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A SIMPLE APPROACH TO THE INVARIANT SUBSPACE STRUCTURE OF ANALYTIC CROSSED PRODUCTS

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1. INTRODUCTION

The invariant subspace structure of analytic crossed products was studied in [1]–[11]. These algebras are certain subalgebras of von Neumann algebras constructed as crossed products of von Neumann algebras by $*$-automorphisms.

Let M be a finite von Neumann algebra with a faithful normal tracial state φ and let α be a $*$-automorphism of M such that $\varphi \circ \alpha = \varphi$. Form the Hilbert space $\mathsf{L}^2 = \ell^2(\mathbf{Z}) \otimes L^2(M, \varphi)$, where $L^2(M, \varphi)$ is the noncommutative L^2-space associated by M and φ and let $\mathfrak{L}$ be the crossed product on L^2 determined by M and α and let $\mathfrak{L}_+$ be the analytic crossed product by M and α. Before, we called this algebra to be a nonselfadjoint crossed product. Put $\mathsf{H}^2 = \ell^2(\mathbf{Z}_+) \otimes L^2(M, \varphi)$.

We shall denote by $\mathrm{Lat}(\mathfrak{L}_+)$ the lattice of closed subspaces invariant under $\mathfrak{L}_+$ which are left-pure. If every subspace $\mathfrak{M}$ in $\mathrm{Lat}(\mathfrak{L}_+)$ is of the form $\mathfrak{M} = V\mathsf{H}^2$, where V is a partial isometry in the commutant $\mathfrak{R}$ of $\mathfrak{L}$, we shall say that the Beurling-Lax-Halmos theorem (hereafter abbreviated as the BLH theorem) is valid. In [5, Theorem 3.2], it is shown that α acts trivially on the center of M if and only if the BLH theorem is valid. In general, the BLH theorem is not valid. If α does not fix the center $\mathfrak{Z}(M)$ of M elementwise, then the invariant subspace structure is very complicated. In [3], M. McAsey studied a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ which consists of two-sided invariant subspaces of L^2. On the other hand, in [11], Solel studied a complete set of canonical models for $\mathrm{Lat}(\mathfrak{L}_+)$ using the center trace of $L(M)'$ (see the definition of $L(M)'$ to Section 2).

Our aim in this note is to try a simple approach to invariant subspace structure of analytic crossed products. As a generalization of BLH theorem, we shall show that every $\mathfrak{M}$ in $\mathrm{Lat}(\mathfrak{L}_+)$ is of the form $\sum_{n=0}^{\infty} \oplus V_n \mathbb{H}^2$, where $\{V_n\}_{n=0}^{\infty}$ is a family of partial isometries in $\mathfrak{R}$ such that $\{V_n V_n^*\}_{n=0}^{\infty}$ is mutually orthogonal. Further, for any m such that $1 \leqslant m \leqslant \infty$, we consider the necessary and sufficient conditions that every $\mathfrak{M} \in \mathrm{Lat}(\mathfrak{L}_+)$ is of the form $\sum_{n=0}^{m-1} \oplus V_n \mathbb{H}^2$, where $\{V_n\}_{n=0}^{m-1}$ is a family of partial isometries in $\mathfrak{R}$ such that $\{V_n V_n^*\}_{n=0}^{m-1}$ is mutually orthogonal (cf. Theorems 3.8 and 3.9).

2. PRELIMINARIES

Let M be a σ-finite finite von Neumann algebra. That is, there exists a faithful normal tracial state φ on M. Let $L^2(M,\varphi)$ be the noncommutative L^2-space associated with M and φ. For every $x \in M$, let l_x (resp. r_x) be the left (resp. right) multiplication on $L^2(M,\varphi)$: that is, $l_x y = xy$ (resp. $r_x y = yx$). Put $l(M) = \{l_x : x \in M\}$ and $r(M) = \{r_x : x \in M\}$, respectively. Also, we fix once and for all a $*$-automorphism α of M which preserves φ: i.e., $\varphi \circ \alpha = \varphi$. Then there is a unitary operator u on $L^2(M,\varphi)$ induced by α. To construct a crossed product, we consider the Hilbert space $\mathbb{L}^2$ defined by $\{f : \mathbb{Z} \to L^2(M,\varphi) : \sum_{n\in\mathbb{Z}} \|f(n)\|_2^2 < \infty\}$, where $\|\cdot\|_2$ is the norm of $L^2(M,\varphi)$. For $x \in M$, we define operators L_x, R_x, L_δ and R_δ on $\mathbb{L}^2$ by the formulae $(L_x f)(n) = l_x f(n)$, $(R_x f)(n) = r_{\alpha^n(x)} f(n)$, $(L_\delta f)(n) = u f(n-1)$ and $(R_\delta f)(n) = f(n-1)$. Put $L(M) = \{L_x : x \in M\}$ and $R(M) = \{R_x : x \in M\}$. We set $\mathfrak{L} = \{L(M), L_\delta\}''$ and $\mathfrak{R} = \{R(M), R_\delta\}''$ and define the left (resp. right) analytic crossed product $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) to be the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by $L(M)$ (resp. $R(M)$) and L_δ (resp. R_δ). Let E_n be the projection on $\mathbb{L}^2$ defined by the formula $(E_n f)(k) = f(n)$, if $k = n$, and 0, if $k \neq n$. Furthermore, we define $\mathbb{H}^2 = \{f \in \mathbb{L}^2 : f(n) = 0,\ n < 0\}$. We refer the reader to [4, 5, 6, etc.] for discussions of these algebras including some of their elementary properties.

DEFINITION 2.1. Let $\mathfrak{M}$ be a closed subspace of $\mathbb{L}^2$. We shall say that $\mathfrak{M}$ is: *left-invariant*, if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$; *left-reducing*, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; *left-pure*, if $\mathfrak{M}$ contains no non-trivial left-reducing subspaces; and *left-full*, if the smallest left-reducing subspace containing $\mathfrak{M}$ is $\mathbb{L}^2$. The right-hand versions of these concepts are defined similarly, and a closed subspace which is both left and right invariant will be called *two-sided invariant.*

As in [4], we have to study the wandering subspaces for the bilateral shifts L_δ. Let $\mathfrak{M}$ be a left-invariant subspace of $\mathbb{L}^2$ and let $P_{\mathfrak{F}}$ bet the projection of $\mathbb{L}^2$ onto

$\mathfrak{M} \ominus L_\delta \mathfrak{M} (= \mathfrak{F})$. By [5, Theorem 3.2], $P_\mathfrak{F}$ lies in $L(M)'$. Further, by [7, Proposition 2.2] we have:

LEMMA 2.2. *Let $\mathfrak{M}$ be a left-invariant subspace of $\mathbf{L}^2$ and let $P_\mathfrak{F}$ be the projection of $\mathbf{L}^2$ onto $\mathfrak{F} = \mathfrak{M} \ominus L_\delta \mathfrak{M}$. Then $P_\mathfrak{F}$ is a finite projection in $L(M)'$.*

LEMMA 2.3. $E_0 L(M)' E_0 = R(M) E_0$.

Proof. Let $T \in L(M)'$. Since $L(M)' = (l(M) \otimes 1)' = r(M) \otimes B(\ell^2(\mathbf{Z}))$, T has the matricial representation $(r_{x_{ij}})_{i,j=-\infty}^{\infty}$, where $x_{ij} \in M$. Thus it is clear that $E_0 T E_0 = R_{x_{00}} E_0$. This completes the proof.

3. INVARIANT SUBSPACE STRUCTURE

In this section, we study the form of left-pure, left-invariant subspaces of $\mathbf{L}^2$. Let $\{p_n\}_{n=0}^{\infty}$ be a family of mutually orthogonal projections in M. We define a closed subspace $\mathbf{H}^2(\{p_n\})_{n=0}^{\infty}$ of $\mathbf{H}^2$ by

$$\mathbf{H}^2(\{p_n\}_{n=0}^{\infty}) = \sum_{n=0}^{\infty} \oplus R_{p_n} R_\delta^n \mathbf{H}^2.$$

Then it is clear that $\mathbf{H}^2(\{p_n\}_{n=0}^{\infty})$ is a left-pure, left-invariant subspace of $\mathbf{H}^2$ with the wandering projection $\sum_{n=0}^{\infty} R_{p_n} E_n$.

First, we have the following theorem.

THEOREM 3.1. *Let $\mathfrak{M}$ be a non-zero left-pure, left-invariant subspace of $\mathbf{L}^2$. Then there exists a partial isometry W in $\mathfrak{R}$ such that $\mathfrak{M} = W\mathbf{H}^2(\{p_n\}_{n=0}^{\infty})$, where $\{p_n\}_{n=0}^{\infty}$ is a family of mutually orthogonal projections in M and $W^*W = \sum_{n=0}^{\infty} R_{p_n}$. Therefore, putting $V_n = W R_{p_n} R_\delta^n$, then $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus V_n \mathbf{H}^2$ and $\{V_n\}_{n=0}^{\infty}$ is a family of partial isometries in $\mathfrak{R}$ such that $\{V_n V_n^*\}_{n=0}^{\infty}$ is mutually orthogonal and $V_n V_n^*$ is in $R(M)$.*

Proof. Put $\mathfrak{F} = \mathfrak{M} \ominus L_\delta \mathfrak{M}$. Let $P_\mathfrak{F}$ be the orthogonal projection of $\mathbf{L}^2$ onto $\mathfrak{F}$. Let Z be the center of M. Since $L(Z)$ is the center of $L(M)'$, by the comparability theorem ([13, Theorem 4.6]), there exists a central projection z_0 in M such that $L_{z_0} P_\mathfrak{F} \precsim L_{z_0} E_0$ and $L_{1-z_0} E_0 \precsim L_{1-z_0} P_\mathfrak{F}$. Thus, there exist projections R_0, $R_0' \in L(M)'$ such that $L_{1-z_0} E_0 \sim R_0 \leqslant L_{1-z_0} P_\mathfrak{F}$ and $L_{z_0} P_\mathfrak{F} \sim R_0' \leqslant L_{z_0} E_0$. By Lemma 2.3, there exists a projection r_0 in M such that $R_0' = R_{r_0} E_0$. We set $p_0 = r_0 + (1 - z_0)$ and $Q_0 = L_{z_0} P_\mathfrak{F} + R_0$, rspectively. Since $R_{p_0} E_0 = (R_{r_0} + R_{1-z_0}) E_0 = R_{r_0} E_0 + L_{1-z_0} E_0$, it is clear that $P_{\mathfrak{F}} \geqslant Q_0 \sim R_{p_0}E_0 \leqslant E_0$. Further, due to the maximality of the chosen pair and due to [13, Corollary 4.5], it follows that $c(P_{\mathfrak{F}} - Q_0)R_{1-p_0}E_0 = 0$, where $c(P_{\mathfrak{F}} - Q_0)$ is the central support projection of $P_{\mathfrak{F}} - Q_0$ in $L(M)'$.

Next we consider the projections $P_{\mathfrak{F}} - Q_0$ and $R_{1-p_0}E_1$. Again, by the comparability theorem, there exist a projection Q_1 in $L(M)'$ and a projection p_1 in M such that $P_{\mathfrak{F}} - Q_0 \geqslant Q_1 \sim R_{p_1}E_1 \leqslant R_{1-p_0}E_1$ and $c(P_{\mathfrak{F}} - Q_0 - Q_1)R_{1-p_0-p_1} = 0$, where $c(P_{\mathfrak{F}} - Q_0 - Q_1)$ is the central support projection of $(P_{\mathfrak{F}} - Q_0 - Q_1)$ in $L(M)'$. By the induction, we choose a sequence $\{Q_n\}_{n=0}^{\infty}$ of mutually orthogonal projections in $L(M)'$ and a sequence $\{p_n\}_{n=0}^{\infty}$ of mutually orthogonal projections in M such that

$$P_{\mathfrak{F}} - \sum_{i=0}^{n-1} Q_i \geqslant Q_n \sim R_{p_n}E_n \leqslant R_{1-\sum_{i=0}^{n-1} p_i} E_n.$$

and $c(P_{\mathfrak{F}} - \sum_{i=0}^{n} Q_i)R_{1-\sum_{i=0}^{\infty} p_i} E_n = 0$ We now put $Q = \sum_{n=0}^{\infty} Q_n$ and $p = \sum_{n=0}^{\infty} p_n$, respectively. Then it is clear that $P_{\mathfrak{F}} \geqslant Q \sim \sum_{n=0}^{\infty} R_{p_n}E_n$. Then we shall prove that $P_{\mathfrak{F}} = Q$. At first, there exists a partial isometry v in $L(M)'$ such that $v^*v = Q$ and $v^*v = \sum_{n=0}^{\infty} R_{p_n}E_n$. Put $V = \sum_{n=-\infty}^{\infty} L_\delta^n v L_\delta^{*n}$. Then it is clear that V is a partial isometry in $\mathfrak{R}$ such that $VV^* = R_p$. If $p = 1$, then V is a coisometry in $\mathfrak{R}$. Since $\mathfrak{R}$ is finite, V is unitary and so we have $P_{\mathfrak{F}} = Q$. Therefore, we may suppose that $p \neq 1$. Let L_z be the central support projection of $P_{\mathfrak{F}} - Q$ in $L(M)'$. Since $L_z R_{1-p} E_n = 0$ for every $n = 0, 1, 2, \ldots$, we have $l_z r_{1-\alpha^n(p)} f(n) = r_z r_{1-\alpha^n(p)} f(n) = 0$ for every $f \in \mathbb{L}^2$ and $n \geqslant 0$. This implies that $z(1 - \alpha^n(p)) = 0$. Then $z \leqslant \alpha^n(p)$ for $n \geqslant 0$. Put $z_\alpha = \bigvee_{n=0}^{\infty} \alpha^{-n}(z) \leqslant p$. Since $\alpha^{-1}(z_\alpha) \leqslant z_\alpha$ and $\varphi(\alpha^{-1}(z_\alpha)) = \varphi(z_\alpha)$, by the finiteness of M, we have $\alpha(z_\alpha) = z_\alpha$. Since $L_{z_\alpha} = R_{z_\alpha}$, L_{z_α} is a central projection of $\mathfrak{R}(= \mathfrak{L}')$. Put $F_1 = \sum_{k=-\infty}^{\infty} L_\delta^k P_{\mathfrak{F}} L_\delta^{*k}$ and $F_2 = \sum_{k=-\infty}^{\infty} L_\delta^k Q L_\delta^{*k}$, respectively. Since $\sum_{k=-\infty}^{\infty} L_\delta^k \left(\sum_{n=0}^{\infty} R_{p_n} E_n \right) L_\delta^{*k} = R_p$, we have

$$L_{z_\alpha} \geqslant L_{z_\alpha} F_1 \geqslant L_{z_\alpha} F_2 \sim L_{z_\alpha} R_p = L_{z_\alpha}.$$

By the finiteness of $\mathfrak{R}$, $L_{z_\alpha} = L_{z_\alpha} F_1 = L_{z_\alpha} F_2$. This implies that $L_{z_\alpha} P_{\mathfrak{F}} = L_{z_\alpha} Q$. On the other hand, since L_z is a central support projection of $P_{\mathfrak{F}} - Q$ in $L(M)'$ and $z_\alpha \geqslant z$, we have $L_{z_\alpha}(P_{\mathfrak{F}} - Q) = P_{\mathfrak{F}} - Q$. This implies that $L_{1-z_\alpha} P_{\mathfrak{F}} = L_{1-z_\alpha} Q$. Therefore, we have

$$P_{\mathfrak{F}} = L_{z_\alpha} P_{\mathfrak{F}} + L_{1-z_\alpha} P_{\mathfrak{F}} = L_{z_\alpha} Q + L_{1-z_\alpha} Q = Q.$$

Then we have the theorem. This completes the proof.

Since $\mathfrak{R}$ is finite, by Theorem 3.2, we have

COROLLARY 3.2. *Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of L^2 such that $P_{\mathfrak{F}} \sim \sum_{n=0}^{\infty} R_{p_n} E_n$ for some family $\{p_n\}_{n=0}^{\infty}$ of mutually orthogonal projections in M. Then $\mathfrak{M}$ is left-full if and only if $\sum_{n=0}^{\infty} p_n = 1$.*

PROPOSITION 3.3. Let $\mathfrak{M}$ be a left-pure, left-invariant subspace of L^2. Then there exist a partial isometry W in $\mathfrak{R}$ and a left-full, left-pure, left-invariant subspace $\mathfrak{N}$ of H^2 such that $\mathfrak{M} = W\mathfrak{N}$.

Proof. By Theorem 3.1, there exists a family $\{q_n\}_{n=0}^{\infty}$ of mutually orthogonal projections in M such that $P_{\mathfrak{F}} = \sum_{n=0}^{\infty} R_{q_n} E_n$. Put $q = \sum_{n=0}^{\infty} q_n$. We consider the family $\{p_n\}_{n=0}^{\infty}$ of mutually orthogonal projection of M such that $p_0 = q_0 + 1 - q$ and $p_n = q_n$ $(n \geqslant 1)$. Put $\mathfrak{N} = \mathsf{H}^2(\{p_n\}_{n=0}^{\infty})$. By Corollary 3.2, $\mathfrak{N}$ is left-full. Since $P_{\mathfrak{F}} \precsim \sum_{n=0}^{\infty} R_{p_n} E_n$, this completes the proof.

Next, we consider the following problem.

PROBLEM 3.4. Let $1 \leqslant m < \infty$. When is every left-pure, left-invariance subspace of L^2 of the form $\sum_{n=0}^{m-1} \oplus V_n \mathsf{H}^2$, where V_n is a partial isometry in $\mathfrak{R}$ such that $\{V_n V_n^*\}_{n=0}^{m-1}$ is a family of mutually projections in $\mathfrak{R}$? Equivalently, when is every left-pure, left-invariant subspace of L^2 of the form $V\mathsf{H}^2(\{p_n\}_{n=0}^{m-1})$ for a partial isometry V in $\mathfrak{R}$ and a family $\{p_n\}_{n=0}^{m-1}$ of mutually orthogonal projections in M?

Let Z be the center of M. Since the restriction $\alpha|Z$ of α to Z is a *-automorphism on Z, by Zorn's lemma, there exists a partition of unity $\{q_m\}_{m=0}^{\infty}$ in Z such that, if $m \geqslant 1$, we have $\alpha^m|Zq_m = \mathrm{id}$ and $\alpha^j|Zq \neq \mathrm{id}$ for every central projection q $(0 \lneqq q \leqslant q_m)$ and every $1 \leqslant j < m$; then we have $\alpha^j|Zq \neq \mathrm{id}$ for every central projection q $(0 \lneqq q \leqslant q_0)$ and every $j \geqslant 1$ (cf. [12, 17.22]). It is clear that $\alpha(q_m) = q_m$ for every $m \geqslant 0$. Since L_{q_m} is a central projection of $\mathfrak{L}$, we can reduce the case that $q_m = 1$ for every $m \geqslant 0$. Further, we remark that if $q_1 = 1$, then α acts trivially on the center Z of M, and if $q_0 = 1$, then α is aperiodic on Z.

THEOREM 3.5. *For every m $(1 \leqslant m < \infty)$, suppose that $q_m = 1$. Then every left-pure, left-invariant subspace of L^2 is of the form $V\mathsf{H}^2(\{p_n\}_{n=0}^{m-1})$, where $\{p_n\}_{n=0}^{m-1}$ is a family of mutually orthogonal projections in M and V is a partial isometry in $\mathfrak{R}$ such that $V^*V = \sum_{n=0}^{m-1} R_{p_n}$.*

Proof. As in the proof of Theorem 3.1, we choose a sequence $\{Q_n\}_{n=0}^{m-1}$ of mutually orthogonal projections in $L(M)'$ and a sequence $\{p_n\}_{n=0}^{m-1}$ of mutually orthogonal projections in M such that, for $n = 0, 1, 2, \ldots, m-1$,

$$P_{\mathfrak{F}} - \sum_{i=0}^{n-1} Q_i \geqslant Q_n \sim R_{p_n}E_n \leqslant R_{1-\sum_{i=0}^{n-1} p_i} E_n.$$

and $c(P_{\mathfrak{F}} - \sum_{i=0}^{n} Q_i)R_{1-\sum_{i=0}^{n} p_i} E_n = 0$ where $c(P_{\mathfrak{F}} - \sum_{i=0}^{n} Q_i)$ is the central support projection of $(P_{\mathfrak{F}} - \sum_{i=0}^{n} Q_i)$ in $L(M)'$. We now put $Q = \sum_{n=0}^{m-1} Q_n$ and $p = \sum_{n=0}^{m-1} p_n$, respectively. Then it is clear that $P_{\mathfrak{F}} \geqslant Q \sim \sum_{n=0}^{m-1} R_{p_n}E_n$. Then we shall prove that $P_{\mathfrak{F}} = Q$. Let L_z be the central support projection of $P_{\mathfrak{F}} - Q$ in $L(M)'$. Then $L_z R_{1-p} E_n = 0$ for $0 \leqslant n \leqslant m-1$, and so $z(1-\alpha^n(p)) = 0$. Put $z_\alpha = \bigvee_{n=0}^{m-1} \alpha^{-n}(z)$. Since $z \leqslant \alpha^n(p)$ for $0 \leqslant n \leqslant m-1$, we have $z_\alpha \leqslant q$. Since $q_m = 1$, α^m acts trivially on the center Z of M. Thus $\alpha^{-1}(z_\alpha) \leqslant z_\alpha$. Since $\varphi(\alpha^{-1}(z_\alpha)) = \varphi(z_\alpha)$, we have $\alpha(z_\alpha) = z_\alpha$ by the finiteness of M. As in the proof of Theorem 3.1, we have $P_{\mathfrak{F}} = Q$. This completes the proof.

Since E_0 is finite in $L(M)'$, there exists the unique faithful normal semifinite $L(Z)$-trace Φ that maps E_0 into I (cf. [11]). Let $\mathfrak{M}$ be a left-pure, invariant subspace of L^2. Let $P_{\mathfrak{F}}$ be the projection onto $\mathfrak{F}$ $(= \mathfrak{M} \ominus L_\delta \mathfrak{M})$. Since $P_{\mathfrak{F}}$ is a finite projection in $L(M)'$, there exists an element f in $Z_+^\wedge$ such that $L_f = \Phi(P_{\mathfrak{F}})$, where $Z_+^\wedge$ is the extend positive part of Z. Then we can define, by induction, a sequence $\{f_i\}_{i=0}^\infty$ of functions in $Z_+^\wedge$ as follows in [11]:

$$f_0 = f \wedge 1, \quad f_k = \left(f - \sum_{n=0}^{k-1} f_n\right) \wedge \left(1 - \sum_{n=0}^{k-1} \alpha^{k-n}(f_n)\right).$$

Then, by the simple calculation, we have $\Phi(R_{p_n}E_n) = L_{f_n}$ for every $n \geqslant 0$.

By Theorem 3.5, we have

PROPOSITION 3.6. *Suppose that $q_m = 1$ for some $1 \leqslant m < \infty$. Let $\mathfrak{M}$ be a left--pure, left-invariant subspace of L^2 with the wandering subspace $\mathfrak{F}$. Then $\Phi(P_{\mathfrak{F}}) \leqslant \leqslant mI$.*

Proof. By the prooof of Theorem 3.5, there exists a family $\{p_n\}_{n=0}^{m-1}$ of mutually orthogonal projections in M such that $P_{\mathfrak{F}} \sim \sum_{n=0}^{m-1} R_{p_n}E_n$. Thus, $\Phi(P_{\mathfrak{F}}) =$

$= \sum_{n=0}^{m-1} \Phi(R_{p_n} E_n) \leqslant \sum_{n=0}^{m-1} \Phi(E_n) = \sum_{n=0}^{m-1} \Phi(R_\delta^n E_0 R_\delta^{*n}) = m\Phi(E_0) = mI$. This completes the proof.

PROPOSITION 3.7. Suppose that $q_m = 1$ for some $m \geqslant 0$.

(1) If $1 \leqslant m < \infty$, then there exists a left-pure, left-invariant subspace $\mathsf{H}^2(\{p_n\}_{n=0}^{m-1})$, such that $\{p_n\}_{n=0}^{m-1}$ is a family of mutually orthogonal projections in M, which cannot write the form $V\mathsf{H}^2(\{p'_n\}_{n=0}^{m-2})$ for any partial isomety V in $\mathfrak{R}$ and any family $\{p'_n\}_{n=0}^{m-2}$ of mutually orthogonal projections in M.

(2) If $m = 0$, then there exists a left-pure, left-invariant subspace $\mathsf{H}^2(\{p_n\}_{n=0}^{\infty})$, such that $\{p_n\}_{n=0}^{\infty}$ is a family of mutually orthogonal projections in M, which cannot write the form $V\mathsf{H}^2(\{p'_n\}_{n=0}^{k})$ for any partial isometry V in $\mathfrak{R}$, any family $\{p'_n\}_{n=0}^{k}$ of mutually orthogoanl projections in M and every $k \geqslant 0$.

Proof. (1) Since $q_m = 1$, there exists a maximal projection p in Z such that $p, \alpha(p), \alpha^2(p), \ldots, \alpha^{m-1}(p)$ are mutually orthogoanl. By the maximality of p, we have $\sum_{n=0}^{m-1} \alpha^k(p) = 1$. Put $p_n = \alpha^{-1}(p)$ for every $0 \leqslant n \leqslant m-1$. We consider a left-pure, left-invariant subspace $\mathsf{H}^2(\{p_n\}_{n=0}^{m-1})$ of H^2. Then the wandering projection $P_{\mathfrak{F}}$ of $\mathsf{H}^2\{p_n\}_{n=0}^{m-1}\})$ is $\sum_{n=0}^{m-1} R_{p_n} E_n$. Since $\Phi(P_{\mathfrak{F}}) = \sum_{n=0}^{m-1} \Phi(R_{\alpha^{-n}(p)} E_n) =$
$= \sum_{n=0}^{m-1} \Phi(R_\delta^n R_p R_\delta^{-n} R_\delta^n E_0 R_\delta^{-n}) = \sum_{n=0}^{m-1} \Phi(R_\delta^n R_p E_0 R_\delta^{-n}) = \sum_{n=0}^{m-1} \Phi(R_p E_0) =$
$= m\Phi(L_p E_0) = mL_{p'}$, by Proposition 3.6, $\mathsf{H}^2(\{p_n\}_{n=0}^{m-1}\})$ cannot write the form $V\mathsf{H}^2(\{p'_n 0\}_{n=0}^{m-2}\})$ for any partial isometry V in $\mathfrak{R}$.

(2) Suppose that $q_0 = 1$. Since α is not trivial on Z, there exists a nonzero projection r_1 in Z such that $\alpha(r_1)r_1 = 0$. Since $\alpha^2|Zr_1 \neq \mathrm{id}$, there exists a nonzero projection r'_2 ($\leqslant r_2$) in Z such that $\alpha^2(r'_2) \neq r'_2$. Putting $r_2 = r'_2(1 - \alpha^2(r'_2))$, $\{\alpha^i(r_2)\}_{i=0}^{2}$ is mutually orthogonal. Repeating this method, there exists a family $\{r_n\}_{n=1}^{\infty}$ of projections in Z such that $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n \geqslant \cdots \to 0$ and $\{\alpha^i(r_n)\}_{i=0}^{n}$ is mutually orthogonal for every $n \geqslant 1$. We put $p_0 = 1 - \sum_{n=0}^{m-1} \alpha^n(r_n)$ and $p_n = \alpha^n(r_n)$ for every $n \geqslant 1$. We define a left-pure, left-full, left-invariant subspace $\mathsf{H}^2(\{p_n\}_{n=0}^{\infty})$ of H^2. Then $\mathsf{H}^2(\{p_n\}_{n=0}^{\infty})$ cannot write the form $V\mathsf{H}^2(\{p_n\}_{n=0}^{k})$ for every $k \geqslant 0$, because $\Phi(P_{\mathfrak{F}})$ is not bounded. This completes the proof.

Therefore, by Theorem 3.5 and Propositions 3.6 and 3.7, we have the following theorems.

THEOREM 3.8. *Suppose that $1 \leqslant m < \infty$. Then the following assertions are equivalent:*

(1) *Every left-pure, left-invariant subspace of* L^2 *is of the form* $\sum_{n=0}^{m-1} \oplus V_n \mathsf{H}^2$, *for some family* $\{V_n\}_{n=0}^{m-1}$ *of partial isometries in* $\mathfrak{N}$ *such that* $\{V_n V_n^*\}_{n=0}^{m-1}$ *is mutually orthogonal. Further, there exists a left-pure, left-invariant subspace of* L^2 *which cannot write the form* $\sum_{n=0}^{m-1} \oplus V_n' \mathsf{H}^2$ *for any family* $\{V_n'\}_{n=0}^{m-2}$ *of partial isometries in* $\mathfrak{N}$ *such that* $\{V_n' V_n^*\}_{n=0}^{m-2}$ *is mutually orthogonal.*

(2) *For every left-pure, left-invariant subspace* $\mathfrak{M}$ *of* L^2, $\Phi(P_{\mathfrak{F}}) \leqslant mI$, *where* $P_{\mathfrak{F}}$ *is the wandering projection of* $\mathfrak{M}$. *Further, there exists a left-pure, left-invariant subspace* $\mathfrak{M}$ *of* L^2 *such that* $\Phi(P_{\mathfrak{F}})$ *is not bound by* $(m-1)I$.

(3) $q_m \neq 0$ *and, for every* $k \geqslant m+1$ *and* $k = 0$, $q_k = 0$.

THEOREM 3.9. *The following assertions are equivalent:*

(1) *There exists a left-pure, left-invariant subspace of* L^2 *which cannot write the form* $\sum_{n=0}^{k} \oplus V_n \mathsf{H}^2$, *for any* k $(0 \leqslant k \leqslant \infty)$, *and for any family* $\{V_n\}_{n=0}^{k}$ *of partial isometries in* $\mathfrak{N}$ *such that* $\{V_n V_n^*\}$ *is mutually orthogonal.*

(2) *There exists a left-pure, left-invariant subspace* $\mathfrak{M}$ *of* L^2 *such that* $\Phi(P_{\mathfrak{F}})$ *is not bounded.*

(3) $q_0 \neq 0$.

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