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THE CARATHEODORY AND KOBAYASHI INFINITESIMAL METRICS AND COMPLETELY BOUNDED HOMOMORPHISMS

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Dedicated to the memory of my good friend and colleague John W. Bunce

1. INTRODUCTION

In this note, we prove the following theorem.

THEOREM 1.1. *Let $\Omega \subseteq \mathbf{C}^n$ be a bounded convex domain, and let $z \in \Omega$, $0 \neq \xi \in \mathbf{C}^n$. Then, the Caratheodory (infinitesimal) metric $\gamma(\Omega; z, \xi)$ associated with Ω at (z, ξ) coincides with the (infinitesimal) Kobayashi metric $\kappa(\Omega; z, \xi)$ associated with Ω at (z, ξ).*

For the case that Ω is a strongly convex domain with smooth boundary, the Theorem 1.1 could be proved using the celebrated holomorphic retraction theorem of Lempert (see [13]). We shall actually extend Lempert's holomorphic retraction theorem to the class of bounded convex domains. Our techniques are based on operator theory and they are radically different from Lempert's. In fact, they are variations of those used in [1], where it was shown another consequence of Lempert's holomorphic retraction result (for the case of strongly convex domains with smooth boundaries, see [14]), i.e. the (macro) Caratheodory distance between two distinct points in Ω coincides with the corresponding Kobayashi distance. One of the advantages of the operator theoretical techniques is that no assumptions on either the smoothness of the boudary $\partial\Omega$ or on the strong convexity of Ω are necessary (thus proving a more general result than that produced using Lempert's techniques).

In the course of our discussion, we point out that the Caratheodory and Kobayashi metrics coincide with the completely bounded norm of certain natural (completely bounded) homomorphisms (see Remark 2.2 part (b)). This motivates our introduction of new higher order Caratheodory and Kobayashi types of biholomorphic invariants. We also prove that the directional derivative of the Caratheodory distance coincides with the infinitesimal Caratheodory metric on an arbitrary bounded domain in $\mathbf{C}^n$ (see Theorem 2.9). This result, together with the analogous result for the Kobayashi distance (proved in [20] for the class of Taut pseudoconvex domains) and the theorem for the (macro) Caratheodory and Kobayashi distances on bounded convex sets, give another proof of Theorem 1.1. However, the results of [20] are based on Lempert's Theorem, so that this alternative proof of Teorem 1.1 is not really independent of Lemperts results.

After this paper was written we learned of the existence of the unpublished manu--script of Professors Halsey Royden and Pit-Mann Wong (see [24]). We would like to express our thanks to Professor Myung-Yull Pang for having shared this information with us. In their paper, Professors Royden and Wong proved Theorem 1.1, first for strongly convex domains with smooth boundary (as a direct consenquence of Lempert's theorem) and then for arbitrary convex domains, using an approximation argument. They also extend Lempert's retraction theorem to arbitrary bounded convex sets. Since our approach is completely different from the classical one, we hope that this note can provide still another angle with which to study biholomorphic invariants.

2. PRELIMINARIES

We now introduce some needed notation. Let M be an n-dimensional complex manifold, let $z \in \mathsf{M}$, and let $0 \neq \xi \in T_z(\mathsf{M})$ (i.e., ξ is a holomorphic tangent vector at z).

DEFINITION 2.1. Given $h \in H^\infty(\mathsf{M})$, we let $h_m(\mathsf{M}; z, \xi)$ be the following $m \times m$ matrix:

$$\begin{bmatrix} h(z) & 0 & 0 & 0 & \dots & 0 \\ h'(z)(\xi) & h(z) & 0 & 0 & \dots & 0 \\ \frac{1}{2}h''(z)(\xi,\xi) & h'(z)(\xi) & h(z) & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ \frac{1}{(m-1)!}h^{(m-1)}(z)(\xi,\dots,\xi) & \frac{1}{(m-2)!}h^{(m-2)}(z)(\xi,\dots,\xi) & \dots & \dots & \dots & h(z) \end{bmatrix}.$$

REMARK 2.2. (a) It follows that the map $h \longrightarrow \rho_m(z,\xi)h = h_m(z,\xi)$ is an algebra homomorphism from $H^\infty(\mathsf{M})$ into the algebra $\mathcal{M}_m$ of $m \times m$ matrices. Indeed, choose a system of coordinates $Z = (Z_1, \dots, Z_n)$ with center at z and let $A_m(Z,\xi)$ be the commuting n-tuple of matrices in $\mathcal{M}_m$ given by

$$A_m(Z,\xi) = \begin{pmatrix} Z & 0 & \dots & \dots & 0 \\ \xi & Z & 0 & \dots & 0 \\ 0 & \xi & Z & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \xi & Z \end{pmatrix}.$$

Then, a straightforward verification (see part (b) of the present remark below) shows that $\rho_m(z,\xi)h = h(A_m(z,\xi))$ for every $h \in H^\infty(\mathbf{M})$.

(b) In a different language, the map $\rho_m(z,\xi)$ defines an $H^\infty(\mathbf{M})$ (Hilbert) module on $\mathbf{C}^m$ (see [5]). Actually, we claim that $\rho_m(z,\xi)$ is completely bounded (in the sense of [2], see also [22]). We recall that a homomorphism $\rho : \mathcal{A} \longrightarrow \mathcal{B}$ between the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called completely bounded when the sequence of norms of the induced homomorphisms $\rho \otimes I_k : \mathcal{A} \otimes \mathcal{M}_k \to \mathcal{B} \otimes \mathcal{M}_k$ is bounded. The least upper bound of this sequence is called the c.b. norm of ρ. Our claim can be proved as follows, when $\mathbf{M}$ is a domain $\Omega \subseteq \mathbf{C}^n$ (see [21] for a similar argument): let T_Z be the (bilateral shift) operator multiplication by the coordinate function Z on the L^2-space $L^2(\partial\mathbf{D})$ over the unit circle $\partial\mathbf{D}$. Let $r > 0$ be small enough so that $z + r\xi\mathbf{D} \subseteq \Omega$. Then $z + r\xi T_Z$ is an n-tuple of commuting normal operators, and $\sigma(z + r\xi T_Z) \subseteq \Omega$. Using the analytic functional calculus for the n-tuple $z + r\xi T_Z$ we see that

$$f(z + r\xi T_Z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)(\xi, \dots, \xi)}{k!} r^k T_Z^k, \quad f \in H^\infty(\Omega).$$

Let $\mathcal{F} \simeq \mathbf{C}^m$ be the orthogonal complement in the Hardy space $H^2(\partial\mathbf{D}) \subseteq L^2(\partial\mathbf{D})$ of the subspace consisting of those functions in $H^2(\partial\mathbf{D})$ with vanishing k-th derivatives, $k < m$. Then, the compression of $z + r\xi T_Z$ to $\mathcal{F}$ is unitarily equivalent to the n-tuple $A(z, r\xi)$ defined above. Since $\mathcal{F}$ is an analytic semi-invariant subspace of $z + r\xi T_Z$, such a compression is an algebra homomorphism and we have:

$$(f(z + r\xi T_Z))_{\mathcal{F}} \simeq f(A(z, r\xi)), \quad f \in H^\infty(\Omega).$$

Since compressions give rise to completely bounded maps, the map $f \in H_\infty(\Omega) \longrightarrow$ $\longrightarrow f(A(z,r\xi)) \in \mathcal{M}_m$ is completely bounded. But,

$$f(A(z,r\xi)) = f(RA(z,\xi)R^{-1}) = Rf(A(z,\xi))R^{-1},$$

for every $f \in H^\infty(\Omega)$, where R is the invertible diagonal matrix in $\mathcal{M}_m$ whose k-th diagonal element is r^{k-1}. Therefore, the map $f \longrightarrow \rho_m(z,\xi)f = f(A(z,\xi))$ is also completely bounded, as claimed.

(c) We conjecture that the norm and the c.b. norm of the map $\rho_m(z,\xi)$ are the same.

Since we shall need to consider maps from and to $\mathbf{D}$, we denote by $\mathbf{D}(\mathbf{M})$ the set of holomorphic maps $h : \mathbf{M} \longrightarrow \mathbf{D}$, and by $\mathbf{M}(\mathbf{D})$ the set of holomorphic maps $\varphi : \mathbf{D} \longrightarrow \mathbf{M}$.

DEFINITION 2.3 Let

$$\gamma_m(\mathbf{M}; z, \xi) = \sup_{h \in \mathbf{D}(\mathbf{M}) h^{(j)}(z)(\xi,\ldots,\xi)=0, j<m} \|\rho_{m+1}(z,\xi)h\|^{1/m},$$

$m \in \mathbf{Z}_+$, $z \in \mathbf{M}$, $0 \neq \xi \in T_z(\mathbf{M})$.

REMARK 2.4. (a) It is not hard to check (using the Moebius transformation that sends $h(z)$ to zero) that $\|\rho_2(z,\xi)\| = \gamma_1(\mathbf{M}; z, \xi)$ which is precisely the infinitesimal Caratheodory pseudometric $\gamma(\mathbf{M}; z, \xi)$ of $\mathbf{M}$ at (z,ξ) (see [11], and [12]). Employing a well known property of the Caratheodory pseudometric, it then follows that the above conjecture (in Remark 2.2 part (c)) is valid for $m = 2$. Indeed, substitution in the definition of $\gamma(\mathbf{M}; z, \xi)$ of the unit disk $\mathbf{D}$ by the unit ball of $\mathcal{M}_k$ yields the same value for every $k \in \mathbf{Z}_+$ (see [11]). This fact was already pointed out in [17] (see also [18] and [16]).

(b) Since composition with a holomorphic mapping $\psi : \mathbf{M} \longrightarrow \mathbf{M}'$ is obviously norm decreasing, $\{\gamma_m\}$ is a sequence of biholomorphic invariants.

(c) There is a similar sequence of "Kobayashi-like" biholomorphic invariants. Indeed, we recall that the infinitesimal Kobayashi (Royden) pseudometric $\kappa(\mathbf{M}; z, \xi)$ is defined by $\kappa(\mathbf{M}; z, \xi) = \inf_{\varphi} 1/\alpha$, where the infimum is taken over all holomorphic maps φ in $\mathbf{M}(\mathbf{D})$ such that $\varphi(0) = z$ and $\varphi'(0) = \alpha\xi$ (see [9], [12], and [23]). We denote by $\mathcal{A}_m(\mathbf{M}; z, \xi)$ such analytic disks in $\mathbf{M}$. Notice that the m-th derivative version of Schwartz's lemma yields

$$\gamma(z,\xi) \leqslant \gamma_m(z,\xi) \leqslant \gamma_{2m}(z,\xi) \leqslant \kappa(z,\xi),$$

$m \in \mathbf{Z}_+$, $z \in \mathbf{M}$, $0 \neq \xi \in T_z(\mathbf{M})$.

DEFINITION 2.5. Given $m \in \mathbf{Z}_+$, $z \in \mathbf{M}$ and $0 \neq \xi \in T_z(\mathbf{M})$, as above, we define

$$\kappa_m(\mathbf{M}; z, \xi) = \inf_{\varphi \in \mathcal{A}(\mathbf{M}; z, \xi)} \|(\varphi^{-1})_{m+1}(\varphi(r\mathbf{D}); z, \xi)\|,$$

where $0 < r \leqslant 1$ is any number for which $\varphi'' : r\mathbf{D} \to \mathbf{M}$ is injective, so that φ^{-1} is well defined on $\varphi(r\mathbf{D})$.

REMARK 2.6. (a) Of course, the matrix $\varphi^{-1}_{m+1}(\varphi(r\mathbf{D}); z, \xi)$ in the above definition does not depend on r. It is easy to check that $\kappa(\mathbf{M}; z, \xi) = \kappa_1(\mathbf{M}; z, \xi)$. Furthermore, it follows that $\kappa_m(\mathbf{M}'; \psi(z), \psi'(z)\xi) \leqslant \kappa_m(\mathbf{M}; z, \xi)$ for every holomorphic mapping $\psi'' : \mathbf{M} \longrightarrow \mathbf{M}'$, so that $\{\kappa_m\}$ is an increasing sequence of biholomorphic invariants.

(b) We recall that, if $\Omega \subseteq \mathbf{C}^n$ is bounded then the set of holomorphic maps $\mathbf{D}(\Omega)$ is a normal family (see [11]). On the other hand, if Ω is Taut-pseudoconvex (for instance, when Ω is pseudoconvex and $\partial\Omega$ is C^1, see [7]), the set $\Omega(\mathbf{D})$ is also a normal family. Therefore, the values of $\gamma_m(\Omega; z, \xi)$ and $\kappa_m(\Omega; z, \xi)$ are attained by respective extremal mappings.

(c) From Theorem 1.1 and Remark 2.4 part (c), it follows that $\gamma_m(\Omega; z, \xi) = \kappa(\Omega; z, \xi)$, for every $m = 1, 2, \ldots$, every $z \in \Omega$, and every $0 \neq \xi \in \mathbf{C}^n$, when Ω is a bounded convex domain in $\mathbf{C}^n$. What about the sequence $\{\kappa_m(\Omega; z, \xi)\}$?

(d) Another interesting problem is to find other classes of complex manifolds for which the infinitesimal Caratheodory and Kobayashi pseudometrics coincide. Here is still another interesting question: what is the behavior of the sequence $\{\kappa_m(\Omega; z, \xi)\}$, where Ω is strongly pseudoconvex with smooth boundary, as z tends to some point of $\partial\Omega$ within a non-tangential approach region? It may be worth pointing out that the behavior of the sequence $\{\gamma_m(\Omega; z, \xi)\}$ is the same as the Kobayashi metric $\kappa(\Omega; z, \xi)$. (See Remark 2.4 part (c) and [6].)

We next recall the definitions of the Caratheodory and Kobayashi distances. In doing so, we use the Moebius distance on the unit disk rather than the more standard Poincaré distance in order to simplify calculations. We point out that the latter is the inverse hyperbolic tangent of the former.

DEFINITION 2.7. Let $\delta(\lambda, \mu) = |\lambda - \mu|/|1 - \lambda\bar{\mu}|$, $\lambda, \mu \in \mathbf{D}$, be the Moebius distance on the unit disk (invariant under Moebius transformations). We define the (macro) Caratheodory distance $\gamma(\mathbf{M}; z, w)$ and the (macro) Kobayashi distance $\kappa(\mathbf{M}; z, w)$ between the points z and w in the manifold $\mathbf{M}$ (see [10] and [4]), respectively, as follows:

$$\gamma(\mathbf{M}; z, w) = \sup_{h \in \mathbf{D}(\mathbf{M})} \delta(h(z), h(w))$$

$$\kappa(\mathbf{M}; z, w) = \inf_{\Gamma} \int_0^1 \kappa(\mathbf{M}; \Gamma(t), \Gamma'(t)) \mathrm{d}t,$$

where the last infimum is taken over all continuously differentiable curves $\Gamma : [0,1] \longrightarrow \mathbf{M}$ such that $\Gamma(0) = z$ and $\Gamma(1) = w$.

It follows that

$$\gamma(\mathbf{M}; z, w) \leqslant \kappa(\mathbf{M}; z, w) \leqslant \widetilde{\kappa}(\mathbf{M}; z, w) = \inf_{\Delta} \delta(\lambda, \mu)$$

where the infimum is taken over all analytic disks $\Delta : \mathbf{D} \longrightarrow \mathbf{M}$ in $\mathbf{M}$ such that $\Delta(\lambda) = z$, and $\Delta(\mu) = w$. When z and w are sufficiently close, the last inequality in the last chain of inequalities often becomes an equality (for example, when $\mathbf{M}$ is Kobayashi simple at z in the sense of [20]).

REMARK 2.8. (a) Using the fact that $\mathrm{Aut}(\mathbf{D})$ is transitive, we can assume, in the definition of $\gamma(\mathbf{M}; z, w)$, that the supremum is taken over functions that vanish at one of the points, say at z. Likewise, in the definition of $\widetilde{\kappa}(\mathbf{M}; z, w)$ we may assume that the infimum is taken over pairs (λ, μ) such that $\lambda = 0$.

(b) Now, we obtain matricial definitions of the Caratheodory and Kobayashi distances. Let $z \in \mathbf{M} \subseteq \mathbf{C}^n$ and let $0 \neq \xi \in T_z(\mathbf{M}) \subseteq \mathbf{C}^n$. Assume that for small values of $t \in \mathbf{R}$, $z + t\xi \in \mathbf{M}$. Also, let $B(\mathbf{M}; z, t, \xi)$ be the commuting n-tuple of matrices in $\mathcal{M}_2$ given by

$$B(\mathbf{M}; z, t, \xi) = \begin{pmatrix} z + t\xi & 0 \\ \xi & z \end{pmatrix}.$$

It follows easily that, for every $h \in H^\infty(\mathbf{M})$ and $t \neq 0$, we have:

$$h(B(\mathbf{M}; z, t, \xi)) = \begin{pmatrix} h(z + t\xi) & 0 \\ \dfrac{h(z + t\xi) - h(z)}{t} & h(z) \end{pmatrix}.$$

(See [21] for the case $n = 1$ of the above fact.) Therefore, if $h(z) = 0$, we have: $\|h(B(\mathbf{M}; z, t, \xi))\| = \sqrt{1 + t^2}\|h(z + t\xi)/t\|$. Thus, we see that

$$\sqrt{1 + t^2}\frac{\gamma(\mathbf{M}; z + t\xi, z)}{|t|} = \sup_{h \in \mathbf{D}(\mathbf{M}), h(z) = 0} \|h(B(\mathbf{M}; z, t, \xi))\|,$$

$$\sqrt{1 + t^2}\frac{\widetilde{\kappa}(\mathbf{M}; z + t\xi, z)}{|t|} = \inf_{\varphi} \|\varphi^{-1}(B(\varphi(r\mathbf{D}); z, t, \xi))\|,$$

where $0 < r < 1$ is any number such that $\varphi'' : r\mathbf{D} \longrightarrow \mathbf{M}$ is injective.

(c) In view of the above remarks, the fact that $\lim_{t \to 0} B(\mathbf{M}; z, t, \xi) = A(\mathbf{M}; z, \xi)$, and Theorem 1.1, the following relationships between the macro and infinitesimal Caratheodory and Kobayashi metrics, when $\mathbf{M}$ is a bounded domain Ω in $\mathbf{C}^n$ become feasible:

$$\gamma(\Omega; z, \xi) = \lim_{t \to 0} \frac{\gamma(\Omega; z + t\xi, z)}{|t|},$$

$$\kappa(\Omega; z, \xi) = \lim_{t \to 0} \frac{\widetilde{\kappa}(\Omega; z + t\xi, z)}{|t|}.$$

The last equation was very recently proved by Myung-Yull Pang (see [20]), in the case that Ω is a Taut pseudoconvex (bounded) domain. The first is proved in the following theorem.

THEOREM 2.9. *Let $\Omega \subseteq \mathbf{C}^n$ be a bounded domain, and let $z \in \Omega$, $0 \neq \xi \in \mathbf{C}^n$. Then*

$$\lim_{t \to 0} \frac{\gamma(z + t\xi, z)}{|t|} = \gamma(z, \xi).$$

Proof. Let $\delta > 0$ be such that $z + t\xi \in \Omega$ for every $|t| \leqslant \delta$. Also, let $\{t_n\}$ be a sequence of real numbers such that $|t_n| < \delta$ and $t_n \to 0$. Since the set of functions $\{f \in \mathbf{D}(\Omega) : f(z) = 0\}$ constitutes a normal family, there exists an extremal function f_n such that

$$\sqrt{1+t_n^2}\gamma(z + t_n\xi, z) = \|f_n(B(\Omega; z, t_n, \xi))\|.$$

Let $\{s_n\}$ be a subsequence of $\{t_n\}$ and let $\{g_n\}$ be the sequence of corresponding extremal functions. Since such a sequence constitutes a normal family, we can assume, by dropping to a subsequence (if necessary), that $\{g_n\}$ converges uniformly to a holomorphic function $g \in \mathbf{D}(\Omega)$ on compact subsets of Ω so that $g(z) = 0$. In particular, since the set of matrices $\{B(z,t,\xi) : |t| \leqslant \delta\}$ is compact, we deduce (using the analytic functional calculus for commuting n-tuples in $\mathcal{M}_m$) that

$$\lim_{n\to\infty} \sup_{|t|\leqslant\delta} \|(g_n - g)(B(z,t,\xi))\| = 0.$$

Therefore,

$$\lim_{m\to\infty} \|(g_m - g)(B(z, s_m, \xi))\| = 0.$$

It follows that

$$|g'(z)\xi| = \lim_{m\to\infty} \frac{|g(z + s_m\xi) - g(z)|}{|s_m|} =$$

$$\lim_{m\to\infty} \frac{|g_m(z + s_m\xi) - g_m(z)|}{|s_m|} = \lim_{m\to\infty} \frac{\gamma(z + s_m\xi, z)}{|s_m|}.$$

Since $|g'(z)\xi| \leqslant \gamma(z,\xi)$, we deduce that

$$\lim_{m\to\infty} \frac{\gamma(z + s_m\xi, z)}{|s_m|} \leqslant \gamma(z,\xi).$$

In order to prove the opposite inequality, let $h \in \mathbf{D}(\Omega)$ be an extremal function for $\gamma(z,\xi)$. Then,

$$\gamma(z,\xi) = |h'(z)\xi| = \lim_{m\to\infty} \frac{|h(z + s_m\xi) - h(z)|}{|s_m|} \leqslant \lim_{m\to\infty} \frac{\gamma(z + s_m\xi, z)}{|s_m|}.$$

Since $\{s_m\}$ was a subsequence of $\{t_n\}$, and $\{t_n\}$ was an arbitrary sequence in $[-\delta,\delta]$ such that $t_n \to 0$, we conclude that

$$\lim_{t\to 0} \frac{\gamma(z + t\xi, z)}{|t|} = \gamma(z,\xi),$$

as desired. ■

3. PROOF OF THEOREM 1.1

Theorem 1.1 is an immediate consequence of the following theorem

THEOREM 3.1. *Let $\Omega \subseteq \mathbf{C}^n$ be a bounded domain, and let $z \in \Omega$, $0 \neq \xi \in \mathbf{C}^n$. Then $\kappa(\widehat{\Omega}; z, \xi) \leqslant \gamma(\Omega; z, \xi)$, where $\widehat{\Omega}$ is the convex hull of Ω. In particular,*

$$\gamma(\widehat{\Omega}; z, \xi) = \kappa(\widehat{\Omega}; z, \xi) = \widetilde{\kappa}(\widehat{\Omega}; z, \xi).$$

Furthermore, if f_0 is an extremal map in $\mathbf{D}(\widehat{\Omega})$ for $\gamma(\widehat{\Omega}; z, \xi)$, then there exists $\varphi_0 \in \in \widehat{\Omega}(\mathbf{D})$ such that $\varphi_0(0) = z$, $\varphi_0'(0) = \xi/\gamma(\widehat{\Omega}; z, \xi)$, and $f_0 \circ \varphi_0$ is the identity on $\mathbf{D}$, so that $\mathbf{D}$ is a holomorphic retraction of $\widehat{\Omega}$.

Proof. In order to prove the first assertion, it is enough to show that there exists an analytic disk $\psi : \mathbf{D} \to \widehat{\Omega}$ such that $\psi(0) = z$, $\psi'(0) = \alpha\xi$, and $\beta = 1/\alpha = \gamma(z, \xi)$. Indeed, the existence of ψ implies that $\kappa(\widehat{\Omega}; z, \xi) \leqslant 1/\alpha = \gamma(z, \xi)$. Now, choose an extremal map h_0 in $\mathbf{D}(\Omega)$ that realizes the Caratheodory metric $\gamma(\Omega; z, \xi) = \gamma(z, \xi)$, i.e., $h_0'(z)\xi = \beta$, and $h_0(z) = 0$. Also, let

$$M' = \begin{pmatrix} z & 0 \\ \xi/\beta & z \end{pmatrix}$$

so that

$$M_0 = h_0(M') = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let H^2 be the classical Hardy space over the unit circle, and let $K_\lambda(\mu) = 1/(1 - \overline{\lambda}\mu)$ be the Szegö kernel. Further, let $\mathcal{M} = \operatorname{span}(K_0, K_0')$. Then, $M_0^* = (M_z)^*|_{\mathcal{M}}$. Since M' commutes with M_0, we can consider the n-tuple M' acting on the same space $\mathcal{M}$ as M_0, so that $(M')^* K_0 = \overline{z} K_0$, and $(M')^* K_0' = \overline{\xi} K_0 + \overline{z} K_0'$. We now claim

LEMMA 3.2. *With the above notations, let M be the commuting $n+1$-tuple of matrices in $\mathcal{M}_2$ given by $M = (M_0, M')$. Then, $\|g(M)\| \leqslant \|g\|_\infty$, for every $g \in H^\infty(\mathbf{D} \times \Omega)$.*

Proof. Let $F : \Omega \longrightarrow \mathbf{D} \times \Omega$ be given by $F(\omega) = (h_0(\omega), \omega)$, $\omega \in \Omega$. Then $F(M') = M$. We need only show to the desired inequality for $g \in \mathbf{D}(\mathbf{D} \times \Omega)$, such that $g(0, z) = 0$. But, for such a g, it follows that $g(M) = g \circ F(M'), g \circ F \in \mathbf{D}(\Omega)$, and $g \circ F(z) = 0$. Therefore,

$$\|g(M)\| = \|g \circ F(M')\| = |(g \circ F)'(z)\xi|/\beta \leqslant 1. \qquad \blacksquare$$

Continuing with the proof of Theorem 3.1, let $\mathcal{B}$ be the uniform closure in $C(\partial\mathbf{D} \times \partial\Omega)$ of the set of functions which are holomorphic in neighborhoods of $\overline{\mathbf{D}} \times \overline{\Omega}$.

Then, the map $g \longrightarrow g(M)$, from $\mathcal{B}$ into $\mathcal{M}_2$ is a unital completely contractive homomorphism. By the Arveson dilation theorem (see [2]), there exists a non-degenerate representation $\pi : C(\partial\mathbf{D} \times \partial\Omega) \rightarrow \mathcal{L}(\mathcal{H})$ and an isometry $V : \mathcal{M} \longrightarrow \mathcal{H}$ such that $g(M) = V^*(\pi g)V$ for every $g \in \mathcal{B}$. Furthermore, from Arveson's theorem, it also follows that there are subspaces $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}$ such that $\mathrm{Ran}(V) = \mathcal{H}_1 \ominus \mathcal{H}_0$, and such that $\mathcal{H}_0$ and $\mathcal{H}_1$ are invariant under πg for every $g \in \mathcal{B}$. Let $N = (N_0, N') = \pi\chi$, where χ is the $n+1$-tuple of (global) coordinate functions on $\overline{\mathbf{D}} \times \overline{\Omega}$, and let $S = (S_0, S') = N|_{\mathcal{H}_1}$, so that S is a subnormal $n+1$-tuple. Note that, since $\sigma(N_0) \subseteq \partial\mathbf{D}$, it follows that S_0 is an isometry. It also follows that $\sigma(S) \subseteq \overline{\mathbf{D}} \times \overline{\Omega}$, and $h(M^*) = V^*h(S^*)V$ for every h holomorphic in a neighborhood of $(\overline{\mathbf{D}} \times \overline{\Omega})^*$. Let $\mathcal{N}$ be the subspace of $\mathcal{H}_1$ defined by

$$\mathcal{N} = \left\{ x \in \mathcal{H}_1 : \lim_{k\rightarrow\infty} (S_0^*)^k x = 0 \right\}.$$

From a basic fact of the Wold decomposition (see [19]) of S_0, we deduce that $\mathcal{N}$ is a reducing subspace of S_0, and that $T_0 = (S_0)_{|\mathcal{N}}$ is a pure isometry. Moreover, $\mathcal{N}$ is invariant under $(S')^* = (S_1^*, \ldots, S_n^*)$ because it commutes with S_0^*. Also, since $\mathrm{Ran}(V)$ is invariant under S_0^*, $M_0^* \simeq (S_0)^*|\mathrm{Ran}(V)$, and $M_0^2 = 0$, we see that $\mathrm{Ran}(V) \subseteq \mathcal{N}$. Let T' be the n-tuple on $\mathcal{N}$ defined by $(t')^* = (S')^*_{|\mathcal{N}}$. It follows that $T = (T_0, T')$ is a commuting $n+1$-tuple, and that $M'^* = V^*(T')^*V$. Now we use the Nagy-Foiaş machinary (see [19]) applied to the pure isometry T_0 to deduce the existence of a unitary transformation $U : \mathcal{N} \longrightarrow H^2 \otimes \mathcal{K}$, where $\mathcal{K}$ is another separable Hilbert space, such that $UT_0U^* = M_z \otimes I_{\mathcal{K}}$. Since $UT'U^*$ commutes with $M_z \otimes I_{\mathcal{K}}$, there exists a bounded holomorphic function $\varphi : \mathbf{D} \longrightarrow \mathcal{L}(\mathcal{K}) \otimes \mathbf{C}^n$ such that

$$((UT'U^*)f)(\lambda) = (M_\varphi f)(\lambda) = \varphi(\lambda)f(\lambda), \quad \forall f \in H^2 \otimes \mathcal{K}, \forall \lambda \in \mathbf{D}.$$

It follows that $M_\varphi^*(xK_\mu) = K_\mu(\varphi(\mu)^*x)$, for every $x \in \mathcal{K}$ and every $\mu \in \mathbf{D}$. Notice that

$$\mathrm{Ker}(M_z \otimes I_{\mathcal{K}})^* = \{xK_0 : x \in \mathcal{K}\},$$

$$\mathrm{Ker}((M_z \otimes I_{\mathcal{K}})^2)^* = \{yK_0 + y'K_0' : y, y' \in \mathcal{K}\}.$$

Let $W = UV$. Since $\mathrm{Ran}(V)$ is invariant under S_0^*, it is also invariant under T_0^* $(= (S_0)^*_{|\mathcal{N}})$. Therefore, $T_0^*W = WW^*T_0^*W = WM_0^*$. Since $M_0^*K_0 = 0$, we deduce that there exists $x \in \mathcal{K}$ such that $WK_0 = xK_0$, and since W is an isometry and $\|K_0\| = 1$, it follows that $\|x\| = 1$. Furthermore, observe that $WK_0' \in \mathrm{Ker}((M_z \otimes I_{\mathcal{K}})^2)^*$, so $WK_0 = yK_0 + y'K_0'$, for some $y, y' \in \mathcal{K}$. But,

$$xK_0 = WK_0 = WM_0^*K_0' = T_0^*WK_0' = T_0^*(yK_0 + y'K_0') = y'K_0,$$

so $y' = x$. Also, since $\|x\| = 1$, $\|K_0'\| = 1$, $\langle K_0, K_0'\rangle = 0$, and W is an isometry, we see that in the equation $WK_0' = yK_0 + xK_0'$, y must be zero. We claim that the analytic disk in $\mathbf{C}^n$ defined by

$$\psi(\lambda) = \langle \varphi(\lambda)x, x\rangle, \quad \lambda \in \mathbf{D},$$

satisfies the required properties. To prove our claim, we observe first that

$$\operatorname{Ran}(\psi) \subseteq (\sigma(S'))\hat{} \subseteq \overline{(\widehat{\Omega})}.$$

But, from the maximum modulus principle, $\operatorname{Ran}(\psi) \subseteq \widehat{\Omega}$. Secondly,

$$\psi(0) = \langle x, \varphi^*(0)x\rangle = \langle xK_0, (T')^* xK_0\rangle = \langle WK_0, (T')^* WK_0\rangle = \langle K_0, (M')^* K_0\rangle = z,$$
$$\psi'(0) = \langle x, (\varphi'(0))^* x\rangle = \langle xK_0, (\varphi'(0))^* xK_0\rangle = \langle xK_0, \varphi^*(0)xK_0' + (\varphi'(0))^* xK_0\rangle =$$
$$= \langle xK_0, M_\varphi^* xK_0'\rangle = \langle WK_0, (T')^* WK_0'\rangle = \langle K_0, (M')^* K_0'\rangle = \frac{\xi}{\beta},$$

and the first assertion is proved. For the last assertion, we repeat the above argument but starting with an extremal function f_0 for $\gamma(\widehat{\Omega}; z, \xi)$ (instead of the function h_0) so that $f_0(z) = 0$, and $f_0'(z)\xi = \gamma(\widehat{\Omega}; z, \xi)$. Let $\varphi_0 \in \widehat{\Omega}(\mathbf{D})$ be an analytic disk in $\widehat{\Omega}$ such that $\varphi_0(0) = z$, and $\varphi_0'(0) = \xi/\gamma(\widehat{\Omega}; z, \xi)$ (whose existence is guaranteed by the above proof). We see that

$$(f_0 \circ \varphi_0)'(0) = f_0'(z)\varphi_0'(0) = 1.$$

Since $f_0 \circ \varphi_0 \in \mathbf{D}(\mathbf{D})$, and $f_0 \circ \varphi_0(0) = 0$, it follows, from Schwartz's Lemma, that $f_0 \circ \varphi_0(\lambda) = \lambda$ for every $\lambda \in \mathbf{D}$, as desired. ∎

REMARK 3.3. A similar statement to Theorem 3.1, but for the (macro)Caratheodory and Kobayashi distances is valid. (Its proof uses the arguments of [1] and those in the proof of Theorem 3.1.) Thus, we managed to extend Lempert's holomorphic retraction results to arbitrary bounded convex domains (with no assumptions on their boundaries).

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