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A NOTE ON THE SIMILARITY PROBLEM

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INTRODUCTION

The problem of whether a bounded representation of a C^*-algebra on a Hilbert space is similar to a $*$ representation of that algebra on the space is posed and studied in [7]. A necessary and sufficient condition (somewhat akin to complete boundedness of the representation) is proved there. In [1], Barnes shows that when the image of the representation and its adjoint have a common cyclic vector, a "similarity" with a $*$ representation can be effected by closed, densely defined operator with densely defined inverse. In [2], Bunce improves this by showing that the operator effecting the similarity can be chosen bounded and by removing the condition on the adjoint of the image. In [3], Erik Christensen solves the original problem for irreducible representations (so, each vector is cyclic for the image). Both Bunce and Christensen make crucial use of an inequality conjectured by Ringrose [12] and proved by Pisier [11].

In [6], Haagerup comes tantalizingly close to solving the entire problem when he proves the following result.

THEOREM A. *Each bounded cyclic representation of a C^*-algebra on a Hilbert space is similar to a $*$ representation of the algebra on that space.*

Haagerup's proof makes use of a method that allows him to simplify Pisier's proof of the Pisier-Ringrose inequality and improve the constant (from 6 to 4) in the key inequality proved by Pisier. Haagerup proves a number of other results. For example, he settles the problem completely for C^*-algebras that admit no trace. He observes that the problem reduces to studying representations of von Neumann algebras (by using the universal representation of the C^*-algebra). Thus the problem is settled positively for von Neumann algebras with no central summand of finite type. The hyperfinite (*matricial*) von Neumann algebras fall under the amenable group techniques in [7].

In [4], Christensen settles the problem for factors of type II_1 with property Γ. Since there are such factors that are not matricial and do not come under the special cases of Haagerup's results, Christensen's result is a significant step forward. The key to Christensen's proof is an inequality (Proposition 2.1 of [4]) which is cleverly combined with some work of Connes [5] on central sequences and ultrapowers.

In this paper, we prove (Proposition 2) a version of the Haagerup-Pisier inequality that retains many estimates and analytic options in preliminary form. Playing off these options, carefully, allows us to prove Proposition 3, almost a direct translation of Lemma 1.4 of [6], without invoking ultrapowers. The rest of the argument in [6] applies, yielding a proof of Theorem A from operator-algebra basics (without ultrapowers). Proposition 4 is precisely Haagerup's Lemma 1.5 in his proof of Theorem A. Although we do not present the rest of Haagerup's proof of Theorem A, we include his Lemma 1.5 (our Proposition 4) as a direct application of Proposition 3 and for its use in our proof of Proposition 2.1 of [4] (our Proposition 8).

Playing off other options in Proposition 2 allows us to prove (again, from operator algebra basics) an enhanced form of the Haagerup-Pisier inequality (Proposition 5) needed in the proof of Proposition 2.1 of [4]. We prove the Pisier-Ringrose inequality, Proposition 7, as an easy application of Corollary 6 (an immediate corollary of Proposition 5). We also include a variant of the argument given by Christensen for proving his Proposition 2.1 of [4]. (It appears at the end of this paper.) We use the notation of [10].

THE ARGUMENTS

The next result is an elaboration of a known approximate polar decomposition assertion for elements of a C^*-algebra. As arranged, the proof does not require restricting the element being decomposed to a bounded set.

PROPOSITION 1. *Let A be an element of a C^*-algebra with unit I, V_n be $A\left(A^*A+\frac{1}{n}I\right)^{-\frac{1}{2}}$ for each positive integer n, $|A|$ be $(A^*A)^{\frac{1}{2}}$, and T_n be $A-V_n|A|$. Then $0\leqslant T_n^*T_n\leqslant \frac{1}{n}I$ so that $\lim_{n\to\infty} T_n=0$. At the same time, $0\leqslant S_n\leqslant\sqrt{\frac{1}{n}}I$, where $S_n=|A|-V_n^*A$, so that $\lim_{n\to\infty} S_n=0$.*

Proof. Note that

$$0 \leqslant T_n^* T_n = \left[I - \left(A^*A + \frac{1}{n}I\right)^{-\frac{1}{2}} (A^*A)^{\frac{1}{2}}\right] A^*A \left[I - \left(A^*A + \frac{1}{n}I\right)^{-\frac{1}{2}} (A^*A)^{\frac{1}{2}}\right] =$$

$$= A^*A \left[I - \left(A^*A + \frac{1}{n}I\right)^{-\frac{1}{2}} (A^*A)^{\frac{1}{2}}\right]^2 =$$

$$= A^*A \left[\left(A^*A + \frac{1}{n}I\right)^{\frac{1}{2}} - (A^*A)^{\frac{1}{2}}\right]^2 \left(A^*A + \frac{1}{n}I\right)^{-1} =$$

$$= A^*A \left[2A^*A + \frac{1}{n}I - 2(A^*A)^{\frac{1}{2}} \left(A^*A + \frac{1}{n}I\right)^{\frac{1}{2}}\right] \left(A^*A + \frac{1}{n}I\right)^{-1} \leqslant$$

$$\leqslant \left[2A^*A + \frac{1}{n}I - 2(A^*A)^{\frac{1}{2}}(A^*A)^{\frac{1}{2}}\right] = \frac{1}{n}I.$$

It follows that $\|T_n\|^2 = \|T_n T_n^*\| \leqslant \frac{1}{n}$, whence $\lim_{n\to\infty} T_n = 0$.

We have that

$$S_n = |A| - V_n^* A = (A^*A)^{\frac{1}{2}} - \left(A^*A + \frac{1}{n}I\right)^{-\frac{1}{2}} A^*A =$$

$$= (A^*A)^{\frac{1}{2}} \left[I - \left(A^*A + \frac{1}{n}I\right)^{-\frac{1}{2}} (A^*A)^{\frac{1}{2}}\right].$$

Thus $0 \leqslant S_n$ and $S_n^2 = T_n^* T_n \leqslant \frac{1}{n}I$. Hence $0 \leqslant S_n \leqslant \sqrt{\frac{1}{n}}I$ and $\lim_{n\to\infty} S_n = 0$. ■

PROPOSITION 2. *Let η be a bounded linear mapping of a C^*-algebra $\mathfrak{A}$ into a Hilbert space $\mathcal{H}$, let m be $\inf\{\|\eta(U)\| : U \in \mathcal{U}(\mathfrak{A})\}$, and let M be $\sup\{\|\eta(U)\| : U \in \mathcal{U}(\mathfrak{A})\}$, where $\mathcal{U}(\mathfrak{A})$ is the unitary group of $\mathfrak{A}$. Then $M = \|\eta\|$ and there are sequences $\{U_n\}$, $\{W_n\}$ of unitary elements U_n, W_n in $\mathfrak{A}$ such that $\{\|\eta(U_n)\|\}$ is monotone increasing to M and $\{\|\eta(W_n)\|\}$ is monotone decreasing to m. Let ${}_n\theta(A)$, $\theta_n(A)$, ${}_n\sigma(A)$, $\sigma_n(A)$ be $\eta(W_nA)$, $\eta(AW_n)$, $\langle {}_n\theta(A), {}_n\theta(I)\rangle$, $\langle \theta_n(A), \theta_n(I)\rangle$, respectively, and define ${}_n\mu(A)$, $\mu_n(A)$, ${}_n\rho(A)$, $\rho_n(A)$ as $\eta(U_nA)$, $\eta(AU_n)$, $\langle {}_n\mu(A), {}_n\mu(I)\rangle$, $\langle \mu_n(A), \mu_n(I)\rangle$ respectively, for each A in $\mathfrak{A}$. Then $\|{}_n\sigma\|$, $\|\sigma_n\|$, $\|{}_n\rho\|$, $\|\rho_n\|$ do not exceed $\|\eta\|^2$, the sequences $\{{}_n\sigma\}$, $\{\sigma_n\}$, $\{{}_n\rho\}$, $\{\rho_n\}$ have weak* limit points, and*

$$\lim_n \|\mathrm{Im}\, {}_n\sigma\| = \lim_n \|\mathrm{Im}\, \sigma_n\| = \lim_n \|\mathrm{Im}\, {}_n\rho\| = \lim_n \|\mathrm{Im}\, \rho_n\| = 0.$$

There are functions g and h, with $g(n)$, $h(n) \to 0$ as $n \to \infty$, such that, for each H in $(\mathfrak{A}_h)_1$,

$$\mathrm{Re}\, {}_n\sigma(H^2) \leqslant g(n) + \|{}_n\theta(H)\|^2, \qquad \mathrm{Re}\, \sigma_n(H^2) \leqslant g(n) + \|\theta_n(H)\|^2,$$

$$\mathrm{Re}_n\rho(H^2) \geqslant h(n) + \|_n\mu(H)\|^2, \qquad \mathrm{Re}\rho_n(H^2) \geqslant h(n) + \|\mu_n(H)\|^2.$$

Proof. If A in $\mathfrak{A}$ has norm less than 1, then from [9], there are unitary operators $Y_1, \ldots, Y_r$ such that

$$A = \frac{1}{r}(Y_1 + \cdots + Y_r),$$

whence $\|\eta(A)\| \leqslant \frac{1}{r}(\|\eta(Y_1)\| + \cdots + \|\eta(Y_r)\|)$. It follows that $\|\eta(A)\| \leqslant \|\eta(Y_j)\|$ for some j, that $\|\eta\| = M$, and there is a sequence of unitary operators U_n in $\mathfrak{A}$ such that $\{\,\|\eta(U_n)\|\,\}$ is monotone increasing to M and

$$M\left(\frac{n^3-1}{n^3}\right)^{\frac{1}{2}} \leqslant \|\eta(U_n)\| \leqslant M \qquad (n = 1, 2, \ldots).$$

At the same time, there is a sequence of unitary operators W_n in $\mathfrak{A}$ such that $\{\|\eta(W_n)\|\}$ is monotone decreasing to m and

$$m \leqslant \|\eta(W_n)\| \leqslant \left(m^2 + \frac{1}{n^3}\right)^{\frac{1}{2}}.$$

In the arguments that follow, we shall write $\theta^{(n)}$ in place of both ${}_n\theta$ and θ_n and $\sigma^{(n)}$ in place of ${}_n\sigma$ and σ_n since the arguments and results apply in both cases. Our convention is that $\theta^{(n)}$ and $\sigma^{(n)}$ stand for either ${}_n\theta$ and ${}_n\sigma$ or θ_n and σ_n in a given argument. Similarly, $\mu^{(n)}$ and $\rho^{(n)}$ replace both ${}_n\mu$, μ_n and ${}_n\rho$, ρ_n, respectively, with the same convention applying.

With H a self-adjoint element of the unit ball $(\mathfrak{A})_1$ of $\mathfrak{A}$ and t real, we have that

$$m \leqslant \|\theta^{(n)}(\mathrm{e}^{\mathrm{i}tH})\| = \|\theta^{(n)}(I) + \mathrm{i}t\theta^{(n)}(H) + t^2R^{(n)}(t)\|,$$

where $R^{(n)}(t) = \sum_{j=2}^{\infty} \frac{1}{j!}t^{j-2}\left(\theta^{(n)}((\mathrm{i}H)^j)\right)$. Note that, with t positive,

$$\|R^{(n)}(t)\| \leqslant \|\eta\| \sum_{j=2}^{\infty} \frac{1}{j!}t^{j-2} \leqslant \|\eta\| \sum_{j=2}^{\infty} \frac{1}{(j-2)!}t^{j-2} \leqslant \|\eta\|\mathrm{e}^t.$$

It follows that

$$\begin{aligned}(m - t^2\|R^{(n)}(t)\|)^2 &\leqslant \|\theta^{(n)}(I) + \mathrm{i}t\theta^{(n)}(H)\|^2 = \\ &= \|\theta^{(n)}(I)\|^2 - 2t\mathrm{Im}\,\langle\theta^{(n)}(H), \theta^{(n)}(I)\rangle + t^2\|\theta^{(n)}(H)\|^2 = \\ &= \|\eta(W_n)\|^2 - 2t\mathrm{Im}\,\sigma^{(n)}(H) + t^2\|\theta^{(n)}(H)\|^2 \leqslant \\ &\leqslant m^2 + \frac{1}{n^3} - 2t\mathrm{Im}\,\sigma^{(n)}(H) + t^2\|\theta^{(n)}(H)\|^2.\end{aligned}$$

Hence, with t positive, we have that

$$t^3\|R^{(n)}(t)\|^2 - 2tm\|R^{(n)}(t)\| - t\|\theta^{(n)}(H)\|^2 \leqslant \frac{1}{tn^3} - 2\operatorname{Im}\sigma^{(n)}(H).$$

Letting t be $\frac{1}{n}$, we see that

$$\frac{1}{n^3}\left\|R^{(n)}\left(\frac{1}{n}\right)\right\|^2 + 2\operatorname{Im}\sigma^{(n)}(H) \leqslant \frac{1}{n^2} + \frac{2m}{n}\left\|R^{(n)}\left(\frac{1}{n}\right)\right\| + \frac{1}{n}\|\theta^n(H)\|^2.$$

As $\left\|R^{(n)}(\frac{1}{n})\right\| \leqslant \|\eta\|e^{\frac{1}{n}}$, and $\|\theta^{(n)}(H)\|^2 \leqslant \|\eta\|^2$, we conclude that

$$2\operatorname{Im}\sigma^{(n)}(H) \leqslant \frac{1}{n^2} + \frac{2m}{n}\|\eta\|e^{\frac{1}{n}} + \frac{1}{n}\|\eta\|^2 \qquad (H \in (\mathfrak{A}_h)_1).$$

Let $\operatorname{Im}\sigma^{(n)}$ be the real-linear functional on $\mathfrak{A}_h$ whose value at H is $\operatorname{Im}\sigma^{(n)}(H)$. This same discussion applied to $-H$ shows that

$$2|\operatorname{Im}\sigma^{(n)}(H)| \leqslant \frac{1}{n^2} + \frac{\|\eta\|}{n}(2me^{\frac{1}{n}} + \|\eta\|) \qquad (H \in (\mathfrak{A}_h)_1).$$

whence

$$\lim_n \|\operatorname{Im}\sigma^{(n)}\| = 0 \qquad (n \to \infty).$$

Again, with H in $\mathfrak{A}_h$, $m^2 \leqslant \|\theta^{(n)}e^{\pm itH})\|^2$ and

$$\theta^{(n)}(e^{\pm itH}) = \theta^{(n)}(I) \pm it\theta^{(n)}(H) - \frac{t^2}{2}\theta^{(n)}(H^2) + t^3R_{\pm}^{(n)}(t),$$

where $R_{\pm}^{(n)}(t) = \sum_{j=3}^{\infty}\frac{1}{j!}t^{j-3}(\theta^{(n)}((\pm iH)^j))$. Thus as before, with t positive,

$$\|R_{\pm}^{(n)}(t)\| \leqslant \|\eta\|\,\|H\|^3e^{t\|H\|}.$$

Let $A_{\pm}^{(n)}(t)$ be $\theta^{(n)}(I) \pm it\theta^{(n)}(H) - \frac{t^2}{2}\theta^{(n)}(H^2)$. Then

$$m^2 \leqslant \|A_{\pm}^{(n)}(t)\|^2 + S_{\pm}^{(n)}(t),$$

where $S_{\pm}^{(n)}(t) = 2t^3\operatorname{Re}\langle A_{\pm}^{(n)}(t), R_{\pm}^{(n)}(t)\rangle + t^6\|R_{\pm}^{(n)}(t)\|^2$. It follows from the parallelogram law, with $S^{(n)}$ as $\frac{1}{2}(S_{+}^{(n)}(t) + S_{-}^{(n)}(t))$, that

$$m^2 - S^{(n)}(t) \leqslant \left\|\theta^{(n)}(I) - \frac{t^2}{2}\theta^{(n)}(H^2)\right\|^2 + t^2\|\theta^{(n)}(H)\|^2 =$$

$$= \|\eta(W_n)\|^2 - t^2\operatorname{Re}\langle\theta^{(n)}(H^2), \theta^{(n)}(I)\rangle + \frac{t^4}{4}\|\theta^{(n)}(H^2)\|^2 + t^2\|\theta^{(n)}(H)\|^2 \leqslant$$

$$\leqslant m^2 + \frac{1}{n^3} - t^2\operatorname{Re}\sigma^{(n)}(H^2) + \frac{t^4}{4}\|\theta^{(n)}(H^2)\|^2 + t^2\|\theta^{(n)}(H)\|^2.$$

Thus
$$\operatorname{Re}\sigma^{(n)}(H^2) - t^{-2}S^{(n)}(t) \leqslant \frac{1}{t^2n^3} + \frac{t^2}{4}\|\theta^{(n)}(H^2)\|^2 + \|\theta^{(n)}(H)\|^2.$$
Letting t be $\frac{1}{n}$, we have that
$$\operatorname{Re}\sigma^{(n)}(H^2) \leqslant n^2 S^{(n)}\left(\frac{1}{n}\right) + \frac{1}{n} + \frac{1}{4n^2}\|\theta^{(n)}(H^2)\|^2 + \|\theta^{(n)}(H)\|^2.$$
Since
$$S_{\pm}^{(n)}\left(\frac{1}{n}\right) = \frac{2}{n^3}\operatorname{Re}\left\langle A_{\pm}^{(n)}\left(\frac{1}{n}\right), R_{\pm}^{(n)}\right\rangle + \frac{1}{n^6}\left\|R_{\pm}^{(n)}\left(\frac{1}{n}\right)\right\|^2,$$
and
$$\left\|A_{\pm}^{(n)}\left(\frac{1}{n}\right)\right\| \leqslant \frac{1}{n^2}\|\eta\|\left(n^2 + n\|H\| + \frac{1}{2}\|H\|^2\right),$$
we see that
$$\left|S^{(n)}\left(\frac{1}{n}\right)\right| \leqslant \frac{2}{n^5}\|\eta^2\|\left(n^2 + n\|H\| + \frac{1}{2}\|H\|^2\right)\|H\|^3 e^{\frac{1}{n}\|H\|} + \frac{1}{n^6}\|\eta\|^2\|H\|^6 e^{\frac{2}{n}\|H\|}.$$
Thus
$$\operatorname{Re}\sigma^{(n)}(H^2) \leqslant g(n) + \|\theta^{(n)}(H)\|^2,$$
where $g(n) \to 0$ as $n \to \infty$.

Again, with H self-adjoint in $\mathfrak{A}$, we have that
$$M \geqslant \|\mu^{(n)}(e^{itH})\| = \|\mu^{(n)}(I) + it\mu^{(n)}(H) + t^2 P^{(n)}(t)\|,$$
where $P^{(n)}(t) = \sum_{j=2}^{\infty} \frac{1}{j!} t^{j-2}(\mu^{(n)}((iH)^j))$. As before, with t positive,
$$\|P^{(n)}(t) \leqslant \|\eta\|\,\|H\|^2 e^{t\|H\|}.$$
Hence
$$\begin{aligned}(M + t^2\|P^{(n)}(t)\|)^2 &\geqslant \|\mu^{(n)}(I) + it\mu^{(n)}(H)\|^2 = \\ &= \|\eta(U_n)\|^2 - 2t\operatorname{Im}\rho^{(n)}(H) + t^2\|\mu^{(n)}(H)\|^2 \geqslant \\ &\geqslant M^2 - \frac{M^2}{n^3} - 2t\operatorname{Im}\rho^{(n)}(H).\end{aligned}$$
Letting t be $\frac{1}{n}$, we have that
$$\frac{2M}{n}\left\|P^{(n)}\left(\frac{1}{n}\right)\right\| + \frac{1}{n^3}\left\|P^{(n)}\left(\frac{1}{n}\right)\right\|^2 + \frac{M^2}{n^2} \geqslant -2\operatorname{Im}\rho^{(n)}(H).$$

Thus, when $\|H\| \leqslant 1$,

$$\left(\frac{2M}{n} + \frac{1}{n^3}\|\eta\| e^{\frac{1}{n}}\right) \|\eta\| e^{\frac{1}{n}} + \frac{M^2}{n^2} \geqslant -2\mathrm{Im}\,\rho^{(n)}(H).$$

This conclusion applied to $-H$ in place of H, yields

$$|2\mathrm{Im}\,\rho^{(n)}(H)| \leqslant \left(\frac{2M}{n} + \frac{1}{n^3}\|\eta\| e^{\frac{1}{n}}\right) \|\eta\| e^{\frac{1}{n}} + \frac{M^2}{n^2},$$

whence

$$\lim \|\mathrm{Im}\,\rho^{(n)}\| = 0 \qquad (n \to \infty).$$

Again, with H self-adjoint in $\mathfrak{A}$, $M^2 \geqslant \|\mu^{(n)}(e^{\pm itH})\|^2$ and

$$\mu^{(n)}(e^{\pm itH}) = \mu^{(n)}(I) \pm it\mu^{(n)}(H) - \frac{t^2}{2}\mu^{(n)}(H^2) + t^3 P_{\pm}^{(n)}(t),$$

where $P_{\pm}^{(n)}(t) = \sum_{j=3}^{\infty} \frac{1}{j!} t^{j-3}(\mu^{(n)}((\pm iH)^j))$. As before, with t positive,

$$\|P_{\pm}^{(n)}(t)\| \leqslant \|\eta\|\, \|H\|^3 e^{t\|H\|}.$$

Let $B_{\pm}^{(n)}(t)$ be $\mu^{(n)}(I) \pm it\mu^{(n)}(H) - \frac{t^2}{2}\mu^{(n)}(H^2)$. Then

$$M^2 \geqslant \|B_{\pm}^{(n)}(t)\|^2 + Q_{\pm}^{(n)}(t),$$

where $Q_{\pm}^{(n)}(t) = 2t^3 \mathrm{Re}\,\langle B_{\pm}^{(n)}(t), P_{\pm}^{(n)}(t)\rangle + t^6\|P_{\pm}^{(n)}(t)\|^2$. It follows from the parallelogram law, with $Q^{(n)}(t)$ as $\frac{1}{2}(Q_{+}^{(n)}(t) + Q_{-}^{(n)}(t))$, that

$$M^2 - Q^{(n)}(t) \geqslant \left\|\mu^{(n)}(I) - \frac{t^2}{2}\mu^{(n)}(H^2)\right\|^2 + t^2\|\mu^{(n)}(H)\|^2 \geqslant$$
$$\geqslant M^2 - \frac{M^2}{n^3} - t^2 \mathrm{Re}\,\rho^{(n)}(H^2) + \frac{t^4}{4}\|\mu^{(n)}(H^2)\|^2 + t^2\|\mu^{(n)}(H)\|^2.$$

Thus

$$\mathrm{Re}\,\rho^{(n)}(H^2) \geqslant -\frac{M^2}{t^2 n^3} + \frac{t^2}{4}\|\mu^{(n)}(H^2)\|^2 + \|\mu^{(n)}(H)\|^2 + t^{-2}Q^{(n)}(t).$$

Letting t be $\frac{1}{n}$, we have that

$$\mathrm{Re}\,\rho^{(n)}(H^2) \geqslant n^2 Q^{(n)}\left(\frac{1}{n}\right) - \frac{M^2}{n} + \|\mu^{(n)}(H)\|^2.$$

From the form of $Q_{\pm}^{(n)}\left(\frac{1}{n}\right)$ and the fact that

$$\left\|B_{\pm}^{(n)}\left(\frac{1}{n}\right)\right\| \leqslant \frac{1}{n^2}\|\eta\|\left(n^2+n\|H\|+\frac{1}{2}\|H\|^2\right),$$

we see that

$$\left|Q^{(n)}\left(\frac{1}{n}\right)\right| \leqslant \frac{2}{n^5}\|\eta\|^2\left(n^2+n\|H\|+\frac{1}{2}\|H\|^2\right)\|H\|^3 e^{\frac{1}{n}\|H\|}+\frac{1}{n^6}\|\eta\|^2\|H\|^6 e^{\frac{2}{n}\|H\|}.$$

Thus

$$\operatorname{Re}\rho^{(n)}(H^2) \geqslant h(n)+\|\mu^{(n)}(H)\|^2,$$

where $h(n) \to 0$ as $n \to \infty$. ■

In the next proposition, we apply the results of Proposition 2 to the situation where the mapping η is constructed from a bounded representation of a C^*-algebra on a Hilbert space. We adopt the notation of the statement of Proposition 2 in the following proposition.

PROPOSITION 3. *Let π be a bounded, non-degenerate representation of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$ and $\eta(A)$ be $\pi(A)x_0$ for each A in $\mathfrak{A}$, where x_0 is a vector in $\mathcal{H}$. Define $\psi_n(A)$ to be $\|\pi\|^{-2}{}_n\sigma(A)$ and $\varphi_n(A)$ to be $\|\pi\|^2{}_n\rho(A)$, for each A in $\mathfrak{A}$. Then $\{\psi_n\}$ and $\{\varphi_n\}$ have weak* limit points in the (norm) dual of $\mathfrak{A}$ and each pair ψ and φ of such limit points consists of hermitian functionals on $\mathfrak{A}$ satisfying the relations*

$$\varphi(I) \leqslant \|\pi\|^6\psi(I)$$
$$\psi(A^*A) \leqslant \|\pi(A)x_0\|^2 \leqslant \varphi(A^*A) \qquad (A \in \mathfrak{A}).$$

Proof. As defined, η is a bounded linear mapping of $\mathfrak{A}$ into $\mathcal{H}$ and $\|\eta\| \leqslant \|\pi\|\,\|x_0\|$. From Proposition 2, $\{\|\pi\|^{-2}{}_n\sigma\}$ and $\{\|\pi\|^2{}_n\rho\}$ have weak* limit points $\psi(=\|\pi\|^{-2}\sigma)$ and $\varphi(=\|\pi\|^2\rho)$. Since

$$\lim_n\|\operatorname{Im}{}_n\sigma\| = \lim_n\|\operatorname{Im}{}_n\rho\| = 0,$$

we have that $\sigma(H)$ and $\rho(H)$ are limits points of $\{\operatorname{Re}{}_n\sigma(H)\}$ $\{\operatorname{Re}{}_n\rho(H)\}$ for each self-adjoint H in $\mathfrak{A}$. Thus $\psi(H)$ and $\varphi(H)$ are real.

Note that, with B in $\mathfrak{A}$ and $\{V_k'|B|\}$ the "approximate" polar decomposition of B in $\mathfrak{A}$ (see Proposition 1), we can deduce that for each y in $\mathcal{H}$,

$$\|\pi(B)y\| = \lim_k\|\pi(V_k'|B|)y\| \leqslant \|\pi\|\,\|\pi(|B|)y\|.$$

With A in $\mathfrak{A}$, let $\{V_k|A|\}$ be the approximate polar decomposition of A. Then $A = \lim_k V_k|A|$ and $|A| = \lim_k V_k^* A$. We have that

$$\operatorname{Re}\psi_n(A^*A) = \|\pi\|^{-2}\operatorname{Re}{}_n\sigma(A^*A) =$$
$$= \|\pi\|^{-2}\operatorname{Re}\langle\eta(W_nA^*AW_n^*W_n), \eta(W_n)\rangle.$$

Thus

$$\operatorname{Re}\psi_n(A^*A) = \|\pi\|^{-2}\operatorname{Re}\sigma_n((AW_n^*)^*AW_n^*) \leqslant \|\pi\|^{-2}(g(n) + \|\theta_n(|AW_n^*|)\|^2) =$$
$$= \|\pi\|^{-2}(g(n) + \|\eta((W_nA^*AW_n^*)^{\frac{1}{2}}W_n)\|^2) = \|\pi\|^{-2}(g(n) + \|\eta(W_n(A^*A)^{\frac{1}{2}}W_n^*W_n\|^2) =$$
$$= \|\pi\|^{-2}(g(n) + \|\pi(W_n|A|)x_0\|^2) = \|\pi\|^{-2}(g(n) + \lim_k \|\pi(W_nV_k^*A)x_0\|^2) \leqslant$$
$$\leqslant \|\pi\|^{-2}(g(n) + \|\pi\|^2\|\pi(A)x_0\|^2) = g_0(n) + \|\pi(A)x_0\|^2 =$$
$$= g_0(n) + \|\pi(AU_n^*)\pi(U_n)x_0\|^2 \leqslant g_0(n) + \|\pi\|^2\|\pi(|AU_n^*|)\pi(U_n)x_0\|^2 =$$
$$= g_0(n) + \|\pi\|^2\|\pi(U_n|A|U_n^*)\pi(U_n)x_0\|^2 = g_0(n) + \|\pi\|^2\|{}_n\mu(|A|)\|^2 \leqslant$$
$$\leqslant g_0(n) + \|\pi\|^2(\operatorname{Re}{}_n\rho(A^*A) - h(n)) = l(n) + \operatorname{Re}\varphi_n(A^*A),$$

where $g_0(n) \to 0$ and $l(n) \to 0$ as $n \to \infty$. By choice of ψ and φ and the fact that they are hermitian functionals on $\mathfrak{A}$, we conclude that

$$\psi(A^*A) \leqslant \|\pi(A)x_0\|^2 \leqslant \varphi(A^*A).$$

From the definitions of φ_n and ψ_n, we have that

$$\varphi_n(I) = \|\pi\|^2{}_n\rho(I) = \|\pi\|^2\|\pi(U_n)x_0\|^2$$
$$\psi_n(I) = \|\pi\|^{-2}{}_n\sigma(I) = \|\pi\|^{-2}\|\pi(W_n)x_0\|^2.$$

At the same time,

$$\|\pi(U_n)x_0\|^2 = \|\pi(U_nW_n^*)\pi(W_n)x_0\|^2 \leqslant$$
$$\leqslant \|\pi\|^2\|\pi(W_n)x_0\|^2.$$

Thus $\varphi_n(I) \leqslant \|\pi\|^6\psi_n(I)$, whence $\varphi(I) \leqslant \|\pi\|^6\psi(I)$. ∎

We give a variant of the proof of Proposition 2.1 of [4]. We shall need another result from [6] (Lemma 1.5 of that paper). Its proof is included for completeness and as an application of Proposition 3.

PROPOSITION 4. *Let π be a bounded non-degenerate representation of a unital C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. If $\sum_{j=1}^m A_j^*A_j \leqslant \sum_{k=1}^n B_k^*B_k$ with A_j and B_k in $\mathfrak{A}$, then*

$$\sum_{J=1}^m \pi(A_j)^*\pi(A_j) \leqslant \|\pi\|^6 \sum_{k=1}^n \pi(B_k)^*\pi(B_k).$$

Proof. We begin with some reductions. Suppose that we have the conclusion with the additional assumption that

$$\sum_{j=1}^{m} A_j^* A_j = \sum_{k=1}^{n} B_k^* B_k.$$

With C_j and D_k in $\mathfrak{A}$ and

$$\sum_{j=1}^{m} C_j^* C_j \leqslant \sum_{k=1}^{n} D_k^* D_k,$$

let C_0 be $\left[\sum_{k=1}^{n} D_k^* D_k - \sum_{j=1}^{m} C_j^* C_j\right]^{\frac{1}{2}}$. Then

$$\sum_{j=0}^{m} C_j^* C_j = \sum_{k=1}^{n} D_k^* D_k.$$

By our present assumption, we have that

$$\sum_{j=1}^{m} \pi(C_j)^* \pi(C_j) \leqslant \sum_{j=0}^{m} \pi(C_j)^* \pi(C_j) \leqslant \|\pi\|^6 \sum_{k=1}^{n} \pi(D_k)^* \pi(D_k).$$

[A]ssume, henceforth, that $\sum_{j=1}^{m} A_j^* A_j = \sum_{k=1}^{n} B_k^* B_k$.

We show, next, that we can assume that

$$\sum_{j=1}^{m} A_j^* A_j = \sum_{k=1}^{n} B_k^* B_k = I.$$

Let A_0 and B_0 be εI where $\varepsilon > 0$. Then

$$\varepsilon^2 I \leqslant \sum_{j=0}^{m} A_j^* A_j = \sum_{k=0}^{n} B_k^* B_k.$$

Thus $\sum_{j=0}^{m} A_j^* A_j (= A)$ is positive and invertible. Let C_j be $A_j A^{-\frac{1}{2}}$ and D_k be $B_k A^{-\frac{1}{2}}$. Then

$$\sum_{j=0}^{m} C_j^* C_j = A^{-\frac{1}{2}} \left(\sum_{j=0}^{m} A_j^* A_j\right) A^{-\frac{1}{2}} = \sum_{k=0}^{n} D_k^* D_k$$

and

$$A^{-\frac{1}{2}} \left(\sum_{j=0}^{m} A_j^* A_j\right) A^{-\frac{1}{2}} = A^{-\frac{1}{2}} A A^{-\frac{1}{2}} = I.$$

If we have our conclusion for these C_j and D_k, then

$$\pi(A^{-\frac{1}{2}})^* \left(\sum_{j=0}^{m} \pi(A_j)^*\pi(A_j)\right) \pi(A^{-\frac{1}{2}}) \leqslant$$
$$\leqslant \|\pi\|^6 \pi(A^{-\frac{1}{2}})^* \left(\sum_{k=0}^{n} \pi(B_k)^*\pi(B_k)\right) \pi(A^{-\frac{1}{2}}).$$

Thus

$$\pi(A^{\frac{1}{2}})^*\pi(A^{-\frac{1}{2}})^* \left(\sum_{j=0}^{m} \pi(A_j)^*\pi(A_j)\right) \pi(A^{-\frac{1}{2}})\pi(A^{\frac{1}{2}}) \leqslant$$
$$\leqslant \|\pi\|^6 \pi(A^{\frac{1}{2}})^*\pi(A^{-\frac{1}{2}})^* \left(\sum_{k=0}^{n} \pi(B_k)^*\pi(B_k)\right) \pi(A^{-\frac{1}{2}})\pi(A^{\frac{1}{2}}).$$

Hence

$$\varepsilon^2 I + \sum_{j=1}^{m} \pi(A_j)^*\pi(A_j) \leqslant \|\pi\|^6 \left(\varepsilon^2 I + \sum_{k=1}^{n} \pi(B_k)^*\pi(B_k)\right),$$

for all positive ε. We assume, henceforth, that

$$\sum_{j=1}^{m} A_j^* A_j = \sum_{k=1}^{n} B_k^* B_k = I.$$

Given x_0 in $\mathcal{H}$, Proposition 3 guarantees the existence of two hermitian functionals φ and ψ on $\mathfrak{A}$ such that

$$\psi(T^*T) \leqslant \|\pi(T)x_0\|^2 \leqslant \varphi(T^*T)$$

for each T in $\mathfrak{A}$ and $\varphi(I) \leqslant \|\pi\|^6\psi(I)$. We have

$$\sum_{j=1}^{m} \psi(A_j^*A_j) \leqslant \sum_{j=1}^{m} \|\pi(A_j)x_0\|^2 \leqslant \sum_{j=1}^{m} \varphi(A_j^*A_j) = \varphi\left(\sum_{j=1}^{m} A_j^*A_j\right) = \varphi(I) \leqslant$$
$$\leqslant \|\pi\|^6\psi(I) = \|\pi\|^6 \sum_{k=1}^{n} \psi(B_k^*B_k) \leqslant \|\pi\|^6 \sum_{k=1}^{n} \|\pi(B_k)x_0\|^2.$$

As the inequality

$$\sum_{j=1}^{m} \|\pi(A_j)x_0\|^2 \leqslant \|\pi\|^6 \sum_{k=1}^{n} \|\pi(B_k)x_0\|^2$$

holds for each x_0 in $\mathcal{H}$, we have that

$$\sum_{j=1}^{m} \pi(A_j)^*\pi(A_j) \leqslant \|\pi\|^6 \sum_{k=1}^{n} \pi(B_k)^*\pi(B_k).$$ ∎

We apply Proposition 2, once again, to prove the Haagerup-Pisier inequality (see [6; Theorem 3.2.],[11], and [8]) and from it the Pisier-Ringrose inequality (see [11], [6], and [12]).

PROPOSITION 5. *With the notation of Proposition 2, suppose $\eta \neq 0$. There are a weak* limit points ρ and ρ' of $\{\|\eta\|^{-2}{}_n\rho\}$ and $\{\|\eta\|^{-2}\rho_n\}$, respectively; ρ and ρ' are states of $\mathfrak{A}$, and*

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}).$$

If V and W are unitary operators in $\mathfrak{A}$ and η' is the mapping that assigns $\eta(VAW)$ in $\mathcal{H}$ to A in $\mathfrak{A}$, then

$$\|\eta'(A)\|^2 \leqslant \|\eta\|^2[\rho_W(A^*A) + \rho'_{V^*}(AA^*)] \qquad (A \in \mathfrak{A}),$$

where $\rho_W(T) = \rho(W^*TW)$ and $\rho'_{V^*}(S) = \rho'(VSV^*)$.

Proof. By definition, ${}_n\rho$ and ρ_n lie in the ball of radius $\|\eta\|^2$ in the dual $\mathfrak{A}^\#$ of $\mathfrak{A}$. Since the unit ball of $\mathfrak{A}^\#$ is weak* compact, there are weak* limit points ρ and ρ' in the unit ball of $\mathfrak{A}^\#$. At the same time,

$$\rho_n(I) = {}_n\rho(I) = \langle \eta(U_n), \eta(U_n)\rangle = \|\eta(U_n)\|^2 \to \|\eta\|^2,$$

by choice of U_n. Hence, $\rho(I) = \rho'(I) = 1$. It follows that ρ and ρ' are states of $\mathfrak{A}$ from [10; Theorem 4.3.2].

From Proposition 2,

$$\|\eta(U_nH)\|^2 \leqslant \mathrm{Re}\,{}_n\rho(H^2) - h(n) \qquad (H \in \mathfrak{A}_{\mathrm{h}}).$$

For arbitrary A in $\mathfrak{A}$, from the parallelogram law,

$$\begin{aligned}\|\eta(U_nA)\|^2 + \|\eta(U_nA^*)\|^2 &= \frac{1}{2}[\|\eta(U_n(A+A^*))\|^2 + \|\eta(U_n(\mathrm{i}(A-A^*)))\|^2] \leqslant \\ &\leqslant \frac{1}{2}[\mathrm{Re}\,({}_n\rho((A+A^*)^2 + (\mathrm{i}(A-A^*))^2)) - 2h(n)] = \\ &= \mathrm{Re}\,{}_n\rho(A^*A + AA^*) - h(n),\end{aligned}$$

whence

$$\|\eta(U_nA)\|^2 \leqslant \mathrm{Re}\,\langle \eta(U_nA^*A + U_nAA^*), \eta(U_n)\rangle - h(n)$$

for all A in $\mathfrak{A}$. Replacing A by U_n^*A, we have that

$$\begin{aligned}\|\eta(A)\|^2 &\leqslant \mathrm{Re}\,\langle \eta(U_nA^*U_nU_n^*A + AA^*U_n, \eta(U_n)\rangle - h(n) = \\ &= \mathrm{Re}\,[{}_n\rho(A^*A) + \rho_n(AA^*)] - h(n).\end{aligned}$$

Recalling that $h(n) \to 0$, it follows that

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2 \mathrm{Re}\,[\rho(A^*A) + \rho'(AA^*)] = \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \quad (A \in \mathfrak{A}).$$

Let $\alpha(A)$ be VAW for A in $\mathfrak{A}$. Then α is a linear isometry of $\mathfrak{A}$ onto $\mathfrak{A}$. Note that $\eta' = \eta \circ \alpha$, whence $\|\eta'\| = \|\eta\|$. Replacing A by $\alpha(A)$ in the inequality just proved, we have the second inequality of the statement. ■

COROLLARY 6. *With η a bounded linear mapping of a C^*-algebra $\mathfrak{A}$ into a Hilbert space $\mathcal{H}$, there is a state ρ_0 of $\mathfrak{A}$ such that*

$$\|\eta(A)\|^2 \leqslant 2\|\eta\|^2 \rho_0(A^*A + AA^*) \qquad (A \in \mathfrak{A}).$$

Proof. From Proposition 5, there are states ρ and ρ' of $\mathfrak{A}$ such that

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \qquad (A \in \mathfrak{A}).$$

Let ρ_0 be $\frac{1}{2}(\rho + \rho')$. Then $\rho \leqslant 2\rho_0$ and $\rho' \leqslant 2\rho_0$. Thus

$$\|\eta(A)\|^2 \leqslant \|\eta\|^2[\rho(A^*A) + \rho'(AA^*)] \leqslant 2\|\eta\|^2\rho_0(A^*A + AA^*) \quad (A \in \mathfrak{A}).$$ ■

PROPOSITION 7. *If γ is a bounded, linear mapping of one C^*-algebra $\mathfrak{A}$ into another C^*-algebra $\mathcal{B}$, then*

$$\left\| \sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) + \gamma(A_j)\gamma(A_j)^* \right\| \leqslant 4\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^*A_j + A_jA_j^* \right\|$$

for each finite set $\{A_1, \ldots, A_n\}$ of elements A_j of $\mathfrak{A}$.

Proof. We may assume that $\mathcal{B}$ acts faithfully on a Hilbert space $\mathcal{H}$. Let x be a unit vector in $\mathcal{H}$ and $\eta(A)$ be $\gamma(A)x$. With A in $\mathfrak{A}$ and $\|A\| \leqslant 1$,

$$\|\eta(A)\| = \|\gamma(A)x\| \leqslant \|\gamma(A)\|\,\|x\| \leqslant \|\gamma\|.$$

Thus $\|\eta\| \leqslant \|\gamma\|$. From Corollary 6, there is a state ρ_x of $\mathfrak{A}$ such that

$$\|\eta(A_j)\|^2 = \langle \gamma(A_j)^*\gamma(A_j)x, x\rangle \leqslant 2\|\eta\|^2\rho_x(A_j^*A_j + A_jA_j^*).$$

Summing, we have

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) \right) x, x \right\rangle \leqslant 2\|\eta\|^2 \rho_x \left(\sum_{j=1}^{n} A_j^*A_j + A_jA_j^* \right) \leqslant$$

$$\leqslant 2\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^*A_j + A_jA_j^* \right\|$$

This result applied to the mapping η^* of $\mathfrak{A}$ into $\mathcal{H}$, where $\eta^*(A) = \gamma^*(A)x$ (and $\gamma^*(A) = \gamma(A^*)^*$), and to the elements $A_1^*, \ldots, A_n^*$ yields

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j)\gamma(A_j)^* \right) x, x \right\rangle \leqslant 2\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|.$$

Thus

$$\left\langle \left(\sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) + \gamma(A_j)\gamma(A_j)^* \right) x, x \right\rangle \leqslant 4\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|.$$

As this holds for each unit vector in $\mathcal{H}$ and $\sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) + \gamma(A_j)\gamma(A_j)^*$ is positive, we have that

$$\left\| \sum_{j=1}^{n} \gamma(A_j)^*\gamma(A_j) + \gamma(A_j)\gamma(A_j)^* \right\| \leqslant 4\|\gamma\|^2 \left\| \sum_{j=1}^{n} A_j^* A_j + A_j A_j^* \right\|. \qquad \blacksquare$$

PROPOSITION 8. *Let π be a bounded representation of a factor $\mathcal{M}$ of type II_1 on a Hilbert space $\mathcal{H}$, n be a positive integer, and* tr *be the normalized trace on $\mathcal{M}$. Then, with ι_n the identity mapping on $\mathcal{M}_n(\mathbb{C})$, the mapping $\pi \otimes \iota_n$ on $\mathcal{M} \otimes \mathcal{M}_n(\mathbb{C})$ satisfies*

$$\|(\pi \otimes \iota_n)(T)\| \leqslant \|\pi\|^4 [\|T\|^2 + n\operatorname{tr}(T^*T)]^{\frac{1}{2}}$$

for each T in $\mathcal{M} \otimes \mathcal{M}_n(\mathbb{C})$.

Proof. Let S be the matrix in the algebra $\mathcal{M} \otimes \mathcal{M}_n(\mathbb{C})$ whose first row is $(S_1, \ldots, S_n)$ and all of whose other rows are 0, where $S_1, \ldots, S_n$ are arbitrary elements of $\mathcal{M}$. We have that

$$\|(\pi \otimes \iota_n)(S)(\pi \otimes \iota_n)(S)^*\| = \left\| \sum_{j=1}^{n} \pi(S_j)\pi(S_j)^* \right\|.$$

Now $\sum_{j=1}^{n} S_j S_j^* \leqslant \left\| \sum_{j=1}^{n} S_j S_j^* \right\| I$, from which, by Proposition 4,

$$(*) \qquad \|(\pi \otimes \iota_n)(S)\|^2 = \left\| \sum_{j=1}^{n} \pi(S_j)\pi(S_j)^* \right\| \leqslant \|\pi\|^6 \left\| \sum_{j=1}^{n} S_j S_j^* \right\| = \|\pi\|^6 \|S\|^2.$$

This same arguments applies, as well, to matrices S with just one non-zero row, not necessarily the first row.

Let $(x_1, x_2, \ldots, x_n)(= x)$ be a unit vector in $\mathcal{H}_n$, the n-fold direct sum of $\mathcal{H}$ with itself, and let $\eta_j(T)$ be $\sum_{k=1}^{n} \pi(T_{jk})x_k$ for j in $\{1, \ldots, n\}$, where $(T_{j1}, \ldots, T_{jn})$ is the jth row of T (in $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$). We write T_{j-} for the element of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ with the same jth row as T and 0 at all other entries. Let U_j be the (unitary) element of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ with I at the $(1, j)$, $(j, 1)$, and (k, k) entries $(k \neq 1, j)$, and 0 at all other entries. Note that $U_j = U_j^*$, and U_jT is the element of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ having the same entries as T with the first and jth rows of T interchanged. From $(*)$, we have that

$$\|\eta_j(T)\|^2 = \|(\pi \otimes \iota_n)(T_{j-})x\|^2 \leqslant \|\pi\|^6 \|T_{j-}\|^2 \leqslant \|\pi\|^6 \|T\|^2$$

since $T_{j-} = E_{jj}T$, where E_{jk} is the (matrix-unit) element of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ whose (j, k) entry is I and all of whose other entries are 0.

Thus η_j is a mapping of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ into $\mathcal{H}_n$ with bound not exceeding $\|\pi\|^3$. From Proposition 5, there are states ρ_j and σ_j of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$ such that

$$\|\eta_j(T)\|^2 \leqslant \|\pi\|^6[\rho_j(T^*T) + \sigma_j(TT^*)] \qquad (T \in \mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})).$$

With U a unitary element of $\mathcal{M}$ and denoting by $\tilde{U}$ the element $U \otimes I_n$ of $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$, we have that

$$\eta_j(\tilde{U}T) = \sum_{k=1}^{n} \pi(UT_{jk})x_k = \pi(U)\left(\sum_{k=1}^{n} \pi(T_{jk})x_k\right) = \pi(U)\eta_j(T).$$

Thus

$$\|\eta_j(\tilde{U}T)\|^2 \leqslant \|\pi\|^2 \|\eta_j(T)\|^2$$

and

$$\|\eta_j(T)\|^2 = \|\eta_j(\tilde{U}^*\tilde{U}T)\|^2 \leqslant \|\pi\|^2 \|\eta_j(\tilde{U}T)\|^2 \leqslant$$
$$\leqslant \|\pi\|^8(\rho_j(T^*T) + \sigma_j(\tilde{U}TT^*\tilde{U}^*)).$$

Now

$$\|\eta_j(T)\|^2 = \|\eta_j(T_{j-}\|^2 \leqslant$$
$$\leqslant \|\pi\|^8 \left(\rho_j([T_{js}^*T_{jt}]) + \sigma_j\left(\tilde{U}\left(\sum_{k=1}^{n} T_{jk}T_{jk}^*\right)E_{jj}\tilde{U}^*\right)\right).$$

Using the Dixmier process (see [10; Theorem 8.3.6.]), we have that

$$\|\eta_j(T)\|^2 \leqslant \|\pi\|^8 \left(\rho_j([T_{js}^*T_{jt}]) + \operatorname{tr}\left(\sum_{k=1}^{n} T_{jk}T_{jk}^*\right)\sigma_j(E_{jj})\right).$$

Note that $(U_jT)_{1-} = U_jT_{j-}$. Thus

$$\eta_1(U_jT) = \eta_1((U_jT)_{1-}) = \eta_j(T_{j-}) = \eta_j(T) \qquad (T \in \mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})).$$

From Proposition 5, $\rho_j = \rho_1$ and $\sigma_j = \sigma_{1U_j^*}$. Thus

$$\|\eta_j(T)\|^2 \leqslant \|\pi\|^8 \left(\rho_1([T_{js}^*T_{jt}]) + \operatorname{tr}\left(\sum_{k=1}^n T_{jk}T_{jk}^*\right)\sigma_1(U_jE_{jj}U_j^*)\right) =$$

$$= \|\pi\|^8 \left(\rho_1([T_{js}^*T_{jt}]) + \operatorname{tr}\left(\sum_{k=1}^n T_{jk}T_{jk}^*\right)\sigma_1(E_{11})\right).$$

It follows that

$$\|(\pi \otimes \iota_n)(T)x\|^2 = \sum_{j=1}^n \|\eta_j(T)\|^2 \leqslant$$

$$\leqslant \|\pi\|^8 \left(\rho_1\left(\sum_{j=1}^n [T_{js}^*T_{jt}]\right) + n\operatorname{tr}(T^*T)\sigma_1(E_{11})\right) =$$

$$= \|\pi\|^8 \left(\rho_1\left(\left[\sum_{j=1}^n (T^*)_{sj}T_{jt}\right]\right) + n\operatorname{tr}(T^*T)\sigma_1(E_{11})\right) =$$

$$= \|\pi\|^8(\rho_1(T^*T) + n\operatorname{tr}(T^*T)\sigma_1(E_{11})) \leqslant \|\pi\|^8(\|T\|^2 + n\operatorname{tr}(T^*T)).$$

Since this inequality holds for each unit vector x in $\mathcal{H}_n$,

$$\|(\pi \otimes \iota_n)(T)\| \leqslant \|\pi\|^4(\|T\|^2 + n\operatorname{tr}(T^*T))^{\frac{1}{2}},$$

for each T in $\mathcal{M} \otimes \mathcal{M}_n(\mathbf{C})$. ■

REFERENCES

1. BARNES, B.,The similarity problem for representations of a B^*-algebra, *Michigan Math. J.* **22**(1975), 25–32.
2. BUNCE, J., The similarity problem for representations of C^*-algebras, *Proc. Amer. Math. Soc.* **81**(1981), 409–414.
3. CHRISTENSEN, E., On non self-adjoint representations of C^*-algebras, *Amer. J. Math.* **103**(1981), 817–833.
4. CHRISTENSEN, E., Similarities of II_1 factors with property Γ, *J. Operator Theory* **15** (1986), 281–288.
5. CONNES, A., Classification of injective factors, Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104**(1976), 73–115.
6. HAAGERUP, U., Solution of the similarity problem for cyclic representations of C^*--algebras, *Ann. of Math.* **118**(1983), 215–240.

7. KADISON, R., On the orthogonalization of operator representations, *Amer. J. Math.* **77**(1955), 600–620.
8. KADISON, R., On a inequality of Haagerup-Pisier, *J. Operator Theory*, to appear.
9. KADISON, R.; PEDERSEN, G. K., Means and convex combinations of unitary operators, *Math. Scand.* **57**(1985), 249–266.
10. KADISON, R.; RINGROSE, J., *Fundamentals of the Theory of operator Algebras*, Academic Press, Orlando, Vol. **I.**, 1983, Vol. **II.**, 1986.
11. PISIER, G., Grothendieck's theorem for non-commutative C^*-algebras with an appendix on Grothendieck's constants, *J. Functional Analysis* **29**(1978), 397–415.
12. RINGROSE, J., Linear mappings between operator algebras, *Symposia Math. Istituto Nazionale di Alta Matematica* **20**(*1976*), *297–315.*

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