J. OPERATOR THEORY
27(1992), 309–323

DUAL OPERATOR ALGEBRAS AND THE CLASSES $\mathbb{A}_{m,n}$, I

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1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $1_{\mathcal{H}}$ and is closed in the weak* topology on $\mathcal{L}(\mathcal{H})$. Dual algebras were introduced by S. Brown [8], where he proved that every subnormal operator has a nontrivial invariant subspace. In the past ten years, many functional analysts have made contributions to the theory of dual algebras that have been applied to the topics of invariant subspaces, dilation theory, and reflexivity. Bercovici–Chevreau–Foiaş–Pearcy [3], [4] studied the problem of solving systems of simultaneous equations in the predual of a dual algebra; that problem is also the main topic of this paper. The theory of dual algebras is deeply related to the study of the classes $\mathbb{A}$ and $\mathbb{A}_{m,n}$ (m and n are any cardinal numbers with $1 \leqslant m,\ n \leqslant \aleph_0$) to be defined below.

This paper consists of six sections. In Section 2 we introduce some fundamental notation and terminology concerning dual algebras. In Section 3 we establish a dilation theorem for operators in the classes $\mathbb{A}_{m,n}$, which provides an important tool in many situations. In Section 4 we study the Jordan model of a $C_{\cdot 0}$-contraction operator in some class $\mathbb{A}_{n,1}$ and what we learn is applied in the later sections. In Section 5 we show that if $S^{(n)}(=\underbrace{S \oplus \cdots \oplus S}_{(n)})$ is the unilateral shift of multiplicity n, where n is a positive integer and $A \in C_0$ with $d_{A^*} < \infty$, then $S^{(n)} \oplus A \in \mathbb{A}_{n,\aleph_0}(1) \setminus \mathbb{A}_{n+1,1}$. In the last section, using this result, we distinguish the classes $\mathbb{A}_{m,n}$ one from another, in the sense that $\mathbb{A}_{m,n} \neq \mathbb{A}_{p,q}$ if and only if $m \neq p$ or $n \neq q$.

This paper and its sequel are portions of a dissertation submitted in partial fulfillment of the degree requirements of the University of Michigan. The author owes

a debt of gratitude to his advisor, Professor Carl Pearcy. Also the author wishes to thank Professor Hari Bercovici for helpful discussions.

2. NOTATION AND PRELIMINARIES

The notation and terminology employed agree with those in [1], [5], [6] and [18]. Throughout this paper we denote by $\mathcal{H}$ a separable, infinite dimensional, complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It follows from [12, p. 42] that $\mathcal{L}(\mathcal{H})$ is the dual space of the class $\mathcal{C}_1(\mathcal{H})$, where $\mathcal{C}_1(\mathcal{H})$ is the Banach space of trace-class operators on $\mathcal{H}$ equipped with the trace norm $\|\ \|_1$. The duality is defined by $\langle T, L\rangle = \operatorname{tr}(TL) = \sum_{i=1}^{\infty}(TLe_i, e_i), T \in \mathcal{L}(\mathcal{H}), L \in \mathcal{C}_1(\mathcal{H})$, where $\{e_i\}_{i=1}^{\infty}$ is any orthonormal basis for $\mathcal{H}$. Let $\mathcal{A}$ be a dual algebra. Then it follows from [9] that $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\!\mathcal{A}$, where ${}^{\perp}\!\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing $\langle T, [L]_{\mathcal{A}}\rangle = \operatorname{tr}(TL)$, $T \in \in \mathcal{A}$, $[L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}$. The Banach space $\mathcal{Q}_{\mathcal{A}}$ is called a *predual* of $\mathcal{A}$. We write $[L]$ for $[L]_{\mathcal{A}}$ when there is no possibility of confusion. If x and y are vectors in $\mathcal{H}$, and if we define $(x \otimes y)(u) = (u, y)x$ for all u in $\mathcal{H}$, then the rank one operator $x \otimes y$ belongs to $\mathcal{C}_1(\mathcal{H})$ and satisfies $\operatorname{tr}(x \otimes y) = (x, y)$ and $\|x \otimes y\|_1 = \|x \otimes y\| = \|x\|\|y\|$. As noted above, we write $[x \otimes y]$ for $[x \otimes y]_{\mathcal{A}}$ when there is no possibility of confusion.

DEFINITION 2.1 (cf. [5]). Suppose m and n are cardinal numbers such that $1 \leqslant m,\ n \leqslant \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leqslant i < m,\ 0 \leqslant j < n, \tag{2.1}$$

where $\{[L_{ij}]\}_{\substack{0\leqslant i<m\\ 0\leqslant j<n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0\leqslant i<m}$, $\{y_j\}_{0\leqslant j<n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. Furthermore, if m and n are positive integers and r is a fixed real number satisfying $r \geqslant 1$, a dual algebra $\mathcal{A}$ (with property $(\mathbb{A}_{m,n})$) is said to have property $(\mathbb{A}_{m,n}(r))$ if for every $s > r$ and every $m \times n$ array $\{[L_{ij}]\}_{\substack{0\leqslant i<m\\ 0\leqslant j<n}}$ from $\mathcal{Q}_{\mathcal{A}}$, there exist sequences $\{x_i\}_{0\leqslant i<m}$, $\{y_j\}_{0\leqslant j<n}$ that satisfy (2.1) and also satisfy the following conditions:

$$\|x_i\|^2 \leqslant s \sum_{0\leqslant j<n} \|[L_{ij}]\|, \quad 0 \leqslant i < m, \tag{2.2a}$$

and

$$\|y_j\|^2 \leqslant s \sum_{0\leqslant i<m} \|[L_{ij}]\|, \quad 0 \leqslant j < n. \tag{2.2b}$$

Finally, a dual algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has property $(\mathbb{A}_{m,\aleph_0}(r))$ (for some real number $r \geqslant 1$) if for every $s > r$ and every array $\{[L_{ij}]\}_{\substack{0\leqslant i<m \\ 0\leqslant j<\infty}}$ from $Q_{\mathcal{A}}$ with summable rows, there exist sequences $\{x_i\}_{0\leqslant i<m}$ and $\{y_j\}_{0\leqslant j<\infty}$ of vectors from $\mathcal{H}$ that satisfy (2.1) and (2.2a,b) with the replacement of n by $\aleph_0$. Properties $(\mathbb{A}_{\aleph_0,m}(r))$ and $(\mathbb{A}_{\aleph_0,\aleph_0}(r))$ are defined similarly.

For a brief notation, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the boundary of $\mathbb{D}$. A contraction $T \in \mathcal{L}(\mathcal{H})$ (i.e., $\|T\| \leqslant 1$) is *absolutely continuous* if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely nonunitary contraction, T_1 is either absolutely continuous or acts on the space (0). The space $L^p = L^p(\mathbb{T}), 1 \leqslant p \leqslant \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on $\mathbb{T}$. The space $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, is the Hardy function space. It is well known (cf. [14]) that H^∞ is a weak*-closed subspace of L^∞ and ${}^\perp(H^\infty)$ is the subspace $H_0^1 = \left\{ f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}\mathrm{d}t = 0, \text{ for } n = 0,1,2,\ldots \right\}$. The space H^∞ is the dual space of L^1/H_0^1, and the duality is given by the pairing

$$\langle f, [g] \rangle = \int_{\mathbb{T}} fg \,\mathrm{d}m, \quad f \in H^\infty, \ [g] \in L^1/H_0^1.$$

We denote by $\mathcal{A}_T$ the dual algebra generated by T. The following Foiaş–Sz.-Nagy functional calculus provides a relationship between the function space H^∞ and a dual algebra $\mathcal{A}_T$.

THEOREM 2.2 [5, Theorem 4.1]. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ that has the following properties:*

(a) $\Phi_T(1) = 1_{\mathcal{H}}, \ \Phi_T(\zeta) = T$,

(b) $\|\Phi_T(f)\| \leqslant \|f\|_\infty, \ f \in H^\infty$,

(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*

(d) *the range of Φ_T is weak*-dense in $\mathcal{A}_T$,*

(e) *there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L_1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*

(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L_1/H_0^1.*

We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foiaş–Sz.-Nagy functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if m and n are any cardinal numbers such that $1 \leqslant m, \ n \leqslant \aleph_0$, we denote by $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$ (cf. [4],[5]). It is easy to see that if Φ_T is an invertible operator, then Φ_T is an isometry. A subset Λ of $\mathbb{D}$ is said to be *dominating* for $\mathbb{T}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence from Λ. It follows from [7] that $\Lambda \subset \mathbb{D}$ is dominating for $\mathbb{T}$ if and only if $\sup_{\lambda\in\Lambda} |\tilde{f}(\lambda)| = \|f\|_\infty$, $f \in H^\infty$, where $\tilde{f}$ is the analytic extension of f to $\mathbb{D}$. For $T \in \mathcal{L}(\mathcal{H})$, the subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be a *semi-invariant subspace* for T if there exist $\mathcal{N}_1, \mathcal{N}_2 \in \operatorname{Lat}(T)$ with $\mathcal{N}_1 \supset \mathcal{N}_2$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2 (= \mathcal{N}_1 \cap \mathcal{N}_2^{\perp})$. The operator $T_\mathcal{M} = P_\mathcal{M} T|\mathcal{M}$, where $P_\mathcal{M}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$, is called the *compression* of T to $\mathcal{M}$, and T is called a *dilation* of $T_\mathcal{M}$. Now we recall some notation and terminology from [17], [18], [19] and [20]. An operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is called an *injection* if it is one-to-one (or $\ker X = (0)$). A family $\{X_\alpha\}$ of injections in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ is called *complete* if $\bigvee_\alpha X_\alpha \mathcal{H} = \mathcal{H}'$. Suppose $T \in \mathcal{L}(\mathcal{H})$ and $T' \in \mathcal{L}(\mathcal{H}')$. The operator T is said to be *injected* into T' if there is an injection $X : \mathcal{H} \to \mathcal{H}'$ such that $T'X = XT$, and we write $T' \overset{i}{\succ} T$, (or $T \overset{i}{\prec} T'$). The operator T is said to be *completely injected* into T' if there exists a complete family $\{X_\alpha\}$ of injections in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $T'X_\alpha = X_\alpha T$ for each α, and we write $T' \overset{c.i.}{\succ} T$ (or $T \overset{c.i.}{\prec} T'$). The operator T is said to be a *quasi-affine transform* of T' if there exists a quasi-affinity $X : \mathcal{H} \to \mathcal{H}'$ (i.e., X is one-to-one and has a dense range) such that $T'X = XT$, and we write $T' \succ T$ (or $T \prec T'$). The operator T is said to be *quasi-similar* to T' if $T \prec T'$ and $T' \prec T$, and we write $T \sim T'$ in this case.

For the unilateral shift S of multiplicity one, we recall that the function $S(\theta)$ defined by $S(\theta) = (S^*|(H^2 \ominus \theta H^2))^*$, for an inner function θ is called a *Jordan block*. Recall (cf. [19]) that any operator of the form $S(\theta_1) \oplus S(\theta_2) \oplus \cdots \oplus S(\theta_k) \oplus S^{(l)}$, where $\theta_1, \theta_2, \ldots, \theta_k$ are nonconstant (scalar valued) inner functions, each of which is a divisor of its predecessor, and $0 \leqslant k < \infty$, $0 \leqslant l \leqslant \infty$, is called a *Jordan operator*. Sz.-Nagy [17] showed that if $T \in C_{\cdot 0}(\mathcal{H})$ with $d_T < \infty$, where $d_T = \dim\{\overline{(I - T^*T)^{\frac{1}{2}}\mathcal{H}}\}$, then there exists a uniquely determined Jordan operator $J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(l)}$ such that $J_T \overset{c.i.}{\prec} T \prec J_T$. Moreover, we have $k \leqslant d_T$ and $l = d_{T^*} - d_T$. The operator J_T is called the *Jordan model* of T. Let $\mathcal{K}_i$ be a Hilbert space and let $T_i \in \mathcal{L}(\mathcal{K}_i)$, $i = 1, 2$. Throughout this paper we shall write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

3. A DILATION THEOREM FOR OPERATORS IN $\mathbb{A}_{m,n}$

We recall that a set $\{e_i\}_{0 \leqslant i < n}$ of vectors in a Hilbert space $\mathcal{K}$ is an *n-cyclic set* for an operator A in $\mathcal{L}(\mathcal{K})$ if $\mathcal{K}$ is the smallest invariant subspace for A containing $\{e_i\}_{0 \leqslant i < n}$. If X is a (unbounded) linear transformation, we write $\mathcal{D}(X)$ for the domain of X and $\mathcal{R}(X)$ for the range of X. The following dilation theorem will be a good tool in many situations.

THEOREM 3.1. *Suppose $T \in \mathbb{A}_{m,n}(\mathcal{H})$ for some positive integers m and n. Let A be any absolutely continuous contraction on a Hilbert space $\mathcal{K}$. If A possesses an m-cyclic set $\{e_1, \ldots, e_m\}$ of vectors in $\mathcal{K}$ and its adjoint operator A^* has an n-cyclic set $\{f_1, \ldots, f_n\}$ of vectors in $\mathcal{K}$, then there exist $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ and a closed one-to-one linear transformation*

$$X : \mathcal{D}(X) \to \mathcal{M} \ominus \mathcal{N}$$

such that

(a) *the linear manifold $\mathcal{D}(X)$ is dense in $\mathcal{K}$ and contains $\{e_1, \ldots, e_m\}$,*
(b) *the range $\mathcal{R}(X)$ of X is dense in $\mathcal{M} \ominus \mathcal{N}$, and*
(c) *$A\mathcal{D}(X) \subset \mathcal{D}(X)$ and $T_{\mathcal{M}\ominus\mathcal{N}}Xz = XAz$ for all z in $\mathcal{D}(X)$.*

Proof. The idea of this proof comes from Hari Bercovici and that of [5, Theorem 5.3]. For each pair $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we have $[L_{ij}]_T \in \mathcal{Q}_T$ such that $\phi_A([e_i \otimes f_j]_A) = \phi_T([L_{ij}]_T)$. Then, by the hypothesis, there exist sequences $\{x_i\}_{i=1}^m$ and $\{y_j\}_{j=1}^n$ of vectors in $\mathcal{H}$ such that $[L_{ij}]_T = [x_i \otimes y_j]_T$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. Thus we have $(A^k e_i, f_j) = (T^k x_i, y_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $k = 0, 1, 2, \ldots$. We define

$$\mathcal{M} = \bigvee_{i=1}^{m} \{p_i(T)x_i \ : \ p_i \text{ is any polynomial}, \ 1 \leqslant i \leqslant m\}$$

and

$$\mathcal{M}_* = \bigvee_{j=1}^{n} \{q_j(T^*)y_j \ : \ q_j \text{ is any polynomial}, 1 \leqslant j \leqslant n\}.$$

If we write $\mathcal{N} = \mathcal{M} \ominus \mathcal{M}_*$, then obviously we have $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ and $\mathcal{N} \subset \mathcal{M}$. Let $x_i = z_i + w_i$, where $z_i \in \mathcal{M} \ominus \mathcal{N}$, $w_i \in \mathcal{N}$. Since $T^{*k} y_j \in \mathcal{M}_*$ for all j and k, we have $(T^k w_i, y_j) = (w_i, T^{*k} y_j) = 0$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $k = 0, 1, 2, \ldots$. Therefore we have $(A^k e_i, f_j) = (T^k x_i, y_j) = (T^k z_i, y_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $k = 0, 1, 2, \ldots$. If we write $\tilde{T}$ for the compression $T_{\mathcal{M}\ominus\mathcal{N}}$, then we have $(A^k e_i, f_j) = (\tilde{T}^k z_i, y_j) + (v_{ik}, y_j) = (\tilde{T}^k z_i, y_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $k = 0, 1, 2, \ldots$. Now we consider the correspondence

$$X_0 : \sum_{i=1}^{m} p_i(A)e_i \to \sum_{i=1}^{m} p_i(\tilde{T})z_i,$$

where $p_i(\lambda)$, $1 \leqslant i \leqslant m$, are any polynomials. We shall show the closure of X_0 is the required map. First, to show that X_0 is a well-defined linear transformation, let $p_i(\lambda)$, $1 \leqslant i \leqslant m$ be polynomials such that

$$\sum_{i=1}^{m} p_i(A)e_i = 0.$$

Then for any polynomials, $q_j(\lambda)$, $1 \leqslant j \leqslant n$, we have

$$\left(\sum_{i=1}^{m} p_i(A)e_i, \sum_{j=1}^{n} q_j(A^*)f_j\right) = 0.$$

For any polynomial $q_j(\lambda)$, $1 \leqslant j \leqslant n$, this is equivalent to

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(p_i(A)e_i, q_j(A^*)f_j) = 0,$$

$$\text{i.e.,}\quad \sum_{i=1}^{m}\sum_{j=1}^{n}(\tilde{q}_j(A)p_i(A)e_i, f_j) = 0, \quad \text{where } \tilde{q}_j(\lambda) = \overline{q_j(\bar{\lambda})},$$

$$\text{i.e.,}\quad \sum_{i=1}^{m}\sum_{j=1}^{n}(\tilde{q}_j(T)p_i(T)x_i, y_j) = 0,$$

$$\text{i.e.,}\quad \left(\sum_{i=1}^{m} p_i(T)x_i, \sum_{j=1}^{n} q_j(T^*)y_j\right) = 0,$$

$$\text{i.e.,}\quad \left(\sum_{i=1}^{m} p_i(T)(z_i + w_i), \sum_{j=1}^{n} q_j(T^*)y_j\right) = 0,$$

$$\text{i.e.,}\quad \left(\sum_{i=1}^{m} p_i(\tilde{T})z_i, \sum_{j=1}^{n} q_j(T^*)y_j\right) = 0,$$

which is equivalent to

$$\sum_{i=1}^{m} p_i(\tilde{T})z_i = 0.$$

Thus X_0 is a well-defined one-to-one linear transformation. Second, to show X_0 is closable, we shall claim that if

$$\sum_{i=1}^{m_k} p_i^{(k)}(A)e_i \to e' \ (k \to \infty)$$

and

$$\sum_{i=1}^{m_k} p_i^{(k)}(\tilde{T})z_i \to z' \ (k \to \infty),$$

then we have $e' = 0$ if and only if $z' = 0$. To show this, a similar method to the above proof is used. Indeed, $e' = 0$ if and only if $\left(e', \sum_{j=1}^{n} q_j(A^*)f_j\right) = 0$, for any polynomials $q_j(\lambda)$,

$$\text{i.e.,}\quad \lim_k \left(\sum_{i=1}^{m_k} p_i^{(k)}(A)e_i, \sum_{j=1}^{n} q_j(A^*)f_j\right) = 0, \quad \text{for any polynomials } q_j(\lambda),$$

$$\text{i.e.,}\quad \lim_k \left(\sum_{i=1}^{m_k} p_i^{(k)}(T)x_i, \sum_{j=1}^{n} q_j(T^*)y_j\right) = 0, \quad \text{for any polynomials } q_j(\lambda),$$

$$\text{i.e.,}\quad \left(z', \sum_{j=1}^{n} q_j(T^*)y_j\right) = 0, \quad \text{for any polynomials } q_j(\lambda).$$

This means $z' = 0$. Therefore X_0 is closable and its closure X is one-to-one. Since $\mathcal{R}(X_0)$ is dense in $\mathcal{M} \ominus \mathcal{N}, \mathcal{R}(X)$ is dense in $\mathcal{M} \ominus \mathcal{N}$. Finally, for any polynomials $p_i(\lambda)$, $1 \leqslant i \leqslant m$, we have

$$\tilde{T}X_0\left(\sum_{i=1}^{m} p_i(A)e_i\right) = \tilde{T}\left(\sum_{i=1}^{m} p_i(\tilde{T})z_i\right) = X_0A\left(\sum_{i=1}^{m} p_i(A)e_i\right).$$

Hence we have $XAz = \tilde{T}Xz$ for all z in $\mathcal{D}(X)$. The theorem is proved. ∎

REMARK 3.2. The preceding proof also implies that this theorem is valid when $n = \aleph_0$.

4. OPERATORS IN $A_{m,n}$ WITH APPLICATIONS TO JORDAN MODELS

We recall (cf. [18, Theorem I. 4.1]) that for every contraction T on a Hilbert space $\mathcal{H}$ there exists an isometric dilation B_T on some Hilbert space $\mathcal{K}(\supset \mathcal{H})$, where $\mathcal{K} = \bigvee_{n=0}^{\infty} B_T^n\mathcal{H}$. Moreover, if B' is another isometric dilation of T, it follows from the proof of [18, Theorem I. 4.1] that B' is unitarily equivalent to an extension of B_T.

LEMMA 4.1. *Suppose T is a contraction operator on $\mathcal{H}$ and $T \in C_{\cdot 0}$. Let*

$$T = \begin{pmatrix} * & * & * \\ 0 & \tilde{T} & * \\ 0 & 0 & * \end{pmatrix}$$

relative to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then we have $d_{\tilde{T}^*} \leqslant d_{T^*}$.

Proof. By the previous remark, there exists a minimal isometric dilation $B_T \in \in \mathcal{L}(\mathcal{K})$ of T such that

$$B_T = \begin{pmatrix} * & * \\ 0 & T \end{pmatrix},$$

relative to the decomposition $\mathcal{K} = \mathcal{H}^\perp \oplus \mathcal{H}$. Since $T \in C_{\cdot 0}$, it follows from [1, Corollary I.2.11] that B_T is the unilateral shift of multiplicity d_{T^*}. Moreover, since B_T has a matrix

$$B_T = \left(\begin{array}{c:ccc} * & & * & \\ \hdashline & * & * & * \\ 0 & 0 & \tilde{T} & * \\ & 0 & 0 & * \end{array}\right)$$

relative to the decomposition $\mathcal{K} = \mathcal{H}^\perp \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, it is obvious that B_T is an isometric dilation of $\tilde{T}$. On the other hand, if we write $B_{\tilde{T}}$ for the minimal isometric dilation of $\tilde{T}$, then $B_{\tilde{T}}$ is the unilateral shift of multiplicity $d_{\tilde{T}^*}$. According to the minimality, since B_T is an extension of $B_{\tilde{T}}$, it is obvious that $d_{T^*} \geqslant d_{\tilde{T}^*}$. Hence the proof is complete. ∎

The following lemma, which comes from [20, Theorem 2.3] is a useful tool for finding the Jordan models of operators in $\mathbf{A}_{m,n} \cap C_{\cdot 0}$.

LEMMA 4.2. *Let T_1 and T_2 be contractions in $C_{\cdot 0}(\mathcal{H})$ with $d_{T_1^*} < \infty$ and $d_{T_2} < \infty$. If $T_1 \prec T_2$, then we have $J_{T_1} = J_{T_2}$.*

It is well known that if T is a $C_{\cdot 0}$ contraction, then $d_T \leqslant d_{T^*}$. Note that the linear transformation X in Theorem 3.1 was not necessarily bounded.

LEMMA 4.3. *In Theorem 3.1, if we denote $\tilde{T} = T_{\mathcal{M}\ominus\mathcal{N}}$ and define $B = (A \oplus \oplus\tilde{T})G(X)$, where $G(X)$ is the graph of X, then $B \in \mathcal{L}(G(X))$ and satisfies $B \prec A$ and $B \prec \tilde{T}$.*

Proof. The technique of this proof comes from H. Bercovici (cf. [1, p.64]). Since X is closed, $G(X)$ is closed in $\mathcal{K} \oplus (\mathcal{M} \ominus \mathcal{N})$. So B is a bounded operator acting on $G(X)$. To show B maps $G(X)$ into itself, i.e., that $B \in \mathcal{L}(G(X))$, let $x \oplus y \in G(X)$ so that $y = Xx$. Then $B(x \oplus y) = (A \oplus \tilde{T})(x \oplus Xx) = Ax \oplus \tilde{T}Xx$. Moreover, since $\tilde{T}Xz = XAz$ for every $z \in \mathcal{D}(X)$, we have $B(x \oplus y) = Ax \oplus X(Ax)$. Hence $B(x \oplus y) \in \in G(X)$ and $B \in \mathcal{L}(G(X))$. Now we define $Y_1 = P_1|G(X)$ and $Y_2 = P_2|G(X)$, where P_1 is the projection of $\mathcal{K} \oplus (\mathcal{M} \ominus \mathcal{N})$ onto $\mathcal{K}$ and P_2 is the projection of $\mathcal{K} \oplus (\mathcal{M} \ominus \mathcal{N})$ onto $\mathcal{M} \ominus \mathcal{N}$. Since X is one-to-one, it is obvious that the operators $Y_1 \in \mathcal{L}(G(X), \mathcal{K})$ and $Y_2 \in \mathcal{L}(G(X), \mathcal{M} \ominus \mathcal{N})$ are one-to-one. It follows from (a) and (b) in Theorem 3.1 that $\overline{Y_1 G(X)} = \mathcal{K}$ and $\overline{Y_2 G(X)} = \mathcal{M} \ominus \mathcal{N}$. Hence it is sufficient to show that $AY_1 = Y_1 B$ and $\tilde{T}Y_2 = Y_2 B$. To do so, let $x \oplus Xx \in G(X)$ for $x \in \mathcal{D}(X)$. Then $(AY_1)(x \oplus Xx) = = Ax = Y_1(Ax \oplus X(Ax)) = Y_1(Ax \oplus \tilde{T}Xx) = Y_1(A \oplus \tilde{T})(x \oplus Xx) = Y_1 B(x \oplus Xx)$ and $(\tilde{T}Y_2)(x \oplus Xx) = \tilde{T}Xx = Y_2(Ax \oplus \tilde{T}Xx) = Y_2(A \ominus \tilde{T})(x \oplus Xx) = Y_2 B(x \oplus Xx)$.

Therefore $B \prec A$ and $B \prec \tilde{T}$. The lemma is proved. ■

PROPOSITION 4.4. *Suppose $T \in C_{\cdot 0} \cap \mathbb{A}_{n,1}(\mathcal{H})$, for some positive integer n and $d_{T^*} < \infty$. If $A \in C_{\cdot 0}(\mathcal{K})$ has an n-cyclic set of vectors, A^* has a cyclic vector, and $d_{A^*} < \infty$, then there exist $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $J_A = J_{\tilde{T}}$, where $\tilde{T} = T_{\mathcal{M} \ominus \mathcal{N}}$.*

Proof. Since A is completely nonunitary contraction, it follows from Theorem 3.1 that there exist $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ and a one-to-one closed linear transformation $X : \mathcal{D}(X) \to \mathcal{M} \ominus \mathcal{N}$ such that $\overline{\mathcal{D}(X)} = \mathcal{K}$, $\overline{\mathrm{Range}(X)} = \mathcal{M} \ominus \ominus \mathcal{N}$, $A\mathcal{D}(X) \subset \mathcal{D}(X)$, and $T_{\mathcal{M} \ominus \mathcal{N}} X z = XAz$ for all z in $\mathcal{D}(X)$. If we define $B = = (A \ominus \tilde{T})|G(X)$, it follows from Lemma 4.3 that $B \prec A$ and $B \prec \tilde{T}$. To show $\tilde{T} \in \in C_{\cdot 0}(\mathcal{M} \ominus \mathcal{N})$, let $x \in \mathcal{M} \ominus \mathcal{N}$. Then since T^{k^*} has a matrix of the form

$$\begin{pmatrix} * & 0 & 0 \\ * & \tilde{T}^{*k} & 0 \\ * & * & * \end{pmatrix}$$

relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, for all positive integer k, and since $\|T^{k*}x\| \to 0$ $(k \to \infty)$, it follows from a simple calculation that $\|\tilde{T}^{*k}\| \to 0$. Hence $\tilde{T} \in C_{\cdot 0}$. Moreover, since $A \in C_{\cdot 0}$, and since $A \oplus \tilde{T}$ is an extension of B, we can show easily that $B \in C_{\cdot 0}$. Now, to apply Lemma 4.2, to show that $J_B = J_A$ and $J_B = J_{\tilde{T}}$, we shall show that the defect indices of $\tilde{T}$ and B are finite. But according to Lemma 4.1, we have $d_{\tilde{T}} \leqslant d_{\tilde{T}^*} \leqslant d_{T^*} < \infty$ and $d_{B^*} \leqslant d_{(A \oplus \tilde{T})^*} \leqslant d_{(A \oplus T)^*} < \infty$. So the proof is complete. ■

Now we address the problem of the Jordan models of operators in $C_{\cdot 0} \cap \mathbb{A}_{n,1}$.

THEOREM 4.5. *Suppose $T \in C_{\cdot 0} \cap \mathbb{A}_{n,1}(\mathcal{H})$ for some positive integer n and satisfies $d_{T^*} < \infty$. Then there exists a positive integer r with $n \leqslant r + d_T$ such that $0 \leqslant k \leqslant d_T$ and $J_T = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(r)}$, in the sense that if $k = 0$, then $J_T = S^{(r)}$.*

Proof. Since $T \in C_{\cdot 0}$ we know that T has a Jordan model J_T. Let $J_T = = S(\theta_1) \oplus \cdots \oplus S(\theta_k) \oplus S^{(r)}$ be the Jordan model of T with $0 \leqslant k \leqslant d_T$, $r = d_{T^*} - d_T$. Let us consider the unilateral shift $S^{(n)}(= A)$ of multiplicity n. It follows from [13, p. 281] that A has an n cyclic set and A^* has a cyclic vector. Moreover since $d_{A^*} = n$, according to Proposition 4.4, there exist $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $J_{\tilde{T}} = J_{S^{(n)}} = S^{(n)}$, where $\tilde{T} = T_{\mathcal{M} \ominus \mathcal{N}}$. Hence $n = d_{\tilde{T}^*} - d_{\tilde{T}}$, and according to Lemma 4.1 we have $0 \leqslant d_{\tilde{T}} = d_{\tilde{T}^*} - n \leqslant d_{T^*} - n = r + d_T - n$. So $n \leqslant r + d_T$, and the proof is complete. ■

The following corollary gives good information about the defect indices of operators in $\left(\bigcap_{n=1}^{\infty} \mathbb{A}_{n,1}(\mathcal{H})\right) \cap C_{\cdot 0}$.

COROLLARY 4.6. *If* $T \in \left(\bigcap_{n=1}^{\infty} \mathbb{A}_{n,1}(\mathcal{H})\right) \cap C_{\cdot 0}$, *then* $d_{T^*} = \aleph_0$.

Proof. Suppose $d_{T^*} < \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $d_{T^*} < n_0$ and $T \in \mathbb{A}_{n_0,1}$. By Theorem 4.5, there exists a positive integer r such that $n_0 \leqslant r + d_T$ and $d_{T^*} - d_T = r$. Therefore we have $d_T + r = d_{T^*} < n_0 \leqslant r + d_T$. This contradiction proves the corollary. ■

The following proposition is a generalization of [16, Proposition 2.4].

PROPOSITION 4.7. *Suppose that for* $0 \leqslant i < n$, $T_i \in \mathbb{A}_{1,m}(r_i)$ *for some real number* $r_i \geqslant 1$ *and some cardinal number* m *satisfying* $1 \leqslant m \leqslant \aleph_0$. *Then:*

(a) *if* $n < \aleph_0$, *then* $\bigoplus_{0 \leqslant i < n} T_i \in A_{n,m}(r)$, *where* $r = \max_{0 \leqslant i < n} r_i$, *and*

(b) *if* $n = \aleph_0$, *then* $\bigoplus_{0 \leqslant i < n} T_i \in A_{\aleph_0}(1)$.

Proof. Assume $n < \aleph_0$. Let $s > \max_{0 \leqslant i < n} r_i$ and let $\{[L_{ij}]_{\hat{T}}\}_{\substack{0 \leqslant i < n \\ 0 \leqslant j < m}} \subset \mathcal{Q}_{\hat{T}}$, where $\hat{T} = \bigoplus_{0 \leqslant i < n} T_i$. Let us consider the usual isometric isomorphic weak* homeomorphism ϕ_i from $\mathcal{Q}_{\hat{T}}$ onto $\mathcal{Q}_{T_i}$ (cf. [5,p. 34]). Since $\phi_i([L_{ij}]_{\hat{T}}) \in \mathcal{Q}_{T_i}$ and $s > r_i$ for any i with $0 \leqslant i < n$, by definition there exist vectors x_i and $y_j^{(i)}$ in $\mathcal{H}$, $0 \leqslant i < n$, $0 \leqslant j < m$, such that

$$\phi_i([L_{ij}]_{\hat{T}}) = [x_i \otimes y_j^{(i)}]_{T_i},$$
$$\|x_i\|^2 \leqslant s \sum_{0 \leqslant j < m} \|[L_{ij}]_{\hat{T}}\|,$$

and

$$\|y_j^{(i)}\|^2 \leqslant s\|[L_{ij}]_{\hat{T}}\|.$$

Now if we set

$$\tilde{x}_i = \underbrace{\overbrace{0 \oplus \cdots \oplus 0}^{(i)} \oplus x_i \oplus 0 \oplus \cdots \oplus 0}_{(n)}, \quad 0 \leqslant i < n$$

and

$$\tilde{y}_j = \underbrace{y_j^{(0)} \oplus y_j^{(1)} \oplus \cdots \oplus y_j^{(i)} \oplus \cdots \oplus y_j^{(n-1)}}_{(n)}, \quad 0 \leqslant j < m,$$

then we have $\phi_i([\tilde{x}_i \otimes \tilde{y}_j]_{\hat{T}}) = [x_i \otimes y_j^{(i)}]_{T_i} = \phi_i([L_{ij}]_{\hat{T}})$, $0 \leqslant i < n$, $0 \leqslant j < m$. So we get

$$[\tilde{x}_i \otimes \tilde{y}_j]_{\hat{T}} = [L_{ij}]_{\hat{T}}, \quad 0 \leqslant i < n, \quad 0 \leqslant j < m,$$

$$\|\tilde{x}_i\|^2 = \|x_i\|^2 \leqslant s \sum_{0\leqslant j<m} \|[L_{ij}]_{\hat{T}}\|, \quad 0 \leqslant i < n,$$

and

$$\|\tilde{y}_j\|^2 = \sum_{0\leqslant i<n} \|y_j^{(i)}\|^2 \leqslant s \sum_{0\leqslant i<n} \|[L_{ij}]_{\hat{T}}\|.$$

Therefore $\hat{T} \in A_{n,m}(r)$. Next if $n = \aleph_0$, according to [5, Proposition 5.8], we have $\hat{T} \in$ $\in A_{\aleph_0}$. Moreover since $A_{\aleph_0} = A_{\aleph_0}(1)$ (cf. [5, Theorem 6.3]), we have this proposition. ∎

H. Bercovici, C. Foiaş and C. Pearcy (cf. [4]) showed that if we write $S^{(n)}$ for the unilateral shift of multiplicity n and $S^{(1)} = S$, then we have $S^{(n)} \in A_n \setminus A_{n+1}$. The following corollary is a generalization.

COROLLARY 4.8. *For any $n \in \mathbb{N}$, we have:*
(a) $S^{(n)} \in A_{n,\aleph_0}(1) \setminus A_{n+1,1}$, *and*
(b) $S^{(n)*} \in A_{\aleph_0,n}(1) \setminus A_{1,n+1}$.

Proof. It is easy to show that $S \in A_{1,\aleph_0}(1)$ (cf. [11, Theorem 6.2]). Hence by Proposition 4.7 we have $S^{(n)} \in A_{n,\aleph_0}(1)$. To show $S^{(n)} \notin A_{n+1,1}$, we suppose $S^{(n)}$ belongs to $A_{n+1,1}$. Since $S^{(n)}$ satisfies all conditions in Theorem 4.5 and $d_{S^{(n)}} = 0$, there exists $r \in \mathbb{N}$ with $n+1 \leqslant r$ such that $J_{S^{(n)}} = S^{(r)}$. This contradiction proves that $S^{(n)} \notin A_{n+1,1}$. Furthermore, since (b) is the dual of (a), obviously we have (b). Hence the proof is complete. ∎

This corollary will be used as a lemma for Theorem 5.3.

5. JORDAN OPERATORS AND MEMBERSHIP IN $A_{m,n}$

Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$ and let

$$T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$$

be its (unique) triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. If the Jordan model of T is $S(\theta_1)\oplus\cdots\oplus S(\theta_k)\oplus S^{(l)}$, then the Jordan models of T_1 and T_2 are $S(\theta_1)\oplus\cdots\oplus S(\theta_k)$ and $S^{(l)}$, respectively (cf. [20, Lemma 2.7]). We obtain immediately the following lemma.

LEMMA 5.1. *Let $T \in A_{n,1}(\mathcal{H}) \cap C_{\cdot 0}$ with $d_{T^*} < \infty$. Then $T \in C_{10}$ if and only if there exists $r \in \mathbb{N}$ with $n \leqslant r + d_T$ such that $J_T = S^{(r)}$.*

Proof. Assume $T \in C_{10}$. By the previous remark, there exists $r \in \mathbb{N}$ such that $J_T = S^{(r)}$. Furthermore, according to Theorem 4.5 we have $n \leqslant r + d_T$. Conversely, let $J_T = S^{(r)}$ and let

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}$$

be its unique triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. Then it is sufficient to show that $T_1 = 0$ on (0). To do so, let us consider any nontrivial Jordan block $S(\theta)$ and set

$$\tilde{T} = \left(\begin{array}{c:cc} S(\theta) & 0 & \\ \hdashline & T_1 & A \\ 0 & 0 & T_2 \end{array}\right) \cong \left(\begin{array}{cc:c} S(\theta) & 0 & * \\ 0 & T_1 & \\ \hdashline & & \\ & 0 & T_2 \end{array}\right).$$

Then this matrix has the triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. Since $J_{\tilde{T}} = J_{S(\theta)\oplus T} = J_{S(\theta)} \oplus J_T = S(\theta) \oplus S^{(r)}$, by the previous remark, we have $J_{S(\theta)\oplus T_1} = S(\theta)$ and $J_{S(\theta)} \oplus J_{T_1} = S(\theta) \oplus J_{T_1}$. Hence $J_{T_1} = 0$ on (0) and $T_1 = 0$ on (0). The lemma is proved. ■

LEMMA 5.2. *If $T \in \mathbf{A}_{n,1}(\mathcal{H}) \cap C_{\cdot 0}$ with $d_{T^*} < \infty$, then there exist $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $\dim(\ker \tilde{T}^*) = n$ and $J_{\tilde{T}} = S^{(n)}$, where $\tilde{T} = T_{\mathcal{M}\ominus\mathcal{N}}$.*

Proof. Let us consider $A = S^{(n)}$. It follows from Proposition 4.4 that there exist $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $J_{\tilde{T}} = S^{(n)}$ and $d_{\tilde{T}^*} - d_{\tilde{T}} = n$. By Lemma 5.1, we have $\tilde{T} \in C_{10}$. Now applying [15, Lemma 8], we obtain $\dim(\ker \tilde{T}^*) = n$. ■

Now we are ready to establish new elements in $\mathbf{A}_{m,n}$.

THEOREM 5.3. *If B is an operator of class C_0 (i.e., there exists $u \in H^\infty$, $u \not\equiv 0$, such that $u(B) = 0$) acting on a separable Hilbert space with $d_{B^*} < \infty$, then we have $S^{(n)} \oplus B \in \mathbf{A}_{n,\aleph_0}(1) \setminus \mathbf{A}_{n+1,1}$, $n \in \mathbb{N}$.*

Proof. Without loss of generality, we can assume that B is nontrivial (otherwise, this is exactly Corollary 4.8). Let us denote $T = S^{(n)} \oplus B$. Since $T \in \mathbf{A}_{n,\aleph_0}(1)$ obviously, it is sufficient to show that $T \notin \mathbf{A}_{n+1,1}$. Suppose $T \in \mathbf{A}_{n+1,1}$. Since $d_{T^*} = n + d_{B^*} < \infty$, it follows from Lemma 5.2 that there exist $\mathcal{M}, \mathcal{N} \in \operatorname{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $\dim(\ker \tilde{T}^*) = n + 1$ and $J_{\tilde{T}} = S^{(n+1)}$, where $\tilde{T} = T_{\mathcal{M}\ominus\mathcal{N}}$. Let us say

$$T = \begin{pmatrix} E & * & * \\ 0 & \tilde{T} & * \\ 0 & 0 & * \end{pmatrix}$$

relative to the appropriate decomposition of the Hilbert space of T. If we denote $T_1 = T|\mathcal{M}$, we have

$$T_1 = \begin{pmatrix} E & * \\ 0 & \tilde{T} \end{pmatrix}$$

and $T_1 \in C_{\cdot 0}$. Since B is a nontrivial operator of class C_0 acting on a separable Hilbert space, it follows from [1, Theorem 5.1, p. 56] that there exists the Jordan operator $J_B = S(\theta_1) \oplus \cdots \oplus S(\theta_k)$, $k \in \mathbb{N}$, such that $B \sim J_B$ (i.e. $B \prec J_B$ and $B \succ J_B$). Then we have $J_{T_1} \overset{c.i.}{\prec} T_1 \overset{i}{\prec} T = S^{(n)} \oplus B \prec J_{S^{(n)} \oplus B} \overset{c.i.}{\prec} S^{(n)} \oplus B \prec S^{(n)} \oplus J_B =$ $= S^{(n)} \oplus S(\theta_1) \oplus \cdots \oplus S(\theta_k)$. Now according to [19, Theorem 4], there exist nonnegative integers n', k' with $n' \leqslant n$, $k' \leqslant k$ (at least one of n' and k' is positive) and nonconstant (scalar valued) inner functions θ'_i, $i \leqslant k'$, such that θ'_i is a divisor of θ_i and

$$J_{T_1} = S^{(n')} \oplus S(\theta'_1) \oplus \cdots \oplus S(\theta'_{k'}). \tag{5.1}$$

Note that $E \in C_{\cdot 0}$. Let

$$E = \begin{pmatrix} E_1 & * \\ 0 & E_2 \end{pmatrix}$$

be its (unique) triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. Then we say

$$T_1 = \begin{pmatrix} E & * \\ 0 & \tilde{T} \end{pmatrix} \cong \left(\begin{array}{c:cc} E_1 & * & * \\ \hdashline 0 & E_2 & * \\ 0 & 0 & \tilde{T} \end{array}\right).$$

Let us denote

$$\tilde{E} = \begin{pmatrix} E_2 & * \\ 0 & \tilde{T} \end{pmatrix} \in \mathcal{L}(\mathcal{K}).$$

Since $J_{\tilde{T}} = S^{(n+1)}$, by Lemma 5.1, we have $\tilde{T} \in C_{10}$. Now we shall claim $\tilde{E} \in C_{10}$. Let $z = x \oplus y \in \mathcal{K}$ with $z \neq 0$, and let

$$\tilde{E}^n = \begin{pmatrix} E_2^n & R_n \\ 0 & \tilde{T}^n \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$

Then we have $\|\tilde{E}^n z\|^2 = \|E_2^n x + R_n y\|^2 + \|\tilde{T}^n y\|^2$ for all $n \in \mathbb{N}$. If $y = 0$ and $x \neq 0$, we have $\|E_2^n x\| \not\to 0$. If $y \neq 0$, then $\|\tilde{T}^n y\| \not\to 0$. Hence $\|\tilde{E}^n z\| \not\to 0$ and $\tilde{E} \in C_{10}$. So

$$T_1 = \begin{pmatrix} E_1 & * \\ 0 & \tilde{E} \end{pmatrix}$$

is the triangular form of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$. By (5.1), we get $J_{\tilde{E}} = S^{(n')}$, $0 \leqslant n' \leqslant n$, and $d_{\tilde{E}^*} - d_{\tilde{E}} = n'$. Applying [15, Lemma 8], we obtain $\dim(\ker \tilde{E}^*) = n' \leqslant n$. Recall that $\dim(\ker \tilde{T}^*) = n + 1$. Then the relation

$$\tilde{E}^* = \begin{pmatrix} E_2^* & 0 \\ * & \tilde{T}^* \end{pmatrix}$$

induces a contradiction, and the proof is complete. ∎

We now can obtain immediately the following corollary, which is an improvement of Corollary 4.8.

COROLLARY 5.4. *Let J be the Jordan operator of the form $S(\theta_1) \oplus \cdots \oplus \oplus S(\theta_k) \oplus S^{(n)}$, $0 \leqslant k < \infty$, $1 \leqslant n < \infty$. Then we get $J \in \mathbb{A}_{n,\aleph_0}(1) \setminus \mathbb{A}_{n+1,1}$.*

6. THE CLASSES $\mathbb{A}_{m,n}$ ARE DISTINCT

H. Bercovici, C. Foiaş and C. Pearcy [4] showed that $\mathbb{A}_1 \supsetneq \mathbb{A}_2 \supsetneq \cdots \supsetneq \mathbb{A}_{\aleph_0}$. Furthermore, H. Bercovici (cf. [23]) and B. Chevreau (cf. [10]) showed independently that $\mathbb{A} = \mathbb{A}_1(1)$. If $p, q, m, n \in \mathbb{N}$ with $p \geqslant m$, $q \geqslant n$, then obviously we have $\mathbb{A}_{p,q} \subset \mathbb{A}_{m,n}$. We recall from the previous section that $S^{(n)} \in \mathbb{A}_{n,\aleph_0} \setminus \mathbb{A}_{n+1,1}$ for each positive integer n. Using this result, we shall distinguish the classes $\mathbb{A}_{m,n}$ one from another as follows:

THEOREM 6.1. *Suppose m, n, p and q are cardinal numbers such that $1 \leqslant \leqslant m, n, p, q \leqslant \aleph_0$. Then $\mathbb{A}_{m,n} \neq \mathbb{A}_{p,q}$ if and only if $m \neq p$ or $n \neq q$. Furthermore, we have $\mathbb{A}_{m,n} \subset \mathbb{A}_{p,q}$ if and only if $m \geqslant p$ and $n \geqslant q$.*

Proof. Assume that $m \neq p$ or $n \neq q$. If $m \neq p$, without loss of generality, we can assume $m < p$. Since $S^{(m)} \in \mathbb{A}_{m,\aleph_0}$, we have $S^{(m)} \in \mathbb{A}_{m,n}$. But since $\mathbb{A}_{p,q} \subset \mathbb{A}_{p,1} \cap \mathbb{A}_{m+1,1}$ and since $S^{(m)} \notin \mathbb{A}_{m+1,1}$, we have $S^{(m)} \notin \mathbb{A}_{p,q}$. Therefore $\mathbb{A}_{m,n} \neq \mathbb{A}_{p,q}$. On the other hand, if we assume $n \neq q$ and $n < q$, since $S^{(n)*} \in \mathbb{A}_{\aleph_0,n}$, we have $S^{(n)*} \in \mathbb{A}_{m,n}$. Moreover, since $S^{(n)*} \notin \mathbb{A}_{1,n+1}$ and since $\mathbb{A}_{p,q} \subset \mathbb{A}_{1,q} \cap \mathbb{A}_{1,n+1}$, we have $S^{(n)*} \notin \mathbb{A}_{p,q}$. Thus $\mathbb{A}_{m,n} \neq \mathbb{A}_{p,q}$. For the second part, if $m \geqslant p$ and $n \geqslant q$, then we know that $\mathbb{A}_{m,n} \subset \mathbb{A}_{p,q}$. Conversely, suppose $m < p$ or $n < q$. Without loss of generality we can assume $m < p$. Based on an examination of the above arguments, we obtain $S^{(m)} \in \mathbb{A}_{m,n} \setminus \mathbb{A}_{p,q}$, so that $\mathbb{A}_{m,n}$ cannot be contained in $\mathbb{A}_{p,q}$. Therefore the proof is complete. ∎

The author was partially supported by a grant from KOSEF 903-0102-014-1.

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Received June 12, 1990.