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A COMPLETE ISOMORPHISM INVARIANT FOR A CLASS OF TRIANGULAR UHF ALGEBRAS

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1. INTRODUCTION

Let $\mathfrak{A}$ be a unital AF algebra and $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ an increasing sequence of unital finite dimensional subalgebras of $\mathfrak{A}$ such that $\bigcup_n \mathfrak{A}_n$ is dense in $\mathfrak{A}$. Suppose for each n we have a maximal abelian self-adjoint subalgebras (masa) $\mathcal{D}_n$ of $\mathfrak{A}_n$ such that $\mathcal{D}_n \subsetneq \mathcal{D}_{n+1}$. Then $\mathcal{D} = \overline{\bigcup_n \mathcal{D}_n}$ is a masa in $\mathfrak{A}$. A norm closed subalgebra $\mathcal{T}$ of $\mathfrak{A}$ is said to be triangular with diagonal $\mathcal{D}$ if $\mathcal{T} \cap \mathcal{T}^* = \mathcal{D}$. This class of triangular subalgebras of AF algebra (TAF algebras [6]) has been the subject of many recent studies, e.g. [1], [4–15]. A major question in this area is to classify such algebras up to isometric isomorphism. Complete isomorphism invariants for TAF algebras have been given by Power [11] and Ventura [14] and are also described implicitly in the diagonal extension theorem in Peters and Wagner ([7], 1.10) and the subgroupoid of Muhly and Solel [4]. These invariants have been used quite successfully in the study of certain types of TAF algebras, e.g. Power [10], [11], Thelwall [12], [13]. However, for most TAF algebras, it is quite difficult to determine if two TAF algebras have the same invariant. Thus, the problem of finding more explicit and computable isomorphism invariants still remains open.

Suppose $\mathcal{T}$ is a TAF algebra with diagonal $\mathcal{D}$. Let $\mathcal{P}(\mathcal{D})$ be the projections in $\mathcal{D}$ and $\mathcal{W}_{\mathcal{T}}$ the set of partial isometries v in $\mathcal{T}$ such that $v\mathcal{D}v^* \subseteq \mathcal{D}$ and $v^*\mathcal{D}v \subseteq \mathcal{D}$. Given $e, f \in \mathcal{P}(\mathcal{D})$, we write $e \prec_{\mathcal{T}} f$ if there exists $v \in \mathcal{W}_{\mathcal{T}}$ such that $vv^* = e$ and $v^*v = f$. This diagonal ordering is defined by Peters, Poon and Wagner in [6]. Given two TAF algebras $(\mathcal{T}, \mathcal{D})$ and $(\mathcal{S}, \mathcal{E})$, the orderings $\prec_{\mathcal{T}}$ and $\prec_{\mathcal{S}}$ are said to be isomorphic if there exists a C^*-isomorphism $\Theta : \mathcal{D} \to \mathcal{E}$ such that for each $e, f \in \mathcal{P}(\mathcal{D})$, we have $e \prec_{\mathcal{T}} f$ if and only if $\Theta(e) \prec_{\mathcal{S}} \Theta(f)$. It is shown in [6] that if $\Theta : \mathcal{T} \to \mathcal{S}$ is an isometric isomorphism, then the restriction of Θ to $\mathcal{D}$ gives an isomorphism of $\prec_T$ and $\prec_S$. Although this ordering is not a complete invariant for isomorphism classes of TAF algebras, it is more computable and has been proven quite useful in the study of TAF algebras in [6], [7]. We are going to prove that the ordering $\prec_T$ is a complete invariant for isometric isomorphism within a class of TAF algebras.

Let M_k denote the $k \times k$ complex matrices and T_k (respectively, D_k) the upper triangular (respectively, diagonal) matrices in M_k. I_k will denote the identity matrix in M_k. Suppose we are given a sequence of positive integers $\{k_i\}_{i=0}^{\infty}$ with $k_0 = 1$ and for each $i \geqslant 1$ a unital embedding $\varphi_i : M_{k_{i-1}} \to M_{k_i}$. Then the C^*-limit [2], $\mathfrak{A} = \varinjlim(M_{k_i}, \varphi_i)$ is called a UHF algebra [3]. In [3], Glimm showed that the generalized natural number $\prod_{i=1}^{\infty} n_i$ ([3], [2]), where $n_i = \frac{k_i}{k_{i-1}}$ for $i \geqslant 1$, is a complete isomorphism invariant for $\mathfrak{A}$ in the class of UHF algebras. Suppose, in addition, that each φ_i satisfies $\varphi_i(T_{k_{i-1}}) \subseteq T_{k_i}$ and, consequently $\varphi_i(D_{k_{i-1}}) \subseteq D_{k_i}$. Then the algebras $\mathcal{T} = \varinjlim(T_{k_i}, \varphi_i)$, $\mathcal{D} = \varinjlim(D_{k_i}, \varphi_i)$ can be regarded as subalgebras of $\mathfrak{A}$ and $\mathcal{T}$ is triangular in $\mathfrak{A}$ with diagonal $\mathcal{D}$. The class of $\mathcal{T}$ with $\varphi_i(x) = x \otimes I_{n_i}$ for every i has been studied by Power in [9]. In [1], Baker studied the class where $\varphi_i(x) = I_{n_i} \otimes x$ and showed that the generalized natural number $\prod_{i=1}^{\infty} n_i$ is also a complete invariant for isometric isomorphism, within this class of algebras. In this paper, we study the class where each $\varphi_i(x) = I_{n_i} \otimes x$ or $x \otimes I_{n_i}$. We will give a complete invariant for isometric isomorphisms within this class of algebras.

Recall that (Glimm [3] or Effros [2], p. 28) a generalized natural number $\mathbf{n} = \prod_{i=1}^{\infty} n_i$ is a mapping $f : \mathbf{P} \to \{0, 1, \ldots, \infty\}$, where $\mathbf{P} = \{p_i : p_i \text{ is the } i^{\text{th}} \text{ prime number}\}$, such that

$$f(p) = \sup\{j \geqslant 0 : p^j \text{ divides} \prod_{i=1}^{m} n_i \text{ for some } m\}.$$

If $f(p_i) = r_i$, we can write $\mathbf{n} = 2^{r_1}3^{r_2}5^{r_3}\ldots$. An (ordinary) natural number $n = p_1^{r_1}\ldots p_m^{r_m}$ can be regarded as a generalized natural number with $r_i = 0$ for $i > m$. Two generalized natural numbers can be multiplied as follows

$$(2^{r_1}3^{r_2}\ldots)(2^{s_1}3^{s_2}\ldots) = 2^{r_1+s_1}3^{r_2+s_2}\ldots, \text{ where we let } \infty + t = t + \infty = \infty.$$

To simplify notations for later discussion, let $\sigma_n(x) = I_n \otimes x$ and $\nu_n(x) = x \otimes I_n$. σ_n and ν_n are called [6] the standard embedding and nest embedding respectively. Let $\Phi = \{\varphi : \varphi = \sigma_n \text{ or } \nu_n \text{ for some } n \geqslant 1\}$. Suppose $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ is a sequence in Φ. Define

$$\sigma(\varphi) = \prod\{n_i : \varphi_i = \sigma_{n_i}\},$$

$$\nu(\varphi) = \prod\{n_i : \varphi_i = \nu_{n_i}\} \text{ and}$$
$$\tau(\varphi) = \sigma(\varphi)\nu(\varphi).$$

For $\varphi_1, \ldots, \varphi_n \in \Phi$ and π a permutation on $\{1, \ldots, n\}$, we note that $\varphi_1 \circ \cdots \circ \varphi_n = \varphi_{\pi(1)} \circ \cdots \circ \varphi_{\pi(n)} = \nu_r \circ \sigma_s$ where $r = \prod\{n_i : \varphi_i = \nu_{n_i}\}$ and $s = \prod\{n_i : \varphi_i = \sigma_{n_i}\}$.

The main result in this paper is the following

THEOREM 1.1. *Let $\{k_i\}_{i=0}^{\infty}$, $\{l_i\}_{i=0}^{\infty}$ be two sequences of positive integers with $k_0 = l_0 = 1$ and $n_i = \dfrac{k_i}{k_{i-1}}$, $m_i = \dfrac{l_i}{l_{i-1}}$ for $i \geqslant 1$. Suppose $\varphi = \{\varphi_i\}_{i=1}^{\infty}$, $\psi = \{\psi\}_{i=1}^{\infty}$ are sequences in Φ such that $\varphi_i = \sigma_{n_i}$ or ν_{n_i} and $\psi_i = \sigma_{m_i}$ or ν_{m_i} for each i. For each $i \geqslant 0$, let $\mathcal{T}_i = T_{k_i}, \mathcal{D}_i = D_{k_i}, \mathcal{S}_i = T_{l_i}$ and $\mathcal{E}_i = \mathcal{D}_{l_i}$. Since $\varphi_i(\mathcal{T}_{i-1}) \subseteq \mathcal{T}_i, \varphi_i(\mathcal{D}_{i-1}) \subseteq \mathcal{D}_i, \psi_i(\mathcal{S}_{i-1}) \subseteq \mathcal{S}_i$ and $\psi_i(\mathcal{E}_{i-1}) \subseteq \mathcal{E}_i$, we can form the direct limits $\mathcal{T} = \varinjlim(\mathcal{T}_i, \varphi_i), \mathcal{D} = \varinjlim(\mathcal{D}_i, \varphi_i), \mathcal{S} = \varinjlim(\mathcal{S}_i, \psi_i)$ and $\mathcal{E} = \varinjlim(\mathcal{E}_i, \psi_i)$. Then $\mathcal{D}, \mathcal{E}$ are the diagonals of $\mathcal{T}, \mathcal{S}$ respectively. Let $\prec_{\mathcal{T}}$ (and $\prec_{\mathcal{S}}$) be the diagonal orderings on $\mathcal{D}$ (and $\mathcal{E}$) defined by $\mathcal{T}$ (and $\mathcal{S}$). Then the following conditions are equivalent.*

1) *$\mathcal{T}$ is isometrically isomorphic to $\mathcal{S}$.*

2) *The orderings $\prec_{\mathcal{T}}$ and $\prec_{\mathcal{S}}$ are isomorphic.*

3) *There exist positive integers n and m such that $n\sigma(\varphi) = m\sigma(\psi)$ and $m\nu(\varphi) = n\nu(\psi_i)$.*

The proof of this theorem will be given in Section 2. We end this section with some simple corollaries.

Given a generalized natural number $\mathbf{m}$, let $P(\mathbf{m}) = \{p \in \mathbf{P} : p|\mathbf{m}\}$ and $P_0(\mathbf{m}) = \{p \in P(\mathbf{m}) : p^{\infty} \not| \mathbf{m}\}$. Suppose $\mathcal{S}$ and $\mathcal{T}$ are constructed from sequences $\varphi = \{\varphi_i\}$ and $\psi = \{\psi_i\}, \varphi_i, \psi_i \in \Phi$ such that $\tau(\varphi) = \tau(\psi) = \mathbf{m}$. Then condition 3 in Theorem 1.1 holds if and only if there exists a finite subset $\mathbf{Q} \subseteq P_0(\mathbf{m})$ such that for every prime $p \notin \mathbf{Q}$ and every positive integer d, we have $p^d \mid \sigma(\varphi)$ if and only if $p^d \mid \sigma(\psi)$, and $p^d \mid \nu(\varphi)$ if and only if $p^d \mid \nu(\psi)$. We note that since $\tau(\varphi) = \mathbf{m}$, therefore, for each prime number p such that $p^{\infty} \mid \mathbf{m}$, exactly one of the following three conditions holds:

i) $p^{\infty} \mid \sigma(\varphi)$ and $p^{\infty} \mid \nu(\varphi)$

ii) $p^{\infty} \mid \sigma(\varphi)$ and $p^{\infty} \not| \nu(\varphi)$

iii) $p^{\infty} \not| \sigma(\varphi)$ and $p^{\infty} \mid \nu(\varphi)$

Thus, for a fixed generalized natural number $\mathbf{m}$, if we let $\mathcal{T}(\mathbf{m})$ be the collection of isomorphism classes of $\mathcal{T}$ constructed from a sequence φ in Φ with $\tau(\varphi) = \mathbf{m}$, we have

COROLLARY 1.2. *For each generalized natural number $\mathbf{m}$, we have*

1) *if $P(\mathbf{m})$ is infinite, then $\mathcal{T}(\mathbf{m})$ is uncountable, and*

2) *if* $P(\mathbf{m})$ *is finite, then* $T(\mathbf{m})$ *has exactly* 3^n *elements, where* n *is the number of* $p \in P(\mathbf{m})$ *such that* $p^\infty \mid \mathbf{m}$.

In particular, if $\mathbf{m} = p^\infty$ for some prime number p, we have the following.

COROLLARY 1.3. *Let* p *be a prime. Then* $T(p^\infty)$ *is equal to the set of isomorphism classes of the algebras constructed from the following sequences*

1) $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \nu_p$ *for all* i,
2) $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \sigma_p$ *for all* i,
3) $\{\varphi_i\}_{i=1}^\infty, \varphi_i = \sigma_p$ *when* i *is even,* $\varphi_i = \nu_p$ *when* i *is odd.*

For $p = 2$, the fact that all three algebras defined in Corollary 1.3 are non-isomorphic has been proven by Peters, Poon and Wagner in [6] (Examples 1.1 and 3.27).

2. THE PROOF

Let $\{e_{ij}^n : i, j = 1, \ldots, n\}$ be the usual matrix units for M_n. Thus, T_n and D_n are spanned by $\{e_{ij}^n : 1 \leqslant i \leqslant j \leqslant n\}$ and $\{e_{ii}^n : 1 \leqslant i \leqslant n\}$ respectively. We will write e_i^n for e_{ii}^n. Then, for non-zero projections $e, f \in \mathcal{P}(D_n)$, we have $e \prec_{T_n} f$ if and only if $e = \sum_{t=1}^{k} e_{i_t}^n, f = \sum_{t=1}^{k} e_{j_t}^n$ for some $1 \leqslant i_1 < \cdots < i_k \leqslant n, 1 \leqslant j_1 < \cdots < j_k \leqslant n$ and $i_t \leqslant j_t$ for every $1 \leqslant t \leqslant k$.

Let $\{k_i\}, \{l_i\}, \{\varphi_i\}, \{\psi_i\}, \mathcal{D}, \mathcal{E}, \mathcal{S}$ and $\mathcal{T}$ be as given in Theorem 1.1. If $e, f \in \mathcal{P}(\mathcal{D})$ such that $e \prec_{\mathcal{T}} f$, then there exists some i such that $e, f \in \mathcal{D}_i$. By Corollary 3.7 in [6], every v in $W_{\mathcal{T}}$ is of the form $v = dw$ for some unitary d in $\mathcal{D}$ and $w \in W_{\mathcal{T}_j}$ for some j. Thus, we may assume $e \prec_{\mathcal{T}_i} f$. Hence, the discussion in the previous paragraph gives an explicit description of $\prec_{\mathcal{T}}$. Clearly, similar description holds for $\prec_{\mathcal{S}}$.

Suppose $\Theta : \mathcal{D} \to \mathcal{E}$ is an isomorphism between the orderings $\prec_{\mathcal{T}}$ and $\prec_{\mathcal{S}}$. Then for each n, there exist $i(n)$ and $j(n)$ such that $\Theta(\mathcal{D}_n) \subseteq \mathcal{E}_{j(n)}$ and $\Theta^{-1}(\mathcal{E}_n) \subseteq \mathcal{D}_{i(n)}$. Furthermore, we may assume that for each n and e, f in $\mathcal{D}_n$ (respectively, $\mathcal{E}_n$) $e \prec_{\mathcal{T}_n} f \Rightarrow \Theta(e) \prec_{\mathcal{S}_{j(n)}} \Theta(f)$ (respectively, $e \prec_{\mathcal{S}_n} f \Rightarrow \Theta^{-1}(e) \prec_{\mathcal{T}_{i(n)}} \Theta^{-1}(f)$).

In the following three lemmas, Θ (and Θ_i) : $D_n \to D_m$ will always denote a 1-1 unital C^*-homomorphism such that $e \prec_{T_n} f \Rightarrow \Theta(e) \prec_{T_m} \Theta(f)$. To simplify notations, we will write $e^n[k, l] = \sum_{i=k}^{l} e_i^n$ for $1 \leqslant k \leqslant l \leqslant n$. If r, s are positive integers and $\nu_r \circ \sigma_s : D_n \to D_m$, then direct computation shows that for $1 \leqslant i \leqslant n$, $\nu_r \circ \sigma_s(e_i^n) = \sum_{t=0}^{s-1} e^m[1 + (i-1)r + nrt, ir + nrt]$.

LEMMA 2.1. *Let* $\Theta : D_n \to D_m$. *For each* $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m$, *let* $\mu(i,j)$ *be the number of* k *such that* $1 \leqslant k \leqslant j$ *and* $\Theta(e_i^n) \geqslant e_k^m$ *(the usual ordering of projections in* D_m*). Then, we have*

1) $\mu(i,j) \geqslant \mu(i',j)$ *and* $\mu(i,j) \leqslant \mu(i,j')$ *for* $1 \leqslant i \leqslant i' \leqslant n$ *and* $1 \leqslant j \leqslant j' \leqslant m$.

2) *For each* $1 \leqslant j \leqslant m, \sum_{i=1}^{n} \mu(i,j) = j$.

3) *Suppose* $\Theta(e^n[1,i]) \geqslant e^m[1,j]$ *and* $\Theta(e^n[1,i])e_{j+1}^m = 0$. *Then* $\Theta(e_{i+1}^n) \geqslant e_{j+1}^m$.

Proof.

1) For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant j' \leqslant m$, it follows from definitions that $\mu(i,j) \leqslant \leqslant \mu(i,j')$. Suppose $1 \leqslant i \leqslant i' \leqslant n$. Then $e_i^n \prec_{T_n} e_{i'}^n \Rightarrow \Theta(e_i^n) \prec_{T_m} \Theta(e_{i'}^n)$. So

$$\Theta(e_i^n) = \sum_{t=1}^{k} e_{j_t}^m \text{ and } \Theta(e_{i'}^n) = \sum_{t=1}^{k} e_{j'_t}^m$$

for some $1 \leqslant j_1 < \cdots < j_k \leqslant m, 1 \leqslant j'_1 < \cdots < j'_k \leqslant m$ and $j_t \leqslant j'_t$ for every $1 \leqslant t \leqslant k$. Hence, for each $1 \leqslant j \leqslant m$ we have $\mu(i,j) \geqslant \mu(i',j)$

2) follows from $\sum_{i=1}^{n} \Theta(e_i^n) \geqslant e^m[1,j]$.

3) From the given conditions, we have $\sum_{k=1}^{i} \mu(k,j) = j = \sum_{k=1}^{i} \mu(k,j+1)$ and $\mu(i',j) = 0$ for $1 \leqslant i < i' \leqslant n$. From 1) and 2) we have $\mu(i+1,j+1) \geqslant \mu(i',j+1)$ for all $i' > i+1$ and $\sum_{k=i+1}^{n} \mu(k,j+1) = 1$. Therefore, $\mu(i+1,j+1) = 1$. So, it follows from $\mu(i+1,j) = 0$ that $\Theta(e_{i+1}^n) \geqslant e_{j+1}^m$.

LEMMA 2.2. *Let* $\Theta_1 : D_n \to D_m, \Theta_2 : D_m \to D_p$ *and* $\Theta_2 \circ \Theta_1 = \nu_r \circ \sigma_s$ *for some positive integers* r, s. *Then there exists a positive integer* k *such that* $k|r$ *and for* $1 \leqslant i \leqslant n$

$$(*) \qquad \Theta_1(e_i^n) \geqslant e^m[1+(i-1)k, ik],$$

$$\Theta_2(e^m[1+(i-1)k, ik]) \geqslant e^p[1+(i-1)r, ir].$$

Furthermore, if $nk < m$, *then* $\Theta_1(e_1^n) \geqslant e_{nk+1}^m$.

Proof. We first note that for $1 \leqslant i \leqslant n$,

$$\Theta_2(\Theta_1(e_i^n))e^p[1,nr] = \nu_r \circ \sigma_s(e_i^n)e^p[1,nr] = e^p[1+(i-1)r, ir].$$

Hence, for every $1 \leqslant j \leqslant m$, if $\Theta_2(e_j^m)e^p[1+(i-1)r, ir] \neq 0$, then $\Theta_1(e_i^n) \geqslant e_j^m$.

From 1) and 2) in Lemma 2.1 with $j = 1$, we have $\Theta_1(e_1^n) \geqslant e_1^m$ and $\Theta_2(e_1^m) \geqslant e_1^p$. Let k_1 be the greatest integer k such that $\Theta_2(e_k^m)e^p[1,r] \neq 0$. By putting $j = 1, \ldots, r$

successively in 3) of Lemma 2.1, we have $\Theta_2(e^m[1,k_1])\geqslant e^p[1,r]$ and $\Theta_2(e_l^m)e^p[1,r]\neq 0$ for all $1\leqslant l\leqslant k_1$. Hence, we have $\Theta_1(e_1^n)\geqslant e^m[1,k_1]$. Furthermore, $\Theta_2\circ\Theta_1(e_2^n)\geqslant e_{r+1}^m$. Therefore, $\Theta_2(e^m[1,k_1])e_{r+1}^p = 0$. Hence, again by 3) of Lemma 2.1, $\Theta_2(e_{k_1+1}^m)\geqslant e_{r+1}^p$. This gives $\Theta_1(e_2^n)\geqslant e_{k_1+1}^m$. Applying Lemma 2.1 inductively on i and j, we can get integers $0=k_0<k_1<\cdots<k_n$ such that

$$\Theta_2(e^m[1+k_{i-1},k_i])\geqslant e^p[1+(i-1)r,ir]$$

and

$$\Theta_1(e_i^n)\geqslant e^m[1+k_{i-1},k_i]$$

for $1\leqslant i\leqslant n$.

Applying 1) in Lemma 2.1 for Θ_1 we have

$$k_i-k_{i-1}\geqslant k_{i'}-k_{i'-1} \text{ for } 1\leqslant i\leqslant i'\leqslant n.$$

For each $1\leqslant j\leqslant k_n$, let $\mu_j=\mu(j,nr)$ as defined in Lemma 2.1 for Θ_2. We have $\mu_j\geqslant\mu_{j'}$ for $1\leqslant j\leqslant j'\leqslant k_n$. But, for every $1<i\leqslant n$ we have

$$r=\sum_{j=1}^{k_1}\mu_j\geqslant k_1\mu_{k_1}\geqslant(k_i-k_{i-1})\mu_{k_{i-1}}\geqslant\sum_{j=k_{i-1}+1}^{k_i}\mu_j=r.$$

This gives $k_i-k_{i-1}=k_1=k$ for some constant k and $\mu_j=\dfrac{r}{k}$ for all $1\leqslant j\leqslant k_n=nk$. Hence, $k\mid r$ and $k_i=ik$ for $1\leqslant i<n$. Furthermore, if $nk<m$, applying 1) and 2) of Lemma 2.1 to Θ_1 with $j=nk+1$, we have $\Theta_1(e_1^n)\geqslant e_{nk+1}^m$.

LEMMA 2.3. *Suppose $n_1>1$. Let $\Theta_i: D_{n_i}\to D_{n_{i+1}}$ for $i=1,2,3$ such that $\Theta_2\circ\Theta_1=\nu_{r_1}\circ\sigma_{s_1}$ and $\Theta_3\circ\Theta_2=\nu_{r_2}\circ\sigma_{s_2}$ for some positive integers r_1,r_2,s_1,s_2. Then there exist k_1 and k_2 dividing r_1 such that, with $q_1=n_2/(n_1k_1)$, $\Theta_1=\nu_{k_1}\circ\sigma_{q_1}$ and*

$$\Theta_2(e_j^{n_2})\geqslant e^{n_3}[1+(j-1)k_2,jk_2] \quad \text{for } 1\leqslant j\leqslant n_2.$$

If, in addition, there exists $\Theta_4: D_{n_4}\to D_{n_5}$ such that $\Theta_4\circ\Theta_3=\nu_{r_3}\circ\sigma_{s_3}$ for some positive integers r_3,s_3, then $\Theta_2=\nu_{k_2}\circ\sigma_{q_2}$, where $q_2=s_1/q_1$.

Proof. We apply Lemma 2.2 to $\Theta_2\circ\Theta_1$ and $\Theta_3\circ\Theta_2$ respectively, and from (*), we have k_1, and k_2 such that

$$\Theta_1(e_i^{n_1})\geqslant e^{n_2}[1+(i-1)k_1,ik_1] \quad 1\leqslant i\leqslant n_1$$

and

$$\Theta_2(e_j^{n_2})\geqslant e^{n_3}[1+(j-1)k_2,jk_2] \quad 1\leqslant j\leqslant n_2.$$

Since

$$\Theta_2 \circ \Theta_1(e_i^{n_1}) = \nu_{r_1} \circ \sigma_{s_1}(e_i^{n_1}) = \sum_{t=0}^{s_1-1} e^{n_3}[1+(i-1)r_1+n_1r_1t, ir_1+n_1r_1t]$$

and $n_1 > 1$, we have $k_1k_2 = r_1$. For each $1 \leqslant i \leqslant n_1$ and $1 \leqslant j \leqslant n_2$ such that $\Theta_1(e_i^{n_1}) \geqslant e_j^{n_2}$, we have

$$\Theta_2 \circ \Theta_1(e_i^{n_1}) \geqslant \Theta_2(e_j^{n_2}) \Rightarrow$$

$$\Rightarrow \Theta_2 \circ \Theta_1(e_i^{n_1})e^{n_3}[1, k_2n_2] \geqslant e^{n_3}[1+(j-1)k_2, jk_2] \Rightarrow$$

$$\Rightarrow \sum_{t=0}^{q_1-1} e^{n_3}[1+(i-1)r_1+n_1r_1t, ir_1+n_1r_1t] \geqslant e^{n_3}[1+(j-1)k_2, jk_2],$$

where $q_1 = \dfrac{n_2k_2}{n_1r_1} = \dfrac{n_2}{n_1k_1}$

$$\Rightarrow 1+(i-1)r_1+n_1r_1t \leqslant 1+(j-1)k_2 \leqslant jk_2 \leqslant ir_1+n_1r_1t$$

for some $0 \leqslant t \leqslant q-1$

$$\Rightarrow 1+(i-1)k_1+n_1k_1t \leqslant j \leqslant ik_1+n_1k_1t \text{ for some } 0 \leqslant t \leqslant q_1-1.$$

Hence, $\Theta_1(e_i^{n_1}) = \sum_{t=0}^{q_1-1} e^{n_2}[1+(i-1)k_1+n_1k_1t, ik_1+n_1k_1t] = \nu_{k_1} \circ \sigma_{q_1}(e_i^{n_1})$ for $1 \leqslant i \leqslant n_1$.

If, in addition, there exists $\Theta_4 : T_{n_4} \to T_{n_5}$ such that $\Theta_4 \circ \Theta_3 = \nu_{r_3} \circ \sigma_{s_3}$ for some positive integers r_3, s_3, then we can apply the results in the first part of this lemma to Θ_2, Θ_3 and Θ_4 and use

$$q_2 = \frac{n_3}{n_2k_2} = \frac{n_1r_1s_1}{n_1k_1q_1k_2} = \frac{s_1}{q_1}.$$

Proof of Theorem 1.1. Since the result is trivial if either $\mathcal{T}$ or $\mathcal{S}$ is finite dimensional, we may assume that both $\tau(\varphi)$ and $\tau(\psi)$ are infinite.

1) $\Rightarrow$ 2) is proved in Proposition 3.20 of [6].

For 2) $\Rightarrow$ 3), suppose that $\Theta : \mathcal{D} \to \mathcal{E}$ is an isomorphism of $\prec_{\mathcal{T}}$ and $\prec_{\mathcal{S}}$. Then for every natural number $n|\tau(\varphi)$, there exist mutually orthogonal projections $e_1, \ldots, e_n$ in $\mathcal{D}$ such that

$$e_i \prec_{\mathcal{T}} e_{i+1} \text{ for } 1 \leqslant i < n \text{ and } \sum_{i=1}^{n} e_i = 1_{\mathcal{D}}, \text{ the identity in } \mathcal{D}$$

$$\Rightarrow \Theta(e_i) \prec_{\mathcal{S}} \Theta(e_{i+1}) \text{ for } 1 \leqslant i < n \text{ and } \sum_{i=1}^{n} \Theta(e_i) = 1_{\mathcal{E}}, \text{ the identity in } \mathcal{E}.$$

Also, $\Theta(e_1), \dots, \Theta(e_n)$ are mutually orthogonal projections in $\mathcal{E}$. Therefore, $n \mid \tau(\psi)$. Applying the same argument for Θ^{-1}, we have $\tau(\varphi) = \tau(\psi)$.

In order to prove 3), we may assume, without loss of generality, that $n_1, m_1 > > 1$. Choose $j(1)$ and $i(1)$ such that

i) $\Theta(\mathcal{D}_1) \subseteq \mathcal{E}_{j(1)}$ and for $e, f \in \mathcal{D}_1, e \prec_{\mathcal{T}_1} f \Rightarrow \Theta(e) \prec_{\mathcal{S}_{j(1)}} \Theta(f)$, and

ii) $\Theta^{-1}(\mathcal{E}_1) \subseteq \mathcal{D}_{i(1)}$ and for $e, f \in \mathcal{E}_1, e \prec_{\mathcal{S}_1} f \Rightarrow \Theta^{-1}(e) \prec_{\mathcal{T}_{i(1)}} \Theta^{-1}(f)$.

Let $\mathbf{Q}$ be the set of all primes p such that $p \mid k_{i(1)} l_{j(1)}$ and p^∞ does not divide anyone of $\sigma(\varphi), \sigma(\psi), \nu(\varphi)$ and $\nu(\psi)$. Given a prime $p \notin \mathbf{Q}$ and positive integer d such that $p^d \mid \nu(\psi)$, we can choose $l > j(1)$ such that $p^d \mid \prod\{m_i : j(1) < i \leqslant l, \psi_i = \nu_{m_i}\}$. Then $\psi_l \circ \psi_{l-1} \circ \cdots \circ \psi_{j(1)+1} = \nu_t \circ \sigma_u$ for some positive integers t, u with $p^d | t$. Choose $k > 1$ such that $\Theta^{-1}(\mathcal{E}_l) \subseteq \mathcal{D}_k$ and for $e, f \in \mathcal{E}_l, e \prec_{\mathcal{S}_l} f \Rightarrow \Theta^{-1}(e) \prec_{\mathcal{T}_k} \Theta^{-1}(f)$. Then the embedding $\mathcal{D}_1 \to \mathcal{D}_k$ is given by $\nu_r \circ \sigma_s$ where $r = \prod\{n_i : \varphi_i = \nu_{n_i}, 1 < i \leqslant k\}$. Applying Lemma 2.2 for $\Theta_1 = \nu_t \circ \sigma_u \circ \Theta$ on $\mathcal{D}_1$ and $\Theta_2 = \Theta^{-1}$ on $\mathcal{E}_l$, we have $p^d | r$ and consequently, $p^d | \nu(\varphi)$.

By repeating the above argument with Θ^{-1}, we can show that for every prime $p \notin \mathbf{Q}$ and positive integer $d, p^d | \nu(\varphi)$ if and only if $p^d | \nu(\psi)$. Similarly, by applying Lemma 2.3, we can prove the same condition for $\sigma(\varphi)$ and $\sigma(\psi)$.

For 3) $\Rightarrow$ 1), suppose φ and ψ satisfies

$$n\nu(\varphi) = m\nu(\psi) \quad \text{and} \quad m\sigma(\varphi) = n\sigma(\psi)$$

for some positive integers n and m. Without loss of generality, we may assume n and m are relatively prime and all n_i and m_i are prime numbers. For each prime p dividing n (or m), we have $p | \sigma(\varphi)$ (or $p | \nu(\varphi)$ respectively). Since switching a finite number of σ_{n_i} to ν_{n_i} (or ν_{n_i} to σ_{n_i}) will not change the isomorphism class of $\mathcal{T}$, we may assume that $\sigma(\varphi) = \sigma(\psi)$ and $\nu(\varphi) = \nu(\psi)$. Hence, there exists a permutation π of the positive integers such that $\psi_i = \varphi_{\pi(i)}$.

So the result follows from the next lemma.

LEMMA 2.4. *Let $\{\varphi_i\}_{i=1}^\infty, \{\psi_i\}_{i=1}^\infty, \mathcal{T}$ and $\mathcal{S}$ be as given in Theorem 1.1. Suppose there is a permutation π of the positive integers such that $\psi_i = \varphi_{\pi(i)}$. Then $\mathcal{T}$ and $\mathcal{S}$ are isomorphic.*

Proof. Let $E = \{i_1, \dots, i_r\}$ be a non-empty finite subset of positive integers. Then from the discussion in the paragraph preceding Theorem 1.1, the map $\varphi_E = = \varphi_{i_1} \circ \cdots \circ \varphi_{i_r}$ is well defined on M_n for every fixed n. If E_1 and E_2 are disjoint subsets, then we have $\varphi_{E_1} \circ \varphi_{E_2} = \varphi_{E_1 \cup E_2}$.

We note that the sequence $\{l_i\}$ of $\mathcal{S}$ is determined by $\{k_i\}$ and the permutation π. Specifically, we have $m_i = n_{\pi(i)}$ and $l_i = \prod_{j=1}^{i} m_j$. We are going to construct two sequences

$$i(0) < i(1) < i(2) < \cdots,\ j(1) < j(2) < \cdots$$

and mappings Θ_i such that the following diagram commutes

$$\begin{array}{cccccccc} \mathcal{T}_0 & \to & \mathcal{T}_{i(0)} & \to & \mathcal{T}_{i(1)} & \to & \mathcal{T}_{i(2)} & \to \ldots \\ \| & & \Theta_1 & \searrow & \uparrow \Theta_2 & \searrow \Theta_3 & \uparrow \Theta_4 & \\ \mathcal{S}_0 & \to & & & \mathcal{S}_{j(1)} & \to & \mathcal{S}_{j(2)} & \to \ldots \end{array}$$

For each $i \leqslant j$, let $[i,j] = \{k : i \leqslant k \leqslant j\}$. First choose $i(0) = 1$ and $j(1) > \pi^{-1}(1)$. Let $E_1 = \pi([1,j(1)]) \backslash \{1\}$ and $\Theta_1 = \varphi_{E_1}$. Next, we choose $i(1) > \max\{\pi(i) : 1 \leqslant i \leqslant j(1)\}$ and set $E_2 = [1,i(1)] \backslash \pi([1,j(1)])$, $\Theta_2 = \varphi_{E_2}$. We have

1) $E_1 \cup \{1\} = \pi([1,j(1)]) \Rightarrow \Theta_1 \circ \varphi_1 = \varphi_{\pi(j(1))} \circ \cdots \circ \varphi_{\pi(1)}$ and

2) $E_2 \cup E_1 = [2,i(1)] \Rightarrow \Theta_2 \circ \Theta_1 = \varphi_{i(1)} \circ \cdots \circ \varphi_2$.

Suppose we have chosen sequences

$$1 = i(0) < i(1) < \cdots < i(r),$$
$$0 = j(0) < j(1) < \ldots < j(r)$$

and subsets $E_0 = \{1\}, E_1, E_2, \cdots, E_{2r}$ such that

$$(**) \qquad \begin{aligned} E_{2t} \cap E_{2t-1} &= \emptyset \\ E_{2t-1} \cap E_{2t-2} &= \emptyset \\ E_{2t} \cup E_{2t-1} &= [i(t-1)+1, i(t)] \\ E_{2t-1} \cup E_{2t-2} &= \pi([j(t-1)+1, j(t)]) \end{aligned}$$

for $1 \leqslant t \leqslant r$. Then, we can choose $j(r+1) > \max\{\pi^{-1}(t) : 1 \leqslant t \leqslant i(r)\}, i(r+1) > \max\{\pi(t) : 1 \leqslant t \leqslant j(r+1)\}$ and put $E_{2r+1} = \pi([j(r)+1, j(r+1)]) \backslash E_{2r}, E_{2r+2} = [i(r)+1, i(r+1)] \backslash E_{2r+1}$. Then $(**)$ holds for $1 \leqslant t \leqslant (r+1)$. Thus $\{i(t)\}, \{j(t)\}$ and E_t can be defined inductively and the diagram commutes with $\Theta_i = \varphi_{E_i}$.

REMARK 2.5. After this paper had been submitted, we learned that the equivalence of 1) and 3) in Theorem 1.1 has also been obtained by Hopenwesser and Power in [16]. Their proof uses the invariant defined by Power in [11] and is quite different from the one given here.

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