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INTERPOLATION BY PROJECTIONS IN C^*-ALGEBRAS OF REAL RANK ZERO

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Dedicated to the memory of John Bunce

It is known that a C^*-algebra A has a real rank zero (invertible elements of $\tilde{A}_{sa}$ are dense in $\tilde{A}_{sa}$, where $\tilde{A}_{sa}$ is the set of self-adjoint elements of $\tilde{A}$) if and only if A has HP (every hereditary C^*-subalgebra of A has an approximate identity of projections) if and only if A has FS (elements with finite spectrum are dense in A_{sa}). We show that A has real rank zero if and only if orthogonal closed projections in A^{**}, one of which is compact, can be separated by a projection in A. A similar condition characterizes $M(A)$ having real rank zero.

A topological space X has strong inductive dimension zero if and only if whenever F_1 and F_2 are disjoint closed subsets of X, there is a clopen set U such that $F_1 \subset U$ and $U \cap F_2 = \emptyset$. The most closely analogous property for C^*-algebras is the *interpolation by multiplier projections* property (IMP), which is defined as follows: If p and q are closed projections in A^{**} such that $pq = 0$, then there is a projection r in $M(A)$ such that $p \leqslant r \leqslant 1 - q$. Since (IMP) requires that one look more at $M(A)$ than at A, we also define the *interpolation by projections* property (IP) as follows: If p and q are projections in A^{**} such that p is compact, q is closed, and $pq = 0$, then there is a projection r in A such that $p \leqslant r \leqslant 1 - q$. Of course there is no distinction between (IP) and (IMP) if A is unital. Above, A^{**} is the enveloping W^*-algebra of A.

The fact that (FS) implies (HP) was proved by Pedersen [8], and the fact that (HP) implies (FS) was proved by Blackadar [2]. The fact that these are equivalent to real rank zero is proved in [4]. Since (IP), in our opinion, is a more powerful condition that (HP) or (FS), we belive the fact that it is equivalent to these is useful. In Theorem 1 below we reprove the above-quoted result of Blackadar because we belive that use of the (IP) property smooths the exposition, although our proof is not fundamentally different from Blackadar's.

THEOREM 1. *Let A be a C^*-algebra. Then the following are equivalent:*

1. *A has* HP.
2. *A has* IP.
3. *A has* FS.

Proof. $1 \Rightarrow 2$: Let p be a compact projection in A^{**} and q a closed projection such that $pq = 0$. By Akemann's "Urysohn lemma", Lemma III.1 of [1], there is a in A_{sa} such that $p \leqslant a \leqslant 1 - q$. Choose an approximate identity (s_n), consisting of projections, for $\mathrm{her}(1-q)$, the hereditary C^*-subalgebra of A whose open projection is $1-q$, where n lies in some directed set. Consider $x_n = (1-a)^{\frac{1}{2}}(1-s_n) \in \tilde{A}$, where $\tilde{A} = A + \mathbb{C}\cdot 1 \subset A^{**}$. Then $1-s_n-x_n^* x_n = (1-s_n)a(1-s_n) \leqslant \|(1-s_n)a(1-s_n)\|(1-s_n)$. If n is so large that $\|(1-s_n)a\| < 1$, then x_n has a polar decomposition, $x_n = u_n|x_n|$, where u_n is in $1 + \mathrm{her}(1-q) \subset \tilde{A}$ and $u_n^* u_n = 1 - s_n$. Then $r_n = u_n u_n^*$ is a projection in $\tilde{A}$ such that $\mathrm{her}_{\tilde{A}}(r_n) = \mathrm{her}_{\tilde{A}}(x_n x_n^*) = \mathrm{her}_{\tilde{A}}\left((1-a)^{\frac{1}{2}}(1-s_n)(1-a)^{\frac{1}{2}}\right)$, where $\mathrm{her}_{\tilde{A}}(y)$ denotes the smallest hereditary C^*-subalgebra of $\tilde{A}$ containing y. Since $(1-a) \perp p$, this shows that $r_n \perp p$. Since also r_n is in $1 + \mathrm{her}(1-q)$, $r_n \geqslant q$. Thus we may take $r = 1 - r_n$ to achieve that $r \in A$ and $p \leqslant r \leqslant 1 - q$.

$2 \Rightarrow 3$: Let $h \in A_{sa}$. In order to approximate h by self-adjoint elements with finite spectrum, we first assume that $h \geqslant 0$. Then there is no loss of generality in assuming also that $\|h\| \leqslant 1$. Consider the spectral projections $p_{k,n} = E_{[\frac{k}{n},1]}(h)$ and $q_{k,n} = E_{[0,\frac{k-1}{n}]}(h)$, $1 \leqslant k \leqslant n$. By the (IP) property there are projections $r_{k,n}$ in A such that $p_{k,n} \leqslant r_{k,n} \leqslant 1 - q_{k,n}$. Clearly, $r_{k,n} \leqslant r_{k-1,n}$ for $k > 1$. Thus if we define $x_n = \dfrac{1}{n}\sum_{k=1}^{n} r_{k,n}$, then x_n has finite spectrum and $\|x_n - h\| \leqslant \dfrac{1}{n}$. Note also that $x_n \in \mathrm{her}\,(h) = \mathrm{her}\,(1 - q_{1,n})$.

Now for general h, write $h = h_+ - h_-$, and use the above construction to find x_n, y_n in A_{sa} such that x_n and y_n have finite spectra, $x_n \longrightarrow h_+$, and $y_n \longrightarrow h_-$. Since $x_n \in \mathrm{her}(h_+)$ and $y_n \in \mathrm{her}(h_-)$, $x_n - y_n$ also has finite spectrum.

$3 \Rightarrow 1$ is part of Proposition 14 of [8].

REMARK. It is easy to see that (IMP) implies (IP). If p is a compact projection and q a closed projection such that $pq = 0$, choose a as above so that $p \leqslant a \leqslant 1 - q$. Let q' be the spectral projection $E_{[0,\frac{1}{2}]}(a)$, and let f be a continuous function such that $f = 1$ on $[\frac{1}{2}, 1]$ and $f(0) = 0$. Using (IMP), choose a projection r in $M(A)$ such that $p \leqslant r \leqslant 1 - q'$. Then $f(a)r = r$, and hence $r \in A$. Since $q' \geqslant q$, (IP) follows.

COROLLARY 2. *If A is a σ-unital C^*-algebra, then A has* IMP *if and only if $M(A)$ has real rank zero.*

Proof. Let $M(A)$ have real rank zero (equivalently $M(A)$ has (HP)), and let p and q be closed projections in A^{**} such that $pq = 0$. Consider $A^{**} \subset M(A)^{**} \cong$ $\cong A^{**} \oplus (M(A)/A)^{**}$. By 3.33 of [3] (a corollary of the author's "Urysohn lemma"), there are closed projections p_1 and q_1 in $M(A)^{**}$ such that $p_1 \geqslant p$, $q_1 \geqslant q$, and $p_1 q_1 = 0$. Then $1 \Rightarrow 2$ of Theorem 1, applied to $M(A)$, gives a projection r in $M(A)$ such that $p_1 \leqslant r \leqslant 1 - q_1$. Then in A^{**} we have $p \leqslant r \leqslant 1 - q$.

If A has (IMP), the same proof as for $2 \Rightarrow 3$ above shows that $M(A)$ has (FS).

REMARK. Zhang [11] exhibits some other properties equivalent to (IMP).

We say that A has property (O) if whenever $x, y \in M(A)$ and $xy \in A$ there is a projection p in $M(A)$ such that $x(1-p), py \in A$. (Cf. Theorem 2.3 of [7].) It was suggested to us by S. Zhang that property (O) might be equivalent to $M(A)$ having real rank zero. Regarding this we mention that results of Murphy [6] and Zhang [10,11] show that $M(A)$ has real rank zero if and only if every element of $M(A)_{sa}$ is quasidiagonal. Zhang's conjecture is proved as Corollary 4 below. Corollary 3 is a similar result for $M(A)/A$ which sometimes can be applied in connection with the concept of weak equivalence of C^*-algebra extensions (see [5]).

COROLLARY 3. *Let A be a σ-unital C^*-algebra and $C(A) = M(A)/A$. Then $C(A)$ has real rank zero if and only if the following is true:*

(O′) *Whenever $x, y \in C(A)$ and $xy = 0$, there is a projection p in $C(A)$ such that $xp = x$ and $py = 0$.*

Proof. First assume $C(A)$ has real rank zero and $xy = 0$ in $C(A)$. By the SAW^* property of $C(A)$ (Theorem 13 of [9], applied to x^*x and yy^*), there is t in $C(A)$ such that $0 \leqslant t \leqslant 1$ and $x(1-t) = ty = 0$. Let $q_1 = E_{\{1\}}(t)$ and $q_0 = E_{\{0\}}(t)$, spectral projections in $C(A)^{**}$. By Theorem 1, applied to $C(A)$, there is a projection p in $C(A)$ such that $q_1 \leqslant p \leqslant 1 - q_0$. Clearly p has the required properties.

Next asssume (O′). We will show that $C(A)$ has (IP). Thus assume q_1 and q_2 are orthogonal closed projections in $C(A)^{**}$. By [1] there is $h \in C(A)$ such that $q_1 \leqslant h \leqslant 1 - q_2$. Let $x = f_1(h)$ and $y = f_2(h)$ where f_1 and f_2 are continuous functions such that $f_1(1) = 1 = f_2(0)$ and $f_1 \cdot f_2 = 0$. Then if p is a projection in $C(A)$ such that $xp = x$ and $py = 0$, it follows that $q_1 \leqslant p \leqslant 1 - q_2$.

REMARK. Clearly, in Corollary 3 $C(A)$ could be replaced by any unital SAW^*--algebra.

COROLLARY 4. *If A is a σ-unital C^*-algebra of real rank zero, then A has property (O) if and only if $M(A)$ has real rank zero.*

REMARK. What is actually shown in the proof is that A satisfies (O) if and only if $C(A)$ has real rank zero and every projection in $C(A)$ is the image of a projection in $M(A)$, and the hypothesis that A have real rank zero is not used to show this. However, it is known that if I is an ideal of a C^*-algebra B, then B has real rank zero if and only if I and B/I have real rank zero and projections lift; and it seems more interesting to state the result as given. The assertion just made is Theorem 3.14 of [4], Lemma 2.4 of [12], and Proposition 2.3 of [13]. The earliest proof of the fact that $M(A)$ has real rank zero implies lifting of projections results from combining Theorem 9 of [6] with Lemma 2.2 of [10] or [11]. The rest of the assertion was noticed jointly by the three people involved.

Proof. Assume A has property (O). Clearly (O′) is true, and hence $C(A)$ has real rank zero. Let p be a projection in $C(A)$, and let x and y be elements of $M(A)$ whose images are p and $1-p$, respectively. By property (O), there is a projection $\tilde{p}$ in $M(A)$ such that $x(1-\tilde{p}), \tilde{p}y \in A$. Clearly $\tilde{p}$ lifts p.

Now assume $M(A)$ has real rank zero. Then clearly $C(A)$ has real rank zero, and hence (O′) is true. Let $\pi: M(A) \longrightarrow C(A)$ be the natural map. If $x, y \in M(A)$ and $xy \in A$, then by (O′) there is a projection $\overline{p}$ in $C(A)$ such that $\pi(x)\overline{p} = \pi(x)$ and $\overline{p}\pi(y) = 0$. If p is a projection in $M(A)$ such that $\pi(p) = \overline{p}$, then $x(1-p), py \in A$, and hence A has property (O).

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