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ON THE PREDUAL OF DUAL ALGEBRAS

ZHONG-JIN RUAN

1. INTRODUCTION

Let $B(H)$ be the space of bounded linear operators on a Hilbert space H. A *dual algebra* on H is a σ-weakly closed subalgebra of $B(H)$ containing the identity operator 1_H (see details in [3]). It is clear that von Neumann algebras and $H^\infty(D)$, the algebra of all bounded analytic functions on the unit disc D, are dual algebras.

Grothendieck [17] has shown that $L^\infty(X,\mu)$, the commutative von Neumann algebra of all essentially bounded measurable complex functions on an appropiate measure space (X,μ), has a unique predual $L^1(X,\mu)$. Generalizing Grothendieck's result to non-commutative (resp., commutative but non-self adjoint) case, Sakai [30], (resp., Ando [2]) has shown that every von Neumann algebra (resp., $H^\infty(D)$) has a unique predual. The purpose of this paper is to study the predual of certain dual algebras. We note that the operator space structure on preduals of dual algebras plays an important role in this paper.

Given a Hilbert space H, it is known that there is a natural family of norms on the $n\times n$ matrix spaces $M_n(B(H))$ over $B(H)$ by identifying $M_n(B(H))$ with $B(H^n)$. We call this family of norms the *operator matrix norm* on $B(H)$. An (concrete) *operator space* is a linear subspace of $B(H)$, together with the operator matrix norm inherited from $B(H)$. For our convenience, we assume that every operator space E is complete, i.e., $M_n(E)$ is a Banach space for each $n \in \mathbb{N}$.

Given operator spaces E and F, we let $CB(E,F)$ denote the space of all completely bounded maps from E to F with the completely bounded norm. Using the abstract matrix norm characterization for operator spaces in [28], we may regard $CB(E,F)$ as an operator space by identifying $M_n(CB(E,F))$ with $CB(E,M_n(F))$

(see [8]). Let E be an operator space. Then $E^* = CB(E, C)$, the dual space of E, with the operator matrix norm obtained by identifying $M_n(E^*)$ with $CB(E, M_n)$ is again an operator space, which is called the *operator dual* of E (cf. [5] and [10]). An operator space F is called a *dual operator space* if there is an operator space E such that F is completely isometric to E^*. The operator space E is called an *operator predual* of F. Given a dual operator space F, the operator predual(s) of F can be completely isometrically (canonically) embedded into F^*, and thus can be regarded as norm closed subspace(s) of F^*. We say that a dual operator space F has a (strongly) unique operator predual if all of its operator preduals coincide in F^*, i.e., the operator preduals of F have a unique position in F^*.

It is clear that every σ-weakly closed subspace F of $B(H)$ is a dual operator space with a canonical operator predual $B(H)_*/F_\perp$, where $B(H)_*$ is the space of all bounded normal linear functionals on $B(H)$, and $F_\perp$ is the preannihilator of F in $B(H)_*$. We use the notation F_{H_*}, or simply F_* if there is no confusion, to indicate this canonical operator predual $B(H)_*/F_\perp$. On the other hand, if F is a dual operator space with a given operator predual E, there is a Hilbert space H such that F is weak* homeomorphically completely isometric to a σ-weakly closed subspace $\overline{F}$ of $B(H)$ (see [8], [11] and also [4]). In this case, E is completely isometric to $\overline{F}_{H*}$ and we may identify F with the σ-weakly closed subspace $\overline{F}$ in $B(H)$.

In general, dual operator spaces and dual algebras might have more than one operator preduals. To see this, we note that the dual Banach space $\ell^1(N)$ possesses a natural operator space structure by regarding it as the operator dual of the commutative C^*-algebra $C_0(\mathbb{N})$. The operator matrix norms on the preduals of $\ell^1(N)$ can be obtained from that on their second dual $\ell^\infty(N)$. D. Westwood pointed out to the author that $\ell^1(N)$ can be regarded as a σ-weakly closed off-diagonal subspace of a dual algebra. This can easily be derived from the following fact. If E is a σ-weakly closed subspace of $B(H)$, we get a dual subalgebra $C \oplus E$ of $M_2(B(H))$ by letting $C = C(1_H \oplus 1_H)$ and by putting E on the upper right corner in the 2×2 matrix space $M_2(B(H))$. This shows that for general dual algebras, we might have more than one operator preduals.

In section 2, we study the unique operator predual of certain dual algebras. Motivated by Ando [2] and Sakai [30], we prove in Theorem 2.5 that if A is a dual algebra such that the subalgebra of compact operators in A is σ-weakly dense in A, then A has a unique operator predual. Therefore nest algebras, atomic CSL algebras, and more generally, completely distributive CSL algebras have unique operator preduals (Theorem 2.7).

In section 3, we study the operator projective tensor product of operator preduals of dual operator spaces and dual algebras. Given dual operator spaces E and F

on Hilbert spaces H and K, respectively, we denote by $E\overline{\otimes}F$ the σ-weak closure of $E \otimes F$ in $B(H \otimes K)$, where $H \otimes K$ is the Hilbert space tensor product of H and K. If A and B are dual algebras (resp., von Neumann algebras) on Hilbert spaces H and K, respectively, then $A\overline{\otimes}B$ is a dual algebra (resp., von Neumann algebra) on $H \otimes K$. We show in Theorem 3.4 that there is a complete quotient map θ from $E_* \otimes^\wedge F_*$, the operator projective tensor product of E_* and F_*, onto $(E\overline{\otimes}F)_*$. Furthermore, we have the complete isometry $E_* \otimes^\wedge F_* \cong (E\overline{\otimes}F)_*$ if and only if $E\overline{\otimes}F = \mathcal{F}(E, F; B(H), B(K))$, where the latter is the Fubini product of E and F with respect to $B(H)$ and $B(K)$ (Corollary 3.5). Finally we show in Theorem 3.8 that if A (resp., B) is a dual algebra such that the subalgebra of finite rank (resp., compact) operators in A (resp., in B) is σ-weakly dense, then $A\overline{\otimes}B$ has a unique operator predual, which is completely isometric to $A_* \otimes^\wedge B_*$. It follows that if $A_i =$ $=$ Alg$(\mathcal{L}_i)$ $(i = 1, \ldots, n)$ are completely distributive CSL algebras, the dual algebra Alg$(\mathcal{L}_1)\overline{\otimes}\cdots\overline{\otimes}Alg(\mathcal{L}_n)$ has a unique operator predual which is completely isometric to Alg$(\mathcal{L}_1)_* \otimes^\wedge \cdots \otimes^\wedge$ Alg$(\mathcal{L}_n)_*$.

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2. THE OPERATOR PREDUAL OF DUAL ALGEBRAS

Given a dual operator space F, the operator matrix norms on its operator dual and operator predual(s) are uniquely determined by that on F. Thus to show that F has a unique operator predual, it sufficies to show that, as Banach spaces, all of its operator preduals coincide in F^*. Owing to this fact, we, sometimes, only need to consider the norm structure on the operator spaces in this section. First, let us recall the definition of L-summands (resp., M-summands) and M-ideals for Banach spaces introduced by Alfsen and Effros [1].

A closed subspace E_1 of a Banach space E is called an *L-summand* (resp., *M-summand*) in E if there is another norm closed subspace E_2 of E such that

$$E = E_1 \oplus E_2$$

and

$$\|x_1 + x_2\| = \|x_1\| + \|x_2\|$$

(resp.,

$$\|x_1 + x_2\| = \max\{\|x_1\|, \|x_2\|\})$$

for all $x_i \in E_i$. We write $E = E_1 \oplus_L E_2$ (resp., $E = E_1 \oplus_M E_2$). A closed subspace E_1 of E is called an *M-ideal* in E if $E_1^\perp$, the annihilator of E_1, is an L-summand in

E^*. Let E_1 be an L-summand (resp., M-summand) in E with the complement E_2. There is a canonical bounded linear map P from E onto E_1 defined by

$$P(x_1 + x_2) = x_1$$

for all $x_i \in E_i$. The linear map P is contractive with $P^2 = P$, and is called the *L-projection* (resp., *M-projection*) from E onto E_1. In this case, $I - P$ is an L-projection (resp., M-projection) from E onto E_2.

It is known that M-ideals in C^*-algebras are just norm closed two-sided ideals (see [1] and [31]). In particular, $\mathcal{K}(H)$, the space of compact operators on a Hilbert space H, is an M-ideal in $B(H)$. The space $B(H)_*$ is an L-summand in $B(H)^*$ with the complement $B(H)_s$, the space of all singular linear functionals on $B(H)$. It has been shown in [9] that for a unital (non-self adjoint) operator algebra A, a closed subspace J is an M-ideal in A if and only if J is a two-sided ideal of A with a contractive approximate identity. If J is separable, the standard argument shows that the contractive approximate identity can be taken to be a sequence. This is, indeed, a non-self adjoint generalization of C^*-algebra results since every two-sided ideal of a C^*-algebra has a contractive approximate identity. If the operator algebra A is a dual algebra and the subspace J is σ-weakly closed, we have the following proposition which follows immediately from the proof of [9] Theorem 2.2.

PROPOSITION 2.1. *Let A be a dual algebra on H and let J be a σ-weakly closed subspace of A. Then J is an M-summand in A if and only if there is a central projection p in A such that $J = pA$.*

LEMMA 2.2. *Let A be a dual algebra, E an operator predual of A and π the complete contraction from A^{**} onto A induced by the canonical embedding $i : E \to A^*$. Then π is a completely contractive homomorphism from A^{**} onto A such that*

$$\pi(axb) = a\pi(x)b$$

for all $a, b \in A$ and $x \in A^{**}$, and the kernel of π, denoted by $\ker \pi$, is a $\sigma(A^{**}, A^*)$-closed two-sided ideal of A^{**}.

Proof. Let $I(A)$ be the operator injective envelope of A (cf. [29] and [18]). Then $I(A)$ is a unital injective C^*-algebra containing A as a unital subalgebra. By taking the second duals, we may regard A^{**} as a unital subalgebra of $I(A)^{**}$. Hence π can be extended to a complete contraction $\overline{\pi}$ from $I(A)^{**}$ into $I(A)$. Since $\overline{\pi}\,|_A = id\,|_A$ and $I(A)$ is the injective envelope of A, we must have $\overline{\pi}\,|_{I(A)} = id\,|_{I(A)}$. This shows that $\overline{\pi}$ is a projection of norm one from the C^*-algebra $I(A)^{**}$ onto C^*-subalgebra $I(A)$. Hence we have

$$\overline{\pi}(axb) = a\overline{\pi}(x)b$$

for all $a, b \in I(A)$ and $x \in I(A)^{**}$. This implies that

$$\pi(axb) = a\pi(x)b$$

for all $a, b \in A$ and $x \in A^{**}$.

Given $x, y \in A^{**}$, there is a net (x_α) in A which converges to x in $(A^{**}, \sigma(A^{**}, A^*))$. We conclude that

$$\begin{aligned}\langle \pi(xy), i(\varphi)\rangle = \langle xy, i(\varphi)\rangle = \lim\langle x_\alpha y, i(\varphi)\rangle = \\ = \lim\langle \pi(x_\alpha y), i(\varphi)\rangle = \lim\langle x_\alpha \pi(y), i(\varphi)\rangle = \\ = \langle x\pi(y), i(\varphi)\rangle = \langle \pi(x\pi(y)), i(\varphi)\rangle = \langle \pi(x)\pi(y), i(\varphi)\rangle\end{aligned}$$

for all $\varphi \in E$. This shows that π is a homomorphism from A^{**} onto A, and thus $\ker\pi = E^\perp$ is a $\sigma(A^{**}, A^*)$-closed two-sided ideal in A^{**}. ■

DEFINITION 2.3. A dual algebra A on H is called *compactly dense*, or *local* as in [14], (resp., *finitely dense*) if $\mathcal{K}(A) = A \cap \mathcal{K}(H)$ (resp., $\mathcal{F}(A) = A \cap \mathcal{F}(H)$) is σ-weakly dense in A, where $\mathcal{F}(H)$ denote the algebra of finite rank operators on H. A dual algebra A is called *rank-one dense* if $\mathcal{R}_1(A)$, the linear span of rank-one operators in A, is σ-weakly dense in A.

REMARK 2.4. If A is a compactly dense dual algebra on H, then $\mathcal{K}(A)$ is an M-ideal in A (cf. [6] and [7]). Hence we may identify $\mathcal{K}(A)^*$ as a subspace of A^* such that

$$A^* \cong \mathcal{K}(A)^* \oplus_L \mathcal{K}(A)^\perp.$$

On the other hand, if we let Q be the L-projection from $B(H)^*$ onto $B(H)_*$ with the kernel $B(H)_s$ then Q maps $A^\perp$ into $A^\perp$ and thus we have

$$A^\perp = A_n^\perp \oplus_L A_s^\perp,$$

where $A_n^\perp = Q(A^\perp)$ and $A_s^\perp = (I-Q)(A^\perp)$ (see [6] and [7]). Since $A_n^\perp = A^\perp \cap B(H)_*$ and $A_s^\perp = A^\perp \cap B(H)_s$, it is a simple matter to check that

$$A^* = A_* \oplus_L A_s,$$

where $A_* = B(H)_*/A_n^\perp$ and $A_s = B(H)_s/A_s^\perp$. Easy calculation shows that $A_s = \mathcal{K}(A)^\perp$, and thus we have that

$$A_* \cong A^*/\mathcal{K}(A)^\perp \cong \mathcal{K}(A)^*.$$

So compactly dense dual algebras A are the second duals of $\mathcal{K}(A)$, and thus the second operator duals of $\mathcal{K}(A)$.

THEOREM 2.5. *Every compactly dense dual algebra has a (strongly) unique operator predual.*

Proof. Since

$$A^* = A_* \oplus_L A_s,$$

we have

$$A^{**} = A_*^{\perp} \oplus_M A_s^{\perp}$$

by Remark 2.4 and Alfsen and Effros [1], Part II, Proposition 2.5. It follows from Proposition 2.1 that there is a central projection p in A^{**} such that $pA^{**} = A_*^{\perp}$ and $(1-p)A^{**} = A_s^{\perp} = \mathcal{K}(A)^{\perp\perp}$. This implies that

$$A_* = (A_*^{\perp})_{\perp} = (pA^{**})_{\perp} = (1-p)A^*$$

and

$$A_s = (A_s^{\perp})_{\perp} = ((1-p)A^{**})_{\perp} = pA^*.$$

Since $pf \in A_s$ for every $f \in A^*$, it is easy to check that $px = 0$ for all $x \in \mathcal{K}(A)$.

Suppose that E is another operator predual of A. Then E can be regarded as a subspace of A^* by the standard embedding i. This induces, by Lemma 2.2, a completely contractive homomorphism π from A^{**} onto $E^* = A$ such that $\ker(\pi)$ is a $\sigma(A^{**}, A^*)$-weakly closed two-sided ideal of A^{**}.

We claim that $\pi(p) = 0$. It is clear that $\pi(p)$ is a central projection of A such that

$$\pi(p)x = \pi(px) = 0$$

for all $x \in \mathcal{K}(A)$. Thus for every $\varphi \in A_*$, we have

$$\langle \pi(p), \varphi \rangle = \lim \langle x_\alpha, \pi(p)\varphi \rangle = \lim \langle x_\alpha \pi(p), \varphi \rangle = 0$$

where $x_\alpha \in \mathcal{K}(A)$ converges to $\pi(p)$ in the $\sigma(A, A_*)$-topology. This shows that p and thus $(A_*)^{\perp} = pA^{**}$ are contained in $\ker(\pi) = E^{\perp}$. Hence E is contained in A_* and thus we must have $E = A_*$ by the Hahn-Banach Theorem. ∎

REMARK 2.6. After the first draft of this paper was completed and circulated, we found that for dual algebras on separable Hilbert spaces Theorem 2.5 can be proved by using the following recent results of G. Godefroy [15] Theorem V.3 and G. Godefroy and D. Li [16] Proposition 5. We thank A. Arias for pointing this out to the author.

Let X be a Banach space. A sequence (x_n) in X is called a *weakly unconditionally convergent series* (w.u.c. series) if for every $\varphi \in X^*$ we have

$$\sum_{n=0}^{k} |\varphi(x_n)| < \infty.$$

A Banach space Y is said to have property (X) if the following holds: $z \in Y^{**}$ belongs to Y if and only if for every w.u.c. series (y_n) in Y^*, we have

$$z\left(\sum\nolimits_* y_n\right) = \sum z(y_n)$$

where $\left(\sum_* y_n\right)$ denotes the limit of the sequence $\left\{s_k = \sum_{n=1}^{k} y_n\right\}$ in $(Y^*, \sigma(Y^*, Y))$.

G. Godefroy [15] Theorem V.3. *Let Y be a Banach space. If Y has property (X), then Y is the (strongly) unique isometric predual of Y^*.*

G. Godefroy and D. Li [16] Proposition 5. *If a separable Banach space X is an M-ideal in its second bidual, then X^* has the property (X).*

Finally let us recall the definitions for certain CSL algebras. The reader is referred to Davidson [6] for the details. A *subspace lattice* $\mathcal{L}$ on a separable Hilbert space H is a collection of projections on H that is strongly closed, contains 0 and 1_H, and is a lattice under the operations $\vee$ (closed linear span) and $\wedge$ (intersection). We call $\mathcal{L}$ a *commutative subspace lattice* (CSL) if the elements of $\mathcal{L}$ commute. A *nest* is a totally ordered subspace lattice. Hence every nest is a CSL. Given a subspace lattice $\mathcal{L}$, we let $\mathcal{L}''$ denote the double commutant of $\mathcal{L}$. A CSL is called *atomic* if the von Neumann algebra $\mathcal{L}''$ is atomic. A subspace lattice is called a *completely distributive subspace lattice* (CDSL) if it has a distributive law for arbitrary sets, i.e. for any non-empty index sets I and J, we have

$$\wedge_{\alpha\in I}(\vee_{\beta\in J} e_{\alpha,\beta}) = \vee_{\varphi\in J^I}(\wedge_{\alpha\in I} e_{\alpha,\varphi(\alpha)})$$

and its dual holds, where $e_{\alpha,\beta} \in \mathcal{L}$ for all $\alpha \in I$, $\beta \in J$ and J^I is the set of all functions from I to J. The reader may find further details in [25]. We say that a subspace lattice is a *completely distributive* CSL (CDCSL) if it is completely distributive and commutative. It is known that every nest (resp., every atomic CSL) is a CDCSL.

If $\mathcal{L}$ is a subspace lattice on H, we let $\mathrm{Alg}(\mathcal{L})$ denote the set of operators in $B(H)$ that leave the elements in $\mathcal{L}$ invariant. It is easy to see that for every subspace lattice $\mathcal{L}$, $\mathrm{Alg}(\mathcal{L})$ is a dual algebra on H. We call $\mathrm{Alg}(\mathcal{L})$ a *CSL algebra* (resp., *nest algebra, atomic CSL algebra, CDSL algebra* and *CDCSL algebra*) if the corresponding subspace lattice $\mathcal{L}$ is a CSL (resp., nest, atomic CSL, CDSL and CDCSL).

Since every finitely dense (resp., rank-one dense) dual algebra must be compactly dense, every such dual algebra has a unique operator predual. It is clear that every rank-one dense dual algebra is finitely dense. Conversely, there are examples of finitely dense dual algebras which are not rank-one dense (see [24] and [22] example 2.4). But if $\mathcal{L}$ is a CSL, the CSL algebra $A = \mathrm{Alg}(\mathcal{L})$ is finitely dense if and only if A is rank-one dense (see [20] and [6] Theorem 23.16) if and only if A is a CDCSL algebra (see [26] and [27]). Since nest algebras and atomic CSL algebras are CDCSL algebras, we have

THEOREM 2.7. *Nest algebras, atomic CSL algebras, and more generally, CDCSL algebras have (strongly) unique operator preduals.*

REMARK 2.8. It follows from Theorem 2.7 that every isometric linear isomorphism between CDCSL algebras must be σ-weakly continuous. Hence every unital isometric isomorphism between nest algebras is σ-weakly continuous and thus completely isometric by [6] Corollary 20.17.

3. THE TENSOR PRODUCT OF OPERATOR PREDUALS

In this section, we study the tensor product of operator preduals of dual operator spaces and dual algebras. First let us recall the definition of the operator projective tensor product of operator spaces, which was discovered independently in [5] and [10] (see also [11]).

Let E and F be two operator spaces. Given $x = [x_{ij}] \in M_p(E)$ and $y = [y_{k,l}] \in$ $\in M_q(F)$, we define the $pq \times pq$ matrix $x \otimes y \in M_{pq}(E \otimes F)$ by

$$(x \otimes y)_{(i,k),(j,l)} = x_{i,j} \otimes y_{k,l},$$

where i, k and (i, k) indicate the row indices, and j, l and (j, l) indicate the column indices. The *operator projective tensor norm* $\|\ \|_\wedge$ *on* $E \otimes F$ is defined as follows. Given $u \in M_n(E \otimes F)$, we let

$$\|u\|_\wedge = \inf\{\|\alpha\|\,\|x\|\,\|y\|\,\|\beta\|\}$$

where the infimum is taken over all possible representations $u = \alpha(x \otimes y)\beta$ with $\alpha \in M_{n,pq}$, $x \in M_p(E)$, $y \in M_q(F)$, and $\beta \in M_{pq,n}$ for any $p, q \in \mathbb{N}$. By using the abstract L^∞-matrix norm characterization for operator spaces in [28] Theorem 3.1, it is easy to check that this gives an operator space structure on $E \otimes F$. We let $E \otimes_\wedge F$ denote the algebraic tensor product of E and F with the operator projective tensor norm. We let $E \otimes^\wedge F$ denote its completion and call $E \otimes^\wedge F$ the *operator projective tensor product* of E and F.

The operator projective tensor product is commutative and for any given operator spaces E and F we have the complete isometries (cf. [5] and [10])

$$(E \otimes^\wedge F)^* \cong CB(E, F^*) \cong CB(F, E^*).$$

A *complete quotient* map from one operator space E onto another operator space F is a complete contraction which maps the open unit balls of $M_n(E)$ onto those of

$M_n(F)$ for all $n \in \mathbb{N}$. If π is a complete quotient map from E onto F, the adjoint map π^* of π gives a complete injection from F^* into E^*. It is known by Effros and Ruan [12] that the operator projective tensor product preserves complete quotient maps in the following sense.

PROPOSITION 3.1. *Let E_0 and F_0 be closed subspaces of operator spaces E and F. Then the corresponding map*

$$E \otimes^\wedge F \to E/E_0 \otimes^\wedge F/F_0$$

is a complete quotient map.

Another remarkable property of the operator projective tensor product is the following result, which is a special case of Effros and Ruan [11] Theorem 3.2.

PROPOSITION 3.2. *Let H and K be Hilbert spaces. We have the complete isometry*

$$B(H)_* \otimes^\wedge B(K)_* \cong (B(H)\overline{\otimes}B(K))_*.$$

In fact this result can be proved directly by using [12] Corollary 4.4. To see this, we note that

$$B(H)\overline{\otimes}B(K) = B(H \otimes K)$$

and thus we have

$$\begin{aligned} B(H)_* \otimes^\wedge B(K)_* &\cong H_r^* \otimes^\wedge H_c \otimes^\wedge K_r^* \otimes^\wedge K_c \cong \\ &\cong (H \otimes K)_r^* \otimes^\wedge (H \otimes K)_c \cong \\ &\cong B(H \otimes K)_* = (B(H)\overline{\otimes}B(K))_*. \end{aligned}$$

We need the notion of slice maps, which was first introduced by Tomiyama [32] to study the tensor product of von Neumann algebras. The slice maps for dual operator spaces, i.e. σ-weakly closed subspaces of operators on Hilbert spaces, were studied by Jon Kraus [21], [22] and [23]. Let us recall these definitions.

If $\varphi \in B(H)_*$, there is a unique σ-weakly continuous linear map $R_\varphi : B(H)\overline{\otimes} \overline{\otimes}B(K) \to B(K)$ defined by

$$\langle R_\varphi(u), \psi \rangle = \langle u, \varphi \otimes \psi \rangle$$

for all $\psi \in B(K)_*$. Similarly, for each $\psi \in B(K)_*$, there is a unique σ-weakly continuous map $L_\psi : B(H)\overline{\otimes}B(K) \to B(H)$ defined by

$$\langle L_\psi(u), \varphi \rangle = \langle u, \varphi \otimes \psi \rangle$$

for all $\varphi \in B(H)_*$.

Let E and F be dual operator spaces with operator preduals E_* and F_*. If we identify E and F with the σ-weakly closed subspaces of $B(H)$ and $B(K)$ for some Hilbert space H and K, respectively, the Fubini product of E and F with respect to $B(H)$ and $B(K)$ is defined by

$$\mathcal{F}(E, F, B(H), B(K)) = \{u \in B(H \otimes K); R_\varphi(u) \in F \text{ and } L_\psi(u) \in E\}$$

for all $\varphi \in B(H)_*$ and $\psi \in B(K)_*$. It is easy to see that $\mathcal{F}(E, F; B(H), B(K))$ is a σ-weakly closed subspace of $B(H \otimes K)$, which is the closure of $E \otimes F$ with respect to the topology determined by the algebraic tensor product $B(H)_* \otimes B(K)_*$.

PROPOSITION 3.3. *Let E and F be dual operator spaces with operator preduals E_* and F_*. If we identify E and F with the σ-weakly closed subspaces of $B(H)$ and $B(K)$, respectively, we have the complete isometry*

$$(E_* \otimes^\wedge F_*)^* \cong \mathcal{F}(E, F; B(H), B(K)).$$

Proof. By the hypothesis, we may assume that the dual operator spaces E and F have the canonical operator preduals $E_* = E_{H*}$ and $F_* = F_{K*}$, respectively. Thus $E_* \otimes^\wedge F_*$ is a complete quotient of $B(H \otimes K)_* = B(H)_* \otimes^\wedge B(K)_*$ by Proposition 3.1 and Proposition 3.2. It follows that $(E_* \otimes^\wedge F_*)^*$ can be regarded as a σ-weakly closed subspace of $B(H \otimes K)$, which contains the algebraic tensor product $E \otimes F$. Since we have the complete isometries

$$(E_* \otimes^\wedge F_*)^* \cong CB(E_*, F) \cong CB(F_*, E),$$

for every $u \in (E_* \otimes^\wedge F_*)^*$, it is easy to see that $R_\psi(u) \in F$ and $L_\psi(u) \in E$ for all $\varphi \in B(H)_*$ and $\psi \in B(K)_*$. Hence $(E_* \otimes^\wedge F_*)^*$ is contained in $\mathcal{F}(E, F; B(H), B(K))$.

On the other hand, we show that $\mathcal{F}(E, F; B(H), B(K)) \subseteq (E_* \otimes^\wedge F_*)^*$. To this end, given an element $u \in \mathcal{F}(E, F; B(H), B(K))$, it follows from Proposition 3.2 that $u \in CB(B(H)_*, F)$. We claim that $R_\varphi(u) = 0$ for all $\varphi \in E_\perp$. To see this, we have

$$\langle R_\varphi(u), \psi\rangle = \langle u, \varphi \otimes \psi\rangle = \\ = \langle L_\psi(u), \varphi\rangle = 0$$

since $L_\psi(u) \in E$ for all $\psi \in B(K)_*$. This shows that $E_\perp \subseteq \ker u$. Since $E_* = = B(H)_*/E_\perp$, u induces a map $\overline{u}$ from E_* into F such that $\|\overline{u}\|_{cb} = \|u\|_{cb}$. Hence u can be regarded as an element in $CB(E_*, F) = (E_* \otimes^\wedge F_*)^*$. ∎

Proposition 3.3 shows that the Fubini product of dual operator spaces E and F is, in fact, independent to the choice of Hilbert spaces H and K. This fact has also been observed by Jon Kraus [21] Remark 1.2. For our convenience, we may simply write the Fubini product of E and F as $\mathcal{F}(E, F)$.

THEOREM 3.4. *Let E_i be σ-weakly closed subspaces of $B(H_i)$ for $i = 1, 2$. The map $\theta : E_{1*} \otimes^\wedge E_{2*} \to (E_1 \overline{\otimes} E_2)_*$ defined by*

$$\theta(\omega_1, \omega_2) = \omega_1 \otimes \omega_2$$

is a complete quotient map.

Proof. First we prove that θ is a well defined complete contraction. Given $\omega_i \in E_{i*}$ with $\|\omega_i\| < 1$, there are extensions $\tau_i \in B(H_i)_*$ of ω_i such that $\|\tau_i\| < 1$. Since $\tau_1 \otimes \tau_2 \in B(H_1)_* \otimes^\wedge B(H_2)_* = B(H_1 \otimes H_2)_*$ by Proposition 3.2, we have

$$\omega_1 \otimes \omega_2 = \tau_1 \otimes \tau_2 \left.\right|_{E_1 \overline{\otimes} E_2} \in (E_1 \overline{\otimes} E_2)_*.$$

If $\overline{\tau}_i$ are any other extensions of ω_i, we also have

$$\omega_1 \otimes \omega_2 = \overline{\tau}_1 \otimes \overline{\tau}_2 \left.\right|_{E_1 \overline{\otimes} E_2} .$$

Hence θ is a well defined map from $E_{1*} \otimes_\wedge E_{2*}$ into $(E_1 \overline{\otimes} E_2)_*$.

Given $\omega_1 \in M_p(E_{1*})$ and $\omega_2 \in M_q(E_{2*})$ with $\|\omega_i\| < 1$, there are extensions τ_i of ω_i with $\|\tau_i\| < 1$, respectively, such that $\tau_1 \otimes \tau_2$ is a contraction in $M_{pq}(B(H_1 \otimes H_2)_*)$. Hence $\theta_{pq}(\omega_1, \omega_2) = \omega_1 \otimes \omega_2 = \tau_1 \otimes \tau_2 \left.\right|_{E_1 \overline{\otimes} E_2}$ is a contraction in $M_{pq}((E_1 \overline{\otimes} E_2)_*)$. This shows that θ is a complete contraction from $E_{1*} \otimes_\wedge E_{2*}$ into $(E_1 \overline{\otimes} E_2)_*$. Therefore θ can be extended to a complete contraction from $E_{1*} \otimes^\wedge E_{2*}$ into $(E_1 \overline{\otimes} E_2)_*$.

Finally we show that θ maps the open unit balls of $M_n(E_{1*} \otimes^\wedge E_{2*})$ onto those of $M_n((E_1 \overline{\otimes} E_2)_*)$. For any $\omega \in M_n((E_1 \overline{\otimes} E_2)_*)$ with $\|\omega\| < 1$, there is an extension $\tau \in M_n(B(H_1 \otimes H_2)_*)$ such that $\|\tau\| < 1$. Then we can represent τ as $\tau = \alpha(\tau_1 \otimes \otimes \tau_2)\beta \in M_n(B(H_1)_* \otimes^\wedge B(H_2)_*)$ for some $\alpha \in M_{n,\infty^2}$, $\tau_i \in K_\infty(B(H_i)_*)$ and $\beta \in M_{\infty^2,n}$ with norms less than one (cf. Effros and Ruan [11]). Let $\omega_i = \tau_i \left.\right|_{E_i}$. We have that $\overline{\omega} = \alpha(\omega_1 \otimes \omega_2)\beta \in M_n(E_{1*} \otimes^\wedge E_{2*})$ with $\|\overline{\omega}\| < 1$ such that $\theta_n(\overline{\omega}) = \omega$. Hence θ is a complete quotient map from $E_{1*} \otimes^\wedge E_{2*}$ onto $(E_1 \overline{\otimes} E_2)_*$. ∎

We note that given σ-weakly closed subspaces E and F of $B(H)$ and $B(K)$, respectively, we have

$$E \overline{\otimes} F \subseteq \mathcal{F}(E, F).$$

This is because that $\mathcal{F}(E, F)$ is a σ-weakly closed subspace of $B(H \otimes K)$ containing $E \otimes F$, thus it contains $E \overline{\otimes} F$. The equality is always true when both E and F are von Neumann algebras (cf. Tomiyama [32]), but if may fail in general (cf. Kraus [23]). Following the definition in [21], a dual operator space E has *Property S_σ* if we have

$$E \overline{\otimes} F = \mathcal{F}(E, F)$$

for every dual operator space F.

COROLLARY 3.5. *Let E and F be dual operator spaces with operator preduals E_* and F_*. Then the following are equivalent where we identify E and F with the σ-weakly closed subspaces of $B(H)$ and $B(K)$, respectively:*

(1) *We have the complete isometry $\mathcal{F}(E, F) \cong E\overline{\otimes}F$*

(2) *We have the complete isometry $E_* \otimes^\wedge F_* \cong (E\overline{\otimes}F)_*$.*

Proof. (2) implies (1) followed easily from Proposition 3.3. (1) implies (2) by Theorem 3.4 and the Hahn-Banach Theorem. ■

The following result is the Theorem 3.2 in [11], which can be regarded as a consequence of Corollary 3.5 and the Tomiyama's Fubini Theorem for von Neumann algebras.

COROLLARY 3.6. *Let R and S be von Neumann algebras with the unique operator preduals R_* and S_*, respectively. We have the complete isometry*

$$R_* \otimes^\wedge S_* \cong (R\overline{\otimes}S)_*.$$

COROLLARY 3.7. *Let E be a dual operator space with an operator predual E_*. Then the following are equivalent*

(1) *E has Property S_σ*

(2) *We have the complete isometry $E_* \otimes^\wedge F_* \cong (E\overline{\otimes}F)_*$ for every dual operator space F.*

THEOREM 3.8. *Let A and B be finitely dense dual algebras (resp., let B be a compactly dense dual algebra). Then $A\overline{\otimes}B$ is finitely dense (resp., compactly dense) with a unique operator predual, which is completely isometric to $A_* \otimes^\wedge B_*$.*

Proof. It has been discussed in section 2 that if B is a compactly dense dual algebra, then B is the second operator dual of $\mathcal{K}(B)$. Hence the unit balls of $M_n(\mathcal{K}(B))$ are σ-weakly dense in those of $M_n(B)$. Similarly, if B is finitely dense, it is easy to show that the unit balls of $M_n(\mathcal{F}(B))$ are norm dense in those of $M_n(\mathcal{K}(B))$ and thus are σ-weakly dense in those of $M_n(B)$. Hence $A\overline{\otimes}B$ must be a finitely dense (resp., compactly dense) dual algebra if A and B are finitely dense (resp., B is compactly dense). It follows from Theorem 2.5 that A, B and $A\overline{\otimes}B$ have unique operator preduals. Finally since A has Property S_σ by [22] Theorem 2.1, the operator predual $(A\overline{\otimes}B)_*$ is completely isometric to $A_* \otimes^\wedge B_*$. ■

REMARK 3.9. Theorem 3.8 can be generalized to the multi-tensor product case. If A_i $(i = 1, \ldots, n)$ are finitely dense dual algebras, then $A_1\overline{\otimes}\cdots\overline{\otimes}A_n$ is finitely dense and thus has a unique operator predual, which is completely isometric to $A_{1*} \otimes^\wedge \cdots \cdots \otimes^\wedge A_{n*}$. Especially if $A_i = \mathrm{Alg}(\mathcal{L}_i)$ $(i = 1, \ldots, n)$ are CDCSL algebras, it is known by [19] that the dual algebra $A_1 \overline{\otimes} \cdots \overline{\otimes} A_n = \mathrm{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ is again a CDCSL algebra. Since each A_i is finitely dense, the dual algebra $A_1 \overline{\otimes} \cdots \overline{\otimes} A_n$ has a unique operator predual, which is completely isometric to $A_{1*} \otimes^\wedge \cdots \otimes^\wedge A_{n*}$.

REMARK 3.10. There are examples of CSL algebras which contain no non-zero compact operators (cf. [6] Example 23.14). At this time, we do not know if CSL algebras, or more generally, reflexive algebras have unique operator preduals or not. We also do not know if compactly dense CSL algebras have Property S_σ or not.

Added in proofs: After the first draft was completed and a number of preprints were circulated, we received a preprint from David Blecher entitled: Tensor products of operator spaces II. The proof of Theorem 2.5 in Blecher's preprint coincides with our Proposition 3.2, Proposition 3.3 and Corollary 3.6.

We also note that in a recent paper (January 1992) of Edward G. Effros, Jon Kraus and the author entitled: On two quantized tensor products, we have shown many other equivalent conditions related to those in Corollary 3.7.

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ZHONG-JIN RUAN
Department of Mathematics,
University of Illinois,
Urbana, IL 61801
U.S.A.

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