J. OPERATOR THEORY
28(1992), 51–64

APPROXIMATELY HYPERREFLEXIVE ALGEBRAS

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1. INTRODUCTION

Let $B(H)$ denote the set of operators on a Hilbert space H. If $\mathcal{S}$ is a linear subspace of $B(H)$, then Ref$\mathcal{S}$ consists of those operators T for which $Tx \in [\mathcal{S}x]^-$ for every x in H. The subspace $\mathcal{S}$ is *reflexive* if $\mathcal{S} = \mathrm{Ref}\mathcal{S}$. If $\mathcal{A}$ is a unital subalgebra of $B(H)$, then $\mathrm{Ref}\mathcal{A} = \mathrm{AlgLat}\mathcal{A}$, where Lat$\mathcal{A}$ is the lattice of (closed) invariant subspaces (projections) for $\mathcal{A}$, and $\mathrm{AlgLat}\mathcal{A} = \{T \in B(H)\colon \mathrm{Lat}\mathcal{A} \subset \mathrm{Lat}T\}$. The von Neumann double commutant theorem implies that every von Neumann algebra is reflexive, and it is reasonable to view the reflexive subalgebras of $B(H)$ as non-selfadjoint analogues of von Neumann algebras.

Suppose $\mathcal{S}$ is a reflexive linear subspace of $B(H)$. We define a seminorm $\mathrm{d}(\ ,\mathcal{S})$ on $B(H)$ by $\mathrm{d}(T,\mathcal{S}) = \sup\{\mathrm{dist}(Tx, \mathcal{S}x)\colon x \in H, \|x\| \leqslant 1\}$. Alternatively, $\mathrm{d}(T,\mathcal{S}) = \sup\{\|PTQ\|\colon P, Q \text{ projections}, P\mathcal{S}Q = \{0\}\} = \sup\{\|ATB\|\colon \|A\|, \|B\| \leqslant 1, A\mathcal{S}B = \{0\}\}$. It is clear that $\mathrm{d}(T,\mathcal{S}) \leqslant \mathrm{dist}(T,\mathcal{S})$, and since $\mathcal{S}$ is reflexive, we have $\mathrm{d}(T,\mathcal{S}) = 0 \iff \mathrm{dist}(T,\mathcal{S}) = 0$. The subspace $\mathcal{S}$ is *hyperreflexive* if the seminorms $\mathrm{d}(\ ,\mathcal{S})$ and $\mathrm{dist}(\ ,\mathcal{S})$ are equivalent, i.e., if there is a constant K such that $\mathrm{dist}(\ ,\mathcal{S}) \leqslant K\mathrm{d}(\ ,\mathcal{S})$. The smallest such K is called the *constant of hyperreflexivity* of $\mathcal{S}$ and is denoted by $K(\mathcal{S})$; we say $K(\mathcal{S}) = \infty$ if $\mathcal{S}$ is not hyperreflexive.

If $\mathcal{A}$ is a reflexive subalgebra of $B(H)$, we have another description of $\mathrm{d}(\ ,\mathcal{A})$, namely, $\mathrm{d}(T,\mathcal{A}) = \sup\{\|(1-P)TP\|\colon P \in \mathrm{Lat}\mathcal{A}\}$. The first result on hyperreflexivity was the well-known Arveson distance formula [3], which says $K(\mathcal{A}) = 1$ whenever $\mathcal{A}$ is a nest algebra. Further results on hyperreflexivity appear in [4], [7], [8], [22], [19], [18], [9], and [15]. It is known that many von Neumann algebras are hyperreflexive, and it was proved by E. Christensen [7] that the problem of hyperreflexivity for von Neumann algebras is equivalent to the problem of whether every derivation from a von

Neumann subalgebra of $B(H)$ into $B(H)$ is inner. It is known [12] that an affirmative answer to R. Kadison's similarity problem (Is every bounded homomorphism from a C^*-algebra into $B(H)$ similar to a $*$-homomorphism?) implies an affirmative answer to the inner derivation problem. Another related problem, due to J. Dixmier, is whether the range of an operator being invariant for a von Neumann algebra implies it is the range of an operator in the commutant of the algebra (See [5]).

We are interested in asymptotic analogues of reflexivity and hyperreflexivity. If $\mathcal{S}$ is a linear subspace of $B(H)$, we define ApprRef$\mathcal{S}$ as the set of those operators T for which $\|P_\lambda T Q_\lambda\| \to 0$ whenever $\{P_\lambda\}$ and $\{Q_\lambda\}$ are nets of projections such that $\|P_\lambda S Q_\lambda\| \to 0$ for every S in $\mathcal{S}$. Equivalently, $T \in$ ApprRef$\mathcal{S}$ if $\|A_\lambda T B_\lambda\| \to 0$ whenever $\{A_\lambda\}$ and $\{B_\lambda\}$ are bounded nets of operators such that $\|A_\lambda S B_\lambda\| \to 0$ for every S in $\mathcal{S}$. A third formulation is that $T \in$ ApprRef$\mathcal{S}$ if and only if $(Te_\lambda, f_\lambda) \to 0$ whenever $\{e_\lambda\}$ and $\{f_\lambda\}$ are bounded nets of vectors such that $(Se_\lambda, f_\lambda) \to 0$ for every S in $\mathcal{S}$. We say that the subspace $\mathcal{S}$ is *approximately reflexive* if $\mathcal{S} =$ ApprRef$\mathcal{S}$.

Another version of approximate reflexivity was defined in [1,2] and [13] for unital subalgebras of $B(H)$. If $\mathcal{A}$ is such an algebra, we define ApprAlgLat$\mathcal{A}$ to be the set of those operators T for which $\|(1-P_\lambda)TP_\lambda\| \to 0$ whenever $\{P_\lambda\}$ is a net of projections such that $\|(1-P_\lambda)SP_\lambda\| \to 0$ for every S in $\mathcal{A}$. An equivalent definition is obtained when the condition that the P_λ's are projections is replaced with the condition that they form a bounded net of idempotents. In the case in which $\mathcal{A}$ is norm separable, an equivalent definition is obtained when "net" is replaced with "sequence". In [13] it was shown that, for every unital C^*-subalgebra $\mathcal{A}$ of $B(H)$, we have $\mathcal{A} =$ ApprAlgLat$\mathcal{A}$; this is an asymptotic analogue for C^*-algebras of von Neumann's double commutant theorem. Moreover, it was shown in [13] that if $\mathcal{A}$ is a unital C^*-subalgebra of $B(H)$, then $\mathcal{A}$ is the set of those operators T for which $\|U_\lambda T - TU_\lambda\| \to 0$ whenever $\{U_\lambda\}$ is a net of unitary operators such that $\|U_\lambda S - SU_\lambda\| \to 0$ for every S in $\mathcal{A}$. Also $\mathcal{A}$ is the set of those operators T for which $\|A_\lambda T - TA_\lambda\| \to 0$ whenever $\{A_\lambda\}$ is a bounded net of operators such that $\|A_\lambda S - SA_\lambda\| \to 0$ for every S in $\mathcal{A}$.

Unlike the case of Ref and AlgLat, it is not clear that ApprRef$\mathcal{A}$ = ApprAlgLat$\mathcal{A}$ when $\mathcal{A}$ is a unital algebra. This is our first main result (Theorem 9, Corollary 10); the proof involves a characterization (Proposition 7, Lemma 8) of certain completely bounded maps that is analogous to the characterization in [14] of those unital completely positive maps that are approximate compressions of a given representation of a C^*-algebra.

We then investigate an asymptotic notion of hyperreflexivity, *approximate hyperreflexivity*, and we prove analogues of several of the known results on hyperreflexivity. In particular, our proof that ApprAlgLat$\mathcal{A}$ = ApprRef$\mathcal{A}$ for every unital algebra $\mathcal{A}$ of operators also shows that the analogous notions of hyperreflexivity for subspaces and unital algebras coincide. We also introduce a new operator topology that is to the weak (i.e., $\sigma(B(H), B(H)^{\#})$) topology on $B(H)$ what the weak operator topology is to the weak*-topology (ultraweak topology) on $B(H)$. This topology is very much related to approximate reflexivity and approximate hyperreflexivity.

Using the notion of relative approximate hyperreflexivity, we prove that every C^*-algebra is approximately hyperreflexive.

We define the seminorm $\mathrm{d}_a(\ ,\mathcal{S})$ on $B(H)$ by $\mathrm{d}_a(T,\mathcal{S}) = \sup\{\limsup_\lambda \|P_\lambda T Q_\lambda\| : \{P_\lambda\}, \{Q_\lambda\}$ are nets of projections, $\|P_\lambda S Q_\lambda\| \to 0$ for every S in $\mathcal{S}\}$. It is easy to show that $\mathrm{d}_a(T,\mathcal{S}) = \sup\{\limsup_\lambda |(Te_\lambda, f_\lambda)| : \{e_\lambda\}, \{f_\lambda\}$ are nets of unit vectors, $(Se_\lambda, f_\lambda) \to 0$ for every S in $\mathcal{S}\}$. It is clear that $\mathrm{d}_a(T,\mathcal{S}) = 0$ precisely when $T \in \mathrm{ApprRef}\mathcal{A}$.

We say that a linear subspace $\mathcal{S}$ of $B(H)$ is *approximately hyperreflexive* if there is a (smallest) constant $K = K_a(\mathcal{S})$ such that, for every T in $B(H)$, $\mathrm{dist}(T,\mathcal{S}) \leqslant K \mathrm{d}_a(T,\mathcal{S})$.

The following elementary lemma shows how $\mathrm{d}_a(\ ,\mathcal{S})$ and $\mathrm{dist}(\ ,\mathcal{S})$ share some common properties that help to reduce problems of approximate reflexivity and hyperreflexivity to the case of a separable subspace acting on a separable Hilbert space.

LEMMA 1. *Suppose $\mathcal{S}$ is a linear subspace of $B(H)$, and $T \in B(H)$ and $\mathcal{M}$ is the collection of all separable subspaces of H that are reducing for $\mathcal{S} \cup \{T\}$. Then*

$$(1) \qquad \mathrm{d}_a(T,\mathcal{S}) = \inf\{\mathrm{d}_a(T,\mathcal{T}) : \mathcal{T} \text{ is a norm separable subspace of } \mathcal{S}\},$$

and

$$(2) \qquad \text{if } \mathcal{S} \text{ is norm separable, then } \mathrm{d}_a(T,\mathcal{S}) = \sup\{\mathrm{d}_a(T|M, \mathcal{S}|M) : M \in \mathcal{M}\}.$$

The following lemma contains a useful characterization of approximate hyperreflexivity.

LEMMA 2. *Suppose $\mathcal{S}$ is a linear subspace of $B(H)$. The following are equivalent.*

(1) $K_a(\mathcal{S}) \leqslant K$

(2) *For every $\varepsilon > 0$, for every T in $B(H)$ and every finite subset $\mathcal{F}$ of $\mathcal{S}$, there are projections P, Q in $B(H)$ such that*

$$\text{(a)} \qquad (K+\varepsilon)\|PTQ\| \geqslant \mathrm{dist}(T,\mathcal{S}),$$

and

$$\text{(b)} \qquad \|PSQ\| < \varepsilon \text{ for each } S \text{ in } \mathcal{F}.$$

(3) For every $\varepsilon > 0$, for every T in $B(H)$ and every finite subset $\mathcal{F}$ of $\mathcal{S}$, there are unit vectors e, f in H such that

(a) $$(K+\varepsilon)|(Te,f)| \geqslant \operatorname{dist}(T,\mathcal{S}),$$

and

(b) $$|(Se,f)| < \varepsilon \text{ for each } S \text{ in } \mathcal{F}.$$

Proof: The implication (1) $\Rightarrow$ (2) is obvious. The equivalence of (2) and (3) comes from considering the projections $e \otimes e$ and $f \otimes f$. The proof of (2) $\Rightarrow$ (1) is obtained by defining the directed set Λ of all pairs $(\varepsilon, \mathcal{F})$ with $\varepsilon > 0$ and $\mathcal{F}$ a finite subset of $\mathcal{F}$ so that larger elements of Λ have smaller ε's and larger $\mathcal{F}$'s. Suppose $T \in B(H)$, and for each $\lambda = (\varepsilon, \mathcal{F})$ in Λ, choosing projections $P = P_\lambda$ and $Q = Q_\lambda$ so that the conditions in statement (2) hold. It is clear that, for every S in $\mathcal{S}$, we have $\|P_\lambda S Q_\lambda\| \to 0$ and that $K \limsup_\lambda \|P_\lambda T Q_\lambda\| \geqslant \operatorname{dist}(T, \mathcal{S})$. This clearly implies (1). ■

COROLLARY 3. *If $\{\mathcal{S}_i : i \in I\}$ is an increasingly directed family of subspaces of $B(H)$ whose union is dense in $\mathcal{S}$, then*

$$K_a(\mathcal{S}) \leqslant \sup_i K_a(\mathcal{S}_i).$$

Let $\mathcal{K}(H)$ denote the algebra of all compact operators on H. In [1] it was shown that ApprAlgLat$\mathcal{A} \subset [\mathcal{A} + \mathcal{K}(H)]^-$ for every separable unital subalgebra of $B(H)$ when H is separable. In [15] it was shown that ApprRef$\mathcal{S} \subset [\mathcal{S} + \mathcal{K}(H)]^-$ for every linear subspace $\mathcal{S}$ of $B(H)$. One key ingredient of the proof is the following lemma, which is a generalization of a theorem of Glimm [11] characterizing states on $B(H)$ that annihilate $\mathcal{K}(H)$. This lemma can be proved using Voiculescu's theorem on approximate equivalence [23]; the analogue for $l_p, 1 < p < \infty$, is proved in [16].

LEMMA 4. *If φ is a continuous linear functional on $B(H)$ such that $\|\varphi\| = 1$ and $\mathcal{K}(H) \subset \ker\varphi$, then there are nets $\{e_\lambda\}$ and $\{f_\lambda\}$ of unit vectors in H, both converging weakly to 0, such that*

$$\varphi(T) = \lim_\lambda (Te_\lambda, f_\lambda) \text{ for every } T \text{ in } B(H).$$

COROLLARY 5. *If $\mathcal{S}$ is a linear subspace of $B(H)$ and $T \in B(H)$, then* $\operatorname{dist}(T, \mathcal{S} + \mathcal{K}(H)) \leqslant \mathrm{d}_a(T, \mathcal{S})$.

COROLLARY 6. *If $\mathcal{S}$ is a norm closed linear subspace of $B(H)$ that contains $\mathcal{K}(H)$, then $\mathcal{S}$ is approximately hyperreflexive and $K_a(\mathcal{S}) = 1$.*

Our next result is an analogue for completely bounded maps of the characterization in [14] of completely positive maps that are approximate compressions of a given representation of a C^*-algebra. Let $\mathcal{K}(H)$ denote the algebra of all compact operators on H. Suppose $\mathcal{A}$ is a separable C^*-algebra, H is a separable Hilbert space and π and ρ are unital representations of $\mathcal{A}$ in $B(H)$. We say that π and ρ are *approximately equivalent* if there is a sequence $\{U_n\}$ of unitary operators such that:

(i) $\|U_n^*\pi(a)U_n - \rho(a)\| \to 0$ for every a in $\mathcal{A}$, and

(ii) $U_n^*\pi(a)U_n - \rho(a) \in \mathcal{K}(H)$ for every a in $\mathcal{A}$ and every $n \geqslant 1$.

In [23] D. Voiculescu gave a very simple characterization of approximate equivalence. In particular, Voiculescu's theorem implies that if $\pi^{-1}(\mathcal{K}(H)) \subset \ker\rho$, then π and $\pi \oplus \rho$ are approximately equivalent.

If $\varphi: \mathcal{A} \to B(H)$ is a linear map, then, for each positive integer n, we let $\mathcal{M}_n(\mathcal{A})$ denote the C^*-algebra of all $n \times n$ matrices over $\mathcal{A}$, and we define $\varphi_n: \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(B(H))$ by $\varphi_n((a_{ij})) = (\varphi_n(a_{ij}))$. We say that the map φ is *completely positive* if each φ_n is positive and we say that φ is *completely bounded* if $\|\varphi\|_{cb} = \sup_n\|\varphi_n\| < \infty$. It was proved by Wittstock [24] that if φ is completely bounded, then there is a $*$-homomorphism π and operators V, W with $\|V\| \cdot \|W\| = \|\varphi\|_{cb}$, such that $\varphi(a) = V\pi(a)W$ for every a in $\mathcal{A}$. A beautiful account of completely bounded maps is contained in [20] (see also [6] and [10]).

PROPOSITION 7. *Suppose $\mathcal{A}$ is a separable unital C^*-algebra, M and H are separable Hilbert spaces, $\pi: \mathcal{A} \to B(H)$ is a unital representation, and $\varphi: \mathcal{A} \to B(M)$ is a linear map. The following are equivalent:*

(1) *There is a unital representation $\rho: \mathcal{A} \to B(H)$ that is approximately equivalent to π, and operators V and W such that, for every a in $\mathcal{A}$, $\varphi(a) = V\rho(a)W$.*

(2) *There are bounded nets $\{V_\lambda\}$ and $\{W_\lambda\}$ of operators such that, for every a in $\mathcal{A}$, $V_\lambda\pi(a)W_\lambda \to \varphi(a)$ in the weak operator topology.*

(3) *The map φ is completely bounded and there are operators A and B such that $\varphi(a) = A\pi(a)B$ for every a in $\pi^{-1}(\mathcal{K}(H))$.*

Proof. (1) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (3). By choosing appropriate subnets, we can assume that there are operators V and W such that $V_\lambda \to V$ and $W_\lambda \to W$ in the weak operator topology. It follows that, for each compact operator T, we have $V_\lambda TW_\lambda \to VTW$. Hence (3) holds.

(3) $\Rightarrow$ (1). Define ψ on $\mathcal{A}$ by $\psi(a) = A\pi(a)B$. Then ψ is completely bounded and $\sigma = \varphi - \psi$ is a completely bounded map that annihilates the ideal $\pi^{-1}(\mathcal{K}(H))$. It follows from Wittstock's theorem [24], applied to the map induced by σ on $\mathcal{A}/\pi^{-1}(\mathcal{K}(H))$, that there is a representation τ on $\mathcal{A}$ and operators C, D such that $\pi^{-1}(\mathcal{K}(H)) \subset \ker\tau$ and $\sigma(a) = C\tau(a)D$ for every a in $\mathcal{A}$. It follows from Voiculescu's theorem [23] that

π is approximately equivalent to $\sigma = \pi \oplus \tau$. Define $Wx = (Bx, Dx)$ and $V(x,y) = Ax + Cy$. We then have statement (1) above. ■

REMARK. It is clear from the proof of the preceding theorem that, in going from (2) to (1), it is possible to choose V and W in part (1) so that $\|V\| \cdot \|W\| \leqslant 2\liminf_\lambda \|V_\lambda\| \cdot \|W_\lambda\|$. Vern Paulsen has provided the author with an idea that allows the construction of V and W so that $\|V\| \cdot \|W\| \leqslant \liminf_\lambda \|V_\lambda\| \cdot \|W_\lambda\|$. To see this, suppose that $\|V_\lambda\|, \|W_\lambda\| \leqslant 1$ for every λ. There is no harm in assuming that $M = H$. For each λ we can define a unitary element U_λ in $\mathcal{M}_4(B(H))$ whose 2×2 upper lefthand corner is $\begin{bmatrix} V_\lambda & 0 \\ 0 & W_\lambda^* \end{bmatrix}$. By choosing an appropriate subnet, we can assume that $\{U_\lambda T U_\lambda^*\}$ converges in the weak operator topology to an operator $\alpha(T)$ for every T in $B(H)$. It follows from [14] that there is a representation σ of $\mathcal{M}_4(\mathcal{A})$ that is approximately equivalent to π_4 and an isometry Y such that $\alpha(\pi_4(A)) = Y^*\sigma(A)Y$ for every A in $\mathcal{M}_4(\mathcal{A})$. However, there must be a representation ρ that is approximately equivalent to π such that σ is unitarily equivalent (hence, we can assume equal) to ρ_4. If $a \in \mathcal{A}$, let A_a be the element of $\mathcal{M}_4(\mathcal{A})$ whose (1,2)-entry is a and whose remaining entries are 0. It follows that the (1,2)-entry of $\alpha(A_a)$ is $\varphi(a)$, and, since $\rho_4(A_a)$ has (1,2)-entry $\rho(a)$ and all other entries 0, it follows that there are contractions V and W such that $\varphi(a) = V\rho(a)W$ for all a in $\mathcal{A}$. Note that the result in this remark shows how the results in [14] can be extended to nonunital completely positive maps. ■

In the case in which φ is a linear functional, the result in the preceding remark can be obtained more easily.

LEMMA 8. *Suppose $\mathcal{A}$ is a separable unital C^*-algebra, H is a separable Hilbert space, $\pi: \mathcal{A} \to B(H)$ is a unital representation, and $\varphi: \mathcal{A} \to \mathbf{C}$ is a linear map for which there are bounded nets $\{e_\lambda\}$ and $\{f_\lambda\}$ in H such that, for each a in $\mathcal{A}$, $\varphi(a) = \lim(\pi(a)e_\lambda, f_\lambda)$. Then there is a representation $\rho: \mathcal{A} \to B(H)$ that is approximately equivalent to π and vectors u and v such that*

(1) $$\|u\|^2 = \|v\|^2 \leqslant \liminf_\lambda \|e_\lambda\| \cdot \|f_\lambda\|,$$

and

(2) $$\varphi(a) = (\rho(a)u, v) \quad \textit{for every } a \textit{ in } \mathcal{A}.$$

Proof. By choosing appropriate subnets, we can assume that there are vectors e, f such that $e_\lambda \to e$ weakly and $f_\lambda \to f$ weakly and such that, for every T in $B(H)$, $\lim_\lambda(Te_\lambda, f_\lambda) = \alpha(T)$ exists. It follows that $(Te_\lambda, f_\lambda) \to (Te, f) = \alpha(T)$ for every compact operator T in $\mathcal{K}(H)$. Since the functional $\beta(T) = \alpha(T) - (Te, f)$ annihilates $\mathcal{K}(H)$, it follows from Wittstock's theorem [24] that there is a representation τ of

$B(H)$ and vectors x, y with $\|x\|^2 = \|y\|^2 = \|\beta\|$ such that $\mathcal{K}(H) \subset \ker\tau$ and such that $\beta(T) = (\tau(T)x, y)$ for every T in $B(H)$.

We now want to prove that $\|\alpha\| = \|e\| \cdot \|f\| + \|x\| \cdot \|y\|$. First choose an operator A whose restriction to $\mathrm{sp}\{e, f\}$ is a unitary operator and whose restriction to $\{e, f\}^\perp$ is 0 such that $(Ae, f) = \|e\| \cdot \|f\|$. Let Q be the projection onto $\{e, f\}^\perp$, and choose a sequence $\{B_n\}$ of operators with norm 1 such that $0 \leqslant \beta(B_n) \to \|\beta\| = \|x\| \cdot \|y\|$. Note that since $\mathcal{K}(H) \subset \ker\beta$, we can assume that $B_n = QB_nQ$. Since $\beta(A) = 0$, and $(B_n e, f) = 0$ and $\|A + B_n\| = 1$ for each n, we have $\|\alpha\| \geqslant \sup_n \alpha(A + B_n) = (Ae, f) + \sup\beta(B_n) = \|e\| \cdot \|f\| + \|x\| \cdot \|y\|$. The reverse inequality is obvious; whence $\|\alpha\| = \|e\| \cdot \|f\| + \|x\| \cdot \|y\|$.

We now let $\rho = \pi \oplus (\tau \circ \pi)$, $u = e \oplus x$ and $v = f \oplus y$. Then $\varphi(T) = (\rho(a)u, v)$ for every a in $\mathcal{A}$ and $\|u\|^2 = \|v\|^2 = \|e\| \cdot \|f\| + \|x\| \cdot \|y\| \leqslant \|\alpha\| \leqslant \limsup_\lambda \|e_\lambda\| \cdot \|f_\lambda\|$. ■

REMARK. If X is a Banach space with dual $X^\sharp$, there is a natural embedding of X into $X^{\sharp\sharp}$. In general X is not complemented in $X^{\sharp\sharp}$, e.g., c_0 is not complemented in l^∞ [21]. However, $X^\sharp$ is always complemented in $X^{\sharp\sharp\sharp}$, with a norm idempotent $P: X^{\sharp\sharp\sharp} \to X^\sharp$ defined by $P(\varphi) = \varphi|X$. In general, $\|1 - P\|$ is not 1. In the case $X = \mathcal{K}(H)$, we have $X^\sharp = \mathcal{C}_1(H)$, the set of trace class operators on H, and the above proof that $\|\alpha\| = \|e\| \cdot \|f\| + \|x\| \cdot \|y\|$ can be easily adapted to show that $\|1 - P\| = 1$, more precisely, for every φ in $B(H)^\sharp$, $\|\varphi\| = \|P\varphi\| + \|(1 - P)\varphi\|$. ■

The following theorem shows the equivalence of ApprAlgLat and ApprRef for unital algebras. This answers a question raised in [15].

THEOREM 9. *Suppose that $\mathcal{A}$ is a unital subalgebra of $B(H)$ and $T \in B(H)$. Then*

(1) $\mathrm{d}_a(T, \mathcal{A}) = \sup\{\limsup_\lambda \|(1 - P_\lambda)TP_\lambda\| : \{P_\lambda\}$ *is a net of projections,* $\|(1 - P_\lambda)AP_\lambda\| \to 0$ *for every A in $\mathcal{A}$*$\}$,

(2) *if $\mathcal{A}$ is a C^*-algebra, then* $\mathrm{d}_a(T, \mathcal{A}) = \sup\{\limsup_\lambda \|P_\lambda T - TP_\lambda\| : \{P_\lambda\}$ *is a net of projections,* $\|P_\lambda A - AP_\lambda\| \to 0$ *for every A in $\mathcal{A}$*$\}$.

Proof. (1). It follows from Lemma 1 that we can assume that $\mathcal{A}$ and H are both norm separable. First suppose that $\{P_\lambda : \lambda \in \Lambda\}$ is a net of projections for which $\|(1 - P_\lambda)AP_\lambda\| \to 0$ for every A in $\mathcal{A}$. Let $\Gamma = \Lambda \times \{1, 2, 3, \ldots\}$ directed so that $(\alpha, m) \leqslant (\beta, n)$ means $\alpha \leqslant \beta$ and $m \leqslant n$. For each $\gamma = (\lambda, n) \in \Gamma$ choose a unit vector e_γ in $\mathrm{ran}P_\lambda$ and a unit vector f_γ in $\mathrm{ran}(1 - P_\lambda)$ so that $|((1 - P_\lambda)TP_\lambda e_\gamma, f_\gamma)| \geqslant \left(1 - \frac{1}{n}\right) \|(1 - P_\lambda)TP_\lambda\|$. It follows, for every A in $\mathcal{A}$ and every γ in Γ, that $|(Ae_\gamma, f_\gamma)| = |((1 - P_\lambda)AP_\lambda e_\gamma, f_\gamma)| \leqslant \|(1 - P_\lambda)AP_\lambda\|$. It follows that $(Ae_\gamma, f_\gamma) \to 0$ for every A in $\mathcal{A}$. On the other hand, it follows that $\lim_\gamma |(Te_\gamma, f_\gamma)| = \limsup_\lambda \|(1 - P_\lambda)TP_\lambda\|$. It follows that $\mathrm{d}_a(T, \mathcal{A}) \geqslant \sup\{\limsup_\lambda \|(1 - P_\lambda)TP_\lambda\| : \{P_\lambda\}$ is a net of projections, $\|(1-P_\lambda)AP_\lambda\| \to 0$ for every A in $\mathcal{A}\}$.

To prove the reverse inequality, suppose that $\{e_\lambda\}$ and $\{f_\lambda\}$ are nets of unit vectors in H such that $(Ae_\lambda, f_\lambda) \to 0$ for every A in $\mathcal{A}$, and let $\delta = \limsup_\lambda |(Te_\lambda, f_\lambda)|$. By choosing an appropriate subnet, we can assume that $\lim_\lambda |(Te_\lambda, f_\lambda)| = \delta$. Again, by choosing an appropriate subnet, we can assume that $\varphi(S) = \lim_\lambda (Se_\lambda, f_\lambda)$ exists for every S in $B(H)$.

It follows from Lemma 8 that there is a representation $\rho: C^*(\mathcal{A} \cup \{T\}) \to B(H)$ that is approximately equivalent to the identity representation on $C^*(\mathcal{A} \cup \{T\})$ and vectors u,v in H such that $\|u\|^2 = \|v\|^2 \leqslant \liminf_\lambda \|e_\lambda\| \cdot \|f_\lambda\| = 1$ and $\varphi(A) = (\rho(A)u, v)$ for every A in $C^*(\mathcal{A} \cup \{T\})$. Since ρ is approximately equivalent to the identity representation, there is a sequence $\{U_n\}$ of unitary operators such that, for every A in $C^*(\mathcal{A} \cup \{T\})$, $\|U_n^*\pi(A)U_n - \rho(A)\| \to 0$. Let P be the projection onto $[\rho(\mathcal{A})u]^-$. It follows that $(1-P)\rho(A)P = 0$ for every A in $\mathcal{A}$, and since $(\rho(A)u, v) = \varphi(A) = 0$ for every A in $\mathcal{A}$, we conclude that $(1-P)v = v$. Thus $\|(1-P)\rho(T)P\| \geqslant |((1-P)\rho(T)Pu, v)| = |(\rho(T)u, v)| = \varphi(T) = \delta$. For each positive integer n, let $P_n = U_n P U_n^*$. For each A in $C^*(\mathcal{A} \cup \{T\})$, we have $\|(1-P_n)AP_n\| = \|(1-P)U_n^*AU_nP\| \to \|(1-P)\rho(A)P\|$. It follows that $\|(1-P_n)AP_n\| \to 0$ for every A in $\mathcal{A}$ and $\limsup_n \|(1-P_n)TP_n\| \geqslant \delta$. It therefore follows that $d_a(T, \mathcal{A}) \leqslant \sup\{\limsup_\lambda \|(1-P_\lambda)TP_\lambda\| : \{P_\lambda\}$ is a net of projections, $\|(1-P_\lambda)AP_\lambda\| \to 0$ for every A in $\mathcal{A}\}$.

(2). This follows from the fact that if $\mathcal{A}$ is a C^*-algebra and $\{P_\lambda\}$ is a net of projections such that $\|(1-P_\lambda)AP_\lambda\| \to 0$ for every A in $\mathcal{A}$, then $\|P_\lambda A - AP_\lambda\| \to 0$ for every A in $\mathcal{A}$, and, for each λ, we have $\|P_\lambda T - TP_\lambda\| = \max(\|(1-P_\lambda)TP_\lambda\|, \|P_\lambda T(1-P_\lambda)\|)$. ∎

REMARKS. 1. The techniques of the preceding theorem, combined with the techniques in [22], can be used to show that if $\mathcal{A}$ is a C^*-subalgebra of $B(H)$ and $T \in B(H)$, then $d_a(T, \mathcal{A}) \leqslant \sup\{\limsup_\lambda \|U_\lambda T - TU_\lambda\| : \{U_\lambda\}$ is a net of unitaries, $\|U_\lambda A - AU_\lambda\| \to 0$ for every A in $\mathcal{A}\} \leqslant \sup\{\limsup_\lambda \|S_\lambda T - TS_\lambda\| : \{S_\lambda\}$ is a net of contractions, $\|S_\lambda A - AS_\lambda\| \to 0$ for every A in $\mathcal{A}\} \leqslant 2d_a(T, \mathcal{A})$.

2. Note that if $\mathcal{S}$ is a norm separable linear subspace of $B(H)$ with H separable, and if $T \in B(H)$ and ψ is any continuous linear functional on $B(H)$, then there is a representation $\rho: C^*(\mathcal{S} \cup \{T\}) \to B(H)$ that is approximately equivalent to the identity representation and unit vectors $f, g \in H \oplus H \oplus \cdots$ such that $\|f\|^2 = \|g\|^2 = \|\psi\|$ and $\psi(S) = (\rho^{(\infty)}(S)f, g)$ for every S in $C^*(\mathcal{S} \cup \{T\})$ (see the proof of Theorem 2.2 in [13]). In particular, if $\mathcal{S} = \mathcal{A}$ is a C^*-algebra, $\psi|\mathcal{A} = 0$ and $\psi(T) = \mathrm{dist}(T, \mathcal{A})$, and $\|\psi\| = 1$, then it follows that $\mathrm{dist}(\rho(T), \rho(\mathcal{A})'') = \mathrm{dist}(\rho^{(\infty)}(T), \rho^{(\infty)}(\mathcal{A})'') \geqslant \mathrm{dist}(T, \mathcal{A})$. If P is any projection in $\rho(\mathcal{A})'$, then the proof of the preceding theorem (and the theorem itself) shows that $d_a(T, \mathcal{A}) \geqslant \|(1-P)\rho(T)P\|$. It follows that if $\rho(\mathcal{A})''$ is hyperreflexive, then $\mathcal{A}$ is approximately hyperreflexive and $K_a(\mathcal{A}) \leqslant K(\rho(\mathcal{A})'')$. In particular,

it follows from [7, Thm. 2.3], [22, Thm. 2.1] that if $\mathcal{A}$ is nuclear, then, since $\rho(\mathcal{A})''$ is injective, $K_a(\mathcal{A}) \leqslant K(\rho(\mathcal{A})'') \leqslant 4$. With a larger estimate of $K_a(\mathcal{A})$, we prove (Theorem 13) that every C^*-algebra is approximately hyperreflexive. ■

COROLLARY 10. *If $\mathcal{A}$ is a unital subalgebra of $B(H)$, then*

$$\operatorname{ApprRef}\mathcal{A} = \operatorname{ApprAlgLat}\mathcal{A}.$$

COROLLARY 11. *If $\mathcal{A}$ is a unital subalgebra of $B(H)$, then $K_a(\mathcal{A}) \leqslant K$ if and only if, for every $\varepsilon > 0$, for every T in $B(H)$ and every finite subset $\mathcal{F}$ of $\mathcal{A}$, there is a projection P in $B(H)$ such that*

(a) $$\|(1-P)TP\| \geqslant (K-\varepsilon)\operatorname{dist}(T,\mathcal{A}),$$

and

(b) $$\|(1-P)SP\| < \varepsilon \ \text{ for each } \ S \ \text{ in } \ \mathcal{F}.$$

We now define an analogue of the notion of relative hyperreflexivity introduced in [15]. Suppose that $\mathcal{S}$ and $\mathcal{T}$ are linear subspaces of $B(H)$ and $\mathcal{S} \subset \mathcal{T}$. We say that $\mathcal{S}$ is *relatively approximately hyperreflexive* in $\mathcal{T}$ if there is a smallest constant $K = K_a(\mathcal{S},\mathcal{T})$ such that $\operatorname{dist}(T,\mathcal{S}) \leqslant K d_a(T,\mathcal{S})$ for every T in $\mathcal{T}$.

The following elementary lemma is contained in [15].

LEMMA 12. *Suppose that $\mathcal{R},\mathcal{S},\mathcal{T}$ are linear subspaces of $B(H)$ and $\mathcal{R} \subset \mathcal{S} \subset \mathcal{T}$. Then $K_a(\mathcal{R},\mathcal{T}) \leqslant (1+K_a(\mathcal{R},\mathcal{S}))(1+K_a(\mathcal{S},\mathcal{T}))-1$.*

We are now ready to prove our main result. If H is a Hilbert space, then $H^{(n)}$ denotes the direct sum of n copies of H; if $T \in B(H)$, then $T^{(n)}$ denotes the direct sum of n copies of T acting on $H^{(n)}$. Similarly, if ρ is a representation of a C^*-algebra, then $\rho^{(n)}$ denotes the direct sum of n copies of ρ.

THEOREM 13. *If $\mathcal{A}$ is a unital C^*-algebra of $B(H)$, then $\mathcal{A}$ is approximately hyperreflexive and $K_a(\mathcal{A}) \leqslant 29$.*

Proof. Let $M = \{h \in H : [\mathcal{K}(H) \cap \mathcal{A}]h = \{0\}\}$, let Q be the projection onto M and let $P = 1-Q$. The identity representation of $\mathcal{A} \cap \mathcal{K}(H)$ is unitarily equivalent to $0 \oplus \sum_i^{\oplus} \tau_i^{(n_i)}$ relative to $H = M \oplus \sum_i^{\oplus} H_i^{(n_i)}$, with each τ_i irreducible. The identity representation on $\mathcal{A}$ is unitarily equivalent to $\pi \oplus \sum_i^{\oplus} \pi_i^{(n_i)}$. Also $P\mathcal{A}''P$ is the set of operators of the form $0\oplus\sum_i^{\oplus} T_i^{(n_i)}$ with each T_i in $B(H_i)$. Let $\mathcal{B} = QB(H)Q+P\mathcal{A}''P$ be the set of all operators of the form $T \oplus \sum_i^{\oplus} T_i^{(n_i)}$ with T in $B(M)$ and each T_i in $B(H_i)$. Then $\mathcal{B}$ is a type I von Neumann algebra, and by [7, Thm. 2.4], [22, Thm. 2.1] $\mathcal{B}$ is hyperreflexive with $K(\mathcal{B}) \leqslant 4$. Clearly, $K_a(\mathcal{B}) \leqslant K(\mathcal{B})$.

Next let $\mathcal{D} = (\mathcal{A}+\mathcal{K}(H))\cap\mathcal{B}$. Then $\mathcal{D} = P\mathcal{K}(H)P+\mathcal{A}$. Suppose $T \in \mathcal{B}$ and choose a continuous linear functional φ with norm 1 so that $\varphi|\mathcal{D} = 0$ and $\varphi(T) = \text{dist}(T,\mathcal{D})$. Since $\mathcal{B}\cap\mathcal{K}(H) \subset \ker\varphi$, and $[\mathcal{B}+\mathcal{K}(H)]/\mathcal{K}(H)$ is isomorphic to $\mathcal{B}/\mathcal{K}(H)$, it is possible to extend φ to a continuous linear functional ψ on $B(H)$ with norm 1 such that $\mathcal{K}(H) \subset \ker\psi$. It follows from Lemma 4 that there are nets $\{e_\lambda\}$ and $\{f_\lambda\}$ of unit vectors converging weakly to 0 such that $\psi(S) = \lim_\lambda(Se_\lambda, f_\lambda)$ for every S in $B(H)$. It follows that $\text{dist}(T,\mathcal{D}) \leqslant \mathrm{d}_a(T,\mathcal{D})$. Hence $K_a(\mathcal{D},\mathcal{B}) = 1$.

It follows from Lemma 12 that $\mathcal{D}$ is approximately hyperreflexive and $K_a(\mathcal{D}) \leqslant$ $\leqslant (4+1)(1+1)-1 = 9$.

Next suppose that $T \in \mathcal{D}$. Then $T = A+B$ with $A \in \mathcal{A}$ and $B \in P\mathcal{K}(H)P$. It follows that $\text{dist}(T,\mathcal{A}) = \text{dist}(B,\mathcal{A})$ and $\mathrm{d}_a(T,\mathcal{A}) = \mathrm{d}_a(B,\mathcal{A})$. Hence we can assume that $T \in P\mathcal{K}(H)P$. Suppose e and f are unit vectors in M. The formula $\varphi(A) =$ $= (Ae, f)$ defines a continuous linear functional on $\mathcal{A}$ that annihilates $\mathcal{A}\cap\mathcal{K}(H)$. Arguing as before, we obtain nets $\{u_\lambda\}$ and $\{v_\lambda\}$ of vectors in the unit ball of H converging weakly to 0 such that $\varphi(A) = \lim_\lambda(Au_\lambda, v_\lambda)$ for every A in $\mathcal{A}$. Clearly, we can choose the u_λ's and v_λ's to be in $\{e,f\}^\perp$. Define $e_\lambda = (e+u_\lambda)/\sqrt{2}$ and $f_\lambda = (f-v_\lambda)/\sqrt{2}$ for each λ. Then $(Ae_\lambda, f_\lambda) \to 0$ for every A in $\mathcal{A}$ and $(Te_\lambda, f_\lambda) \to (Te,f)/2$. It follows that $|(Te,f)| \leqslant 2\mathrm{d}_a(T,\mathcal{A})$. Since $T = PTP$, we conclude that $\|T\| \leqslant 2\mathrm{d}_a(T,\mathcal{A})$. Thus $\text{dist}(T,\mathcal{A}) \leqslant 2\mathrm{d}_a(T,\mathcal{A})$. It follows that $K_a(\mathcal{A},\mathcal{D}) \leqslant 2$, and by Lemma 12, we conclude that $\mathcal{A}$ is approximately hyperreflexive and $K_a(\mathcal{A}) \leqslant (9+1)(2+1)-1 =$ $= 29$. ∎

The techniques of the preceding proof yield three more useful results for arbitrary linear subspaces. Perhaps the most remarkable of these is the following characterization of approximate hyperreflexivity.

PROPOSITION 14. *Suppose that $\mathcal{S}$ is a norm closed linear subspace of $B(H)$. Then $\mathcal{S}$ is approximately hyperreflexive if and only if there is a constant K, such that, for every finite-rank operator T in $B(H)$,*

$$\text{dist}(T,\mathcal{S}) \leqslant K\mathrm{d}_a(T,\mathcal{S}).$$

Proof. Suppose there is a constant K, such that, for every finite-rank operator T in $B(H)$ $\text{dist}(T,\mathcal{S}) \leqslant K\mathrm{d}_a(T,\mathcal{S})$. If $S \in \mathcal{S}$ and T has finite rank, it follows that

$\operatorname{dist}(S + T, \mathcal{S}) = \operatorname{dist}(T, \mathcal{S}) \leqslant K d_a(T, \mathcal{S}) = K d_a(S + T, \mathcal{S})$. Since the seminorms $\operatorname{dist}(\ , \mathcal{S})$ and $d_a(\ , \mathcal{S})$ are norm continuous, we conclude that $\operatorname{dist}(T, \mathcal{S}) \leqslant K d_a(T, \mathcal{S})$ holds for every T in $[\mathcal{S} + \mathcal{K}(H)]^-$. However, we know from Corollary 6 that $K_a([\mathcal{S} + \mathcal{K}(H)]^-) = 1$. Thus, by Lemma 12, $\mathcal{S}$ is approximately hyperreflexive.

The reverse implication is obvious. ∎

Another result shows the relation between Ref$\mathcal{S}$ and ApprRef$\mathcal{S}$ when $\mathcal{S} \subset \mathcal{K}(H)$.

PROPOSITION 15. *Suppose* $\mathcal{S} \subset \mathcal{K}(H)$. *Then* ApprRef$\mathcal{S} = \mathcal{K}(H) \cap$ Ref$\mathcal{S}$, *and* ApprAlgLat$\mathcal{S} = C^*(\mathcal{K}(H)) \cap$ AlgLat$\mathcal{S}$.

Proof. Suppose $T \in \mathcal{K}(H) \cap$ Ref$\mathcal{S}$. Suppose via contradiction $\{e_\lambda\}$ and $\{f_\lambda\}$ are nets of unit vectors such that $(Se_\lambda, f_\lambda) \to 0$ for each S in $\mathcal{S}$, but $(Te_\lambda, f_\lambda) \not\to 0$. By choosing an appropriate subnet we can assume that there is an $\varepsilon > 0$ and vectors e, f such that $e_\lambda \to e$ weakly and $f_\lambda \to f$ weakly and, for every λ, $|(Te_\lambda, f_\lambda)| \geqslant \varepsilon$. However, for every compact operator A, we have $(Ae_\lambda, f_\lambda) \to (Ae, f)$. Hence $(Se, f) = 0$ for every S in $\mathcal{S}$, but $|(Te, f)| \geqslant \varepsilon$. This contradicts the fact that $T \in$ Ref$\mathcal{S}$.

For the converse, it is clear that ApprRef$\mathcal{S} \subset$ Ref$\mathcal{S}$ and ApprRef$\mathcal{S} \subset$ ApprRef$\mathcal{K}(H) \subset \mathcal{K}(H)$. ∎

Corollary 6 states that $\mathcal{S}$ is approximately hyperreflexive whenever $\mathcal{K}(H) \subset \mathcal{S}$. The following result shows what happens at the other extreme.

PROPOSITION 16. *Suppose* $\mathcal{S}$ *is a linear subspace of* $B(H)$. *If* $\mathcal{S} \cap \mathcal{K}(H) = 0$ *and* $\mathcal{S} + \mathcal{K}(H)$ *is norm closed, then* $\mathcal{S}$ *is approximately hyperreflexive.*

Proof. The hypotheses imply that if η is the restriction to $\mathcal{S}$ of the quotient map from $B(H)$ to $B(H)/\mathcal{K}(H)$, then η is 1-1 and $\eta(\mathcal{S})$ is closed. Thus $\eta^{-1}: \eta(\mathcal{S}) \to \mathcal{S}$ is a bounded linear map. Suppose that T has finite rank and choose unit vectors e, f in H so that $(Te, f) = \|T\|$. The mapping $s \to (\eta^{-1}(s)e, f)$ on $\eta(\mathcal{S})$ extends to a linear functional β on $B(H)/\mathcal{K}(H)$ with $\|\beta\| \leqslant \|\eta^{-1}\|$. The functional $\varphi = \beta \circ \eta$ defines a continuous linear functional on $B(H)$ that annihilates $\mathcal{K}(H)$. Hence there are nets $\{e_\lambda\}$ and $\{f_\lambda\}$ of vectors converging weakly to 0 such that $\|e_\lambda\|^2 = \|f_\lambda\|^2 = \|\varphi\| \leqslant \|\eta^{-1}\|$ for every λ, and such that $\varphi(A) = \lim_\lambda (Ae_\lambda, f_\lambda)$ for every A in $B(H)$. We can assume that the e_λ's and f_λ's are all in $\{e, f\}^\perp$. Let $u_\lambda = e + e_\lambda$ and $v_\lambda = f - f_\lambda$ for each λ. Then $\|u_\lambda\|^2 = \|v_\lambda\|^2 = 1 + \|\varphi\|$ for every λ, $(Su_\lambda, v_\lambda) \to 0$ for every S in $\mathcal{S}$ and $(Tu_\lambda, v_\lambda) \to (Te, f) = \|T\| \geqslant \operatorname{dist}(T, \mathcal{S})$. It follows that $\operatorname{dist}(T, \mathcal{S}) \leqslant (1 + \|\varphi\|) d_a(T, \mathcal{S}) \leqslant (1 + \|\eta^{-1}\|) d_a(T, \mathcal{S})$ for every finite rank operator. It follows from Proposition 14 that $\mathcal{S}$ is approximately hyperreflexive. Moreover, $K_a(\mathcal{S}) \leqslant (2 + \|\eta^{-1}\|)2 - 1 = 3 + 2\|\eta^{-1}\|$. ∎

Let $\mathcal{E}$ denote the collection of continuous linear functionals on $B(H)$ that are bounded w^*-limits of rank-one tensors, i.e., $\varphi \in \mathcal{E}$ if and only if there are bounded nets $\{e_\lambda\}$ and $\{f_\lambda\}$ in H such that, for every T in $B(H)$, we have $\varphi(T) = \lim_\lambda (Te_\lambda, f_\lambda)$. It follows, for each linear subspace $\mathcal{S}$ of $B(H)$, that

$$\text{ApprRef}\mathcal{S} = (\mathcal{S}^\perp \cap \mathcal{E})_\perp,$$

where $^\perp$ denotes the annihilator in $B(H)^\sharp$, and $_\perp$ denotes the preannihilator in $B(H)$. Moreover, it follows from Lemma 8 that, for every subspace $\mathcal{S}$ of $B(H)$, we have

$$\text{d}_a(T, \mathcal{S}) = \sup\{|\varphi(T)| : \varphi \in \mathcal{S}^\perp \cap \mathcal{E} \text{ and } \|\varphi\| = 1\}.$$

It follows that $\mathcal{S}$ is approximately hyperreflexive if and only if $\mathcal{S}$ is $\mathcal{E}$-hyperreflexive in the sense of [15]. It follows that we can apply all of the relevant results of [15].

We call the weak topology on $B(H)$ induced by the linear span sp$\mathcal{E}$ the *approximate weak operator topology* on $B(H)$; we denote this topology as the *a.w.-topology*. Note, by Proposition 7 and Lemma 8, $\varphi \in \mathcal{E}$ if and only if $\varphi|\mathcal{K}(H)$ can be represented as a rank-one tensor; whence $\varphi \in$ sp$\mathcal{E}$ precisely when $\varphi|\mathcal{K}(H)$ is continuous with respect to the weak operator topology. In particular, $\mathcal{E}$ contains all of the functionals φ in $B(H)^\sharp$ that annihilate $\mathcal{K}(H)$. By the *weak topology* on $B(H)$ we mean the $\sigma(B(H), B(H)^\sharp)$-topology. The map $\pi: B(H) \to B(H)^{(\infty)}$ defined by $\pi(T) = T^{(\infty)} = T \oplus T \oplus \cdots$ is a homeomorphism with the weak* (ultraweak) topology on $B(H)$ and the weak operator topology on $B(H)^{(\infty)}$. Similarly, π is a homeomorphism with the weak topology on $B(H)$ and the a.w. topology on $B(H)^{(\infty)}$.

In [15] it is shown, for a linear subspace $\mathcal{S}$ of $B(H)$ and an operator T in $B(H)$, that T is in the a.w. closure of $\mathcal{S}$ if ad only if $T^{(n)} \in \text{ApprRef}\mathcal{S}^{(n)}$ for each positive integer n. It follows that if $\mathcal{A}$ is a unital subalgebra of $B(H)$ and $T \in B(H)$, then T is in the a.w. closure if and only if $T^{(n)} \in \text{ApprAlgLat}\mathcal{A}^{(n)}$ for every positive integer n. In [13] it was asked if the latter condition implies that T is the norm closure of $\mathcal{A}$. To find a counterexample, it suffices to find an algebra $\mathcal{A}$ that is norm closed but not a.w. closed. Such an algebra is obtained by letting φ be a weak* continuous linear functional on $B(H)$ that is not continuous with respect to the weak operator topology, and defining $\mathcal{A}$ to be the algebra of all operators on $H \oplus H$ with an operator matrix $\begin{bmatrix} \lambda & A \\ 0 & \lambda \end{bmatrix}$ with λ a scalar and A in kerφ.

In [17] a subspace $\mathcal{S}$ was defined to have *property D* (resp. D_σ) if every weak operator (resp. weak*) continuous linear functional agrees on $\mathcal{S}$ with a rank-one tensor. The approximate analogues say that $\mathcal{S}$ has *property* D^a (resp. D^a_σ) if every a.w. (resp. norm) continuous linear functional agrees on $\mathcal{S}$ with an element of $\mathcal{E}$. Intuition might suggest that since there are so many more norm continuous linear functionals than weak*-continuous functionals that it should be more difficult for a subspace to have property D_σ^a than to have property D_σ; however, the opposite is true. The next lemma gives a characterization that relates all four of the above properties.

LEMMA 17. *Suppose $\mathcal{S}$ is a linear subspace of $B(H)$.*

(1) *$\mathcal{S}$ has property D^a if and only if every weak operator continuous linear functional agrees on $\mathcal{S}$ with an element of $\mathcal{E}$.*

(2) *$\mathcal{S}$ has property D_σ^a if and only if every weak* continuous linear functional agrees on $\mathcal{S}$ with an element of $\mathcal{E}$.*

(3) *property $D \Rightarrow$ property D^a, and property $D_\sigma \Rightarrow$ property D_σ^a.*

Proof. (1). Suppose every weak operator continuous linear functional agrees on $\mathcal{S}$ with an element of $\mathcal{E}$. Suppose φ is an a.w. continuous linear functional. Then $\varphi = \alpha + \beta$ where α is a weak operator continuous linear functional and $\beta|\mathcal{K}(H) = 0$. By hypothesis, there is an ξ in $\mathcal{E}$ such that $\alpha - \xi$ annihilates $\mathcal{S}$. Let $\zeta = \xi + \beta$. Then $\varphi - \zeta$ annihilates $\mathcal{S}$, and since $\zeta|\mathcal{K}(H) = \xi|\mathcal{K}(H)$ is a rank-one tensor, it follows that $\zeta \in \mathcal{E}$. The reverse implication is obvious.
(2). This follows by imitating the proof of (1).
(3). This is obvious from (1) and (2). ∎

The following is a direct application of results in [15].

PROPOSITION 18. *Suppose $\mathcal{S}$ is an approximately hyperreflexive linear subspace of $B(H)$.*

(1) *Every approximate weakly closed linear subspace of $\mathcal{S}$ is approximately hyperreflexive if and only if $\mathcal{S}$ has property D^a.*

(2) *Every norm closed linear subspace of $\mathcal{S}$ is approximately hyperreflexive if and only if $\mathcal{S}$ has property D_σ^a.*

(3) *If $\mathcal{T}$ is an approximately reflexive linear subspace of $B(H)$ and $\mathcal{S}+\mathcal{T}$ is norm closed, then $\mathcal{S} \cap \mathcal{T}$ is approximately hyperreflexive.*

The author gratefully acknowledges support from the National Science Foundation while this research was undertaken.

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Received August 6, 1990.