J. OPERATOR THEORY
27(1992), 283–308

TRANSFERENCE OF ALMOST EVERYWHERE CONVERGENCE

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1. INTRODUCTION

Let G be a locally compact abelian group, and let $X_p = L^p(\mathcal{M}, \mu)$, where $1 \leqslant p < \infty$ and $(\mathcal{M}, \mu)$ is an arbitrary measure space (this notation will be fixed throughout all that follows). As shown in [2, Theorem (2.3)] (and sketched in [1, Théorème 1]), a strongly continuous, uniformly bounded representation of G in X_p by separation-preserving operators (for instance, by positivity-preserving operators) will transfer to X_p strong type (p, p) bounds for maximal operators on $L^p(G)$ defined by sequences of convolution operators. This result ceases to be valid if the separation-preserving hypothesis is removed, or if "strong type" is replaced by "weak type" (see [1] for the relevant counterexamples). In [3], [4], we introduced a broad proper subclass of the separation-preserving representations of G in X_p so as to guarantee the transference of weak type bounds for maximal convolution operators ([4, Theorem (4.14)] or, alternatively, [3, Théorème 2]). This subclass, consisting of the μ-distributionally bounded representations of G described in §2 below, includes the representations induced by measure-preserving transformations on $\mathcal{M}$, and consequently results stemming from distributional boundedness generalize their ergodic theory counterparts.

The present article applies the foregoing results, in conjunction with the Banach Principle ([16, Theorem 1.1.1], [23, §3.2]), to the transference of almost everywhere convergence from G to $\mathcal{M}$. More precisely, given a separation-preserving or μ-distributionally bounded representation of G, we develop suitable natural conditions for a sequence in $L^1(G)$ so that when the corresponding sequence of transferred convolution operators is applied to an arbitraty element of X_p, convergence μ-a.e. will ensue. The spirit of these considerations is motivated by the transference viewpoint of R. R. Coifman and G. Weiss ([10], [11]).

In order to make the subsequent discussion more self-contained, §2 reviews the features of transference theory which are essential for our considerations, while fixing some notation in the process. §3 treats the transference of a.e. convergence in a setting wherein the convolution operators have kernels supported in a common compact subset of G. In §4 this requirement on the supports is replaced with a milder condition ((4.8) below) on the convolution kernels reminiscent of the Hörmander condition for singular integrals [27, (2′) on p. 34]. In §5, the foregoing results are applied to the truncates of the classical Calderón-Zygmund singular integral kernels to obtain transferred singular integral operators defined in terms of μ-a.e. convergence. The transferred singular integral operators so obtained generalize their ergodic counterparts.

Throughout what follows, the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$ will denote, respectively, the set of positive integers, the additive group of integers, the additive group of real numbers, and the multiplicative group of the unit circle in the complex plane $\mathbb{C}$. The set-theoretic (respectively, group-theoretic) difference of two sets A and B will be written $A\backslash B$ (respectively, $A - B$).

2. PRELIMINARIES

In this section we collect, for subsequent use, some definitions and prior facts, in the process standardizing some of our notation. Henceforth let $\{k_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1(\lambda)$, where λ denotes a fixed Haar measure on G. We shall write $N_p(\{k_n\})$ (respectively, $N_p^{(w)}(\{k_n\})$) for the strong (respectively, weak) type (p,p) norm of the maximal convolution operator on $L^p(\lambda)$ defined by the sequence $\{k_n\}_{n=1}^{\infty}$. We shall denote by $u \to S_u$ a strongly continuous representation of G in $X_p \equiv L^p(\mu)$ such that

$$\alpha_p \equiv \sup\{\|S_u\| : u \in G\} < \infty. \tag{2.1}$$

For each $\varphi \in L^1(G)$, we use Bochner integration to define the *transferred convolution operator* $H_\varphi^{(p)} : X_p \to X_p$ as follows:

$$H_\varphi^{(p)} f = \int_G \varphi(u) S_{-u} f \, d\lambda(u), \quad \text{for all } f \in X_p. \tag{2.2}$$

In particular, $H_\varphi^{(p)}$ is a bounded linear mapping of X_p into itself. Let $\mathbf{M}_p^{\#}$ be the maximal operator on X_p defined by the sequence $\{H_{k_n}^{(p)}\}$:

$$\mathbf{M}_p^{\#} f = \sup_{n \in \mathbb{N}} |H_{k_n}^{(p)} f| \quad \text{for all } f \in X_p. \tag{2.3}$$

DEFINITION. The representation S is said to be *separation-preserving*, provided that whenever $f \in X_p$, $g \in X_p$ and $f \cdot g = 0$ μ-a.e., then $(S_u f)(S_u g) = 0$ μ-a.e. for all $u \in G$. We remark that S will be separation-preserving provided that S_u is positivity-preserving for each $u \in G$ (see, for instance, [2, Scholium A in §2]). The main advantage of the separation-preserving property for our purposes is that it permits S to transfer strong type bounds for maximal convolution operators. More precisely, the following theorem holds.

THEOREM 2.4. ([2, Theorem (2.3)]). *If S is separation-preserving, then*

$$\left\|\mathbf{M}_p^{\#} f\right\|_p \leqslant \alpha_p^2 N_p(\{k_n\})\|f\|_p, \quad \text{for all } f \in X_p,$$

where α_p is the constant defined in (2.1).

In order to treat the transference of weak type maximal bounds, we shall need the notion of distributionally bounded representation from [3], [4]. For a μ-measurable function f, let $\varphi(f;\cdot)$ denote the *distribution function of f* defined by:

$$\varphi(f;y) = \mu(\{\omega \in \mathcal{M} : |f(\omega)| > y\}), \quad \text{for all } y > 0.$$

DEFINITION. Let $\Gamma(\mu)$ be the group (under composition) of all injective linear mappings of $L^1(\mu) \cap L^\infty(\mu)$ onto itself. A *μ-distributionally bounded representation of G* is an identity-preserving homomorphism $u \to R_u$ of G into $\Gamma(\mu)$ for which there exists a positive real constant c such that:

(2.5) $\varphi(R_u f; y) \leqslant c\varphi(f;y)$, for all $u \in G$, all $f \in L^1(\mu) \cap L^\infty(\mu)$, and all $y > 0$.

We next list the properties of distributionally bounded representations which will be required for our purposes. The demonstrations of these properties can be found in [4]. It follows from (2.5) that there is a unique representation $u \to R_u^{(p)}$ of G in X_p such that $R_u^{(p)} f = R_u f$ for all $u \in G$ and all $f \in L^1(\mu) \cap L^\infty(\mu)$. Moreover,

$$\sup\left\{\left\|R_u^{(p)}\right\| : u \in G\right\} \leqslant c^{1/p},$$

and (2.5) is valid for the representation $R^{(p)}$ and all $u \in G$, $f \in X_p$, $y > 0$. As shown in [4, Corollary (2.20)] (see, alternatively, [3,(3)]), the representation $R^{(p)}$ is separation-preserving. As noted in [4,(2.9)], for $1 \leqslant p_1, p_2 < \infty$,

(2.6) $$R_u^{(p_1)} f = R_u^{(p_2)} f, \quad \text{for all } u \in G,\ f \in X_{p_1} \cap X_{p_2},$$

and it then follows from [4, Proposition (3.2)] that if there is some $s \in [1,\infty)$ such that $R^{(s)}$ is strongly continuous on X_s, then $R^{(t)}$ is strongly continuous on X_t for all

$t \in [1, \infty)$. In this case, we say that the μ-distributionally bounded representation R is *strongly continuous*, and we apply the notation in (2.2) and (2.3) to the representation $R^{(p)}$ in place of S. In particular, if the distributionally bounded representation R is strongly continuous, and $1 \leqslant p_1, p_2 < \infty$, then, as a consequence of (2.6), we have for each $\psi \in L^1(G)$:

$$H_\psi^{(p_1)} f = H_\psi^{(p_2)} f, \quad \text{for all } f \in X_{p_1} \cap X_{p_2}. \tag{2.7}$$

The next theorem describes the role played by distributional boundedness in the transference of weak type bounds for maximal convolution operators [4, Theorem (4.14)] (for an outline of the proof, see [3, Théorème 2]).

THEOREM 2.8. *Let R be a strongly continuous μ-distributionally bounded representation of G. Then*

$$\varphi(\mathbf{M}_p^{\#} f; y) \leqslant c^2 \{ N_p^{(w)}(\{k_n\}) \|f\|_p y^{-1} \}^p, \quad \textit{for all } y > 0, \textit{ and all } f \in X_p,$$

where c is the constant appearing in (2.5).

The next theorem [4,Theorem (2.21)] provides a direct way of testing for distributional boundedness.

THEOREM 2.9. *Let $\mathfrak{S}$ be a representation of G in X_p. Then there is a μ--distributionally bounded representation R of G such that $\mathfrak{S} = R^{(p)}$ if and only if the following two conditions hold:*

(i) $\sup\{ \|\mathfrak{S}_u\|_p : u \in G \} < \infty$;

(ii) $\|\mathfrak{S}_u f\|_\infty \leqslant \|f\|_\infty$, *for all $u \in G$ and all $f \in L^p(\mu) \cap L^\infty(\mu)$.*

REMARKS 2.10. (i) For purposes of illustration, we describe here a natural example of a distributionally bounded representation. Let $\mathfrak{K}$ be a compact abelian group (other than $\{0\}$ or $\mathbb{T}$) with archimedean ordered dual, and with normalized Haar measure θ. In [6] the correspondence of Helson's theory between the cocycles on $\mathfrak{K}$ and the normalized simply invariant subspaces of $L^2(\mathfrak{K})$ was generalized to the setting of $L^p(\mathfrak{K})$ for $1 < p < \infty$. In brief, each unimodular cocycle A on $\mathfrak{K}$ gives rise to a strongly continuous one-parameter group $\{U_t\}$ of isometries on $L^p(\mathfrak{K})$. The group $\{U_t\}$ has a spectral decomposition from which the normalized simply invariant subspace corresponding to A is obtained. The group $\{U_t\}$ has the form

$$(U_t f)(x) = A_t(x) f(x + t), \quad \text{for } t \in \mathbb{R}, \ x \in \mathfrak{K}, \ f \in L^p(\mathfrak{K}).$$

It is clear from this that the group $\{U_t\}$ is the $L^p(\mathfrak{K})$-version of a θ-distributionally bounded representation of $\mathbb{R}$. (ii) It is not difficult to frame examples of strongly continuous distributionally bounded representations such that any constant c in (2.5) must be strictly greater than 1 [4, (2.27)-(ii)]. (iii) An example of a separation--preserving representation $\mathfrak{S}$ of $\mathbb{R}$ in $L^p(\mathbb{R})$ which does not satisfy (2.9)-(ii) is furnished by the strongly continuous one-parameter group of positivity-preserving isometries of $L^p(\mathbb{R})$ defined by

$$(\mathfrak{S}_t f)(x) = \mathrm{e}^{t/p} f(\mathrm{e}^t x) \quad \text{for } t \in \mathbb{R},\ x \in \mathbb{R},\ f \in L^p(\mathbb{R}).$$

3. TRANSFERENCE OF ALMOST EVERYWHERE CONVERGENCE UNDER CONDITIONS OF COMPACTNESS

In order to handle measure-theoretic technicalities of joint measurability without having to assume that $(S_u f)(x)$ is measurable in (u, x) on $G \times \mathcal{M}$, we begin this section with the following technical lemma ([7, proof of Lemma 2.5]).

LEMMA 3.1. *Let $u \to \Theta_u$ be a uniformly bounded strongly continuous representation of G in a closed subspace Y of $L^p(\mu)$. Let K be a compact subset of G, and let V be a relatively compact open subset of G. Let $y \in Y$. Then there exist a σ-finite measurable subset $\mathcal{M}_0$ of $\mathcal{M}$ and a jointly measurable function $F : G \times \mathcal{M} \to \mathbb{C}$ such that:*

(i) *F vanishes off $(V - K) \times \mathcal{M}_0$;*

(ii) *for λ-almost all $u \in V - K$, $F(u, \cdot)$ is a representing function for the equivalence class (modulo equality μ-a.e.) of $\Theta_u y$.*

If, moreover, $k \in L^1(G)$ has support contained in K, then for each $v \in V$,

$$\int_G k(u) F(v - u, \omega) \mathrm{d}\lambda(u)$$

exists for μ-almost all $\omega \in \mathcal{M}$, and, as a function of ω, is a representing function for the equivalence class of

$$\Theta_v \int_G k(u) \Theta_{-u} y \mathrm{d}\lambda(u), \tag{3.2}$$

the integral in (3.2) being a Bochner integral.

The following scholium provides a useful complement to Lemma 3.1. This technical scholium is readily deduced with the aid of [17, Theorem E, p. 139, and Theorem B, p. 27].

SCHOLIUM 3.3. *Let $F(\cdot, \cdot)$ be a measurable compex-valued function on $G \times \mathcal{M}$, and let $k \in L^1(G)$. Then the set $\mathcal{I}$ consisting of all $(v, \omega) \in G \times \mathcal{M}$ such that*

$k(u)F(v-u,\omega)$ is $\mathrm{d}\lambda(u)$-integrable is a measurable subset of $G \times \mathcal{M}$, and the integral $\int_G k(u)F(v-u,\omega)\mathrm{d}\lambda(u)$ defines a jointly measurable function of (v,ω) on $\mathcal{I}$.

The stage is now set for our first result on the transference of a.e. convergence.

THEOREM 3.4. *Let S be a strongly continuous representation of G in X_p satisfying (2.1), and let $\{k_n\}_{n=1}^{\infty}$ be a sequence of functions belonging to $L^1(G)$. Suppose that:*

(i) *there is a compact subset K of G such that, for each $n \in \mathbb{N}$, k_n vanishes λ-a.e. on $G \backslash K$;*

(ii) *for each $\psi \in L^p(G)$, $\{k_n * \psi\}_{n=1}^{\infty}$ converges λ-a.e. on G;*

(iii) *the maximal operator $\mathbf{M}_p^{\#}$ defined in (2.3) is of weak type (p,p) on X_p.*

Then for each $f \in X_p$, the sequence $\left\{H_{k_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$.

Proof. Fix $f \in X_p$, and let V be a relatively compact open neighborhood of the identity element of G. Applying Lemma 3.1 to $\Theta = S$, $Y = X_p$, K, V, and $y = f$, we obtain a σ-finite measurable $\mathcal{M}_0 \subseteq \mathcal{M}$ and a jointly measurable $F: G \times \mathcal{M} \to \mathbb{C}$ satisfying (3.1)-(i), (ii). Moreover, the remaining part of Lemma 3.1 shows that, for $n \in \mathbb{N}$, $v \in V$,

$$\int_G k_n(u)F(v-u,\omega)\mathrm{d}\lambda(u) = \left(H_{k_n}^{(p)} S_v f\right)(\omega), \text{ for } \mu\text{-almost all } \omega \in \mathcal{M}. \tag{3.5}$$

By Fubini's theorem,

$$\int_{\mathcal{M}_0} \left\{ \int_{V-K} |F(u,\omega)|^p \mathrm{d}\lambda(u) \right\} \mathrm{d}\mu(\omega) = \int_{V-K} \|S_u f\|_p^p \mathrm{d}\lambda(u) \leqslant$$
$$\leqslant \lambda(V-K)\alpha_p^p \|f\|_p^p < \infty.$$

Hence, in view of (3.1)-(i), there is a μ-null set $\mathcal{N}$ of $\mathcal{M}$ such that

$$F(\cdot,\omega) \in L^p(G) \text{ for all } \omega \in \mathcal{M} \backslash \mathcal{N}. \tag{3.6}$$

Let Ψ be the subset of $V \times \mathcal{M}_0$ given by

$$\left\{ (v,\omega) \in V \times \mathcal{M}_0 : \int_G k_n(u)F(v-u,\omega)\mathrm{d}\lambda(u) \text{ exists for all } n \in \mathbb{N}, \text{ and} \right.$$
$$\left. \lim_{n\to\infty} \int_G k_n(u)F(v-u,\omega)\mathrm{d}\lambda(u) \text{ exists in } \mathbb{C} \right\}.$$

In view of Scholium 3.3, Ψ is a measurable subset of $G \times \mathcal{M}$. It follows from (3.4)-(ii) and (3.6) that for each $\omega \in \mathcal{M}_0 \backslash \mathcal{N}$, there is a λ-null set G_ω of G such that

$$(V \backslash G_\omega) \times \{\omega\} \subseteq \Psi. \tag{3.7}$$

Denoting by χ_Ψ the characteristic function, defined on $G \times \mathcal{M}$, of Ψ, we see from (3.7) and Fubini's theorem that

$$\int_V \left\{ \int_{\mathcal{M}_0} (1 - \chi_\Psi(u,\omega)) \mathrm{d}\mu(\omega) \right\} \mathrm{d}\lambda(u) = 0.$$

Consequently, for λ-almost all $v \in V$,

$$(3.8) \qquad (v,\omega) \in \Psi \quad \text{for } \mu\text{-almost all } \omega \in \mathcal{M}_0.$$

It follows from (3.1)-(i), (3.5), and (3.8) that, for λ-almost all $v \in V$,

$$\left\{ \left(H_{k_n}^{(p)} S_v f \right)(\omega) \right\}_{n=1}^{\infty} \text{ converges in } \mathbb{C}, \text{ for } \mu\text{-almost all } \omega \in \mathcal{M}.$$

Since S is strongly continuous, and V is a neighborhood of the identity in G, it follows, in particular, that there is a sequence $\{v_j\}$ in V such that $\|S_{v_j} f - f\|_p \to 0$, as $j \to \infty$, and, for each $j \in \mathbb{N}$,

$$\left\{ \left(H_{k_n}^{(p)} S_{v_j} f \right)(\omega) \right\}_{n=1}^{\infty} \text{ converges for } \mu\text{-almost all } \omega \in \mathcal{M}.$$

Hence the set consisting of all $g \in X_p$ such that $\left\{ H_{k_n}^{(p)} g \right\}_{n=1}^{\infty}$ converges μ-a.e. is dense in X_p. In view of the Banach Principle, an application of (3.4)-(iii) completes the proof of Theorem 3.4. ■

COROLLARY 3.9. *Let $\mathfrak{B}$ be a compact abelian group, and suppose that the sequence $\{k_n\}_{n=1}^{\infty} \subseteq L^1(\mathfrak{B})$ has the property that*

$$\textit{for each } \psi \in L^p(\mathfrak{B}),\ \{k_n * \psi\}_{n=1}^{\infty} \textit{ converges } \lambda\textit{-a.e. on } \mathfrak{B}.$$

Let S be a strongly continuous representation of $\mathfrak{B}$ in X_p. If either

(i) *S is separation-preserving, and $N_p(\{k_n\}) < \infty$,*

or

(ii) *$S = R^{(p)}$ for some μ-distributionally bounded representation R of $\mathfrak{B}$, and*

$$N_p^{(w)}(\{k_n\}) < \infty,$$

then, for each $f \in X_p$, the sequence $\left\{ H_{k_n}^{(p)} f \right\}_{n=1}^{\infty}$ converges μ-a.e.

Proof. Use Theorem 3.4. together with Theorems 2.4 and 2.8. (Since $\mathfrak{B}$ is compact, the uniform boundedness of S follows automatically from strong continuity.) ■

Similarly, we have the following result.

COROLLARY 3.10. *Let S be a strongly continuous representation of the locally compact abelian group G in X_p satisfying (2.1), and let $\{k_n\}_{n=1}^{\infty}$ be a sequence of functions belonging to $L^1(G)$. Suppose that:*

(i) *there is a compact subset K of G such that, for each $n \in \mathbb{N}$, k_n vanishes λ-a.e. on $G \backslash K$;*

(ii) *for each $\psi \in L^p(G)$, $\{k_n * \psi\}_{n=1}^{\infty}$ converges λ-a.e. on G.*

If either

(a) *S is separation-preserving, and $N_p(\{k_n\}) < \infty$,*

or

(b) *$S = R^{(p)}$ for some μ-distributionally bounded representation R of G, and*

$$N_p^{(w)}(\{k_n\}) < \infty,$$

then, for each $f \in X_p$, the sequence $\left\{H_{k_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges μ-a.e.

EXAMPLES 3.11. When $\mathfrak{B} = \mathbb{T}$ and the representation S in Corollary 3.9 satisfies (3.9)-(i) (respectively, (3.9)-(ii)), then the corresponding hypotheses on the sequence $\{k_n\}_{n=1}^{\infty}$ in Corollary 3.9 are well-known to hold for the truncates of the periodic Hilbert kernel when $1 < p < \infty$ (respectively, $1 \leqslant p < \infty$) - see, e.g., [28, Theorems V.1.1, V.2.12, and V.2.13]. As a further illustration and application of the foregoing circle of ideas, we note that Corollary 3.10 permits transference from $\mathbb{R}^N$ of the Lebesgue Differentiation Theorem. This transference result (Theorem 3.12) includes Wiener's Local Ergodic Theorem ([29, Theorem III"]), which generalized Lebesgue's theorem to the setting of a finite measure space acted on by a multi--parameter group of measure-preserving transformations. More specifically, let χ_N denote the characteristic function on $\mathbb{R}^N$ of the closed unit ball $B_N = \{y \in \mathbb{R}^N : \|y\| \leqslant 1\}$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers convergent to 0. For each $n \in \mathbb{N}$, define $\kappa_n \in L^1(\mathbb{R}^N)$ by putting $\kappa_n(x) \equiv \dfrac{1}{\varepsilon_n^N \lambda(B_N)} \chi_N\left(\dfrac{x}{\varepsilon_n}\right)$.

THEOREM 3.12. *Let S be a uniformly bounded, strongly continuous representation of $\mathbb{R}^n$ in X_p. If either*

(i) *$1 < p < \infty$, and S is separation-preserving;*

or

(ii) *$1 \leqslant p < \infty$, and $S = R^{(p)}$ for some μ-distributionally representation R of $\mathbb{R}^N$, then, for each $f \in X_p$, $\left\{H_{\kappa_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges to f μ-a.e. on $\mathcal{M}$, where $\{\kappa_n\}_{n=1}^{\infty}$ is the sequence just described.*

Proof. It is well-known that in either of the cases (3.12)-(i), (ii), the sequence $\{\kappa_n\}_{n=1}^{\infty}$ satisfies the corresponding requirements of Corollary 3.10 [27, §I.1.3, Theorem 1-(b), (c) and Corollary 1]. Consequently for each $f \in X_p$, $\left\{H_{\kappa_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges

μ-a.e. on $\mathcal{M}$. To complete the proof it suffices to notice that $\left\|H_{\kappa_n}^{(p)} f - f\right\|_p \to 0$ as $n \to \infty$, which is a consequence of the following standard reasoning. A Hahn-Banach separation argument shows that X_p is the closed linear span of $\left\{H_\varphi^{(p)}(X_p) : \varphi \in L^1(\mathbb{R}^N)\right\}$. Our claim now ensues, since $\{\kappa_n\}_{n=1}^\infty$ is a bounded approximate identity for $L^1(\mathbb{R}^N)$. ■

In special circumstances, (3.4)-(ii) is known to imply that $N_p^{(w)}(\{k_n\}) < \infty$. In such cases the applications of Theorem 3.4 take on a particularly simple form. We close this section with a sample theorem illustrating this state of affairs.

THEOREM 3.13. *Suppose that* $1 \leqslant p \leqslant 2$, *and let* S *be a strongly continuous representation of the compact abelian group* $\mathfrak{B}$ *on* X_p *such that* $S = R^{(p)}$ *for some* μ*-distributionally bounded representation* R *of* $\mathfrak{B}$. *If the sequence* $\{k_n\}_{n=1}^\infty \subseteq L^1(\mathfrak{B})$ *has the property that*

(3.14) *for each* $\psi \in L^p(\mathfrak{B})$, $\{k_n * \psi\}_{n=1}^\infty$ *converges* λ*-a.e. on* $\mathfrak{B}$,

then for each $f \in X_p$, $\left\{H_{k_n}^{(p)} f\right\}_{n=1}^\infty$ *converges* μ*-a.e. on* $\mathcal{M}$.

Proof. Since $\mathfrak{B}$ is compact and $1 \leqslant p \leqslant 2$, the condition (3.14) implies $N_p^{(w)}(\{k_n\}) < \infty$, by [26, Theorem 1, p. 148]. An application of Corollary 3.9 completes the proof of Theorem 3.13. ■

4. GENERAL TRANSFERENCE OF ALMOST EVERYWHERE CONVERGENCE

Broadly speaking, our goal in this section is to transfer a.e.-convergence while relaxing the condition (3.10)-(i). In order to accomplish this, we shall require the following extension of Lemma 3.1 from compact K to σ-compact L.

LEMMA 4.1. *Let* $u \to \Theta_u$ *be a uniformly bounded strongly continuous representation of* G *in a closed subspace* Y *of* $L^p(\mu)$. *Let* L *be a* σ*-compact subset of* G, *and let* V *be a relatively compact open subset of* G. *Let* $y \in Y$. *Then there exist a* σ*-finite measurable subset* $\mathcal{N}$ *of* $\mathcal{M}$ *and a jointly measurable function* $F : G \times \mathcal{M} \to \mathbb{C}$ *such that:*

(i) F *vanishes off* $(V - L) \times \mathcal{N}$;

(ii) *for* λ*-almost all* $u \in V - L$, $F(u, \cdot)$ *is a representing function for the equivalence class (modulo equality* μ*-a.c.) of* $\Theta_u y$.

If, moreover, $k \in L^1(G)$ *vanishes* λ*-a.e. in* $G \backslash L$, *then for each* $v \in V$,

$$\int_G k(u) F(v - u, \omega) \mathrm{d}\lambda(u)$$

exists for μ-almost all $\omega \in \mathcal{M}$, and, as a function of ω, is a representing function for the equivalence class of

$$\Theta_v \int_G k(u)\Theta_{-u} y d\lambda(u),$$

the latter integral being a Bochner integral.

Proof. Let $L = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of compact subsets of G. For each $n \in \mathbb{N}$, apply Lemma 3.1 to K_n to get a σ-finite measurable subset $\mathcal{M}_n$ of $\mathcal{M}$ and a jointly measurable function $F_n : G \times \mathcal{M} \to \mathbb{C}$ such that:

(4.2) F_n vanishes off $(V - K_n) \times \mathcal{M}_n$;

(4.3) for λ-almost all $u \in V - K_n$, $F_n(u, \cdot)$ is a representing function for the equivalence class of $\Theta_u y$.

Put $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$. In particular,

$$F_n \text{ vanishes outside of } (V - K_n) \times \mathcal{N}, \quad \text{for each } n \in \mathbb{N}. \tag{4.4}$$

Suppose now that $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $m > n$. Let

$$\Delta_{m,n} = \{(u,\omega) \in (V - K_n) \times \mathcal{N} : F_n(u,\omega) \neq F_m(u,\omega)\}.$$

Since $V - K_m \supseteq V - K_n$, it follows from (4.3) that for λ-almost all $u \in V - K_n$,

$$F_m(u,\omega) = F_n(u,\omega), \quad \text{for } \mu\text{-almost all } \omega \in \mathcal{M}.$$

Consequently $\Delta_{m,n}$ has product measure 0 in $(V - K_n) \times \mathcal{N}$. Let

$$\mathfrak{A} = \bigcup_{n=1}^{\infty} \bigcup_{m>n} \Delta_{m,n}.$$

Thus $\mathfrak{A}$ has product measure 0 in $\left\{\bigcup_{n=1}^{\infty}(V - K_n)\right\} \times \mathcal{N} = (V - L) \times \mathcal{N}$.

Let $n \in \mathbb{N}$. It follows from the foregoing that if $(u,\omega) \in [(V - K_n) \times \mathcal{N}]\backslash\mathfrak{A}$, then $F_m(u,\omega) = F_n(u,\omega)$ for all $m \geqslant n$. Hence we can unambiguously define a function $F(\cdot,\cdot)$ on $\bigcup_{n=1}^{\infty}\{[(V - K_n) \times \mathcal{N}]\backslash\mathfrak{A}\} = [(V - L) \times \mathcal{N}]\backslash\mathfrak{A}$ by setting

$$F(u,\omega) = F_n(u,\omega) \text{ for } (u,\omega) \in [(V - K_n) \times \mathcal{N}]\backslash\mathfrak{A}. \tag{4.5}$$

Extend F to $G \times \mathcal{M}$ by setting $F = 0$ on $[G \times \mathcal{M}]\backslash\{[(V - L) \times \mathcal{N}]\backslash\mathfrak{A}\}$. Obviously F is jointly measurable on $G \times \mathcal{M}$, and (4.1)-(i) holds.

Since $\mathfrak{A}$ has product measure 0 in $(V - L) \times \mathcal{N}$, we see with the aid of Fubini's theorem (applied to the characteristic function of $\mathfrak{A}$) that for λ-almost all $u \in V - L$, we have

$$(4.6) \qquad (u, \omega) \notin \mathfrak{A}, \text{ for } \mu\text{-almost all } \omega \in \mathcal{N}.$$

For $n \in \mathbb{N}$, it follows from (4.5) and (4.6) that for λ-almost all $u \in V - K_n$,

$$F(u, \omega) = F_n(u, \omega), \text{ for } \mu\text{-almost all } \omega \in \mathcal{N}.$$

Hence by (4.4) and (4.1)-(i), we see that for λ-almost all $u \in V - K_n$,

$$F(u, \omega) = F_n(u, \omega), \text{ for } \mu\text{-almost all } \omega \in \mathcal{M}.$$

The conclusion (4.1)-(ii) follows readily from this, with the aid of (4.3).

To obtain the last assertion of the lemma, let $k \in L^1(G)$ vanish λ-a.e. in $G \backslash L$, and fix an element v of V. Using Bochner integration, we define the bounded linear mapping $T_k : Y \to Y$ by putting

$$T_k x = \int_G k(u) \Theta_{-u} x \, \mathrm{d}\lambda(u), \text{ for all } x \in Y.$$

We have $\Theta_v T_k y = \int_G k(u) \Theta_{v-u} y \, \mathrm{d}\lambda(u)$. Let q be the index conjugate to p. For any $g \in L^q(\mu)$ integration against $g \mathrm{d}\mu$ gives a continuous linear functional on Y. Hence

$$\int_{\mathcal{M}} (\Theta_v T_k y) g \, \mathrm{d}\mu = \int_L \int_{\mathcal{M}} k(u) \big(\Theta_{v-u} y\big)(\omega) g(\omega) \, \mathrm{d}\mu(\omega) \mathrm{d}\lambda(u).$$

Using (4.1)-(i), (ii), we see that

$$\int_{\mathcal{M}} (\Theta_v T_k y) g \, \mathrm{d}\mu = \int_L \int_{\mathcal{N}} k(u) F(v - u, \omega) g(\omega) \, \mathrm{d}\mu(\omega) \mathrm{d}\lambda(u).$$

Applying Fubini's theorem to this, we obtain:

$$(4.7) \qquad \int_{\mathcal{M}} (\Theta_v T_k y) g \, \mathrm{d}\mu = \int_{\mathcal{M}} \int_G k(u) F(v - u, \omega) g(\omega) \, \mathrm{d}\lambda(u) \mathrm{d}\mu(\omega).$$

Since $(\Theta_v T_k y)$ vanishes outside a set of σ-finite measure, and $\int_G k(u) F(v-u, \omega) \mathrm{d}\lambda(u) = 0$ if ω is not in the σ-finite set $\mathcal{N}$, we can complete the proof of Lemma 4.1 by letting g in (4.7) run through the characteristic functions of the subsets of $\mathcal{M}$ having finite measure. ■

Henceforth we shall denote the dual group of G by $\hat{G}$. For $k \in L^1(G)$, the Fourier transform of k will be written $\hat{k}$. Thus, $\hat{k}(\gamma) = \int_G k(u) \overline{\gamma(u)} \mathrm{d}\lambda(u)$, for all $\gamma \in \hat{G}$. In this section we shall develop variants of Corollary 3.10 in which the conditions (3.10)-(i), (ii) on a sequence $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$ will be replaced by the following set of conditions on $\{k_n\}_{n=1}^{\infty}$:

(4.8) for each $s \in G$ and each $\varepsilon > 0$, there is a corresponding relatively compact open neighborhood U of the identity element in G such that

$$\int_{G\setminus U} |k_n(u) - k_n(u+s)| d\lambda(u) < \varepsilon,$$

for all sufficiently large $n \in \mathbb{N}$;

(4.9) $\sup\left\{\left|\hat{k}_n(\gamma)\right| : n \in \mathbb{N},\ \gamma \in \hat{G}\right\} < \infty$;

(4.10) the sequence $\left\{\hat{k}_n(\gamma)\right\}_{n=1}^{\infty}$ converges in $\mathbb{C}$, for each $\gamma \in \hat{G}$.

Before proceeding to develop the results of this section, we comment briefly on the roles of the conditions (4.8)–(4.10). The condition (4.8), reminiscent of a condition of Hörmander for singular integrals [27,(2′) on p. 34], is a weakening of (3.10)-(i) that broadens the scope of our considerations. The conditions (4.9) and (4.10) will replace the condition (3.10)-(ii), which they imply in the presence of a suitable maximal estimate. To be more precise, let $\mathcal{T}(G)$ be the linear space $\{\varphi \in L^1(G) : \hat{\varphi}$ has compact support$\}$. It follows from (4.9) and (4.10) (using Fourier inversion, and dominated convergence on $\hat{G}$) that

(4.11) for each $\varphi \in \mathcal{T}(G)$, the sequence $\{k_n * \varphi\}_{n=1}^{\infty}$ is uniformly bounded and converges pointwise on G.

Since $\mathcal{T}(G)$ is norm-dense in $L^p(G)$, (4.11), taken in conjunction with the Banach Principle, has the following obvious consequence.

SCHOLIUM 4.12. *Suppose that $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$, $1 \leqslant p < \infty$, and $N_p^{(w)}(\{k_n\}) < \infty$. If (4.9) and (4.10) hold, then for each $\psi \in L^p(G)$, the sequence $\{k_n * \psi\}_{n=1}^{\infty}$ converges λ-a.e. on G.*

In order to give effect to the conditions (4.8)–(4.10) on $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$, we shall require the following general result concerning representations.

THEOREM 4.13. *Suppose that $1 < p < \infty$, and S is a uniformly bounded, strongly continuous representation of G in X_p. Let*

$$X_1^{(p)} = \{f \in X_p : S_u f = f, \text{ for all } u \in G\},$$

and let $X_2^{(p)}$ be the linear span in X_p of

$$\left\{f - S_s f : s \in G, f = H_g^{(p)} h \text{ for some } h \in L^p(\mu) \cap L^\infty(\mu), \text{ and some } g \in \mathfrak{C}_{00}(G)\right\},$$

where $\mathfrak{C}_{00}(G)$ is the set of all continuous complex-valued functions on G having compact support. Then

$$D_p \equiv X_1^{(p)} + X_2^{(p)}$$

is norm-dense in X_p. Moreover, if $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$ satisfies (4.10), then

$$(4.14) \qquad \left\{H_{k_n}^{(p)} x\right\}_{n=1}^{\infty} \text{ converges } \mu\text{-a.e. on } \mathcal{M}, \text{ for all } x \in X_1^{(p)}.$$

Proof. Since $1 < p < \infty$, X_p is reflexive, and it follows from [5, Theorem 2.1] that

$$X_p = X_1^{(p)} \oplus \operatorname{clm}\{(I - S_u)X_p : u \in G\},$$

where I denotes the identity operator of X_p, and "clm" stands for "closed linear span of". Since the linear span of $\left\{H_{\psi}^{(p)} X_p : \psi \in L^1(G)\right\}$ is norm-dense in X_p, $\mathfrak{C}_{00}(G)$ is norm-dense in $L^1(G)$, and $L^p(\mu) \cap L^{\infty}(\mu)$ is norm-dense in X_p, the desired conclusion for D_p now follows.

Suppose next that $x \in X_1^{(p)}$. For each $n \in \mathbb{N}$,

$$H_{k_n}^{(p)} x = \left[\int_G k_n(u) d\lambda(u)\right] x = \hat{k}_n(0) x.$$

Application of (4.10) completes the proof of Theorem 4.13. ∎

REMARK. In the case $p = 1$, the linear manifold D_p as defined in the statement of Theorem 4.13 need not be norm-dense in X_p (see [5, proof of Theorem (3.8)-(ii)]).

The stage is now set for the main result of this section, which is expressed in the following theorem.

THEOREM 4.15. *Let R be a strongly continuous μ-distributionally bounded representation of G, and let $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$ satisfy the conditions (4.8)–(4.10). Suppose that $1 < p < \infty$, $N_p(\{k_n\}) < \infty$, and put $S = R^{(p)}$. Then for each $\psi \in L^p(G)$, the sequence $\{k_n * \psi\}_{n=1}^{\infty}$ converges λ-a.e. on G and also in the norm topology of $L^p(G)$. Moreover, in the notation of (2.2) and (2.3), we have that $\mathbf{M}_p^{\#}$ is of strong type (p,p) on X_p, and for each $f \in L^p(\mu)$, the sequence $\left\{H_{k_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$ and in the norm topology of X_p.*

Proof. The asserted convergence λ-a.e. of $\{k_n * \psi\}_{n=1}^{\infty}$ is assumed by Scholium 4.12. The convergence of this sequence in the norm topology of $L^p(G)$ then follows by dominated convergence, since $N_p(\{k_n\}) < \infty$. Since S is separation-preserving, Theorem 2.4 shows that $\mathbf{M}_p^{\#}$ is of strong type (p,p). In order to complete the proof, it now suffices, in view of Theorem 4.13, to show that $\left\{H_{k_n}^{(p)} \Psi\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$, whenever $\Psi \in X_2^{(p)}$ has the form

(4.16) $\Psi = f - S_s f$, where $s \in G$ and $f = H_g^{(p)} h$, for some $g \in \mathfrak{C}_{00}(G)$ and some $h \in L^p(\mu) \cap L^{\infty}(\mu)$.

Fix Ψ, s, f, g, and h as in (4.16). For each $n \in \mathbb{N}$, we choose a fixed representative of the equivalence class (modulo equality μ-a.e. on $\mathcal{M}$) of $H_{k_n}^{(p)} \Psi$. This representative (also denoted $H_{k_n}^{(p)}\Psi$) will be fixed throughout the remainder of the present proof. Let K_0 denote the compact support of g. Henceforth L will be a fixed σ-compact subset of G such that L is symmetric and contains

$$K_0,\ \bigcup_{n=1}^{\infty}\{x \in G : k_n(x) \neq 0\},\ \bigcup_{n=1}^{\infty}\{x \in G : k_n(x) - k_n(x+s) \neq 0\},$$
$$\left[\bigcup_{n=1}^{\infty}\{x \in G : k_n(x) \neq 0\}\right] + K_0, \text{ and}$$
$$\left[\bigcup_{n=1}^{\infty}\{x \in G : k_n(x) - k_n(x+s) \neq 0\}\right] + K_0.$$

Let $\varepsilon > 0$ be given. In accordance with (4.8), choose a symmetric, relatively compact open neighborhood U_ε of the identity element of G such that $s \in U_\varepsilon$, and for some $n_\varepsilon \in \mathbb{N}$, we have

$$\int_{G\setminus U_\varepsilon} |k_n(u) - k_n(u+s)| \mathrm{d}\lambda(u) < \varepsilon, \text{ for all } n \geqslant n_\varepsilon. \tag{4.17}$$

Choose $\mathcal{N}_\varepsilon \subseteq \mathcal{M}$ and $F_\varepsilon : G \times \mathcal{M} \to \mathbb{C}$ for $S,\ L,\ U_\varepsilon$, and h in accordance with Lemma 4.1.

For $n \in \mathbb{N}$, Bochner integration shows easily that

$$H_{k_n}^{(p)}\Psi = H_{[k_n-(k_n)_s]*g}^{(p)} h, \tag{4.18}$$

where $(k_n)_s$ denotes the translate of k_n by s. Notice that for each $n \in \mathbb{N}$,

(4.19) $|k_n| * |g|$ and $|[k_n - (k_n)_s]| * |g|$ each vanish on $G\setminus L$.

It follows from (4.18), (4.19), and the last conclusion of Lemma 4.1 that for each $n \in \mathbb{N}$,

$$(H_{k_n}^{(p)}\Psi)(\omega) = \int_G ([k_n - (k_n)_s] * g)(u) F_\varepsilon(-u, \omega) \mathrm{d}\lambda(u), \tag{4.20}$$

for μ-almost all $\omega \in \mathcal{M}$.

Hence for each $n \in \mathbb{N}$, we have for μ-almost all $\omega \in \mathcal{M}$,

$$(H_{k_n}^{(p)}\Psi)(\omega) = \int_G \int_G [k_n - (k_n)_s](u-t) g(t) F_\varepsilon(-u, \omega) \mathrm{d}\lambda(t) \mathrm{d}\lambda(u). \tag{4.21}$$

It follows from Theorem 2.9-(ii) and Lemma 4.1-(ii) that for λ-almost all $u \in$ $\in U_\varepsilon - L$,

$$\|F_\varepsilon(-u, \cdot)\|_{L^\infty(\mu)} \leqslant \|h\|_{L^\infty(\mu)}. \tag{4.22}$$

Applying (4.22) and Fubini's theorem to the characteristic function defined on $(U_\varepsilon - -L) \times \mathcal{N}_\varepsilon$ of the set

$$\{(u,\omega) \in (U_\varepsilon - L) \times \mathcal{N}_\varepsilon : |F_\varepsilon(-u,\omega)| > \|h\|_\infty\},$$

we find, after taking Lemma 4.1-(i) into account, that for μ-almost all $\omega \in \mathcal{M}$,

$$|F_\varepsilon(-u,\omega)| \leqslant \|h\|_\infty, \quad \text{for } \lambda\text{-almost all } u \in U_\varepsilon - L. \tag{4.23}$$

It follows from (4.23) and (4.19), with the aid of Fubini's theorem for locally compact spaces [20, Theorem 13.9], that for each $n \in \mathbb{N}$, we have for μ-almost all $\omega \in \mathcal{M}$,

$$[k_n - (k_n)_s](u-t)g(t)F_\varepsilon(-u,\omega) \in L^1(\mathrm{d}\lambda(t) \times \mathrm{d}\lambda(u)).$$

It follows by another application of Fubini's theorem for locally compact spaces, this time to (4.21), that for each $n \in \mathbb{N}$, we have for μ-almost all $\omega \in \mathcal{M}$,

$$(H_{k_n}^{(p)}\Psi)(\omega) = \int_G\int_G [k_n - (k_n)_s](u-t)g(t)F_\varepsilon(-u,\omega)\mathrm{d}\lambda(u)\mathrm{d}\lambda(t) =$$

$$= \int_G\int_G [k_n - (k_n)_s](u)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(u)\mathrm{d}\lambda(t) = \tag{4.24}$$

$$= \int_G\int_G [k_n - (k_n)_s](u)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u).$$

Let $\chi_{U_\varepsilon - K_0}$ denote the characteristic function, defined on G, of $U_\varepsilon - K_0$, and, for each $n \in \mathbb{N}$, let $\mathfrak{N}_n = \{x \in G : k_n(x) - k_n(x+s) \neq 0\}$. From (4.24) we see that for each $n \in \mathbb{N}$, the following holds for μ-almost all $\omega \in \mathcal{M}$:

$$(H_{k_n}^{(p)}\Psi)(\omega) =$$

$$= \int_{U_\varepsilon}\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) + \tag{4.25}$$

$$+ \int_{(G\setminus U_\varepsilon)\cap\mathfrak{N}_n}\int_{K_0} [k_n - (k_n)_s](u)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u).$$

By our choice of L, $\mathfrak{N}_n + K_0 \subseteq L$. It follows from this, (4.17), and (4.23) that for $n \geqslant n_\varepsilon$, we have for μ-almost all $\omega \in \mathcal{M}$,

$$\left|\int_{(G\setminus U_\varepsilon)\cap\mathfrak{N}_n}\int_{K_0} [k_n - (k_n)_s](u)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u)\right| \leqslant$$

$$\leqslant \|h\|_{L^\infty(\mu)}\|g\|_{L^1(\lambda)}\varepsilon. \tag{4.26}$$

Similar reasoning based on (4.23) shows that for each $n \in \mathbb{N}$, we have for μ-almost all $\omega \in \mathcal{M}$,

$$[k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega) \in L^1(\mathrm{d}\lambda(t) \times \mathrm{d}\lambda(u)).$$

Consequently for each $n \in \mathbb{N}$, we have that for μ-almost all $\omega \in \mathcal{M}$, we can add

$$\int_{G\setminus U_\varepsilon}\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u)$$

to both sides of (4.25) to get:

$$(H^{(p)}_{k_n}\Psi)(\omega) +$$

$$+\int_{G\setminus U_\varepsilon}\int_G [k_n - (k_n)_s](u)\xi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) =$$

$$= \int_G\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) + \tag{4.27}$$

$$+\int_{(G\setminus U_\varepsilon)\cap\mathfrak{N}_n}\int_{K_0} [k_n - (k_n)_s](u)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u).$$

Another application of (4.23), this time to the iterated integral occurring in the first member of the equation (4.27), shows that for $n \geqslant n_\varepsilon$, we have for μ-almost all $\omega \in \mathcal{M}$ that this iterated integral has absolute value not exceeding $\|h\|_{L^\infty(\mu)}\|g\|_{L^1(\lambda)}\varepsilon$. Combining this fact with (4.26), we see from (4.27) that for each $n \geqslant n_\varepsilon$, the following holds for μ-almost all $\omega \in \mathcal{M}$,

$$\left| (H^{(p)}_{k_n}\Psi)(\omega) - \int_G\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) \right| \leqslant$$

$$\leqslant 2\varepsilon\|h\|_{L^\infty(\mu)}\|g\|_{L^1(\lambda)}. \tag{4.28}$$

Notice that we can also assert that for μ-almost all $\omega \in \mathcal{M}$, we have (4.28) for all $n \geqslant n_\varepsilon$. Applying the translation-invariance of Haar measure and Fubini's theorem to the iterated integral in (4.28), we see that for μ-almost all $\omega \in \mathcal{M}$, we have for $n \geqslant n_\varepsilon$:

$$\int_G\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-u-t)g(t)F_\varepsilon(-u-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) =$$

$$= \int_G\int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-t)g(t-u)F_\varepsilon(-t,\omega)\mathrm{d}\lambda(t)\mathrm{d}\lambda(u) =$$

$$(4.29) \qquad = \int_G \int_G [k_n - (k_n)_s](u)\chi_{U_\varepsilon - K_0}(-t) g(t-u) F_\varepsilon(-t,\omega) \mathrm{d}\lambda(u) \mathrm{d}\lambda(t) =$$

$$= \int_G \chi_{U_\varepsilon + K_0}(t) F_\varepsilon(-t,\omega)\big([k_n - (k_n)_s] * g\big)(t) \mathrm{d}\lambda(t).$$

It is an immediate consequence of the facts established at the outset of the proof of the present theorem that $\{[k_n - (k_n)_s] * g\}_{n=1}^{\infty}$ converges in the norm topology of $L^p(G)$ to a fixed function $\Phi \in L^p(G)$, which is independent of ε, $n \in \mathbf{N}$, and $\omega \in \mathcal{M}$. Moreover, for μ-almost all $\omega \in \mathcal{M}$, it follows from (4.23) that

$$\chi_{U_\varepsilon + K_0}(t) F_\varepsilon(-t,\omega) \in L^q(\mathrm{d}\lambda(t)), \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Applying these facts to (4.29) and (4.28) we find that for μ-almost all $\omega \in \mathcal{M}$, we have for all sufficiently large $m \in \mathbf{N}$, $n \in \mathbf{N}$,

$$\left|\left(H_{k_n}^{(p)}\Psi\right)(\omega) - \left(H_{k_m}^{(p)}\Psi\right)(\omega)\right| < 4\varepsilon \|h\|_{L^\infty(\mu)} \|g\|_{L^1(\lambda)} + 2\varepsilon.$$

We have shown that for each $\eta > 0$, there is a measurable subset $\mathcal{D}_\eta$ of $\mathcal{M}$ such that $\mu(\mathcal{D}_\eta) = 0$ and such that for each $\omega \in \mathcal{M} \setminus \mathcal{D}_\eta$, we have

$$\left|(H_{k_n}^{(p)}\Psi)(\omega) - (H_{k_m}^{(p)}\Psi)(\omega)\right| < \eta, \text{ for all sufficiently large } m \in \mathbf{N}, \ n \in \mathbf{N}.$$

By letting η run through the sequence $\left\{\frac{1}{j}\right\}_{j=1}^{\infty}$, we see that the sequence $\left\{H_{k_n}^{(p)}\Psi\right\}_{n=1}^{\infty}$ is Cauchy μ-a.e. on $\mathcal{M}$. This completes the proof of Theorem 4.15 ∎

COROLLARY 4.30. *Let R be a strongly continuous μ-distributionally bounded representation of G, and let $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$ satisfy the conditions (4.8)-(4.10). Suppose that $N_1^{(w)}(\{k_n\}) < \infty$, and that for some $r \in (1,+\infty)$, we have $N_r^{(w)}(\{k_n\}) <$ $< \infty$. Then for each $\psi \in L^1(G)$, the sequence $\{k_n * \psi\}_{n=1}^{\infty}$ converges λ-a.e. on G, and for each $f \in L^1(\mu)$, the sequence $\left\{H_{k_n}^{(1)} f\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$, where $H_{k_n}^{(1)} f$ is defined by (2.2) for the representation $S \equiv R^{(1)}$.*

Proof. The first conclusion is an immediate consequence of Scholium 4.12. To obtain the second conclusion, fix an index $p \in (1, r)$. By the Marcinkiewicz interpolation theorem [15, Theorem II.2.11], $N_p(\{k_n\}) < \infty$. Hence we can apply Theorem (4.15) to $R^{(p)}$, and thereby infer that for each $f \in L^p(\mu)$, the sequence $\left\{H_{k_n}^{(p)} f\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$. Recalling the general definitions for $R^{(p)}$ and $R^{(1)}$, we see immediately that for each $g \in L^1(\mu) \cap L^\infty(\mu)$, the sequence $\left\{H_{k_n}^{(1)} g\right\}_{n=1}^{\infty}$ converges μ-a.e. on $\mathcal{M}$. By Theorem 2.8, the maximal operator defined by the sequence $\left\{H_{k_n}^{(1)}\right\}_{n=1}^{\infty}$ is of weak type $(1,1)$ on $L^1(\mu)$. Since $L^1(\mu) \cap L^\infty(\mu)$ is norm-dense in $L^1(\mu)$, the proof of the Corollary can now be completed by invoking the Banach Principle. ∎

EXAMPLE AND REMARKS. 4.31. The two general results in this section regarding transference of almost everywhere convergence (Theorem 4.15 and Corollary 4.30) have imposed on the convolution kernels extra conditions in the form of (4.8)–(4.10) together with suitable maximal bounds. It is well-known that in the absence of any extra conditions a sequence of convolution kernels $\{k_n\}_{n=1}^{\infty} \subseteq L^1(G)$ can produce λ-a.e. convergence when applied to each element of $L^p(G)$, whereas some representation of G by measure-preserving transformations of the points of $\mathcal{M}$ will fail to transfer this almost everywhere convergence to X_p (for an example when $p = 1$, see [14, Theorem 1]). As a counterpoint to the results of this section, we include here a simple example of this phenomenon in which such a representation does not transfer the almost everywhere convergence to X_p, for any $p \in [1, +\infty)$. Take $G = \mathbb{R}$, and let λ be Lebesgue measure. Define the sequence $\{k_n\}_{n=1}^{\infty}$ by taking k_n to be the characteristic function of the interval $[n, n+1]$ for each $n \in \mathbb{N}$. Fix $p \in [1, +\infty)$. Given $\psi \in L^p(\mathbb{R})$, it is easy to see from Hölder's inequality that for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\left|(k_n * \psi)(x)\right|^p \leqslant \int_{x-n-1}^{x-n} |\psi(y)|^p \mathrm{d}y,$$

whence $\sum_{n=1}^{\infty} \left|(k_n * \psi)(x)\right|^p < \infty$, and consequently

$$\{k_n * \psi\}_{n=1}^{\infty} \text{ converges to 0 pointwise on } \mathbb{R}, \quad \text{for each } \psi \in L^p(\mathbb{R}). \tag{4.32}$$

Since for each $n \in \mathbb{N}$ and $s \in \mathbb{R}$

$$\hat{k}_n(s) = \begin{cases} 1, & \text{if } s = 0, \\ \dfrac{1 - \mathrm{e}^{-\mathrm{i}s}}{\mathrm{i}s} \mathrm{e}^{-\mathrm{i}ns}, & \text{if } s \neq 0, \end{cases}$$

it follows that

$$\left\{\hat{k}_n(s)\right\}_{n=1}^{\infty} \text{ converges if and only if } s \in 2\pi\mathbb{Z}. \tag{4.33}$$

Clearly (4.9) holds and (4.10) fails in the present circumstances.

For any relatively compact open neighborhood U of 0 in $\mathbb{R}$, it is easily seen that for all sufficiently large $n \in \mathbb{N}$,

$$\int_{\mathbb{R}\setminus U} |k_n(u) - k_n(u+1)| \mathrm{d}u = 2.$$

Hence (4.8) now fails when $s = \varepsilon = 1$.

Next we show that in the present set-up, $N_p^{(w)}(\{k_n\}) = \infty$. Let $\mathfrak{M}$ denote the maximal convolution operator on $L^p(\mathbf{R})$ defined by the sequence $\{k_n\}$. Thus, for $\psi \in L^p(\mathbf{R})$ and $x \in \mathbf{R}$,

$$(\mathfrak{M}\psi)(x) = \sup_{n\in\mathbf{N}} \left| \int_{x-n-1}^{x-n} \psi(y)\mathrm{d}y \right| .$$

It is easily seen that $(\mathfrak{M}\psi)(x+1) \geqslant (\mathfrak{M}\psi)(x)$. Using this we can readily infer that for $n \in \mathbf{Z}$, $\nu \in \mathbf{N}$, and t a positive real number, we have

$$\lambda\left([n, n+1) \cap \{x \in \mathbf{R} : (\mathfrak{M}\psi)(x) > t\}\right) \leqslant$$
$$\leqslant \lambda\left([n+\nu-1, n+\nu) \cap \{x \in \mathbf{R} : (\mathfrak{M}\psi)(x) > t\}\right).$$

So taking $\nu = 1, 2, \dots, m$, we obtain

$$m\lambda\left([n, n+1) \cap \{x \in \mathbf{R} : (\mathfrak{M}\psi)(x) > t\}\right) \leqslant$$
$$\leqslant \lambda\left(\{x \in \mathbf{R} : x \geqslant n, \text{ and } (\mathfrak{M}\psi)(x) > t\}\right) \leqslant$$
$$\leqslant \lambda\left(\{x \in \mathbf{R} : (\mathfrak{M}\psi)(x) > t\}\right).$$

If $N_p^{(w)}(\{k_n\})$ were finite, we could let $m \to +\infty$ to conclude that

$$\mathfrak{M}\psi = 0 \ \lambda-\text{a.e.} \quad \text{for every } \psi \in L^p(\mathbf{R}),$$

which is obviously false. So $N_p^{(w)}(\{k_n\}) = \infty$, as asserted.

To complete the considerations of this example, we shall exhibit a representation of $G = \mathbf{R}$ by measure-preserving transformations of $\mathbf{T}$ such that the induced representation S in $L^p(\mathbf{T})$ does not transfer the almost everywhere convergence expressed in (4.32). Let S be the strongly continuous representation of $\mathbf{R}$ by isometries of $L^p(\mathbf{T})$ defined as follows:

$$(S_t f)(z) = f(e^{\mathrm{i}\,t}z), \quad \text{for } t \in \mathbf{R},\ f \in L^p(\mathbf{T}),\ z \in \mathbf{T}.$$

Take $f_0 \in L^p(\mathbf{T})$ to be the identity mapping of $\mathbf{T}$. Then

$$H_{k_n}^{(p)} f_0 = \hat{k}_n(1) f_0.$$

Since $1 \notin 2\pi\mathbf{Z}$, it follows from (4.33) that $\left\{(H_{k_n}^{(p)} f_0)(z)\right\}_{n=1}^{\infty}$ is divergent for almost all $z \in \mathbf{T}$.

5. SOME APPLICATIONS

Having already described some applications of §3 in (3.11) and (3.12), we now take up some applications of §4 by specializing the considerations therein to classical kernels. As will be noted, these results generalize their classical ergodic theory antecedents to the setting of distributionally bounded representations.

The counterpart of the discrete pointwise ergodic theorem. (5.1) For $n = 0, 1, \ldots$, let χ_n be the characteristic function defined on $G = \mathbb{Z}$ of $\{m \in \mathbb{Z} : -n \leqslant m \leqslant 0\}$, and define $k_n \in \ell^1(\mathbb{Z})$ by putting $k_n = \frac{1}{(n+1)}\chi_n$. It is a well-known result of Hardy and Littlewood that $N_p(\{k_n\}) \leqslant \frac{p}{p-1}$, for $1 < p < \infty$ ([18], [19, Theorems 326 and 394]). It is also well-known from [30] (see [13, Lemma VIII.6.7 and p. 729]) that $N_1^{(w)}(\{k_n\}) < \infty$ (as can be seen, for instance, by applying [13, Lemma VIII.6.7] to the left shift on $\ell^1(\mathbb{Z})$). The conditions (4.8)–(4.10) are readily verified for $\{k_n\}_{n=1}^{\infty}$. Thus Theorem 4.15 and Corollary 4.30 when applied to the present sequence $\{k_n\}_{n=1}^{\infty}$ include Birkhoff's pointwise ergodic theorem [8]. In view of Theorem 2.9, our result in this instance affords a slight extension of the Dunford and Schwartz pointwise ergodic theorem [13, Theorem VIII.6.6] in that we do not require the operator T in the discrete ergodic averages to be contractive on X_p; however, in contrast to the Dunford and Schwartz context, our group setting does require hypotheses regarding the invertibility of T. A precise formulation of the discrete pointwise ergodic theorem resulting from Theorem 4.15 and Corollary 4.30, in conjunction with Theorem 2.9, is as follows.

PROPOSITION. *Let $T : X_1 \to X_1$ be an invertible linear operator such that*

$$\sup\{\|T^m\|_1 : m \in \mathbb{Z}\} < \infty,$$

and suppose that for each $f \in L^1(\mu) \cap L^\infty(\mu)$, we have $\|Tf\|_\infty = \|T^{-1}f\|_\infty = \|f\|_\infty$. Then for each $p \in [1, +\infty)$, T has a unique extension from $L^1(\mu) \cap L^\infty(\mu)$ to a bounded linear mapping of X_p into X_p (also denoted by T), and for each $f \in X_p$, the sequence

$$\left\{ \frac{1}{n+1} \sum_{m=0}^{n} T^m f \right\}_{n=0}^{\infty}$$

converges μ-a.e. on $\mathcal{M}$.

We also remark here that, in particular, the pointwise ergodic theorem has been generalized in the direction of measure-preserving actions of σ-compact amenable locally compact groups in σ-finite measure spaces (see, e.g., [14] and the references therein).

The classical Calderón-Zygmund singular integral kernels. (5.2) In [12] M. Cotlar showed that theorems on singular integrals could be generalized in the context of ergodic theory. Subsequently, in [9], A. P. Calderón showed that the maximal ergodic theorem and the results of Cotlar on ergodic singular integral operators could be made to follow directly from the known properties of the relevant classical operators. Calderón's methods proceeded from multi-parameter groups of measure-preserving transformations acting in a σ-finite measure space, and provided maximal bounds and existence theorems for ergodic singular integral operators. This seminal approach was expanded into a wider framework of transference methods in [10] and [11]. In the same spirit, Corollary 3.10, Theorem 4.15, and Corollary 4.30 above can be viewed as descendants of the results on almost everywhere convergence in [9, Theorems 2 and 3]. We shall now sketch a proof that the truncates of singular integral kernels satisfy the hypotheses of Theorem 4.15 and Corollary 4.30. This will generalize the existence results for ergodic singular integrals in [12] and [9] to the setting of distributionally bounded representations.

We shall consider a Calderón-Zygmund singular kernel $k : \mathbf{R}^n\backslash\{0\} \to \mathbf{C}$ having the form

$$k(x) = \frac{\Omega(x)}{\|x\|^N} \quad (x \in \mathbf{R}^N\backslash\{0\}),$$

for some function Ω on $\mathbf{R}^N\backslash\{0\}$ satisfying:

(5.3) $\Omega(\alpha x) = \Omega(x)$, for $\alpha > 0$, $x \in \mathbf{R}^N\backslash\{0\}$;

(5.4) $\int_{\Sigma_{N-1}} \Omega(x)d\varsigma(x) = 0$, where $\Sigma_{N-1} = \{x \in \mathbf{R}^N : \|x\| = 1\}$, and $d\varsigma$ is the induced Euclidean surface measure on Σ_{N-1};

(5.5) $\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < +\infty$, where

$$\omega(\delta) = \sup_{\substack{\|x-x'\| \leqslant \delta \\ \|x\|=\|x'\|=1}} |\Omega(x) - \Omega(x')|.$$

For $n \in \mathbf{N}$, define $k_n \in L^1(\mathbf{R}^N)$ by putting

$$k_n(x) = \begin{cases} k(x), & \text{for } \frac{1}{n} \leqslant \|x\| \leqslant n; \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

It follows from (5.4) by using polar coordinates that $\hat{k}_n(0) = 0$ for all $n \in \mathbf{N}$, and this fact together with [27, §II.4.3-(i), (ii)] shows that $\{k_n\}_{n=1}^\infty$ satisfies (4.9) and (4.10). Moreover, from [27, Theorem 4 in §II.4.5], we have $N_p(\{k_n\}) < \infty$ for $1 < p < \infty$, and $N_1^{(w)}(\{k_n\}) < \infty$. Thus, in order to show that the present sequence of kernels $\{k_n\}_{n=1}^\infty$ satisfies the hypotheses of Theorem 4.15 and Corollary 4.30, it remains only to verify (4.8). Since the condition (4.8) is roughly akin to a "$o(1)$" variation of

Hörmander's condition for a single kernel ([27, (2′), p. 34]), we proceed from the fact that our kernel k is known to satisfy Hörmander's condition [27, §II.4.2]. This implies that for $y \in \mathbf{R}^N\backslash\{0\}$, $|k(x) - k(x+y)|$ is a Lebesgue integrable function of x over $\{x \in \mathbf{R}^N : \|x\| \geqslant 2\|y\|\}$, and so, given $y \in \mathbf{R}^N\backslash\{0\}$ and $\varepsilon > 0$, there is an $M_0 > \|y\|$ such that:

$$\int_{\|x\|>M} |k(x) - k(x+y)|\mathrm{d}x < \varepsilon, \quad \text{for all } M \geqslant M_0. \tag{5.7}$$

For $n \in \mathbf{N}$ such that $n > M_0$ and $M_0 - \|y\| > \frac{1}{n}$, we have:

$$\begin{aligned}\int_{\|x\|>M_0} |k_n(x+y) - k_n(x)|\mathrm{d}x &= \int_{\|x\|>M_0, \|x+y\| \geqslant n, \|x\| \leqslant n} |k_n(x)|\mathrm{d}x + \\ &+ \int_{\|x\|>M_0, \|x+y\|<n, \|x\| \leqslant n} |k(x+y) - k(x)|\mathrm{d}x + \\ &+ \int_{\|x+y\|<n, \|x\|>n} |k_n(x+y)|\mathrm{d}x.\end{aligned}$$

This shows, with the aid of (5.7), that:

$$\begin{aligned}\int_{\|x\|>M_0} |k_n(x+y) - k_n(x)|\mathrm{d}x &< \varepsilon + \int_{n-\|y\| \leqslant \|x\| \leqslant n} |k_n(x)|\mathrm{d}x + \\ &+ \int_{n-\|y\| \leqslant \|x+y\| \leqslant n} |k_n(x+y)|\mathrm{d}x.\end{aligned}$$

We have found that, for $n > \max\left\{M_0, \frac{1}{M_0 - \|y\|}\right\}$,

$$\int_{\|x\|>M_0} |k_n(x+y) - k_n(x)|\mathrm{d}x < \varepsilon + 2\int_{n-\|y\| \leqslant \|x\| \leqslant n} |k(x)|\mathrm{d}x.$$

Changing to polar coordinates in the last integral, we obtain:

$$\int_{n-\|y\| \leqslant \|x\| \leqslant n} |k(x)|\mathrm{d}x \leqslant \varsigma(\Sigma_{N-1})\|\Omega\|_\infty \log \frac{n}{n - \|y\|}.$$

This establishes (4.8) for our sequence of truncates of k. We summarize the outcome of this discussion in the following.

PROPOSITION 5.8. *Let $k : \mathbf{R}^N\backslash\{0\} \to \mathbf{C}$ have the form*

$$k(x) \equiv \frac{\Omega(x)}{\|x\|^N},$$

for some function Ω on $\mathbf{R}^N\backslash\{0\}$ satisfying (5.3), (5.4), and (5.5). Then the sequence of truncates $\{k_n\}_{n=1}^\infty \subseteq L^1(\mathbf{R}^N)$ defined in (5.6) satisfies the hypotheses of Theorem 4.15 and Corollary 4.30.

We remark that, in particular, Proposition 5.8 includes the sequence of truncates for the Hilbert kernel of $\mathbb{R}$.

The discrete Hilbert transform. (5.9) As would be expected from Proposition 5.8, the sequence of truncated discrete Hilbert kernels on $G = \mathbb{Z}$ satisfies the hypotheses of Theorem 4.15 and Corollary 4.30. In order to avoid repetitious ideas, we shall only describe the bare essentials of the analogous proof for the discrete Hilbert transform. Let $h : \mathbb{Z} \to \mathbb{R}$ be the discrete Hilbert kernel defined by $h(k) = k^{-1}$ for $k \in \mathbb{Z}\backslash\{0\}$, and $h(0) = 0$. For $n \in \mathbb{N}$, let $h_n \in \ell^1(\mathbb{Z})$ be the n-th truncate of h defined by putting $h_n(k) := h(k)$ for $k \in \mathbb{Z}$, $|k| \leqslant n$, and $h_n(k) = 0$ for $k \in \mathbb{Z}$, $|k| > n$. For all $k \in \mathbb{Z}$, $h(k) = \hat{g}_0(k)$, where $g_0 \in \mathbf{BV}(\mathbb{T})$ is the function given by: $g_0(e^{it}) = i(\pi - t)$ for $0 < t < 2\pi$, and $g_0(1) = 0$. It is well-known (see, e.g. the reasoning in [31, Theorem III.3.7]) that for any $f \in \mathbf{BV}(\mathbb{T})$, the partial sum sequence $\{S_n(f, \cdot)\}_{n=0}^{\infty}$ for the Fourier series of f satisfies

$$\|S_n(f,\cdot)\|_\infty \leqslant \frac{\operatorname{var}(f,\mathbb{T})}{\pi} + \sup_{z\in\mathbb{T}}|f(z)|, \text{ for all } n \geqslant 0.$$

It follows from [22, Corollary §II.2.2., p. 53] that

$$\sum_{k=-n}^{n} h(k)z^k \to g_0(z) \text{ as } n \to +\infty, \text{ for all } z \in \mathbb{T}.$$

It is clear from the last two facts that $\{h_n\}_{n=1}^{\infty} \subseteq \ell^1(\mathbb{Z})$ satisfies (4.9) and (4.10). It is well-known that $N_p(\{h_n\}) < \infty$ for $1 < p < \infty$, and that $N_1^{(w)}(\{h_n\}) < \infty$ (see, e.g., [21, Theorem 10] and [24, Lemma 2]). So it remains now only to establish (4.8) for $\{h_n\}_{n=1}^{\infty}$. Given $m \in \mathbb{N}$ and $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N \geqslant 2$, and

$$\sum_{|k|\geqslant N_m} \frac{m}{|k|(|k|-m)} < \frac{\varepsilon}{2}.$$

Elementary calculations show that for $n > (N+1)m$, we have:

$$\sum_{|k|\geqslant N_m} |h_n(k) - h_n(k+m)| \leqslant \sum_{|k|\geqslant N_m} \frac{m}{|k|(|k|-m)} + \sum_{k=-m-n}^{-n+m-1} \frac{1}{|k+m|},$$

whence

$$\sum_{|k|\geqslant N_m} |h_n(k) - h_n(k+m)| < \frac{\varepsilon}{2} + \frac{2m}{n+1-2m},$$

for $n > (N+1)m$. It follows that (4.8) holds for $s = m$ when $m > 0$. The corresponding result for $m < 0$ follows in turn by applying this fact to $(-m)$ and taking into account that h_n is odd for all $n \in \mathbb{N}$. This completes the proof of (4.8) in the present setting. The following proposition describes the generalization of the discrete ergodic Hilbert transform which arises by applying Theorem 2.9, Theorem 4.15, and Corollary 4.30 to $\{h_n\}_{n=1}^{\infty}$.

PROPOSITION 5.10. *Suppose $1 \leqslant p < \infty$, and T is an invertible continuous linear mapping of X_p into X_p such that*

$$\sup\{\|T^m\|_p : m \in \mathbb{Z}\} < \infty,$$

and

$$\|Tf\|_\infty = \|T^{-1}f\|_\infty = \|f\|_\infty, \quad \text{for all } f \in L^p(\mu) \cap L^\infty(\mu). \tag{5.11}$$

Then for each $f \in L^p(\mu)$, the sequence

$$\left\{ \sum_{0<|m|\leqslant n} \frac{1}{m} T^m f \right\}_{n=1}^{\infty} \tag{5.12}$$

converges μ-a.e. on $\mathcal{M}$. If $1 < p < \infty$, then, for each $f \in X_p$, the sequence (5.12) also converges in the norm topology of X_p.

REMARKS 5.13. (i) Recent results of R. Sato [25] describe weaker sets of hypotheses which ensure the conlusions of Proposition 5.10 when μ is σ-finite. Specifically, if $1 < p < \infty$, the two conclusions of Proposition 5.10 remain valid if we replace (5.11) of the hypotheses by the requirement that T be separation-preserving on $L^p(\mu)$ [25, Theorems 1 and 2]. Moreover, when $p = 1$, [25, Theorem 3] states that the μ-a.e. convergence conclusion of Proposition 5.10 holds for each $f \in L^1(\mu)$, provided that T is separation-preserving on $L^1(\mu)$ and that $\{T^m : m \in \mathbb{Z}\}$ is uniformly bounded on $L^1(\mu)$ with respect to the $L^1(\mu)$-norm and uniformly bounded on $L^1(\mu) \cap L^\infty(\mu)$ with respect to the μ-essential supremum norm. (ii) When $1 < p < \infty$, an example of an operator T satisfying the hypotheses of [25, Theorems 1 and 2], but not of Proposition 5.10, is provided by defining $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ as follows: $(Tf)(x) \equiv \beta^{1/p} f(\beta x)$, where β is a positive constant other than 1.

The first author is grateful to the University of Missouri for its Summer Research Fellowship.
The work of the second author was supported by a National Science Foundation grant.

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Received May 14, 1990.