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# DIFFEOMORPHISM OF IRRATIONAL ROTATION $C^*$-ALGEBRAS BY NON-GENERIC ROTATIONS II

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## 1. INTRODUCTION

In 1986 Elliott [2] showed the following result: Any diffeomorphism of irrational rotation $C^*$-algebras by generic rotations is composed by three types of diffeomorphisms induced by a smooth unitary element, an element in the group of $2\times 2$ matrices over integers with determinant 1 and an element in the two dimensional torus.

And we showed in the previous paper [3] that there are an irrational rotation $C^*$-algebra by a non-generic rotation and its diffeomorphism which is not composed of the above three types of diffeomorphisms.

In the present paper we will show that for any irrational rotation $C^*$-algebra by a non-generic rotation there is a diffeomorphism which is not composed of the above types of diffeomorphisms.

## 2. MAIN RESULT

Let $A_\theta$ be an irrational rotation $C^*$-algebra by $\theta$ and let $u$ and $v$ be unitary elements in $A_\theta$ with $uv = e^{2\pi i\theta}vu$ which generate $A_\theta$. Let $A_\theta^\infty$ be the dense$^*$-subalgebra of all smooth elements in $A_\theta$ with respect to the canonical action of the two dimensional torus.

DEFINITION. Let $\alpha$ be an automorphism of $A_\theta$. We say that *it is a diffeomorphism of* $A_\theta$ if $\alpha(A_\theta^\infty) = A_\theta^\infty$.

For any $s,t \in \mathbf{R}$ let $\alpha_{(s,t)}$ be the diffeomorphism of $A_\theta$ defined by $\alpha_{(s,t)}(u) = e^{2\pi is}u$ and $\alpha_{(s,t)}(v) = e^{2\pi it}v$. Let $SL(2,\mathbf{Z})$ be the group of all $2\times 2$ matrices over $\mathbf{Z}$

with determinant 1. For any $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z})$ let $\alpha_h$ be the diffeomorphism of $A_\theta$ defined by $\alpha_h(u) = u^a v^c$ and $\alpha_h(v) = u^b v^d$.

DEFINITION. Let $\theta$ be an irrational number. We say that *it is generic* if there are $C > 0$ and $r > 1$ such that

$$|e^{2\pi i n\theta} - 1| \geqslant \frac{C}{n^r}$$

for any positive integer $n$. That is, $\theta$ is generic if it is not a *Liouville number*.

Throughout the present paper we assume that $\theta$ is non-generic. Let $\tau$ be the unique tracial state on $A_\theta$. For any automorphism $\alpha$ of $A_\theta$ let $\tilde{\tau}$ be a tracial state on the crossed product $A_\theta \times_\alpha \mathbf{Z}$ and $\tilde{\tau}_*$ be the homomorphism of $K_0(A_\theta \times_\alpha \mathbf{Z})$ into $\mathbf{R}$ induced $\tilde{\tau}$. Furthermore for any automorphism $\alpha$ we denote by $\Gamma(\alpha)$ its Connes spectrum.

Now we define a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers in the following way:

We take $n_1$ so that

$$|e^{2\pi i n_1\theta} - 1| < \frac{1}{n_1}.$$

Since $\theta$ is non-generic, we can take $n_{j+1} \in \mathbf{N}$ so that $n_{j+1} > n_j$ and that

$$|e^{2\pi i n_{j+1}\theta} - 1| < \frac{1}{n_{j+1}^{j+1}}.$$

Let $\{a_n\}_{n=-\infty}^{\infty}$ be the sequence defined by

$$a_n = \begin{cases} \frac{1}{j}(1 - e^{2\pi i n_j\theta}) & \text{if } n = n_j \\ \frac{1}{j}(1 - e^{-2\pi i n_j\theta}) & \text{if } n = -n_j \\ 0 & \text{elsewhere.} \end{cases}$$

LEMMA 1. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be as above. For any* $k \in \mathbf{N}$

$$\lim_{|n|\to\infty} |n|^k |a_n| = 0.$$

*Proof:*

$$\lim_{|n|\to\infty} |n|^k |a_n| = \lim_{j\to\infty} n_j^k |a_{n_j}| = \lim_{j\to\infty} n_j^k \frac{1}{j} |1 - e^{2\pi i n_j\theta}| \leqslant \lim_{j\to\infty} \frac{1}{j} \frac{n_j^k}{n_j^j}.$$

As $j \to \infty$, we may assume that $j > k$. Hence

$$\lim_{j\to\infty} \frac{1}{j}\frac{n_j^k}{n_j^j} = 0.$$

Thus we obtain that

$$\lim_{|n|\to\infty} |n|^k |a_n| = 0$$ ■

Let $g$ be the function on $\mathbf{R}$ with period 1 defined by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{2\pi i n t}$$

for any $t \in \mathbf{R}$. By Lemma 1 we can easily see that $g$ is a $C^\infty$-function. Furthermore $g$ is real valued and

$$\int_0^1 g(t)\mathrm{d}t = a_0 = 0.$$

LEMMA 2. *Let $g$ be as above. Then there is no real valued continuous function $k$ on $\mathbf{R}$ with period 1 satisfying that*

$$g(t) = k(t) - k(t+\theta)$$

*for any $t \in \mathbf{R}$.*

*Proof:* We suppose that there is a real valued continuous function $k$ on $\mathbf{R}$ with period 1 satisfying that

$$g(t) = k(t) - k(t+\theta)$$

for any $t \in \mathbf{R}$. Let $\sum_{n=-\infty}^{\infty} b_n \mathrm{e}^{2\pi i n t}$ be the Fourier series of $k$. Then the Fourier series of $k(t) - k(t+\theta)$ is equal to

$$\sum_{n=-\infty}^{\infty} b_n (1 - \mathrm{e}^{2\pi i n\theta})\mathrm{e}^{2\pi i n t}.$$

Hence

$$a_n = b_n(1 - \mathrm{e}^{2\pi i n\theta})$$

for any $n \in \mathbf{Z}$. Thus if $n \neq 0$,

$$b_n = \frac{a_n}{1 - \mathrm{e}^{2\pi i n\theta}}.$$

Hence the Fourier series of $k$ is equal to

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1 - \mathrm{e}^{2\pi i n\theta}} \mathrm{e}^{2\pi i n t} + c$$

where $c$ is a constant number. Since $k$ is continuous,

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}}$$

is Cesáro summable. On the other hand by the definition of $\{a_n\}$,

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}} = 2\sum_{j=1}^{\infty} \frac{1}{j}.$$

Since $\sum \frac{1}{j}$ is not Cesáro summable, neither is

$$\sum_{n=-\infty}^{\infty} \frac{a_n}{1-e^{2\pi i n\theta}}.$$

This is a contradiction. Therefore we obtain the conclusion. ■

REMARK. In [4] H. Furstenberg constructed an example of an analytic diffeomorphism of the two dimensional torus which preserves Haar measure, is minimal, but is not ergodic. Then he showed that there are an irrational number $\theta \in \mathbf{R}$ and a real valued analytic function $g$ on $\mathbf{R}$ with period 1 and $\int_0^1 g(t)dt = 0$ satisfying the conditions in Lemma 2. His result in the above and its proof are explained in Mañé [5, II-7] and we stated them in [3, Lemma 5]. And in the present paper we has proved Lemma 2 in the same arguments as in Mañé [5, II-7].

THEOREM 3. *Let $\theta$ be a non-generic irrational number. Let $A_\theta$ be an irrational rotation $C^*$-algebra by $\theta$. Let $g$ be as in Lemma 2 and let $\alpha$ be an automorphism of $A_\theta$ defined by $\alpha(u) = e^{2\pi i g(v)}u$ and $\alpha(v) = v$. Then $\alpha$ is a diffeomorphism of $A_\theta$ satisfying the following conditions:*

(1) $\alpha_* = \mathrm{id}$ *on* $K_1(A_\theta)$,

(2) $\tilde{\tau}_*(K_0(A_\theta \times_\alpha \mathbf{Z})) = \mathbf{Z} + \mathbf{Z}\theta$,

(3) $\Gamma(\alpha) = \mathbf{T}$.

Proof: Since $g$ is a $C^\infty$-function, $\alpha$ is a diffeomorphism of $A_\theta$ by [3, Lemma 2]. By Lemma 2 and [3, Lemma 4] we can see that $\Gamma(\alpha) = \mathbf{T}$. Clearly $\alpha_* = id$ on $K_1(A_\theta)$ and

$$\mathrm{Ker}\,(id - \alpha_*) = \mathbf{Z}[u] \oplus \mathbf{Z}[v].$$

We can see that $\alpha(v)v^* = 1$ and $\alpha(u)u^* = e^{2\pi i g(v)}$. Let $\xi$ be the continuously differentiable path from 1 to $e^{2\pi i g(v)}$ on [0,1] defined by

$$\xi(r) = e^{2\pi i r g(v)}$$

for any $r \in [0,1]$. Then

$$\frac{1}{2\pi i}\int_0^1 \tau(\xi(r)^* \frac{d}{dr}\xi(r))dr = \int_0^1 \tau(g(v))dr = \tau(g(v)) = \int_0^1 g(r)dr = 0.$$

Therefore we obtain by Pimsner [7, Theorem 3] that

$$\tilde{\tau}_*(K_0(A_\theta \times_\alpha \mathbb{Z})) = \mathbb{Z} + \mathbb{Z}\theta. \qquad \blacksquare$$

COROLLARY 4. *Let $A_\theta$ and $\alpha$ be as in Theorem 3. Then*

$$\alpha \neq \mathrm{Ad}(w) \circ \alpha_k \circ \alpha_{(s,t)}$$

*for any unitary element $w \in A_\theta^\infty$, $h \in SL(2,\mathbb{Z})$ and $s,t \in \mathbb{R}$.*

*Proof:* Immediate by Theorem 3 and [3, Lemma 1]. ■

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## REFERENCES

1. EFFROS, E. G.; HAHAN, F., Locally compact transformation groups and $C^*$-algebras, *Mem. Amer. Math. Soc.*, **75**(1967).
2. ELLIOTT, G. A.; ,The diffeomorphism group of the irrational rotation $C^*$-algebra, *C. R. Math. Rep. Acad. Sci. Canada*, **8**(1986), 329–334.
3. KODAKA, K., A diffeomorphism of an irrational rotation $C^*$-algebra by a non-generic rotation, *J. Operator Theory*, **23**(1990), 73–79.
4. FURSTENBERG, H., Strict ergodicity and transforms of the torus, *Amer. J. Math.*, **83**(1961), 573–601.
5. MAÑÉ, R., *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, 1987.
6. PEDERSEN, G. K., *$C^*$-Algebras and their Automorphism Groups*, Academic Press, 1979.
7. PIMSNER, M. V., Ranges of traces on $K_0$ of reduced crossed products by free groups, *in Operator algebras and their connections with topology and ergodic theory*, Springer Lecture Notes in Math., **1132**(1983), pp.374–408.
9. RIEFFEL, M. A., $C^*$-algebras associated with irrational rotations, *Pacific J. Math.*, **93** (1981), 415–429.

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