(T) = 0$, whence $\lambda(T) \geqslant \frac{1}{2}$ by Theorem 8.3. ∎

Let S denote the *unilateral shift* on ℓ^2, i.e.

$$S(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots).$$

Thus S is an isometry in $\mathfrak{F}$ with index -1. Since $S^n(1-SS^*)$ is the rank one operator that takes the first basis vector to the n'th, it is not hard to see that the C^*-algebra $\mathfrak{T}$ generated by S — the *Toeplitz algebra* — contains the algebra $\mathfrak{K}$ of compact operators. Since the image of S in the Calkin algebra is a unitary with full spectrum, we have a short exact sequence

$$0 \longrightarrow \mathfrak{K} \stackrel{i}{\longrightarrow} \mathfrak{T} \stackrel{q}{\longrightarrow} C(\mathbf{T}) \longrightarrow 0.$$

We choose the identification of $\mathfrak{T}/\mathfrak{K}$ with $C(\mathbf{T})$ such that $q(T_f) = f$ for every f in $C(\mathbf{T})$. Here T_f is the Toeplitz operator on the Hardy space H^2 (identified with ℓ^2); so $T_f = PM_fP$, where P is the projection of $L^2(\mathbf{T})$ onto H^2, and M_f is the ordinary multiplication operator on $L^2(\mathbf{T})$.

COROLLARY 9.2. *The Toeplitz algebra $\mathfrak{T}$ has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Since $\mathfrak{K} \subset \mathfrak{T}$ and the invertible elements are dense in $\mathfrak{T}/\mathfrak{K}$ $(= C(\mathbf{T}))$, the conditions in Theorem 9.1 are met. ∎

Our final example is a curious non-prime C^*-algebra, which is finite in the Murray-von Neumann sense, but which nevertheless contains non-unitary extreme points. For this we shall need the $*$-automorphism θ of $C(\mathbf{T})$ of order two, given by

$$\theta f(t) = f(t^{-1}), \quad t \in \mathbf{T}.$$

PROPOSITION 9.3. *If $\mathfrak{A}$ is the C^*-algebra of operators on $\mathfrak{H} = \ell^2 \oplus \ell^2$, generated by $T = S \oplus S^*$, then $\mathfrak{A}$ consists of those elements in $\mathbf{B}(\mathfrak{H})$ of the form $B \oplus C$, where $B \in \mathfrak{T}$, $C \in \mathfrak{T}$ and $q(B) = \theta q(C)$.*

Proof. The set $\mathfrak{A}_0$ of elements $B \oplus C$ in $\mathfrak{T} \oplus \mathfrak{T}$, such that $q(B) = \theta q(C)$ evidently constitutes a normclosed, $*$-subalgebra of $\mathbf{B}(\mathfrak{H})$, i.e. a C^*-algebra. Since $q(S) = \mathrm{id}$ and $q(S^*) = \overline{\mathrm{id}} = \mathrm{id}^{-1}$ (where $\mathrm{id}(t) = t$ on $\mathbf{T}$), we see that $T = S \oplus S^* \in \mathfrak{A}_0$, whence $\mathfrak{A} \subset \mathfrak{A}_0$.

To prove the converse inclusion, note that

$$T^*T - TT^* = (I \oplus I - P_1) - (I - P_1 \oplus I) = P_1 \oplus -P_1,$$

where P_1 denotes the rank one projection on the first basis vector. Thus $P_1 \oplus 0 \in \mathfrak{A}$ and $0 \oplus P_1 \in \mathfrak{A}$. Moreover, as $T^n(P_1 \oplus 0) = S^nP_1 \oplus 0$ we see, as before, that $\mathfrak{A}$ contains the ideal $\mathfrak{K} \oplus 0$. Similarly $0 \oplus \mathfrak{K} \subset \mathfrak{A}$. The projection $Z = I \oplus 0$ in $\mathbf{B}(\mathfrak{H})$ commutes with

$\mathfrak{A}$ (because it commutes with T), so the map $A \to AZ$ is a $*$-homomorphism of $\mathfrak{A}$. Since $TZ = S \oplus 0$ we see that $\mathfrak{A}Z = \mathfrak{T} \oplus 0$. Now take any element $B \oplus C$ in $\mathfrak{A}_0$. There is an element A in $\mathfrak{A}$ such that $AZ = B$. Since $\mathfrak{A} \subset \mathfrak{A}_0$ we know that $A = B \oplus D$, where $\theta q(D) = q(B)$. But also $\theta q(C) = q(B)$, so $q(D) = q(C)$; i.e. $C - D = K \in \mathfrak{K}$. As $0 \oplus \mathfrak{K} \subset \mathfrak{A}$,

$$A + (0 \oplus K) = B \oplus (D + K) = B \oplus C \in \mathfrak{A},$$

whence $\mathfrak{A}_0 \subset \mathfrak{A}$. ■

PROPOSITION 9.4. *With $\mathfrak{A}$ as in 9.3, the set $\mathfrak{E}$ of extreme points in the unit ball $\mathfrak{B}$ is the disjoint union*

$$\mathfrak{E} = \bigcup_{n \in \mathbb{Z}} \mathfrak{U} T^n \mathfrak{U},$$

where T^{-n} should be interpreted as T^{*n} and $T^0 = I$. In particular, $\mathfrak{E}$ contains no non-unitary isometries, so $\mathfrak{A}$ is Murray-von Neumann finite.

Proof. If $V \in \mathfrak{E}$, then $V = U \oplus W$ for some partial isometries U and W in $\mathfrak{T}$. In fact, since we have $\mathfrak{A}Z = \mathfrak{T} \oplus 0$ and $\mathfrak{A}(I - Z) = 0 \oplus \mathfrak{T}$ (with $Z = I \oplus 0$ as in the proof of Proposition 9.3), both U and W must be extreme in $\mathfrak{T}$. They are therefore either isometries or co-isometries, and in particular they belong to $\mathfrak{F}$. Since the winding number of the function $q(U)$ is $-$ index U, and since θ reverses the direction of its path, we see that

$$\operatorname{index} W = -\operatorname{index} U.$$

Assume now that index $U = n \geqslant 0$. Thus W is an isometry, U a co-isometry, and $S^n U$ and $S^{*n} W$ are partial isometries of index zero. Choose partial isometries A and B of finite rank from $\ker S^n U$ to $\ker U^* S^{*n}$ and from $\ker S^{*n} W$ to $\ker W^* S^n$, respectively. Then

$$U_1 = S^n U + A \oplus S^{*n} W + B$$

is a unitary in $\mathfrak{A}$, because both summands are unitaries in $\mathfrak{T}$ and

$$\theta q(S^n U + A) = \theta q(S^n) \theta q(U) =$$

$$= q(S^{*n}) q(W) = q(S^{*n} W + B).$$

We have

$$T^{*n} U_1 = S^{*n}(S^n U + A) \oplus S^n(S^{*n} W + B) =$$

$$= (U + S^{*n} A) \oplus (S^n S^{*n} W + S^n B) = U \oplus S^n(S^{*n} W + B),$$

because

$$A(\ell^2) = \ker(U^* S^{*n}) = \ker S^{*n}.$$

Both W and $S^n(S^{*n}W+B)$ are isometries in $\mathfrak{T}$ with index $-n$, so

$$V_2 = W(W^*S^n + B^*)S^{*n} + C$$

is unitary in $\mathfrak{T}$ for a partial isometry C of finite rank. Since

$$\ker C = S^n(S^{*n}W + B)\ell^2 = S^n\ell^2,$$

it follows that $CS^n = 0$, so

$$V_2S^n(S^{*n}W+B) =$$
$$= (W(W^*S^n + B^*)S^{*n} + C)S^n(S^{*n}W + B) = W.$$

Finaly, $WW^* = I - Q$ and $S^nS^{*n} = I - P$ for some projections P and Q of rank n, so

$$V_2 = (I-Q)(I-P) + WB^*S^{*n} + C = I + K$$

where $K \in \mathfrak{K}$. Consequently, $U_2 = I \oplus V_2$ is unitary in $\mathfrak{A}$ by Proposition 9.3, and

$$U_2T^{*n}U_1 = (I \oplus V_2)(U \oplus S^n(S^{*n}W + B)) = U \oplus W,$$

as desired.

The case where index $U < 0$ follows from the above by considering $V^* = U^* \oplus W^*$, and the proof is complete. ■

PROPOSITION 9.5. *The C^*-algebra $\mathfrak{A}$ from 9.3 and 9.4 has the uniform λ-property with $\lambda = \frac{1}{2}$.*

Proof. From the Proposition 9.3 we see that there is a short exact sequence

$$(*), \qquad 0 \longrightarrow \mathfrak{K} \oplus \mathfrak{K} \xrightarrow{i} \mathfrak{A} \xrightarrow{p} C(\mathbf{T}) \longrightarrow 0$$

where we choose p such that $p(B \ominus C) = q(B)$ for every $B \oplus C$ in $\mathfrak{A}$ and $q : \mathfrak{T} \to C(\mathbf{T})$ as above. We can therefore use almost the same arguments as in Theorem 9.1.

If $A = B \oplus C \in \mathfrak{A}$ and $\varepsilon > 0$ we can find $E = F \oplus G$ in $\mathfrak{A}$ such that $p(E)$ $(= q(F) = \theta q(G))$ is invertible in $C(\mathbf{T})$, and such that

$$\|B - F\| \vee \|C - G\| = \|(B - F) \oplus (C - G)\| = \|A - E\| < \varepsilon.$$

Regarding F and G as elements in $\mathfrak{T} \cap \mathfrak{F}$ we may assume that

$$\operatorname{index} F = -\operatorname{index} G = n \leqslant 0$$

(considering otherwise A^* and E^*). As in the proof of Theorem 9.1 this implies that

$$F = V|F|, \quad G = W^*|G|,$$

where V and W are isometries in $\mathfrak{T}$. Since

$$\theta q(V) = \theta\left(q(F)q(F^*F)^{-\frac{1}{2}}\right) =$$

$$= q(G)q(G^*G)^{-\frac{1}{2}} = q(W^*),$$

it follows from the Proposition 9.3 that $V^* \oplus W$ and $V \oplus W^*$ are elements in $\mathfrak{A}$. For each element $X \oplus Y$ in $\mathfrak{A}$ define

$$\rho(X \oplus Y) = XV^* \oplus WY, \quad \sigma(X \oplus Y) = XV \oplus W^*Y.$$

Since

$$\theta q(XV^*) = q(Y)q(W) = q(W)q(Y) = q(WY),$$

and similarly $\theta q(XV) = q(W^*Y)$, it follows that ρ and σ are normdecreasing linear maps of $\mathfrak{A}$ into itself. Moreover, $\sigma \circ \rho$ =identity. Now

$$\|\rho(A) - |F^*| \oplus |G|\,\| = \|BV^* \oplus WC - FV^* \oplus WG\| =$$

$$= \|(B-F)V^* \oplus W(C-G)\| = \|\rho(A-E)\| < \varepsilon,$$

and since $q(|F^*|) = q(|F|)$ we see from Proposition 9.3 that $|F^*| \oplus |G|$ $(= \rho(E)) \in \mathfrak{A}$. Evidently $|F^*| \oplus |G| \in (\mathfrak{A}^{-1})^=$, so

$$\operatorname{dist}(\rho(A), \mathfrak{A}^{-1}) < \varepsilon.$$

By Theorem 5.1 this means that $\lambda_u(\rho(A)) > \frac{1}{2}(1-\varepsilon)$, so

$$\rho(A) = \frac{1}{2}(1-\varepsilon)U + \frac{1}{2}(1+\varepsilon)D$$

for some U in $\mathfrak{U}$ and D in $\mathfrak{B}$. Consequently

$$A = \sigma\rho(A) = \frac{1}{2}(1-\varepsilon)\sigma(U) + \frac{1}{2}(1+\varepsilon)\sigma(D).$$

Here $\sigma(U) \in \mathfrak{B}$, whereas, if $U = U_1 \oplus U_2$, we have

$$\sigma(U) = U_1V \oplus W^*U_2 \in \mathfrak{C}.$$

It follows that $\lambda(A) \geqslant \frac{1}{2}(1-\varepsilon)$, and since ε is arbitrary $\lambda(A) \geqslant \frac{1}{2}$. ∎

REMARK 9.6. Using the six term exact sequence in K-theory arising from the short exact sequence (∗) in the previous proof, we find that $K_1(\mathfrak{A}) = 0$ whereas

$K_0(\mathfrak{A}) = \mathbf{Z} \oplus \mathbf{Z}$, one copy of $\mathbf{Z}$ for the finite projections and one copy for the co-finite projections. The interesting part of the diagram reads

$$\begin{array}{ccccccc} K_1(C(\mathbf{T})) & \xrightarrow{\delta} & K_0(\mathfrak{K}\oplus\mathfrak{K}) & \xrightarrow{e} & K_0(\mathfrak{A}) & \xrightarrow{f} & K_0(C(\mathbf{T})) \\ \| & & \| & & \| & & \| \\ \mathbf{Z} & \xrightarrow{\delta} & \mathbf{Z}\oplus\mathbf{Z} & \xrightarrow{e} & \mathbf{Z}\oplus\mathbf{Z} & \xrightarrow{f} & \mathbf{Z} \end{array}$$

where $\delta(n) = (n, n), e(n, m) = (n - m, 0)$ and $f(n, m) = m$.

10. PURELY INFINITE C^*-ALGEBRAS

We now consider C^*-algebras that are infinite in the extreme. Recall from [5] that a C^*-algebra $\mathfrak{A}$ has *real rank zero* if for every pair S, T of elements in $\mathfrak{A}$ such that $ST = 0$, and every $\varepsilon > 0$, there is a projection P in $\mathfrak{A}$ such that $SP = 0$ and $\|(I - P)T\| < \varepsilon$. This condition has a number of equivalent formulations. One is that $\mathfrak{A}^{-1} \cap \mathfrak{A}_{sa}$ should be dense in $\mathfrak{A}_{sa}$. Another, seemingly much stronger, is that the set of self-adjoint elements with finite spectra is dense in $\mathfrak{A}_{sa}$.

Following Cuntz [6] a simple C^*-algebra $\mathfrak{A}$ is said to be *purely infinite* if it has real rank zero and every non-zero projections is infinite (i.e. Murray-von Neumann equivalent to a proper subprojection). This implies that for any pair P, Q of non-zero projections, there is a partial isometry V in $\mathfrak{A}$ such that $V^*V = P$ and $VV^* \leqslant Q$. A number of equivalent formulations are found in [15].

THEOREM 10.1. *If $\mathfrak{A}$ is a purely infinite C^*-algebra, the set of elements T of the form $T = V|T|$, where V is an isometry or a co-isometry in $\mathfrak{A}$, is dense in $\mathfrak{A}$. Thus*

$$(\mathfrak{C}\mathfrak{A}_+)^= = \mathfrak{A}.$$

Proof. If $T \in \mathfrak{A}$ it has a polar decomposition $T = V|T|$, with V in $\mathfrak{A}''$, cf. section 3. It follows from the (Stone-)Weierstrass theorem that $Vf(|T|) \in \mathfrak{A}$ whenever f is a continuous function on $\mathrm{sp}|T|$ vanishing at zero. We also note that $T^* = V^*|T^*|$ is the polar decomposition of T^*, with $|T^*| = V|T|V^*$.

If $|T| \in \mathfrak{A}^{-1}$ then $V = T|T|^{-1}$ is an isometry in $\mathfrak{A}$. Similarly, if $|T^*| \in \mathfrak{A}^{-1}$ then V^* is an isometry, so V is a co-isometry in $\mathfrak{A}$. If 0 is an isolated point both in $\mathrm{sp}|T|$ and in $\mathrm{sp}|T^*|$, let $e(t) = 1$ if $t \in \mathrm{sp}|T| \setminus \{0\}$ and $e(0) = 0$. Then $P = e(|T|)$ and $Q = e(|T^*|)$ are projections in $\mathfrak{A}$ and $V = VP = QV$. As $I - P$ and $I - Q$ are non-zero projections in $\mathfrak{A}$, and $\mathfrak{A}$ is purely infinite, there is a partial isometry W in $\mathfrak{A}$ with $W^*W = I - P$ and $WW^* \leqslant I - Q$. Then $U = W + V$ is an isometry in $\mathfrak{A}$, and $T = U|T|$.

We are left with the case where 0 is an accumulation point both in $\mathrm{sp}|T|$ and in $\mathrm{sp}|T^*|$. Given $\varepsilon > 0$ we define

$$f_1(t) = (t-\varepsilon) \vee 0, \quad f_2(t) = (t-2\varepsilon) \vee 0,$$

$$g_1(t) = (1-\varepsilon^{-1}t) \vee 0, \quad g_2(t) = (1-(2\varepsilon)^{-1}t) \vee 0,$$

for $t \geqslant 0$. Assuming, as we may, that $\|T\| = 1$, we see that

$$f_1(|T^*|)g_1(|T^*|) = f_2(|T|)g_2(|T|) = 0,$$

because $f_i g_i = 0$, $1 \leqslant i \leqslant 2$. Since $\mathfrak{A}$ has real rank zero we can therefore find projections P and Q in $\mathfrak{A}$ such that

$$(I-P)g_2(|T|) = 0, \quad \|Pf_2(|T|)\| \leqslant \varepsilon,$$

$$\|(I-Q)g_1(|T^*|)\| \leqslant \varepsilon, \quad Qf_1(|T^*|) = 0.$$

Evidently P and Q are non-zero, since $g_1(0) = g_2(0) = 1$ and $0 \in \mathrm{sp}|T| \cap \mathrm{sp}|T^*|$. Since $\mathfrak{A}$ is purely infinite we can therefore find a partial isometry W in $\mathfrak{A}$ such that $W^*W = P$ and $WW^* \leqslant Q$. Now define $S = \varepsilon W + Vf_1(|T|)$. Then $S \in \mathfrak{A}$ with

$$\|T - S\| \leqslant \varepsilon + \|V(|T| - f_1(|T|)\| \leqslant \varepsilon + \|\mathrm{id} - f_1\|_\infty = 2\varepsilon.$$

On the other hand

$$\begin{aligned} S^*S &= \varepsilon^2 P + f_1^2(|T|) + 2\varepsilon \mathrm{Re}\, W^*Vf_1(|T|) = \\ &= \varepsilon^2 P + f_1^2(|T|) + 2\varepsilon \mathrm{Re}\, W^*Qf_1(|T^*|)V = \varepsilon^2 P + f_1^2(|T|) \geqslant \\ &\geqslant \varepsilon^2 g_2(|T|) + f_1^2(|T|). \end{aligned}$$

Since $\varepsilon^2 g_2(|t|) + f_1^2(|t|) > 0$ for $0 \leqslant t \leqslant 1$ (in fact $\varepsilon^2 g_2(t) + f_1^2(t) \geqslant \frac{7}{16}\varepsilon^2$) we see that $|S|$ is invertible, whence $S = U|S|$ for the isometry $U = S|S|^{-1}$ in $\mathfrak{A}$. ∎

COROLLARY 10.2. *Every purely infinite C^*-algebra has the uniform λ-property for $\lambda = \frac{1}{2}$.*

Proof. Since $\mathfrak{A}_+ \subset (\mathfrak{A}^{-1})^=$ it follows from Theorem 10.1 that

$$\mathfrak{A} = (\mathbb{C}\mathfrak{A}^{-1})^= = (\mathfrak{A}_\ell^{-1} \cup \mathfrak{A}_r^{-1})^=.$$

Thus $\alpha_q = 0$ for every T in $\mathfrak{B}$, whence $\lambda(T) \geqslant \frac{1}{2}$ by Theorem 8.3. ∎

Again we may ask, as in section 6, whether the supremum in the λ-function is attained for the C^*-algebras considered in sections 8–10; and again the author is inclined to bet that the answer is negative in general. Indeed, on the principle of not being hanged for a sheep, we offer the following companion to Conjecture 6.8.

CONJECTURE 10.3. *For a unital C^*-algebra $\mathfrak{A}$ the following conditions are equivalent:*

(i) $\mathfrak{B} = \frac{1}{2}(\mathfrak{C} + \mathfrak{C})$.

(ii) $\mathfrak{A} = \mathfrak{C}\mathfrak{A}_+$.

(iii) $\mathfrak{A}$ *is an SAW^*-algebra with (?).*

Moreover, if $\mathfrak{A} \subset \mathbf{B}(\mathfrak{H})$, with $\mathfrak{H}$ separable, then already the condition

(iv) $\mathfrak{B} = \frac{1}{2}(\mathfrak{C} + \mathfrak{B})$

will imply that $\mathfrak{A}$ is a von Neumann algebra.

Brief. Obviously (ii) $\Rightarrow$ (i), so the job is to establish (i) $\Rightarrow$ (ii). For condition (iii), note that any sort of weak polar decomposition will imply that $\mathfrak{A}$ is an SAW^*-algebra. Indeed, if $ST = 0$ in $\mathfrak{A}$ we may assume that $S, T \in \mathfrak{A}_+$, replacing them otherwise with S^*S and TT^*. Now let $R = S - T$, and assume that we have a decomposition $R = V|R|$ for some V in $\mathfrak{A}$ with $\|V\| \leqslant 1$. (The norm estimate is crucial here.) Then

$$|R|(I - V^*V)|R| = |R|^2 - R^*R = 0,$$

whence $(I - V^*V)|R| = 0$ because $0 \leqslant I - V^*V$.

In our situation $|R| = S + T$, so $S - T = V(S + T)$, i.e.

$$(I - V)S = (I + V)T.$$

Squaring this equation we get

$$S(I + V^*V - V - V^*)S = T(I + V^*V + V + V^*)T,$$

which equals zero, since the two sides are orthogonal. Since $V^*VB = S$ and $V^*VT = T$ we derive the equations

$$(2 - V - V^*)S = 0 = (2 + V + V^*)T.$$

Let $E = \frac{1}{2}I - \frac{1}{4}(V + V^*)$. Then $0 \leqslant E \leqslant I$ and

$$ES = 0 = (I - E)T,$$

as desired.

We see that condition (ii) is much stronger than demanding that $\mathfrak{A}$ is an SAW^*-algebra. Therefore the enigmatic (?) in condition (iii) is needed for the (hopeful) implication (iii) $\Rightarrow$ (ii). One necessary ingredient in (?), replacing Rieffels stable rank one in Conjecture 6.6, can be given. Recall from [5] that a C^*-algebra $\mathfrak{A}$ has *real rank* n (in symbols $RR(\mathfrak{A}) = n$), if for any $n+1$-tuple $A_1, \dots, A_{n+1}$ of self-adjoint elements and $\varepsilon > 0$ there is an $n+1$-tuple $B_1, \dots, B_{n+1}$ in $\mathfrak{A}_{sa}$, with $\|A_k - B_k\| \leqslant \varepsilon$ for $1 \leqslant k \leqslant n+1$, such that $\sum B_k^2 \in \mathfrak{A}^{-1}$; and such that n is the smallest number for wich this condition is satisfied. If $\mathfrak{A}$ is commutative, i.e. $\mathfrak{A} = C(X)$, it follows from [5, Proposition 1.1] that $RR(\mathfrak{A}) = \dim X$. Moreover, by [5, Proposition 1.2] we always have $RR(\mathfrak{A}) \leqslant 2\mathrm{sr}(\mathfrak{A}) - 1$. Thus if the stable rank of $\mathfrak{A}$ is 1 we know that the real rank of $\mathfrak{A}$ is 0 or 1. The converse is definitely false, since every von Neumann algebra has real rank zero, but infinite stable rank — unless it is finite.

THEOREM 10.4. *If $\mathfrak{A}$ is a C^*-algebra satisfying the uniform λ-property for $\lambda = \frac{1}{2}$, then the real rank of $\mathfrak{A}$ is at most one.*

Proof. Given A_1, A_2 in $\mathfrak{A}_{sa}$ and $\varepsilon > 0$ we lat $T = A_1 + \mathrm{i}A_2$. By scaling the elements we may assume that $\|T\| \leqslant 1$. By assumption we can find V, B in $\mathfrak{E} \times \mathfrak{B}$ such that

$$T = \frac{1}{2}\left(1 - \frac{1}{2}\varepsilon\right) V + \frac{1}{2}\left(1 + \frac{1}{2}\varepsilon\right) B.$$

Let

$$T_0 = \frac{1}{2}\left(1 + \frac{1}{2}\varepsilon\right) V + \frac{1}{2}\left(1 - \frac{1}{2}\varepsilon\right) B.$$

Then $\|T_0 - T\| = \left\|\frac{1}{2}\varepsilon(V - B)\right\| \leqslant \varepsilon$, so if we write $T_0 = B_1 + \mathrm{i}B_2$, with B_1, B_2 in $\mathfrak{A}_{sa}$, then $\|B_k - A_k| \leqslant \varepsilon$ for $k = 1, 2$. Moreover,

$$B_1^2 + B_2^2 = \frac{1}{2}(T_0^* T_0 + T_0 T_0^*).$$

To show that the element above is invertible, consider the multiple S of T_0 given by

$$S = \left(\frac{1}{2}\left(1 + \frac{1}{2}\varepsilon\right)\right)^{-1} T_0 = V + tB,$$

where $t = \left(1 + \frac{1}{2}\varepsilon\right)^{-1}\left(1 - \frac{1}{2}\varepsilon\right) < 1$. Realizing $\mathfrak{A}$ as operators on some Hilbert space $\mathfrak{H}$, we let Z denote the projection on the closure of the subspace $\mathfrak{A}(I - V^*V)\mathfrak{H}$. Since

$$(I - VV^*)\mathfrak{A}(I - V^*V) = 0,$$

it follows that Z belongs to the center of $\mathfrak{A}''$ (the von Neumann algebra generated by $\mathfrak{A}$), and that

$$(I - VV^*)Z = 0 = (I - V^*V)(I - Z).$$

Thus $V(I - Z)$ is an isometry on $(I - Z)\mathfrak{H}$ and VZ is a co-isometry on $Z\mathfrak{H}$. Therefore

$$\begin{aligned} S^*S(I - Z) &= (V^* + tB^*)(V + tB)(I - Z) = \\ &= V^*(I + tVB^*)(I + tBV^*)V(I - Z) \geqslant \\ &\geqslant V^*(1 - t)^2 V(I - Z) = (1 - t)^2(I - Z). \end{aligned}$$

Similarly,

$$\begin{aligned} SS^*Z &= (V+tB)(V^*+tB)Z = \\ &= V(I+tV^*B)(I+tB^*V)V^*Z \geqslant \\ &\geqslant V(1-t)^2V^*Z = (1-t)^2Z. \end{aligned}$$

Consequently,

$$S^*S + SS^* \geqslant S^*S(I-Z) + SS^*Z \geqslant (1-t)^2 I,$$

which proves that $B_1^2 + B_2^2$ is invertible. ∎

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