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ATTAINABLE DISTRIBUTIVE SUBSPACE LATTICES

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By a representation of an abstract lattice $\mathcal{L}$ we mean a homomorphism $\pi : \mathcal{L} \to P(H)$ the lattice of projections on a Hilbert space, such that

$$\pi(0) = 0$$
$$\pi(1) = 1$$
$$\pi(\vee a_i) = \vee \pi(a_i)$$

and

$$\pi(\wedge a_i) = \wedge \pi(a_i).$$

A lattice $\mathcal{L}$ is called attainable if there is a faithful representation π and an operator algebra A such that $\pi(\mathcal{L})$ =Lat(A) the lattice of invariant subspaces of A. An old question of P.R. Halmos [6] asks for a characterization of attainable lattices. On this level of generality only fragmentary results exists [4], [7]. However in special cases definitive results can be obtained. Thus, for example, Arveson [1, Sec. 3.4.] has shown that a complete distributive lattice $\mathcal{L}$ is attainable on a separable Hilbert space iff it is countably generated (as a complete lattice) and admits a positive (faithful) normal valuation. Arveson's proof depends upon an analysis of VAL$(\mathcal{L})$ the space of valuation on $\mathcal{L}$ and a theorem of Kakutani for L-spaces.

Using G.C. Rota's notion of the valuation ring of a distributive lattice, an idea which plays an important role in modern combinatorial theory [8], together with the representation theory of Banach $*$-algebras [2] we are able to offer a proof which is quite close in spirit to the GNS representation of an abstract C^*-algebra as an algebra of operators on some Hilbert space.

Recall that a valuation of a lattice $\mathcal{L}$ is a map $v : \mathcal{L} \to \mathbb{C}$ such that $v(x \wedge y) + v(x \vee \vee y) = v(x) + v(y)$ for all $x, y \in \mathcal{L}$. We say that v is positive (faithful) if $v(x) < v(y)$

whenever $x < y$ and non-negative if $v(x) \leqslant v(y)$ whenever $x \leqslant y$. A valuation is called non-negative definite if for any $a_1, \ldots, a_n \in \mathcal{L}$ the matrix $[v(a_i \wedge a_j] \geqslant 0$. A positive valuation is called normal if whenever $x_n \uparrow x$ then $v(x_n) \uparrow v(x)$ and dually.

The main theorem whose statement and proof we have reformulated is

THEOREM (Arveson). *A complete distributive lattice $\mathcal{L}$ is attainable on a separable Hilbert space iff it is countably generated and has a positive normal valuation.*

LEMMA. *Suppose that v is a non-negative definite valuation on $\mathcal{L}_2$ and that $\varphi : \mathcal{L}_1 \to \mathcal{L}_2$ is a lattice homomorphism. Then $v \cdot \varphi$ is a non-negative definite valuation on $\mathcal{L}_1$.*

Proof. Let $a_1 \ldots, a_n \in \mathcal{L}_1$. Then $[v \cdot \varphi(a_i \wedge a_j)] = [v(\varphi(a_i) \wedge \varphi(a_j))] \geqslant 0$. ■

The lemma implies that if $v_1 : \mathcal{L}_1 \to \mathbf{R}$ $v_2 : \mathcal{L}_2 \to \mathbf{R}$ are non-negative definite then

$$\mathcal{L}_1 \times \mathcal{L}_2 \xrightarrow{\pi_1} \mathcal{L}_1 \xrightarrow{v_1} \mathbf{R}$$

$$\mathcal{L}_1 \times \mathcal{L}_2 \xrightarrow{\pi_2} \mathcal{L}_2 \xrightarrow{v_2} \mathbf{R}$$

are non-negative definite valuations on the product lattice $\mathcal{L}_1 \times \mathcal{L}_2$. This implies that $\mu((a,b)) = v_1(a) + v_2(b)$ is a non-negative definite valuation on $\mathcal{L}_1 \times \mathcal{L}_2$.

We would like to point out that if (X, Σ, μ) is a measure space then the valuation μ is non-negative definite provided $\mu \geqslant 0$. The standard argument runs

$$\int_X |c_1\chi_{E_1} + \cdots + c_n\chi_{E_n}|^2 \,\mathrm{d}\mu \geqslant 0$$

$$\text{so} \int_X \Sigma c_i \bar{c}_j \chi_{E_i}\chi_{E_j} \mathrm{d}\mu \geqslant 0$$

$$\text{whence } \Sigma c_i \bar{c}_j \mu(E_i \wedge E_j) \geqslant 0$$

The next result which has apparently gone unnoticed is a generalization to arbitrary distributive lattices.

PROPOSITION. *Every non-negative valuation is non-negative definite.*

Proof. We first prove the result for finite distributive lattices by induction on the number of elements in $\mathcal{L}$. In case $|\mathcal{L}| = 2$ then $\mathcal{L}$ is the two element chain which can be embedded in the four element Boolean algebra. Furthermore the non-negative valuation can be extended to a non-negative measure on the Boolean algebra. Hence the valuation is non-negative definite.

Next assume the result true for all distributive lattices $\mathcal{L}$ with $|\mathcal{L}| < n$. Let $0 < a < 1$ be an element of $\mathcal{L}$. The intervals $\mathcal{L}_1 = [0, a]$ and $\mathcal{L}_2 = [a, 1]$ are distributive and $|\mathcal{L}_1|, |\mathcal{L}_2| < n$. We restict v to $\mathcal{L}_1$ to obtain a non-negative valuation v_1 on $\mathcal{L}_1$ and we define a non-negative valuation v_2 on $\mathcal{L}_2$ by

$$v_2(x) = v(x) - v(a).$$

By the inductive hypotheses both v_1, v_2 are non-negative definite. Hence the valuation $\mu : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathbb{R}$ defined by $\mu(x, y) = v_1(x) + v_2(y)$ is non-negative definite. Define $\varphi : \mathcal{L} \to \mathcal{L}_1 \times \mathcal{L}_2$ by $\varphi(\ell) = (\ell \wedge a, \ell \vee a)$. Then $\mu \cdot \varphi$ is non-negative definite by the lemma. But

$$\mu\varphi(b) = \mu(b \wedge a, b \vee a) = v_1(b \wedge a) + v_2(b \vee a) = v(b \wedge a) + v(b \vee a) - v(a) = v(b)$$

Thus $v = \mu \cdot \varphi$ is non-negative definite.

For a general distributive lattice consider the sublattice generated by $a_1, a_2, \ldots$ $\ldots, a_n$. It is well known that this is finite. Hence the general result follows. ∎

Let $\mathcal{L}$ be a distributive lattice. If we view $\mathcal{L}$ as a semigroup under the $\wedge$ operation we can consider the semigroup algebra $\ell^1(\mathcal{L})$. An involution is defined under which $\ell^1(\mathcal{L})$ becomes a Banach $*$-algebra as follows

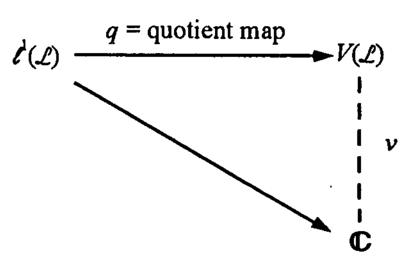
$$\left(\sum_{x \in \mathcal{L}} a_x x\right)^* = \sum_{x \in \mathcal{L}} \overline{a}_x x$$

Let I be the closed (self-adjoint) ideal in $\ell^1(\mathcal{L})$ generated by sums of the form $x \vee y + x \wedge y - x - y$. The Banach $*$-algebra $\ell^1(\mathcal{L})/I \stackrel{\text{def}}{=} V(\mathcal{L})$ is called the normed valuation algebra of $\mathcal{L}$. One should consult [8] for Rota's presentation of his construction.

Let v be a bounded valuation on $\mathcal{L}$. Then v extends to $\ell^1(\mathcal{L})$ by

$$v\left(\sum_{x \in \mathcal{L}} a_x x\right) \stackrel{\text{def}}{=} \sum a_x v(x)$$

Furthermore $v(I) = 0$ so there is a unique map (also called v) which makes the following diagram commute



Conversely, every continuous linear functional on $V(\mathcal{L})$ arises from a bounded valuation.

To prove Arveson's theorem, let $\mathcal{L}$ be a countably generated complete distributive lattice with a positive normal valuation v. Then by the proposition and direct calculation v defines a positive linear functional on $V(\mathcal{L})$. We now apply the GNS construction. The equation $(f,g) = v(f^*g)$ defines a preinner product on $V(\mathcal{L})$. If N is the ideal of all $f \in V(\mathcal{L})$ with $v(f^*f) = 0$ then $V(\mathcal{L})/N$ is a pre-Hilbert space under the inner product. Furthermore for $f \in V(\mathcal{L})$ we can define a bounded operator $\pi_V(f)$ acting on $V(\mathcal{L})/N$ by $\pi_V(f)[g] = [fg]$. The map π_V is a $*$-representation of $V(\mathcal{L})$ on $V(\mathcal{L})/N$ and hence on its completion H. It now follows that the "natural" map

$$\mathcal{L} \longrightarrow \ell^1(\mathcal{L}) \xrightarrow{q} V(\mathcal{L}) \xrightarrow{\pi_V} B(H)$$

is a faithful representation π of $\mathcal{L}$.

Indeed if $\pi(a) = \pi(b)$ then $\pi(a)[1] = \pi(b)[1]$ so $[a] = [b]$ and $[a-b] = 0$ in $V(\mathcal{L})/N$. Thus $(a-b, a-b) = 0$ so $v((a-b)^*(a-b)) = 0$. This implies $v(a)+v(b)-2v(a\wedge b) = 0$. Hence $v(a \wedge b) = v(a \vee b)$. Since v is positive it follows that $a \wedge b = a \vee b$ so $a = b$.

A similar calculation also shows that H is a separable space. The lattice $\mathcal{L}$ together with the metric topology is a separable metric space. See [1, P. 510] for a discussion of this. Hence there is a sequence $\{b_n\} \subseteq \mathcal{L}$ such that for any $b \in \mathcal{L}$ some subsequence $b_{n_i} \to b$. It follows that $([b_{n_i}] - [b], [b_{n_i}] - [b]) \to 0$ in $V(\mathcal{L})/N$. Indeed this inner product is

$$v(b_{n_i}) + v(b) - 2v(b \wedge b_{n_i}) \longrightarrow 0.$$

One checks that $\pi(a)^2 = \pi(a)$. Furthermore $(af, g) = v(afg^*) = v(f(ag)^*) = (f, ag)$ in $V(\mathcal{L})$ so that $\pi(a)$ is a projection. If $a, b \in \mathcal{L}$ then $\pi(a \wedge b) = \pi(a) \cdot \pi(b) = \pi(a) \wedge \pi(b)$. Also $[a \vee b] + [a \wedge b] - [a] - [b] = 0$ in $V(\mathcal{L})$. Thus $\pi(a \vee b) + \pi(a \wedge b) - \pi(a) - \pi(b) = 0$ or $\pi(a \vee b) = \pi(a) + \pi(b) - \pi(a)\pi(b) = \pi(a) \vee \pi(b)$. Therefore π is a lattice homomorphism.

It remains to show that $\pi(\mathcal{L})$ is a complete lattice. Suppose $\{\pi(a_i)\} \subseteq \pi(\mathcal{L})$. Then we can find a countable family $\{\pi(a_{n_i})\}$ (since H is separable) such that

$$\vee \pi(a_{n_i}) = \vee \pi(a_i).$$

Define $b_k = a_{n_1} \vee a_{n_2} \vee \cdots \vee a_{n_k}$. Then $\vee \pi(b_k) = \vee \pi(a_{n_i})$. We have that $b_k \uparrow b$ for some $b \in \mathcal{L}$. Therefore $v(b_k) \uparrow v(b)$. One checks that $\pi(b_k) \xrightarrow{\text{strongly}} \pi(b)$. Hence $\vee \pi(b_k) = \pi(b)$. Dually and we conclude that $\pi(\mathcal{L})$ is complete. That π preserves arbitrary meets and joins follows from the fact that an order preserving bijection between complete lattices must have an order-preserving inverse and thus preserve arbitrary meets and joins. ■

The converse of Arveson's theorem is well-known and straightforward.

CONCLUDING REMARKS. We expect the normed valuation algebra of $\mathcal{L}$ to be of further use in analyzing both $\mathcal{L}$ and Alg($\mathcal{L}$). To indicate the generality of our argument we indicate how to obtain a well known classical representation theorem. By working with the finite sublattices of a distributive lattice one can prove the existence of a separating family $\{v_i\}$ of non-negative valuations on $\mathcal{L}$. For each i one constructs a Hilbert space H_i and a representation π_i of $\mathcal{L}$ on H_i. The representation $\oplus\pi_i$ is faithful. Hence $\mathcal{L}$ is isomorphic to a lattice of commuting projections on $\oplus H_i$. If we apply the Gelfand-transform to the C^*-algebra generated by $\pi(\mathcal{L})$ we can represent $\mathcal{L}$ as a lattice of sets. Thus we obtain a function analytic proof of Stone's classical representation theorem for distributive lattices.

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