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TOEPLITZ OPERATORS ON DISCRETE GROUPS

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1. INTRODUCTION

In this paper we present a generalization of the notion of Toeplitz operator and Toeplitz algebra to arbitrary discrete groups, study the properties of the generalized Toeplitz operators and discuss the extent to which they parallel the properties of classical Toeplitz operators.

Toeplitz operators and Toeplitz algebras represent an important area of modern mathematics. The theory of such operators has been looked at from many different perspectives, giving rise, over the years, to various extensions of the original notions to more general settings.

A Toeplitz operator T_f is defined as the compression to $H^2(\mathbf{T}) \subseteq \mathcal{L}^2(\mathbf{T})$ of the multiplication operator on $\mathcal{L}^2(\mathbf{T})$ by a bounded measurable function f on the unit circle $\mathbb{T}$. The Toeplitz algebra $\mathcal{T} = \mathcal{T}(\mathbf{Z})$ is the C^*-algebra generated by all Toeplitz operators T_f, $f \in C(\mathbf{T})$.

Equivalently, in the Fourier transform space, Toeplitz operators are obtained by compressing to $\ell^2(\mathbf{N})$ the group von Neumann algebra of $\mathbf{Z}$, and $\mathcal{T}$ is the C^*-algebra generated by the compression to $\ell^2(\mathbf{N})$ of the reduced C^*-algebra of $\mathbf{Z}$. On the other hand, $\mathcal{T}$ is also the C^*-algebra (up to $*$-isomorphisms) generated by a nonunitary isometry [5].

Various authors have generalized the latter point of view, and have studied C^*--algebras generated by a nonunitary semigroup of isometries [9], [16], C^*-algebras generated by a commuting family of subnormal operators [2], and C^*-algebras generated by unilateral weighted shifts [20].

Taking another point of view, $\mathcal{T}$ is an extension of $\mathcal{K}(\ell^2(\mathbf{N}))$ by $C(\mathbf{T})$ [5]. This raised the problem of determining all the extensions of the compact operators on a separable infinite-dimensional Hilbert space by $C(\mathbf{T})$, and, more generally, the extension problem for C^*-algebras (see chapter 7 in [11]).

In the previous kinds of generalization of the notion of Toeplitz operator, the algebra structure is paramount, and the Hilbert space on which the operators act in largely disregarded. Conversely, one can emphasize the choice of the underlying Hilbert space, and obtain generalizations by extending the notion of the Hardy space $H^2(\mathbb{T})$. One finds, for example, Hardy spaces defined for a half-line, for finitely connected regions of the plane with analytic boundary, and, more generally, for regions in $\mathbb{C}^n$. We refer to [10] and [11] for a bibliography on the vast literature on this kind of generalization.

Another possibility is to regard $\mathbb{Z}$ as the dual group of $\mathbb{T}$, and $\mathbb{N}$ as a semigroup of $\mathbb{Z}$. To generalize this approach, one considers an abelian locally compact group G, fixes a subsemigroup Σ of $\hat{G}$, and defines the "Hardy space" H^2 to be the space of all $\mathcal{L}^2$-functions on G whose Plancherel-Fourier transforms are supported on Σ [3].

As a special case, one can consider a compact abelian group G such that $\hat{G}$ is a partially ordered group, and choose Σ as the positive cone of $\hat{G}$ ([16], [17], [18], [19]). As one may expect, the Toeplitz algebra $\mathcal{T}^r(G)$ is defined as the C^*-algebra generated by the compression to H^2 of the multiplication algebra of $C(G)$. Equivalently, since in this case $\hat{G}$ is discrete, $\mathcal{T}^r(G)$ can be defined as the C^*-algebra generated by the compression to $\ell^2(\Sigma)$ of the reduced C^*-algebra of $\hat{G}$. If the order on $\hat{G}$ is total, then $\mathcal{T}^r(G)$ is characterized by the universal property of being isomorphic to any C^*--algebra generated by a nonunitary semigroup of isometries $\beta : (\hat{G})^+ \to \mathcal{B}(\mathcal{H})$. In this context, much of the classical Toeplitz theory can be extended, because of the richness of harmonic analysis on compact groups.

Our approach is to look at $\mathbb{N}$ as a subset of $\mathbb{Z}$ with the property that all of its translates, by the action of $\mathbb{Z}$, have the same "boundary at infinity". In general, for any infinite discrete group G, we consider all the infinite subsets of G with the property that they coincide, up to finite sets, with all of their translates by the action of G by left multiplication. We will call these sets almost invariant (in G).

The paper is organized as follows. In Section 2 we point out some of the significant properties of almost invariant sets, and give some examples. In Section 3, given any almost invariant set S, we define generalized Toeplitz operators by compressing to the "Hardy space" $\ell^2(S)$ the operators in $W^*(G)$, the group von Neumann algebra of G. In Section 4 we study the properties of such operators, and find that several basic results for classical Toeplitz operators ([14], [1], [4]) extend to the general setting. The main results in this section are:

(i) the spectral inclusion theorem extends to discrete groups.

(ii) the compression map from $W^*(G)$ into $\mathcal{B}(\ell^2(S))$ is isometric.

(iii) generalized Toeplitz operators can be characterized in terms of Toeplitz matrices.

(iv) generalized Toeplitz operators are extremally noncompact.

In Section 5 we define a generalized Toeplitz algebra $\mathcal{T}(G)$, and show that $\mathcal{T}(G)$ is an extension of $\mathcal{K}(\ell^2(S))$ by $C_r^*(G)$. For a class of groups which contains all torsion--free groups, we prove that $\mathcal{T}(G)$ is the C^*-algebra generated by the compression to $\ell^2(S)$ of the reduced C^*-algebra of G (as long as $S \neq G$). Under more restrictive conditions, we are also able to describe the commutator ideal of $\mathcal{T}(G)$. In Section 6 we generalize the notion of analytic Toeplitz operator, and extend Wintner's theorem on the spectra of analytic Toeplitz operators.

Related results can be found in earlier generalizations of the notion of Toeplitz operator (see e.g. [3] and [16]). We believe, however, that ours are the first such results in a non-commutative context and in such a great generality.

In Section 7 we conclude our study of Toeplitz operators by examining several properties of classical Toeplitz operators which do not extend, in general, to the setting of almost invariant sets. This limitation in our generalization is unavoidable for those parts of the classical theory which rest in an essential way on the commutativity of $\mathbb{Z}$ or on the semigroup property of $\mathbb{N}$ (see also [22]), or on the fact that $\mathbb{Z}$ has a connected dual (cf. [17] and [19]). On the other hand, the success of our generalization in Sections 4, 5 and 6 shows us clearly that a large portion of the classical Toeplitz theory depends entirely on the fact that $\mathbb{N}$ is almost invariant in $\mathbb{Z}$.

2. ALMOST INVARIANT SETS

Let G be an infinite (discrete) group. We want to define a class of subsets of G, which, roughly speaking, determines the possible "points at infinity" of G. For $G = \mathbb{Z}$, the two natural directions $+\infty$ and $-\infty$ are related in an obvious way to the classes of subsets of $\mathbb{Z}$ which differ from $\mathbb{N}$ and $\mathbb{Z} \setminus \mathbb{N}$, respectively, by a finite set. Moreover, any such subset has the property that it changes only by a finite set when translated by an element of $\mathbb{Z}$. With this motivation in mind, one realizes what should be the natural generalization to an arbitrary group G.

DEFINITION. Let G be an infinite group. We say that $S \subseteq G$ is an almost invariant (a.i.) set in G if $S \setminus gS$ is finite ($\forall g \in G$).

Note that

$$S \setminus ghS \subseteq (S \setminus gS) \cup g(S \setminus hS)$$

for all g, h in G. Hence, if H is a generating set for G, then S is a.i. in G if and only if $S \setminus hS$ and $S \setminus h^{-1}S$ are finite sets for all h in H.

In particular, if G is finitely generated, this gives a useful way of checking whether a given subset of G is almost invariant.

Also, note that any infinite almost invariant set S generates G. In fact, $G = \{st^{-1}; s, t \in S\}$. Indeed, if $g \in G$, then $S \cap g^{-1}S = S \setminus (S \setminus g^{-1}S)$ is nonempty. Hence $t = g^{-1}s$, for some $s, t \in S$, and thus $g = st^{-1}$.

Another important property, which we will tacitly use throughout the rest of the paper, is that $S \setminus F$ is almost invariant whenever S is a.i. and $F \subseteq G$ is finite.

For other properties of a.i. sets, and their relation with the space of "directions to infinity" in G, we refer the reader to [21] with no further comments.

For finitely generated groups, almost invariant sets are quite easy to describe. In fact, given any finite generating set H for G, any a.i. set can be described geometrically in terms of the Cayley graph of G with respect to H [21, Theorem 2.1.8]. Here we will not repeat the general result, but simply give a description of a.i. sets for some specific examples.

We also recall that any finitely generated group G falls into one of the three following categories:

(i) the only a.i. sets in G are the finite sets and their complements.

(ii) There exists an infinite set $S \subseteq G$, with infinite complement, such that, up to finite sets, G, $\emptyset$, S and $G \setminus S$ are the only a.i. sets in G.

(iii) There exists a sequence $\{S_n\}_{n=1}^{\infty}$ of a.i. subsets of G such that $S_n \Delta S_m$ is infinite for all $n \neq m$, and any a.i. set S differs from S_n by a finite set for some n.

The first class of groups contains such groups as $\mathbb{Z}^n$ $(n \geqslant 2)$, the triangle groups $T(p, q, r)$ $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\right)$, the Heisenberg group, the group of dyadic rationals and all discrete groups with Kazhdan's property T. The second class consists of all groups which are finite extensions of $\mathbb{Z}$. The third class, which is the most interesting one for our purposes, consists of certain amalgamated free products and HNN extensions of groups, according to Stallings's characterization [26, 5.A.9].

In the latter class, a.i. sets can be easily described algebraically, by means of the normal form of the elements of G. The basic example is the free group on two generators $\mathbb{F}_2$. If a, b are the canonical generators of $\mathbb{F}_2$, then any nontrivial element x in $\mathbb{F}_2$ can be written uniquely as a reduced word in a, b, a^{-1}, b^{-1}. For any nontrivial $x \in \mathbb{F}_2$, we let S_x denote the set of all nontrivial elements of $\mathbb{F}_2$ whose reduced words end in x. If S is an infinite almost invariant set in $\mathbb{F}_2$, then there exist nontrivial $x_1, \ldots, x_n$ in $\mathbb{F}_2$ and a finite set $F \subseteq \mathbb{F}_2$ such that

$$S = F \cup S_{x_1} \cup \ldots \cup S_{x_n} \tag{2.1}$$

[21, 2.1.8]. Similar descriptions can be given for such groups as $SL(2, \mathbb{Z}), PSL(2, \mathbb{Z})$

and $\underbrace{(\mathbf{Z}/2\mathbf{Z}) * \cdots * (\mathbf{Z}/2\mathbf{Z})}_{n \text{ times}}$ $(n \geqslant 3)$.

In the case of groups which are not finitely generated, it is hard to find a general algebraic or geometric characterization of a.i. sets. It seems to us that any such characterization would not be more revealing or helpful than the definition itself.

Let us consider, for example, the free group $\mathbf{F}_\infty$ with countably many generators $a_1, a_2, a_3, \ldots$. For any nontrivial x in $\mathbf{F}_\infty$, the set S_x defined as above is almost invariant. More generally, one can construct fairly complicated a.i. sets, as shown for example by the a.i. set S defined by

$$S = \{a_n\}_{n=1}^{\infty} \cup \bigcup_{n=1}^{\infty} \left(S_{a_n^2} \cup \bigcup_{i=1}^{n-1} \left(S_{a_i a_n} \cup S_{a_i^{-1} a_n} \right) \right).$$

Note that S cannot be expressed as in (2.1) for some finite $F \subseteq \mathbf{F}_\infty$ and some $x_1, \ldots, x_n \in \mathbf{F}_\infty$.

More pathological situations can occur in $\mathbf{F}_\infty$. Indeed, one can construct an infinite a.i. set S with the property that S does not contain any set S_x, $x \in \mathbf{F}_\infty$. In fact, $S_x \setminus S$ is infinite for all x.

Our final example is an infinitely generated abelian group. Let

$$G = \bigoplus_{i=0}^{\infty} \mathbf{Z}/2\mathbf{Z} = \{(n_0, n_1, \ldots, n_r, 0, \ldots, 0, \ldots); r \in \mathbf{N},\ n_i \in \mathbf{Z}/2\mathbf{Z},\ 0 \leqslant i \leqslant r\}.$$

For all $i \in \mathbf{N}$, we define e_i as the element $\{n_j\}_{j=0}^{\infty}$ of G with $n_i = 1$ and $n_j = 0$ for $j \neq i$. For any positive integer n, let

$$S_n = \{e_{2n} + e_{2n+1} + \sum_{i=0}^{2n-1} \lambda_i e_i \ ;\ \lambda_i \in \{0,1\},\ 0 \leqslant i \leqslant 2n-1\}.$$

Then the set $S = \bigcup_{n=1}^{\infty} S_n$ is an infinite almost invariant set in G, with infinite complement.

3. PRELIMINARIES

If $\mathcal{H}$ is a Hilbert space, then we let $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $(\mathcal{B}/\mathcal{K})(\mathcal{H})$ denote respectively the algebra of all bounded linear operators on $\mathcal{H}$ with the usual operator norm, the ideal of all compact operators on $\mathcal{H}$, and their quotient algebra (i.e. the Calkin algebra of $\mathcal{H}$). For any T in $\mathcal{B}(\mathcal{H}), [T]$ will denote the equivalence class of T in $(\mathcal{B}/\mathcal{K})(\mathcal{H})$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Fredholm operator if $[T]$ is invertible in $(\mathcal{B}/\mathcal{K})(\mathcal{H})$, or, equivalently, if T has closed range and finite-dimensional null and defect spaces. For any Fredholm operator T, the integer dim ker T − dim $\mathcal{R}(T)^{\perp}$ is called the index of T and shall be denoted by Ind(T).

If A is a C^*-algebra, we let $K(A)$ denote its commutator ideal, i.e. the closed two-sided ideal generated by all commutators $[a,b] = ab - ba$ $(a, b \in A)$.

Let G be a discrete group, denoted multiplicatively and with identity element e. We shall denote by $\ell^2(G)$ the Hilbert space of all square-summable complex-valued functions on G, and shall denote by $\{\delta_g; g \in G\}$ its canonical orthonormal basis. For every g in G we denote by L_g and R_g the unitary operators on $\ell^2(G)$ defined by

$$L_g(\delta_h) = \delta_{gh}$$

$$R_g(\delta_h) = \delta_{hg^{-1}}$$

for all h in G.

The von Neumann algebra generated by $\{L_g; g \in G\}$ is called the group von Neumann algebra of G and is denoted by $W^*(G)$. The C^*-algebra generated by $\{L_g; g \in G\}$ is called the reduced C^*-algebra of G and is denoted by $C_r^*(G)$. It is a well known fact that $W^*(G)$ is the commutant of the set $\{R_g; g \in G\}$.

For any $S \subseteq G$, we identify $\ell^2(S)$ in the obvious way with a closed linear subspace of $\ell^2(G)$, and denote by P_S the orthogonal projection from $\ell^2(G)$ onto $\ell^2(S)$. If S is an infinite almost invariant set in G, and $T \in W^*(G)$, we define the compression $T^{(S)}$ of T to $\ell^2(S)$ by letting

$$T^{(S)} = P_S T | \ell^2(S).$$

We will refer to $T^{(S)}$ as a Toeplitz operator, and call T the symbol of $T^{(S)}$. We also define generalized "unilateral shifts" T_g, $g \in G$, by letting

$$T_g = P_S L_g | \ell^2(S) \quad (g \in G).$$

Finally, we define a map $\Phi : C_r^*(G) \to (\mathcal{B}/\mathcal{K})(\ell^2(S))$ by letting

$$\Phi(T) = [T^{(S)}] \quad (T \in C_r^*(G)).$$

Note that for $G = \mathbb{Z}$ and $S = \mathbb{N}$ the operators T_g are simply all powers of the unilateral shift and of its adjoint. In this setting, the operators $T^{(S)}$, $T \in C_r^*(G)$, are precisely the Toeplitz operators with continuous symbols.

We refer the reader to [10] for an account of the classical theory of Toeplitz operators.

4. GENERALIZED TOEPLITZ OPERATORS

Throughout the rest of the paper, G will denote a countably infinite discrete group, and S will be an infinite almost invariant set in G.

We begin our study of Toeplitz operators by considering some basic properties of the mapping $T \mapsto T^{(S)}$, $T \in W^*(G)$. In the classical theory, the first systematic study of Toeplitz operators, emphasizing the mapping $f \mapsto T_f$, $f \in \mathcal{L}^\infty(\mathbf{T})$, was made by Brown and Halmos in [1], built on some earlier work of Hartman and Wintner [14], who had proved the spectral inclusion theorem.

LEMMA 4.1. *Let G be a countably infinite group, let $S = \{s(n)\}_{n=1}^\infty$ be an infinite almost invariant set in G, and let I be the identity of the algebra $\mathcal{B}(\ell^2(G))$. Then*

$$R_{s(n)} P_S R^*_{s(n)} \to I \quad \textit{strongly as } n \to \infty.$$

Proof. Let F be a finite subset of G. Then there exists a positive integer n_0 such that

$$Fs(n) \subseteq S \quad (\forall n \geqslant n_0).$$

Indeed, if $F = \{x_1, \ldots, x_m\}$, then

$$\{s \in S; Fs \subseteq S\} = S \cap x_1^{-1} S \cap \ldots \cap x_m^{-1} S =$$

$$= S \setminus ((S \setminus x_1^{-1} S) \cup \ldots \cup (S \setminus x_m^{-1} S))$$

is the complement of a finite set in S.

Let f be a function in $\ell^2(G)$ supported on F, and let n_0 be as above. Then $R^*_{s(n)} f$ is supported on $Fs(n) \subseteq S$ ($n \geqslant n_0$), and thus $P_S R^*_{s(n)} f = R^*_{s(n)} f$ for all $n \geqslant n_0$. Therefore

$$R_{s(n)} P_S R^*_{s(n)} f = f \quad (\forall n \geqslant n_0),$$

from which the conclusion follows immediately. ■

If E is a subset of $\mathbf{C}$, we let hull(E) denote the closed convex hull of E. If T is a bounded linear operator on a Hilbert space $\mathcal{H}$, we let $\sigma(T), \sigma_{\mathrm{e}}(T), \sigma_{\mathrm{ap}}(T)$ and $W(T)$ denote the spectrum, the essential spectrum, the approximate point spectrum and the numerical range of T, respectively.

THEOREM 4.2. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant set in G. Then*

(i) *The "symbol map" $T \mapsto T^{(S)}$ is an isometric $*$-linear mapping from $W^*(G)$ into $\mathcal{B}(\ell^2(S))$.*

(ii) *(spectral inclusion) For all T in $W^*(G)$,*

$$\sigma(T) \subseteq \sigma(T^{(S)}) \text{ and } \sigma_{\mathrm{ap}}(T) \subseteq \sigma_{\mathrm{ap}}(T^{(S)}).$$

For all T in $C_r^*(G)$,

$$\sigma(T) = \sigma_{\mathrm{e}}(T^{(S)}).$$

For all normal T in $W^*(G)$,

$$\sigma(T^{(S)}) \subseteq \mathrm{hull}(\sigma(T)) = \overline{W(T)} = \overline{W(T^{(S)})}.$$

(iii) *$T^{(S)} \geqslant 0$ if and only if $T \geqslant 0$.*

Proof. (i) It is straightforward that $T \mapsto T^{(S)}$ is a $*$-linear mapping. Let $T \in$ $\in W^*(G)$. In particular, T commutes with R_g for all g in G. If $S = \{s(n)\}_{n=1}^{\infty}$, then

$$R_{s(n)} P_S T P_S R_{s(n)}^* = (R_{s(n)} P_S R_{s(n)}^*) T (R_{s(n)} P_S R_{s(n)}^*) \to T$$

strongly as $n \to \infty$, by Lemma 4.1. Hence

$$\|T\| \leqslant \lim_{n\to\infty} \|R_{s(n)} P_S T P_S R_{s(n)}^*\| = \|P_S T P_S\| = \|T^{(S)}\|.$$

Since obviously $\|T^{(S)}\| \leqslant \|T\|$, we have $\|T^{(S)}\| = \|T\|$.

(ii) It follows from the proof of Lemma 4.1 that for any finitely-supported function $f \in \ell^2(G)$ there exists s in S such that $R_s^* f$ is supported on S (this property is the non-commutative version of the lemma in [3, Section 2] and of Lemma 3.1 in [16]). Equivalently, if we let

$$V = \{R_s x;\, s \in S,\ x \in \ell^2(S)\},$$

then V is dense in $\ell^2(G)$.

Let $T \in W^*(G)$, and suppose that, for some $c > 0$,

$$\|T^{(S)} x\| \geqslant c\|x\| \quad (\forall x \in \ell^2(S)).$$

Then for any s in S, and any x in $\ell^2(S)$, we have (in what follows, all norms are taken in $\ell^2(G)$)

$$\|T R_s x\| = \|R_s T x\| = \|T x\| \geqslant$$

$$\geqslant \|P_S T x\| \geqslant c\|x\| = c\|R_s x\|.$$

Hence $\|Ty\| \geqslant c\|y\|$ for all y in $\ell^2(G)$, since V is dense in $\ell^2(G)$. It follows that $\sigma_{\mathrm{ap}}(T) \subseteq \sigma_{\mathrm{ap}}(T^{(S)})$.

If $T^{(S)}$ is invertible, then T is invertible. Indeed, $T^{(S)}$ is bounded below, and so is T by the argument above. Similarly, T^* is bounded below, since $(T^*)^{(S)} = (T^{(S)})^*$ is also invertible, and thus T is invertible (see e.g. [10, 4.9]). Hence $\sigma(T) \subseteq \sigma(T^{(S)})$ for all T in $W^*(G)$.

We will show in Theorem 5.3 that the map Φ is a one-to-one $*$-homomorphism on $C_r^*(G)$. In particular, $\sigma(T) = \sigma_e(T^{(S)})$ for $T \in C_r^*(G)$.

Finally, let T be a normal operator in $W^*(G)$, and let $A = C^*(T, I)$. Then $N \mapsto N^{(S)}$ is an isometric $*$-linear mapping from the abelian C^*-algebra A into $\mathcal{B}(\ell^2(S))$, and thus $\sigma(T^{(S)}) \subseteq \text{hull}(\sigma(T))$ (see e.g. [10, exercise 7.1, p. 203]).

Since T is normal, we have $\text{hull}(\sigma(T)) = \overline{W(T)}$. If $x \in \ell^2(S)$, then $\langle T^{(S)}x, x\rangle = = \langle Tx, x\rangle$; this proves that $W(T^{(S)}) \subseteq W(T)$, and hence $\overline{W(T^{(S)})} \subseteq \overline{W(T)}$. For the reverse inclusion, note that $\overline{W(T^{(S)})}$ includes $\sigma(T^{(S)})$, and therefore it includes $\sigma(T)$. Since $\overline{W(T^{(S)})}$ is convex, it follows that $\overline{W(T)} = \text{hull}(\sigma(T)) \subseteq \overline{W(T^{(S)})}$.

(iii) If $T \geqslant 0$ in $W^*(G)$, then $T^{(S)} \geqslant 0$, as $\langle T^{(S)}x, x\rangle = \langle Tx, x\rangle$ for all x in $\ell^2(S)$. Conversely, if $T^{(S)} \geqslant 0$, then $P_S T P_S$ is a positive operator in $\mathcal{B}(\ell^2(G))$. Hence $T \geqslant 0$, since T is the strong limit of the sequence $R_{s(n)} P_S T P_S R^*_{s(n)}$.

This completes the proof, which was inspired by the arguments in [1, Theorem 5], [3, Theorem 1] and [10, 7.6]. ∎

A consequence of the previous theorem is that a Toeplitz operator with a normal symbol and a real spectrum must be self-adjoint, because if the spectrum of $T^{(S)}$ is real, then the same is true of T, and so T is self-adjoint.

We now want to give an intrinsic characterization of Toeplitz operators on $\ell^2(S)$. A standard basic fact about classical Toeplitz operators is that an operator $A \in \in \mathcal{B}(\ell^2(\mathbb{N}))$ is a Toeplitz operator if and only if the matrix $\{a_{i,j}\}_{i,j\geqslant 0}$ canonically associated with A is a Toeplitz matrix, that is, $a_{i,j} = a_{i+k,j+k}$ [1, Theorem 4]. In fact, for some authors, this is a definition.

In our setting, if $S \subseteq G$, $T \in W^*(G)$, $x, y \in S$, $g \in G$, and $xg, yg \in S$, then

$$\langle T^{(S)}\delta_y, \delta_x\rangle = \langle T\delta_y, \delta_x\rangle = \langle R_g T R_g^* \delta_y, \delta_x\rangle = \langle T\delta_{yg}, \delta_{xg}\rangle = \langle T^{(S)}\delta_{yg}, \delta_{xg}\rangle.$$

This motivates the definition of a Toeplitz matrix as a matrix $\{a_{x,y}\}_{x,y\in S}$ such that

$$a_{x,y} = a_{xg,yg}$$

for all $x, y \in S$, $g \in G$, with $xg, yg \in S$.

THEOREM 4.3. *Let G be a countably infinite group, and let S be an infinite almost invariant set in G. A necessary and sufficient condition that a bounded operator A on $\ell^2(S)$ be a Toeplitz operator is that the matrix associated with A, with respect to* *the orthonormal basis $\{\delta_s; s \in S\}$, be a Toeplitz matrix. Moreover, if A is a Toeplitz operator, then there exists a unique operator $T \in W^*(G)$ such that $A = T^{(S)}$.*

Proof. The proof of necessity was given in the remarks preceding the statement of the theorem. The proof of sufficiency will be modeled after the proof in [1, Theorem 4].

Let A be a bounded operator on $\ell^2(S)$, such that the matrix $\{\langle A\delta_y, \delta_x\rangle\}_{x,y\in S}$ is a Toeplitz matrix. Let $S = \{s(n)\}_{n=1}^{\infty}$, and consider, for each positive integer n, the bounded operator on $\ell^2(G)$ defined by

$$A_n = R_{s(n)} A P_S R^*_{s(n)}.$$

Let $x, y \in G$, and let $n_0 \in \mathbb{N}$ be such that

$$xs(n), ys(n) \in S \quad (\forall n \geqslant n_0).$$

Then $\{\langle A\delta_{ys(n)}, \delta_{xs(n)}\rangle\}_{n\geqslant n_0}$ is a constant sequence, and thus

$$\langle A_n\delta_y, \delta_x\rangle = \langle AP_S\delta_{ys(n)}, \delta_{xs(n)}\rangle = \langle A\delta_{ys(n)}, \delta_{xs(n)}\rangle \tag{4.1}$$

is independent of n for all $n \geqslant n_0$. Therefore, if f, g are finitely supported functions in $\ell^2(G)$, then the sequence $\{\langle A_n f, g\rangle\}_{n=1}^{\infty}$ is convergent.

Since $\|A_n\| \leqslant \|A\|$, it follows by standard arguments that the sequence $\{A_n\}_{n=1}^{\infty}$ is weakly convergent to an operator $T \in \mathcal{B}(\ell^2(G))$ (see e.g. [13]). By (4.1),

$$\langle T\delta_y, \delta_x\rangle = \lim_{n\to\infty} \langle A\delta_{ys(n)}, \delta_{xs(n)}\rangle \tag{4.2}$$

for all x, y in G (in particular, T is uniquely determined by A).

Fix x, y, g in G. Since S is almost invariant in G,

$$(\forall N \in \mathbb{N})(\exists M \in \mathbb{N})(\forall m \geqslant M) \;\; gs(m) \subseteq \{s(n)\}_{n=N}^{\infty}. \tag{4.3}$$

Hence, by (4.2) and (4.3),

$$\langle R_g T R_g^* \delta_y, \delta_x\rangle = \langle T\delta_{yg}, \delta_{xg}\rangle = \lim_{m\to\infty} \langle A\delta_{ygs(m)}, \delta_{xgs(m)}\rangle =$$

$$= \lim_{n\to\infty} \langle A\delta_{ys(n)}, \delta_{xs(n)}\rangle = \langle T\delta_y, \delta_x\rangle.$$

Therefore $R_g T = T R_g$ for all g in G, and thus $T \in W^*(G)$. Using (4.2) again, we get that $A = T^{(S)}$, and so A is a Toeplitz operator.

A similar argument shows that if $T_1, T_2 \in W^*(G)$ and $T_1^{(S)} = T_2^{(S)}$, then

$$\langle T_1\delta_y, \delta_x\rangle = \langle T_2\delta_y, \delta_x\rangle \quad \text{for all } x, y \text{ in } G,$$

and thus $T_1 = T_2$. ∎

Theorem 4.3 shows that for every bounded Toeplitz matrix A there exists a unique $T \in W^*(G)$ such that $T^{(S)} = A$. The matrix $\{\langle T\delta_y, \delta_x\rangle\}_{x,y\in G}$ canonically associated with T can be recaptured from the matrix A by (4.2).

We also showed that T is the weak limit of the sequence $A_n = R_{s(n)}AP_S R^*_{s(n)}$. But $AP_S = P_S T P_S$, and so it follows from the proof of Theorem 4.2 that the weak convergence of A_n to T is, in fact, strong convergence.

Our next result establishes that there exist no nonzero compact Toeplitz operators (cf. [1, p. 94]).

PROPOSITION 4.4. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant set in G. Then, for T in $W^*(G)$,*

$$T^{(S)} \in \mathcal{K}(\ell^2(S)) \Leftrightarrow T = 0.$$

Proof. Let $S = \{s(n)\}_{n=1}^{\infty}$, and let $T \in W^*(G)$ be such that $T^{(S)}$ is compact. Fix x, y in G, and let $n_0 \in \mathbf{N}$ be such that $xs(n), ys(n) \in S$ for all $n \geqslant n_0$. In particular, the sequence $\{\delta_{ys(n)}\}_{n\geqslant n_0}$ converges weakly to 0 in $\ell^2(S)$, and thus $\|T^{(S)}\delta_{ys(n)}\| \to 0$ as $n \to \infty$. Therefore

$$\langle T\delta_y, \delta_x\rangle = \lim_{n\to\infty} \langle T^{(S)}\delta_{ys(n)}, \delta_{xs(n)}\rangle = 0$$

(see (4.2) in the proof of Theorem 4.3). It follows that $\langle T\delta_y, \delta_x\rangle = 0$ for all x, y in G, and hence $T = 0$. ∎

Following Coburn [4], we say that a bounded linear operator A on a Hilbert space $\mathcal{H}$ is extremally noncompact if

$$\|A + K\| \geqslant \|A\| \quad (\forall K \in \mathcal{K}(\mathcal{H})).$$

Coburn showed that Toeplitz operators are extremally noncompact [4, 4.2]. We will now extend this result to arbitrary discrete groups, incidentally obtaining an alternative way of proving the foregoing proposition.

PROPOSITION 4.5. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant set in G. Then*

$$\|T^{(S)} + K\| \geqslant \|T^{(S)}\| \quad (\forall T \in W^*(G)), \ (\forall K \in \mathcal{K}(\ell^2(S))).$$

In particular, the map $T \mapsto [T^{(S)}]$ is an isometric $*$-linear map from $W^*(G)$ into $(\mathcal{B}/\mathcal{K})(\ell^2(S))$.

Proof. Choose $T \in W^*(G)$, $K \in \mathcal{K}(\ell^2(S))$ and $\varepsilon > 0$. Let $S = \{s(n)\}_{n=1}^{\infty}$, and let P_m $(m > 0)$ be orthogonal projection from $\ell^2(S)$ onto $\overline{\operatorname{span}}\{\delta_{s(n)}; 1 \leqslant n \leqslant m\}$. Then $KP_m \to K$ in operator norm as $m \to \infty$. Therefore there exist K' in $\mathcal{K}(\ell^2(S))$ and a finite set $F \subseteq S$ such that

$$\|K - K'\| < \frac{\varepsilon}{2}$$

and, if we let $S' = S \setminus F$,

$$K'f = 0 \quad (\forall f \in \ell^2(S')).$$

Now S' is an infinite almost invariant set in G, so $\|T^{(S')}\| = \|T\| = \|T^{(S)}\|$ by Theorem 4.2. Hence there exists $f \in \ell^2(S')$ such that

$$\|T^{(S')}f\| \geqslant \left(\|T^{(S)}\| - \frac{\varepsilon}{2}\right) \|f\|.$$

Therefore

$$\|(T^{(S)} + K)f\| \geqslant \|(T^{(S)} + K')f\| - \frac{\varepsilon}{2}\|f\| = \|T^{(S)}f\| - \frac{\varepsilon}{2}\|f\| =$$

$$= \|P_S T f\| - \frac{\varepsilon}{2}\|f\| \geqslant \|P_{S'} T f\| - \frac{\varepsilon}{2}\|f\| = \|T^{(S')}f\| - \frac{\varepsilon}{2}\|f\| \geqslant (\|T^{(S)}\| - \varepsilon)\|f\|.$$

Hence $\|T^{(S)} + K\| \geqslant \|T^{(S)}\|$, since $\varepsilon > 0$ was arbitrary. The final statement of the proposition now follows from part (i) of Theorem 4.2. ■

5. TOEPLITZ OPERATORS WITH SYMBOLS IN $C_r^*(G)$ AND TOEPLITZ ALGEBRAS

What might be called the algebra approach to Toeplitz theory emphasizes the view that Toeplitz operators are studied as elements of certain algebras of operators, called Toeplitz algebras.

In this section we will generalize the notion of Toeplitz algebra and extend the relevant theorems of this part of Toeplitz theory to our setting.

The basic theorems about the classical Toeplitz algebra $\mathcal{T}(\mathbf{Z})$ are contained in Coburn's papers [5], [6]. The main result is that $\mathcal{T}(\mathbf{Z})$ contains the full algebra of compact operators $\mathcal{K}$ and that $\mathcal{T}(\mathbf{Z})/\mathcal{K}$ is $*$-isomorphic to $C(\mathbf{T})$, the algebra of all complex-valued continuous function on the unit circle $\mathbf{T}$.

In our context, we will define a C^*-algebra $\mathcal{T}(G)$ which contains $\mathcal{K}(\ell^2(S))$, and show that $\mathcal{T}(G)/\mathcal{K}(\ell^2(S))$ is $*$-isomorphic to $C_r^*(G)$. Under certain additional assumptions, we will show that $\mathcal{T}(G)$ is the C^*-subalgebra of $\mathcal{B}(\ell^2(S))$ generated by all Toeplitz operators $T^{(S)}$, $T \in C_r^*(G)$, thereby generalizing the classical case completely.

We first prove a preliminary result which is interesting in its own right, as it gives an operator-theoretic characterization of almost invariant sets.

For any $S \subseteq G$, we will denote the cardinality of S by $|S|$.

PROPOSITION 5.1. *Let G be a discrete group, and S be a subset of G. Then the following conditions are equivalent:*

(i) *S is an almost invariant set in G.*
(ii) *$TP_S - P_S T \in \mathcal{K}(\ell^2(G))$ $(\forall T \in C_r^*(G))$.*
(iii) *$\Phi : C_r^*(G) \to (\mathcal{B}/\mathcal{K})(\ell^2(S))$ is a unital $*$-homomorphism.*
(iv) *The map $g \mapsto [T_g]$ from G into $(\mathcal{B}/\mathcal{K})(\ell^2(S))$ is multiplicative.*
(v) *$[T_g]$ is a unitary element in $(\mathcal{B}/\mathcal{K})(\ell^2(S))$ for all g in G.*
(vi) *T_g is a Fredholm operator for all g in G.*

Moreover, if the conditions above are satisfied, then

$$\mathrm{Ind}(T_g) = |S \setminus g^{-1}S| - |S \setminus gS| = \mathrm{Trace}([T_g, T_{g^{-1}}]) \ (\forall g \in G).$$

Proof. We will show that (vi)⇒(i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi).
(vi)⇒(i): Choose g in G. Then

$$T_g(\delta_h) = \begin{cases} \delta_{gh} & \text{if } h \in (S \cap g^{-1}S) \\ 0 & \text{if } h \in (S \setminus g^{-1}S) \end{cases}$$

Therefore T_g is a partial isometry with initial space $\ell^2(S \cap g^{-1}S)$, and final space $\ell^2(S \cap gS)$. Hence

$$\ker(T_g) = \ell^2(S \setminus g^{-1}S), \ \mathcal{R}(T_g)^\perp = \ell^2(S \setminus gS).$$

For any $Z \subseteq S$, let I denote the identity operator on $\ell^2(S)$, and let P_Z denote the orthogonal projection from $\ell^2(S)$ onto $\ell^2(Z)$. Then, for all g in G,

$$[T_g, T_{g^{-1}}] = P_{S \cap gS} - P_{S \cap g^{-1}S} =$$
$$= I - P_{S \cap g^{-1}S} - (I - P_{S \cap gS}) = P_{S \setminus g^{-1}S} - P_{S \setminus gS}.$$

The desired implication is now immediate, as well as the formula for the index.
(i)⇒(ii): Since $P_S L_g = L_g P_{g^{-1}S}$, then

$$L_g P_S - P_S L_g = L_g(P_S - P_{g^{-1}S}) = L_g(P_{S \setminus g^{-1}S} - P_{g^{-1}S \setminus S}).$$

Hence $L_g P_S - P_S L_g$ is compact if S is almost invariant, so that (i)⇒(ii) follows by a standard density argument.

(ii)⇒(iii): If (ii) holds, then it is immediate that Φ is a $*$-homomorphism, since

$$\Phi(T)\Phi(T') = [P_S T|\ell^2(S)][P_S T'|\ell^2(S)] = [P_S T P_S T'|\ell^2(S)] =$$

$$= [P_S T T' P_S|\ell^2(S)] = [P_S T T'|\ell^2(S)] = \Phi(TT'),$$

for all T, T' in $C_r^*(G)$.
(iii)⇒(iv): If Φ is a homomorphism, then

$$[T_h][T_g] = \Phi(L_h)\Phi(L_g) = \Phi(L_h L_g) = \Phi(L_{hg}) = [T_{hg}]$$

for all h, g in G.
(iv)⇒(v): If $g \mapsto [T_g]$ is multiplicative, then

$$[T_g]^*[T_g] = [T_{g^{-1}}][T_g] = [T_e] = [I]$$

for all g in G, and similarly $[T_g][T_g]^* = [I]$, that is, $[T_g]$ is unitary ($\forall g \in G$).
(v)⇒(vi): If $[T_g]$ is unitary, then it is invertible in $(\mathcal{B}/\mathcal{K})(\ell^2(S))$, so T_g is a Fredholm operator.

This completes the proof. ■

REMARK. The same proof as above shows that

$$[P_S T|\ell^2(S)][P_S T'|\ell^2(S)] = [P_S T T'|\ell^2(S)]$$

($\forall T' \in C_r^*(G)$), ($\forall T \in W^*(G)$). This generalizes the fact that $T_f T_\psi - T_{f\psi}$ is compact ($\forall f \in C(\mathbf{T})$), ($\forall \psi \in \mathcal{L}^\infty(\mathbf{T})$) (cf. [10, 7.22]). However, the obvious extension of Φ to $W^*(G)$ is not necessarily a homomorphism. In the classical case, this amounts to the fact that $T_f T_\psi - T_{f\psi}$ is not compact for a suitable choice of f, ψ in $\mathcal{L}^\infty(\mathbf{T})$ (cf. [11, Theorem 7] and [25]).

COROLLARY 5.2. *Let G be a finitely generated group, and let S be an a.i. set in G. If G is generated by $g_1, \dots, g_n$, then the set $\{\mathrm{Ind}(T_g); g \in G\}$ is completely determined by the n-tuple*

$$(\mathrm{Ind}(T_{g_1}), \dots, \mathrm{Ind}(T_{g_n})).$$

Proof. For all g in G, $\mathrm{Ind}(T_{g^{-1}}) = \mathrm{Ind}(T_g^*) = -\mathrm{Ind}(T_g)$. Moreover, $[T_{gh}] = [T_g T_h]$ ($\forall g, h \in G$) by Proposition 5.1. Now we simply use the fact that the Fredholm index is multiplicative, and invariant under compact perturbations. ■

DEFINITIONS. In analogy with the classical case $G = \mathbf{Z}$, $S = \mathbf{N}$, we define $\mathcal{T}^r(G)$, the reduced Toeplitz algebra of G with respect to the a.i. set S, to be the

C^*-subalgebra of $\mathcal{B}(\ell^2(S))$ generated by the operators T_g, $g \in G$ (or, equivalently, generated by all operators $T^{(S)}$, $T \in C_r^*(G)$). Moreover, we define $\mathcal{T}(G)$, the Toeplitz algebra of G with respect to S, to be the C^*-subalgebra of $\mathcal{B}(\ell^2(S))$ defined by

$$\mathcal{T}(G) = \{T \in \mathcal{B}(\ell^2(S)); [T] \in \Phi(C_r^*(G))\}.$$

It is clear that $\mathcal{T}^r(G) \subseteq \mathcal{T}(G) = \mathcal{T}^r(G) + \mathcal{K}(\ell^2(S))$, and that

$$\mathcal{T}^r(G) = \mathcal{T}(G) \Leftrightarrow \mathcal{K}(\ell^2(S)) \subseteq \mathcal{T}^r(G)$$

(this is the case for the classical Toeplitz algebra).

We will show that $\mathcal{T}^r(G) = \mathcal{T}(G)$ for a class of groups containing all torsion-free groups, as long as $S \neq G$ (Theorem 5.7).

If A is a separable unital C^*-algebra, by an extension of A we mean a unital one-to-one $*$-homomorphism

$$\sigma : A \to (\mathcal{B}/\mathcal{K})(\mathcal{H})$$

from A into the Calkin algebra of a separable infinite-dimensional Hilbert space $\mathcal{H}$.

THEOREM 5.3. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant set in G. Then $\Phi : C_r^*(G) \to (\mathcal{B}/\mathcal{K})(\ell^2(S))$ is a unital one--to-one $*$-homomorphism.*

In particular, Φ is an extension of $C_r^*(G)$, and there exists a $*$-homomorphism $\rho : \mathcal{T}(G) \to C_r^*(G)$, with continuous cross section $T \mapsto T^{(S)}$, such that

$$0 \to \mathcal{K}(\ell^2(S)) \xrightarrow{i} \mathcal{T}(G) \xrightarrow{\rho} C_r^*(G) \to 0$$

is a short exact sequence of C^*-algebras. If $\mathcal{A}$ is a C^*-algebra, then the sequence of spatial tensor products

$$0 \to \mathcal{K}(\ell^2(S)) \otimes \mathcal{A} \xrightarrow{i\otimes 1} \mathcal{T}(G) \otimes \mathcal{A} \xrightarrow{\rho\otimes 1} C_r^*(G) \otimes \mathcal{A} \to 0$$

is short exact.

Proof. By Proposition 5.1, Φ is a unital $*$-homomorphism, and by Proposition 4.4, Φ is one-to-one. Therefore $C_r^*(G) \simeq \Phi(C_r^*(G)) \simeq \mathcal{T}(G)/\mathcal{K}(\ell^2(S))$, by definition of $\mathcal{T}(G)$. As the map $T \mapsto T^{(S)}$ is completely positive, it extends to a bounded map from $C_r^*(G) \otimes \mathcal{A}$ to $\mathcal{T}(G) \otimes \mathcal{A}$ for every C^*-algebra $\mathcal{A}$. Then the proof of the final statement is the same as that of Proposition 2 in [7]. ∎

COROLLARY 5.4. *Let S be an infinite almost invariant set in G. Then for any A in $\mathcal{T}(G)$ there exist unique $T \in C_r^*(G)$, $K \in \mathcal{K}(\ell^2(S))$ such that*

$$A = T^{(S)} + K.$$

In particular, the class of Fredholm operators in $\mathcal{T}(G)$ consists of all operators of the form $T^{(S)} + K$, with T invertible in $C_r^*(G)$.

By the previous corollary all operators $T^{(S)}$, with T invertible in $C_r^*(G)$, are Fredholm Toeplitz operators in $\mathcal{T}^r(G)$. Moreover, if $S \neq G$, then $\mathcal{T}^r(G)$ contains non-invertible Fredholm Toeplitz operators, as in the classical theory. Indeed, let g be any element of G such that $S \setminus g^{-1}S \neq \emptyset$; then T_g is a Fredholm Toeplitz operator with nontrivial kernel $\ell^2(S \setminus g^{-1}S)$.

This is different from the situation that presents itself in some other generalized Toeplitz theories, where the ordinary Fredholm theory does not help in understanding any near-invertibility of Toeplitz operators. Indeed, in the case of Toeplitz operators on the torus ([12], Corollary 2) or on non-cyclic ordered subgroups of the reals [18], a Toeplitz operator with never-vanishing continuous symbol is Fredholm if and only if it is invertible.

COROLLARY 5.5. *If S is an infinite almost invariant set in G, then the map*

$$g \mapsto [T_g]$$

is a faithful unitary representation of G on the Calkin algebra of $\ell^2(S)$.

Proof. The result follows immediately from Theorem 5.3, together with conditions (iv), (v) of Proposition 5.1. ■

In the classical case, this gives the well-known isomorphism $\mathbb{Z} \simeq \Delta/\Delta_0$, where Δ is the group of invertible elements in $(\mathcal{B}/\mathcal{K})(\ell^2(\mathbb{N}))$, and Δ_0 is the connected component of Δ containing $[I]$.

The commutator ideal of the classical Toeplitz algebra on $\ell^2(\mathbb{N})$ coincides with the algebra of compact operators on $\ell^2(\mathbb{N})$ (cf. [10, 7.23]. The result is implicit in [5],[6]). This cannot be generalized to arbitrary discrete groups, as shown by the following result.

PROPOSITION 5.6. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant set in G. Then $K(\mathcal{T}^r(G)) \subseteq \mathcal{K}(\ell^2(S))$ if and only if G is abelian. In fact, $[T_g, T_h]$ is compact if and only if $gh = hg$.*

Proof. By Corollary 5.5, $[T_g, T_h]$ is compact $(g, h \in G) \Leftrightarrow [T_g][T_h] = [T_h][T_g] \Leftrightarrow$ $\Leftrightarrow [T_{gh}] = [T_{hg}] \Leftrightarrow gh = hg$. In particular, if $K(\mathcal{T}^r(G)) \subseteq \mathcal{K}(\ell^2(S))$, then G is abelian.

Conversely, let G be abelian. Then $[T_g, T_h]$ is compact for all g, h in G. Now $[AB, C] = A[B, C] + [A, C]B$ for all A, B, C in $\mathcal{B}(\ell^2(S))$, so one can easily see by induction that $[S, T]$ is compact for all S, T which are linear combinations of products

$T_{g_1}T_{g_2}\dots T_{g_n}$ $(g_1,\dots,g_n \in G)$. But such linear combinations are dense in $\mathcal{T}^r(G)$ (since $T_g^* = T_{g^{-1}}$ for all g), so $[S,T]$ is compact for all S,T in $\mathcal{T}^r(G)$. Hence $K(\mathcal{T}^r(G)) \subseteq \mathcal{K}(\ell^2(S))$. ∎

THEOREM 5.7. *Let G be a countably infinite discrete group, and let S be an infinite almost invariant proper subset of G. Suppose that G has no nontrivial elements of order $\leqslant n$, where n is the smallest of all $|S\setminus g^{-1}S|$, $g \in G$, $S\setminus g^{-1}S \neq \emptyset$. Then*

$$\mathcal{K}(\ell^2(S)) \subseteq K(\mathcal{T}^r(G)) \subseteq \mathcal{T}^r(G).$$

In particular, $K(\mathcal{T}^r(G))$ and $\mathcal{T}^r(G)$ are irreducible on $\ell^2(S)$, $\mathcal{T}(G) = \mathcal{T}^r(G)$, and one has a short exact sequence

$$0 \to \mathcal{K}(\ell^2(S)) \to \mathcal{T}^r(G) \to C_r^*(G) \to 0.$$

If $C_r^*(G)$ is simple, then $K(\mathcal{T}^r(G)) = \mathcal{T}^r(G)$. If G is abelian, then $K(\mathcal{T}^r(G)) = \mathcal{K}(\ell^2(S))$.

Proof. Choose g in G so that $|S\setminus g^{-1}S| = n$, and let $Y = S\setminus g^{-1}S$. For any subset F of S, let us denote by P_F the orthogonal projection from $\ell^2(S)$ onto $\ell^2(F)$. Then

$$T_e - T_{g^{-1}}T_g = P_S - P_{S\cap g^{-1}S} = P_Y,$$

so P_Y is a nonzero finite-dimensional projection in $\mathcal{T}^r(G)$. We will show that $\mathcal{K}(\ell^2(S))$ is the (closed two-sided) ideal generated by P_Y in $\mathcal{T}^r(G)$.

We claim that $\mathcal{T}^r(G)$ contains a one-dimensional projection. If $|Y| = 1$, then there is nothing to prove, so we may and will assume that $|Y| \geqslant 2$. For any subset $X \subseteq Y$, with $|X| \geqslant 2$, there exists h in G such that $X\cap h^{-1}X$ is a nonempty proper subset of X. Indeed, if $X = \{x_1,\dots,x_m\} \subseteq Y$, $m \geqslant 2$, choose $h = x_2x_1^{-1}$: then $hx_1 = x_2$, and so $X\cap h^{-1}X \neq \emptyset$. Also, $X\cap h^{-1}X \neq X$, since otherwise $h^r x_1 = x_1$ for some $2 \leqslant r \leqslant m \leqslant n$, that is, $h^r = e$.

Now choose any h in G such that $Y\cap h^{-1}Y = \{t \in Y; ht \in Y\}$ is a nonempty proper subset of Y. We have

$$T_{h^{-1}}P_Y T_h P_Y = P_{Y\cap h^{-1}Y},$$

so $P_{Y\cap h^{-1}Y}$ is a nonzero finite-rank projection in $\mathcal{T}^r(G)$, with range $\ell^2(Y\cap h^{-1}Y)$ properly contained in $\ell^2(Y)$. If $|Y\cap h^{-1}Y| \geqslant 2$, then we let $X = Y\cap h^{-1}Y$, we choose k in G such that $X\cap k^{-1}X$ is a nonempty proper subset of X, and so on. After iterating this process a sufficient number of times, we find some x in S (in fact, in Y) such that $P_{\{x\}} \in \mathcal{T}^r(G)$. Now we have

$$T_{y'x^{-1}}P_{\{x\}}T_{xy^{-1}} = \langle\cdot,\delta_y\rangle\delta_{y'} \quad (\forall y,y' \in S).$$

Therefore $\mathcal{T}^r(G)$ contains all matrix units $\langle\cdot,\delta_y\rangle\delta_{y'}$ $(y,y' \in S)$, and thus $\mathcal{K}(\ell^2(S)) \subseteq \mathcal{T}^r(G)$. In particular, $\mathcal{T}^r(G)$ is not commutative, and so $K(\mathcal{T}^r(G))$ is a nonzero (closed two-sided) ideal in $\mathcal{T}^r(G)$. Hence $\mathcal{K}(\ell^2(S)) \subseteq K(\mathcal{T}^r(G))$ (cf. [6, Proposition 1]).

Moreover, $\mathcal{T}(G) = \mathcal{T}^r(G) + \mathcal{K}(\ell^2(S)) = \mathcal{T}^r(G)$, and we can replace $\mathcal{T}(G)$ by $\mathcal{T}^r(G)$ in the exact sequence in Theorem 5.3.

If G is abelian, then $K(\mathcal{T}^r(G)) \subseteq \mathcal{K}(\ell^2(S))$ by Proposition 5.6, and thus $K(\mathcal{T}^r(G)) = \mathcal{K}(\ell^2(S))$.

Finally, suppose that $C_r^*(G)$ is simple. Then $\mathcal{K}(\ell^2(S))$ is a proper ideal in $K(\mathcal{T}^r(G))$. Indeed, if $\mathcal{K}(\ell^2(S)) = K(\mathcal{T}^r(G))$, then $C_r^*(G) \simeq \mathcal{T}^r(G)/K(\mathcal{T}^r(G))$ would be abelian, hence G would be abelian, and thus $C_r^*(G) \simeq C(\hat{G})$ would not be simple.

Therefore $K(\mathcal{T}^r(G))/\mathcal{K}(\ell^2(S))$ is a nonzero ideal in $\mathcal{T}^r(G)/\mathcal{K}(\ell^2(S)) \simeq C_r^*(G)$, and thus $K(\mathcal{T}^r(G)) = \mathcal{T}^r(G)$, since $C_r^*(G)$ is simple. ∎

COROLLARY 5.8. *Let G be a countably-infinite torsion-free discrete group. Then there is a short exact sequence*

$$0 \to \mathcal{K}(\ell^2(S)) \to \mathcal{T}^r(G) \to C_r^*(G) \to 0$$

for any infinite, almost invariant, proper subset S of G.

REMARKS. 1) Theorem 5.7 says that, under certain assumptions, the C^*-algebra $\mathcal{T}^r(G)$ generated by the generalized unilateral shifts T_g, $g \in G$, is an extension of $\mathcal{K}(\ell^2(S))$ by $C_r^*(G)$. Also, $\mathcal{K}(\ell^2(S)) \subseteq \mathcal{I}$ for any nontrivial (closed two-sided) ideal $\mathcal{I}$ in $\mathcal{T}^r(G)$ (cf. [6, Proposition 1]). In particular, the structure of the ideals of $\mathcal{T}^r(G)$ is determined, once the ideal theory for $C_r^*(G)$ is known. All of this completely generalizes Theorems 1 and 2 in [5], which deal with the classical case $G = \mathbf{Z}$, $S = \mathbf{N}$.

Moreover, any $*$-representation of $\mathcal{T}^r(G)$ is equivalent to a direct sum $\pi_1 \oplus \pi_2$, where π_1 is (zero or) a multiple of the identity representation, and π_2 is a representation that vanishes on $\mathcal{K}(\ell^2(S))$ and can therefore be regarded as a $*$-representation of $C_r^*(G)$ (cf. [15, 10.4.7]). In particular, any nonzero irreducible representation of $\mathcal{T}^r(G)$ is either the identity representation or factors through a nonzero irreducible representation of $C_r^*(G)$. Therefore $\mathcal{T}^r(G)$ is type I if and only if $C_r^*(G)$ is type I.

Finally, by Corollary 5.8, $\mathcal{T}^r(G)$ is nuclear if and only if $C_r^*(G)$ is nuclear if and only if G is amenable.

2) If G is abelian, and G, S satisfy the hypotheses of Theorem 5.7, then

$$\mathcal{T}^r(G)/K(\mathcal{T}^r(G)) \simeq C_r^*(G).$$

The same conclusion holds in the setting of Toeplitz operators on abelian partially ordered groups [3], [16], [22], where $\mathcal{T}^r(G)$ is defined in a similar way.

This is opposite to the case of groups G which satisfy the hypotheses of Theorem 5.7, and are such that $C_r^*(G)$ is simple, since in that case $K(T^r(G))$ actually coincides with $T^r(G)$. The reader is referred to [8], [24], for a list of discrete groups whose reduced C^*-algebra is simple.

3) To see why the condition $S \neq G$ is necessary in Theorem 5.7, consider the case $S = G$. Then $T^r(G) = C_r^*(G)$ does not contain any nonzero compact operator. Also, note that the hypotheses of the theorem can be satisfied even when S differs from G by just one point (take $G = \mathbb{Z},\ S = \mathbb{Z} \setminus \{0\}$).

Let us now give an example of a pair G, S which satisfies the hypotheses of Theorem 5.7 and Corollary 5.8. Let $G = \mathbb{F}_2$ and let $S = S_b \cup \{e\}$. It is then immediate that T_a is a unitary operator in $\mathcal{B}(\ell^2(S))$, whereas T_b is an isometry on $\ell^2(S)$ with one--dimensional defect space $\ell^2(\{e\})$. Following the proof of Theorem 5.7, $\mathcal{K}(\ell^2(S))$ is the ideal of $T^r(G)$ generated by $P_{\{e\}} = I - T_b T_{b^{-1}} = [T_{b^{-1}}, T_b]$ (note that $S \setminus bS = \{e\}$). Here T_b plays the same role as the unilateral shift on $\ell^2(\mathbb{N})$.

We are also able to compute the indices of all the Fredholm operators $T_g,\ g \in \mathbb{F}_2$. Indeed, $\mathrm{Ind}(T_a) = 0,\ \mathrm{Ind}(T_b) = -1$, and thus, by Corollary 5.5,

$$\mathrm{Ind}(T_g) = -n(g) \quad \text{for all } g \text{ in } \mathbb{F}_2,$$

where $n(g)$ denotes the arithmetic sum of the exponents of b in the reduced word of g. This index formula generalizes the familiar equality $\mathrm{Ind}(T_{z^n}) = -n$, where T_{z^n} is the Toeplitz operator with symbol z^n ($z \in \mathbb{T},\ n \in \mathbb{Z}$). Note that, unlike in the case $G = \mathbb{Z}$, it is possible that $g \neq e$, and $\mathrm{Ind}(T_g) = 0$.

We may also choose S as the set of all those elements of $\mathbb{F}_2$ whose reduced words end in either a or b. In this case $S \setminus aS = \{a\},\ S \setminus bS = \{b\}$, and so $\mathcal{K}(\ell^2(S))$ is the ideal of $T^r(G)$ generated by either $P_{\{a\}}$ or $P_{\{b\}}$ by Theorem 5.7. Moreover,

$$\mathrm{Ind}(T_g) = -m(g) \quad \text{for all } g \text{ in } \mathbb{F}_2,$$

where $m(g)$ denotes the arithmetic sum of the exponents of a and b in the reduced word of g.

We refer the reader to [21], [23] for a complete discussion on Toeplitz operators on free groups, where we let S range over all possible infinite a.i. sets, and we characterize, in terms of the Fredholm index, the pairs S, S' which determine, in an appropriate sense, the same Toeplitz theory.

6. ANALYTIC TOEPLITZ OPERATORS

In this section we extend the notion of analytic Toeplitz operator to the setting of almost invariant sets. We recall that a Toeplitz operator $T_f,\ f \in \mathcal{L}^\infty(\mathbf{T})$, is analytic if $f \in H^\infty(\mathbb{T})$, that is, if f has an analytic Fourier series. Equivalently, T_f is analytic if $fH^2(\mathbb{T}) \subseteq H^2(\mathbb{T})$ (in particular, T_f is a subnormal operator).

Let S be an infinite a.i. subset of a group G. We define H^∞ to be the Banach algebra

$$H^\infty = \{T \in W^*(G) : T(\ell^2(S)) \subseteq \ell^2(S)\},$$

and we say that $A \in \mathcal{B}(\ell^2(S))$ is an analytic Toeplitz operator if $A = T^{(S)}$ for some T in H^∞.

As in the classical case, a product $A_1 A_2$ of Toeplitz operators is again a Toeplitz operator if either A_2 or A_1^* is analytic. A "generalized unilateral shift" T_g, $g \in G$, is analytic if and only if $L_g \in H^\infty$, that is, if and only if $gs \in S$ for all s in S. This leads us to define the set

$$G^+ = \{g \in G : gS \subseteq S\}.$$

Note that G^+ is a semigroup containing the identity of the group, and that $G^+ = S$ if and only if S is a semigroup and $e \in S$ (this is what happens in the classical case $G = \mathbb{Z}$, $S = \mathbb{N}$). If $G = \mathbb{F}_2$ and $S = S_a$, then G^+ is the semigroup generated by e, a, b and b^{-1}. Neither is G^+ contained in S, nor is S contained in G^+. The Toeplitz operators T_a and T_b are analytic, whereas $T_{ba^{-1}ba}$ is not.

The semigroup G^+ can be used to give an equivalent definition of analytic Toeplitz operator, by generalizing the notion of analytic Fourier series. For any $T \in W^*(G)$ and $g \in G$, let us set

$$a_g = \langle T\delta_e, \delta_g \rangle \quad (\text{“}g\text{-th Fourier coefficient”}).$$

Then one can formally write $T = \sum_{g \in G} a_g L_g$, in the sense that $T(\delta_h) = \sum_{g \in G} a_g \delta_{gh}$ for all h in G. In particular, $T \in H^\infty$ if and only if $\sum_{g \in G} a_g \delta_{gs} \in \ell^2(S)$ for all s in S if and only if $a_g = 0$ for $g \notin G^+$. Hence H^∞ consists of those operators T in $W^*(G)$ which can be written as a formal series

$$T = \sum_{g \in G^+} a_g L_g$$

(generalized analytic Fourier series).

What is interesting here is that G^+, and not S, is the set which plays the role of $\mathbb{N}$. However it is not true, in general, that $G = G^+ \cup (G^+)^{-1}$ or that $G^+ \cap (G^+)^{-1} = \{e\}$, as one can see by taking $G = \mathbb{F}_2$ and $S = S_a$ as above. It then follows that H^∞ does not exhibit an analytic-type behaviour. In the classical case, if f and $\overline{f}$ are in H^∞ then f is constant, but for G, S as above we see that $T = L_b + L_{b^{-1}} = T^*$ is in H^∞, but is not constant.

We are now able to extend Wintner's theorem on the invertibility of analytic Toeplitz operators (cf. [10], Theorem 7.21, and [16], Theorem 3.12). For such operators, we get a more precise version of the spectral inclusion in Theorem 4.2.

PROPOSITION 6.1. *Let G be a countably infinite group, and let S be an infinite almost invariant set in G. If T is an operator in H^∞, then*

$$\sigma(T^{(S)}) = \sigma_{H^\infty}(T).$$

Equivalently, $T^{(S)}$ is invertible if and only if T is invertible in H^∞.

Proof. The map $T \mapsto T^{(S)}$ is unital and multiplicative on H^∞, so, if T is invertible in H^∞, then $T^{(S)}$ is invertible.

Conversely, if T is in H^∞ and $T^{(S)}$ is invertible, then T is invertible in $W^*(G)$, since $\sigma(T) \subseteq \sigma(T^{(S)})$ by Theorem 4.2. But if $T^{(S)} = T|\ell^2(S)$ is invertible in $\mathcal{B}(\ell^2(S))$, then T, as an operator on $\ell^2(S)$, is one-to-one and onto $\ell^2(S)$, and thus the inverse of T in $W^*(G)$ belongs to H^∞. In other words, T is invertible in H^∞, and this completes the proof. ■

7. CLOSING REMARKS

In the setting of classical Toeplitz operators ($G = \mathbf{Z}$, $S = \mathbf{N}$) the commutativity of G is crucial for the development of the theory. In the general case, where G is an arbitrary group and S is an infinite almost invariant set in G, the operators in $W^*(G)$ commute with all right-translation operators R_g, $g \in G$, and this property is sufficient to make up for the commutativity of G and extend the basic theorems of Toeplitz theory to the general setting, as shown by the foregoing results.

However, there are several properties of classical Toeplitz operators which do not carry over to our general setting, as we shall see presently. For such properties, a suitable generalization can be found, instead, in the context of Toeplitz operators on partially ordered abelian groups (cf. [16], [17], [18], [19] and [22]). In that context, the commutativity of G and the semigroup property of S turn out to be essential. Moreover, when the order is total, a key role is played by the fact that an ordered abelian group G has a connected Pontryagin dual $\hat{G}$.

One of the hardest and deepest results of the classical theory is Widom's theorem on the connectedness of the spectra of Toeplitz operators. This result has been extended to ordered abelian groups, in the case of Toeplitz operators with continuous symbols ([17], Theorem 2.3), and to partially ordered abelian groups, in the case of analytic Toeplitz operators ([16], Theorem 3.12). In the general setting of almost invariant sets in discrete groups, we will now show that there exist Toeplitz operators with disconnected spectra, even if we only restrict our attention to the case of analytic Toeplitz operators.

Let $G = \mathbb{F}_2$ and $S = S_a$. Then $\ell^2(S)$ is an invariant subspace for both L_b and $L_b^* = L_{b^{-1}}$, and hence, if $W^*(L_b)$ denotes the von Neumann algebra generated by L_b in $\mathcal{B}(\ell^2(G))$, then

$$W^*(L_b) \subsetneq H^\infty.$$

Moreover, $W^*(L_b) \simeq L^\infty(\mathbb{T})$. Indeed, $\ell^2(G)$ can be decomposed as a direct sum of invariant subspaces for L_b in such a way that L_b becomes unitarily equivalent to an infinite direct sum of countably many copies of the bilateral shift on $\ell^2(\mathbf{Z})$. Therefore H^∞ contains a copy of $L^\infty(\mathbb{T})$. In particular, H^∞ contains a nontrivial idempotent T, and thus $\sigma_{H^\infty}(T) = \{0, 1\}$ is not connected (moreover, H^∞ is not an integral domain). But $\sigma(T^{(S)}) = \sigma_{H^\infty}(T)$ by Proposition 6.1, and hence $T^{(S)}$ is an analytic Toeplitz operator with disconnected spectrum. The previous argument is essentially due to Steve Power.

A harder question to ask is whether there exist analytic Toeplitz operators with disconnected spectra in the case where $G = \mathbb{F}_2$ and $S = S_a \cup S_b$.

Coburn [4, 4.1] showed that the spectrum of any Toeplitz operator T_f, $f \in L^\infty(\mathbb{T})$, coincides with the Weyl spectrum. Equivalently, the only Fredholm Toeplitz operators with index zero are the invertible ones. This is proved by showing that if f is a nonzero function in $L^\infty(\mathbb{T})$, then either T_f or T_f^* must have a trivial kernel.

These properties cease to hold in the case of Toeplitz operators on free groups, as shown by the next result and the following discussion. For simplicity, we will consider the case of a free group on two generators.

PROPOSITION 7.1. *Let $G = \mathbb{F}_2$, and let S be an infinite almost invariant proper subset of G. Then there exists T in $W^*(G)$ such that* $\ker T^{(S)} \neq (0)$ *and* $\ker (T^{(S)})^* \neq (0)$.

Proof. If $T = L_g$, $g \in G$, then $T^{(S)} = T_g$, and

$$\ker T^{(S)} = \ell^2(S \setminus g^{-1}S), \quad \ker(T^{(S)})^* = \ell^2(S \setminus gS).$$

So it suffices to find g in G such that $S \setminus g^{-1}S \neq \emptyset$ and $S \setminus gS \neq \emptyset$.

Let us first consider the case where $e \notin S$. By Theorem 2.1.8 in [21], there exists $x \neq e$ in $\mathbb{F}_2$ such that S contains S_x. Without loss of generality, we may assume that a is the first letter in the reduced word for x. If we let $g = x^{-1}bx$, then $g^{-1} \in S \setminus g^{-1}S$ and $g \in S \setminus gS$.

If $e \in S$, let $z \in G \setminus S$. Then $z^{-1}S$ is an infinite a.i. set in G, and $e \notin z^{-1}S$. By the previous case, there exists h in G such that $z^{-1}S \setminus h^{-1}z^{-1}S \neq \emptyset$

and $z^{-1}S \setminus hz^{-1}S \neq \emptyset$. But

$$z^{-1}S \setminus h^{-1}z^{-1}S = z^{-1}(S \setminus zh^{-1}z^{-1}S)$$

and

$$z^{-1}S \setminus hz^{-1}S = z^{-1}(S \setminus zhz^{-1}S),$$

and thus, if we let $g = zhz^{-1}$, we have $S \setminus g^{-1}S \neq \emptyset$ and $S \setminus gS \neq \emptyset$. ■

Now let $G = \mathbb{F}_2$, $S = S_b$ and $g = b^{-1}ab$. Then $S \setminus g^{-1}S = \{b^{-1}a^{-1}b\}$, and $S \setminus gS = \{b^{-1}ab\}$. In particular, T_g is a non-invertible Fredholm Toeplitz operator with index zero (see Proposition 5.1), showing that Coburn's theorem fails to hold. This is analogous to the situation described in [7], where it is proved that there exist Toeplitz operators on the quarter-plane which are Fredholm operators with index zero, but which are not invertible.

For Toeplitz operators with continuous symbols, a more direct understanding of Coburn's theorem can be obtained by recalling that any invertible f in $C(\mathbb{T})$ can be written as

$$f = e^{\psi} z^n,$$

where ψ is some function in $C(\mathbb{T})$, and n is the winding number of f with respect to the origin. This decomposition can be used to prove the Gohberg-Krein index formula and Coburn's result, by showing that a Toeplitz operator T_f, $f \in C(\mathbb{T})$, is invertible if and only if $f = e^g$ for some g in $C(\mathbb{T})$ (cf. [12], Section 1). The same characterization of invertible Toeplitz operators can be found in the context of Toeplitz operators on ordered groups ([19], Theorem 2.2).

In our setting, given a fixed a.i. $S \subseteq G$, one would wish to determine a necessary and sufficient condition on T in $C_r^*(G)$ for which $T^{(S)}$ is invertible, and find an index formula to compute $\mathrm{Ind}(T^{(S)})$ for all invertible operators T in $C_r^*(G)$. This task turns out to be more difficult than in the classical case, because in general it is not true that if $T \in C_r^*(G)$ and $T^{(S)}$ is invertible, then

$$T = e^A$$

for some $A \in C_r^*(G)$; conversely, we do not know whether in general $T^{(S)}$ is invertible if $T = e^A$ for some $A \in C_r^*(G)$.

Indeed, consider the unitary operator $T = L_a \in C_r^*(\mathbb{F}_2)$. If S_b is defined as above, then $L_a^{(S_b)}$ is a unitary and thus invertible operator. If $L_a = e^A$ for some A in $C_r^*(\mathbb{F}_2)$, then for any infinite a.i. set $S \subseteq \mathbb{F}_2$, the map $t \mapsto (e^{tA})^{(S)}$, $t \in [0,1]$, would be a continuous path connecting I and $L_a^{(S)}$ in the set of Fredholm operators on $\ell^2(S)$, and hence $\mathrm{Ind}(L_a^{(S)}) = 0$ for any infinite a.i. set S. But if $S = S_a$, then $\mathrm{Ind}(L_a^{(S)}) = -1$

by Proposition 5.1, and this gives the desired contradiction. Another way to see this is to note that L_a and e^A belong to two different classes in $K_1(C_r^*(\mathsf{F}_2))$.

The previous example shows us another irregular feature of generalized Toeplitz operators. In the classical theory, and in the context of Toeplitz operators on a half plane, the invertibility of $T^{(S)}$ (T invertible in $C_r^*(G)$) only depends on the class $[T]_1$ of T in $K_1(C_r^*(G))$. More precisely, there is a subset $\Omega \subseteq K_1(C_r^*(G))$ such that, for any invertible T in $C_r^*(G)$, $T^{(S)}$ is invertible if and only if $[T]_1 \in \Omega$ (cf. Theorem 6 in [12]). This fails to be true for $G = \mathsf{F}_2$, $S = S_b$. Indeed, L_a and I are in two different classes of $K_1(C_r^*(\mathsf{F}_2))$, but $L_a^{(S)}$ and $I^{(S)}$ are both invertible, whereas L_a and $L_{b^{-1}ab}$ coincide in $K_1(C_r^*(\mathsf{F}_2))$, and yet $L_{b^{-1}ab}^{(S)}$ is not invertible.

The previous considerations show that, for such groups as free groups, a more sensible problem to study would be to characterize those invertible operators T in $C_r^*(G)$ for which $T^{(S)}$ is invertible for any infinite a.i. $S \subseteq G$, and not just for a given S.

Now let us define, for general G and S,

$$Q_x = T_x^* T_x - T_x T_x^* \quad (x \in S).$$

With the same notation as in the proof of theorem 5.7, note that

$$Q_x = P_{S \cap x^{-1}S} - P_{S \cap xS} = P_{S \setminus xS} - P_{S \setminus x^{-1}S},$$

and so Q_x is a finite-rank operator on $\ell^2(S)$ for all x in S.

If $G = \mathbb{Z}$ and $S = \mathbb{N}$, then $\{Q_n\}_{n \geqslant 1} = \{P_{\{0,1,\ldots,n-1\}}\}_{n \geqslant 1}$ is an approximate identity for $\mathcal{K}(\ell^2(S))$. As we will now show, this property does not extend to free groups. Indeed, consider the case where $G = \mathsf{F}_2$, and $S = S_b = \{x(n)\}_{n=1}^{\infty}$. It is easy to verify that $Q_{b^m} = P_{\{b,b^2,\ldots,b^m\}}$ for all $m \geqslant 1$. If $\{Q_{x(n)}\}_{n=1}^{\infty}$ were an approximate identity for $\mathcal{K}(\ell^2(S))$, we would have

$$Q_{b^m} K \to K \quad \text{as } m \to \infty$$

in operator norm for all K in $\mathcal{K}(\ell^2(S))$. To get a contradiction, it now suffices to choose $K = P_{\{ab\}}$.

Finally, if $G = \mathbb{Z}$ and $S = \mathbb{N}$, $T \in \mathcal{B}(\ell^2(S))$ is a Toeplitz operator if and only if

$$T_x^* T T_x = T \quad \text{for all } x \text{ in } S.$$

In general, this is not necessarily true. Consider any group G such that there exists an infinite a.i. set S and an element $x \in S$ such that $S \setminus x^{-1}S \neq \emptyset$ (see e.g. the example following Proposition 7.1). Then the identity operator $I \in \mathcal{B}(\ell^2(S))$ is a Toeplitz operator, but $T_x^* I T_x = P_{S \cap x^{-1}S} \neq I$.

We conclude our discussion by pointing out that, for a given group G, we always worked with one almost invariant set at a time, and did not bother to establish the relation between Toeplitz operators associated with different almost invariant sets in G. This is done in the paper [23], where we study the class of extensions of the compact operators by $C_r^*(G)$ associated with all infinite almost invariant sets in G. Such class happens to be particularly interesting in the case of free groups, whereas for such group as $SL(2, \mathbb{Z}), PSL(2, \mathbb{Z})$ and D_∞ (the infinite dihedral group) all the extensions are trivial in an appropriate sense. In the latter case, there exists no operator in $C_r^*(G)$ whose compression to $\ell^2(S)$ is a Fredholm Toeplitz operator with nonzero index.

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