J. OPERATOR THEORY
28(1992), 229–250

INTERPOLATION AND EXTRAPOLATION OF WELL-BOUNDED OPERATORS

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1. INTRODUCTION

It has been shown that an operator may exhibit quite different spectral behaviour on the different L^p spaces (see for example [8]). One might ask therefore whether anything positive may be said about the behaviour of an operator on one L^p space knowing its behaviour on another. Questions of this type, concerning spectral interpolation and extrapolation (or spectral permanence), were examined for spectral operators by Oberai [15] (see also [9]) and Krabbe [13]. In this paper we shall give some similar results for well-bounded operators; more specifically

(i) if T defines a well-bounded operator on L^{p_1} and L^{p_2}, must T be well-bounded on L^p for $p_1 < p < p_2$, and,

(ii) if T is well-bounded on L^p for $p_1 < p < p_2$, what can we say about the behaviour of T acting on L^{p_1} and L^{p_2}?

As some of the metods allow alternative proofs of the earlier results on spectral operators (at least for those operators with real spectrum), some of this material is also included.

Well-bounded operators are those which possess a functional calculus for some compact interval of the real line. That is, an operator T acting on some Banach space is well-bounded if there exist real numbers $a < b$ and $K > 0$ such that

$$\|g(T)\| \leqslant K \left\{ |g(b) + \int_a^b \|g'(t)\| \mathrm{d}t \right\}$$

for all absolutely continuous functions on $[a, b]$ (or equivalently for all polynomials). These operators were introduced by Smart [18] to deal with problems of conditionally convergent spectral expansions. The spectral theory for well-bounded operators is now well developed and includes a spectral theory reminiscent of that for self-adjoint operators of Hilbert space. An important subclass of well-bounded operators are those for which the above functional calculus is weakly compact, the so-called well-bounded operators of *type (B)*. Operators of type (B) possess a type of spectral resolution known as a *spectral family* which allows a much simpler spectral theory as well as a larger functional calculus than that available for general well-bounded operators. Furthermore well-bounded operators of type (B) have found several important applications (see, for example, [3, 5]). It should be noted that well-bounded operators on reflexive Banach spaces are automatically of type (B).

Scalar-type spectral operators are those which may be represented as an integral with respect to a countably additive spectral measure. Such operators may also be characterised by the existence of a weakly compact functional calculus for the continuous functions on their spectrum. Consequently, scalar-type spectral operators possess a more powerful spectral theory than do well-bounded operators. The price one pays for this, of course, is the exclusion of many natural operators with conditionally convergent spectral expansions. Details of the theory of well-bounded and spectral operators may be found in [10] or [12].

The main aim of this paper is to give necessary and sufficient conditions to ensure the well-boundedness of operators that act on range of L^p spaces. Interpolation of the well-boundedness property is an easy consequence of the Riesz-Thorin Interpolation Theorem. The extrapolation results are more difficult and rely, as do the ones for spectral operators, on uniform bounds on the norms of the spectral resolutions. A corrolary of the extrapolation results is that if an operator is well-bounded on $L^1(\Omega, \mu)$ and on $L^p(\Omega, \mu)$ for some $p > 1$, and if $\mu(\Omega) < \infty$, then the operator must be of type (B) on $L^1(\Omega, \mu)$. An example will be given in the final section to show that the assumption that $\mu(\Omega) < \infty$ can not be removed.

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space and let $L^0(\Omega, \mathcal{A}, \mu)$ denote the vector space of all equivalence classes of measurable functions on Ω. For $1 \leqslant p \leqslant \infty$, $L^p(\Omega, \mathcal{A}, \mu)$, or simply L^p, will denote the usual Lebesgue space of p-integrable elements of $L^0(\Omega, \mathcal{A}, \mu)$. In the obvious way we shall say that a linear transformation on $L^0(\Omega, \mathcal{A}, \mu)$ defines a bounded operator on L^p if L^p is included in the domain of the transformation and the restriction of the transformation to L^p is bounded. If T is such a linear transformation, we shall use the notation T_p to denote the restricted operator in those situations where is important to specify the appropriate domain. Linear transformations will always be denoted by upper case letters and functions by lower case ones, so there should not be any confusion in the use of $\|T\|_p$ as the norm of an operator on L^p and $\|f\|_p$ as the norm of a function in L^p. If X is a Banach space, $L^p(\Omega, \mathcal{A}, \mu; X)$ will denote the Lebesgue-Bochner space of X-valued functions. The von Neumann-Schatten p-calsses of compact operators will be denoted by C_p. As usual $B(X)$ will denote the algebra of bounded linear operators on X. Throughout the paper the scalar field may be taken to be either the real or complex numbers, although care must be taken to use the appropriate definitions when dealing with spectral operators on real Banach spaces.

2. INTERPOLATION

The simplest results are easy consequences of the Riesz-Thorin Interpolation Theorem. Theorem 1 says that the properties of being scalar-type spectral and well--bounded interpolate. Note that the proof uses just the functional calculus properties of these operators and so is somewhat different to the proofs for spectral operators found in [15]. Recall that a *real scalar-type spectral operator* is a scalar-type spectral operator whose spectrum is contained in the real line. The following essentially well--known lemma shows that on most of the L^p spaces, one may prove that an operator T is scalar-type spectral by constructing a $C(\Delta)$ functional calculus for T for some compact $\Delta \subset \mathbb{C}$.

LEMMA 2.1. *Suppose that X is a Banach space which contains no isomorphic copy of c_0, and that $T \in B(X)$ possesses a $C(\Delta)$ functional calculus for some compact $\Delta \subset \mathbb{C}$. That is, there exists a bounded algebra homomorphism $\psi : \mathbb{C}(\Delta) \to B(X)$ such that if g is a polynomial, then $\psi(g) = g(T)$. Then T is scalar-type spectral.*

Proof. First we shall show that $\sigma(T) \subset \Delta$. Suppose that there exists $\lambda \in \sigma(T)\backslash\Delta$. Then $r_\lambda(z) = (z-\lambda)^{-1} \in C(\Delta)$ and so $\psi(r_\lambda) \in B(X)$. This and the algebra properties of ψ imply that $\psi(r_\lambda^{-1}) = T-\lambda$ is invertible, which is a contradiction. Thus $\sigma(T) \subset \Delta$.

Suppose then that $f \in C(\sigma(T))$. By the Tietze Extension Theorem [11, I.5.3] there exists a continuous extension, $\tilde{f}$, of f to Δ. We can now define $f(T) = \tilde{f}(T)$. To show that this functional calculus is in fact well-defined it suffices to show that if $f = 0$ on $\sigma(T)$, then $f(T) = 0$. However, if $f = 0$ on $\sigma(T)$, then we may write $f = f_1 + f_2$, where $|f_1| < \varepsilon$ on Δ and $f_2 = 0$ on a neighbourhood of $\sigma(T)$. By the usual holomorphic functional calculus, $f_2(T) = 0$, and we may make $\|f_1(T)\|$ as small as we like by letting $\varepsilon \to 0$. Thus $f(T) = 0$. Note that as it is always possible to choose an extension $\tilde{f}$ such that

$$\sup_{\lambda \in \Delta} |\tilde{f}(\lambda)| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|,$$

we have that

$$\|f(T)\| \leqslant M \sup_{\lambda \in \sigma(T)} |f(\lambda)|,$$

where M is the norm of the $C(\Delta)$ functional calculus for T. It follows then that T has a $C(\sigma(T))$ functional calculus on X. As X does not contain a copy of c_0, the proof of Theorem 2 of [7] implies that this functional calculus is weakly compact and so T is scalar-type spectral. ∎

THEOREM 2.2. *Suppose that $1 \leqslant p_1 < p < p_2 \leqslant \infty$ and that $(\Omega, \mathcal{A}, \mu)$ is a positive measure space. Suppose also that T defines a bounded linear operator on $L^{p_1}(\Omega, \mathcal{A}, \mu)$ and $L^{p_2}(\Omega, \mathcal{A}, \mu)$ (and hence on $L^p(\Omega, \mathcal{A}, \mu)$). Then*

(i) *if T is real scalar-type spectral on L^{p_1} and L^{p_2} then it is real scalar-type spectral on L^p;*

(ii) *if T is well-bounded on L^{p_1} and L^{p_2} then it is well-bounded (of type (B)) on L^p.*

Proof. (i) Since T_{p_1} is scalar-type spectral on L^{p_1} it has a $C(\sigma(T_{p_1}))$ functional calculus on this space. Similarly T_{p_2} possesses a $C(\sigma(T_{p_2}))$ functional calculus on L^{p_2}. It follows that T has a $C(\Delta)$ functional calculus on each of this spaces where $\Delta = \sigma(T_{p_1}) \cup \sigma(T_{p_2}) \subset \mathbb{R}$. Furthermore the density of the polynomials in $C(\Delta)$ ensures that these functional calculi agree on $L^{p_1} \cap L^{p_2}$. The Riesz-Thorin interpolation theorem then implies that T_p has a $C(\Delta)$ functional calculus on L^p. The result follows from Lemma 2.1 since L^p does not contain a copy of c_0 when $p < \infty$.

(ii) Essentially the same proof as for part (i) works. Suppose that T has an $AC[a_{p_1}, b_{p_1}]$ functional calculus acting as an operator on L^{p_1} and an $AC[a_{p_2}, b_{p_2}]$ functional calculus acting as an operator on L^{p_2}. Let $a = \min\{a_{p_1}, a_{p_2}\}$ and $b = \max\{b_{p_1}, b_{p_2}\}$. Then by Riesz-Thorin T has an $AC[a, b]$ functional calculus on L^p for all $p \in (p_1, p_2)$. This functional calculus must be weakly compact since L^p is reflexive.

REMARKS. (i) In general an operator may have different spectrum on different L^p spaces (see [2, 1, 17]). However this can not happen for spectral and well-bounded operators. For spectral operators this is shown in [15] (see also [13]), whilst for well-bounded operators this is an easy consequence of Lemma 3.3 below. Note that the proof of the Theorem 2.2(i) is rather easier if one assumes this fact; the point of the above proof is that one does not have to consider the spectral measure for the operator.

(ii) Clearly the same results hold for certain other interpolation spaces as well. For example they hold on the spaces C_p and the spaces $L^p(\Omega, \mathcal{A}, \mu; X)$, as long as say $\mu(\Omega) < \infty$ and X is reflexive (and so $L^p(\Omega, \mathcal{A}, \mu; X)$ is reflexive).

(iii) The difficulty in using these methods to prove interpolation results for scalar--type spectral operators whose spectrum does not lie in the real line is that it does not seem to be easy to show that the $C(\Delta)$ functional calculus for T on L^{p_1} agrees with that on L^{p_2}, without considering the spectral measure for T. If one knows in advance thate these functional calculi agree (for example if an explicit formula for the functional calculus is known), then clearly the same methods show that T_p is scalar-type spectral on L^p for all $p \in (p_1, p_2)$.

3. CONSISTENCY AND DUALITY

In what follows we shall often work with spectral resolution of a well-bounded operator. Of greatest importance will be the case when the well-bounded operator is of type (B), and in this case this spectral resolution is known as a spectral family (of projections).

DEFINITION 3.1. A *spectral family of projections* in a Banach space X is a projection valued function $E : \mathbb{R} \to B(X)$ such that

(i) E is right continuous in the strong operator topology and has a strong left hand limit at each point in $\mathbb{R}$;

(ii) E is uniformly bounded, that is there exists K such that $\|E(\lambda)\| \leqslant K$ for all $\lambda \in \mathbb{R}$;

(iii) E is naturally ordered, that is

$$E(\lambda)E(\mu) = E(\mu)E(\lambda) = E(\min\{\lambda, \mu\})$$

for all $\lambda, \mu \in \mathbb{R}$;

(iv) $E(\lambda) \to 0$ (respectively $E(\lambda) \to I$) in the strong operator topology as $\lambda \to -\infty$ (respectively $\lambda \to \infty$).

If $E(\lambda) = 0$ for all $\lambda < a \in \mathbb{R}$ and $E(\lambda) = I$ for all $\lambda \geqslant b \in \mathbb{R}$, then we say that E is *concentrated on* $[a, b]$.

Spectral families possess a Riemann-Stieltjes type integration theory. There is a one-to-one correspondence between well-bounded operators of type (B) and spectral families concentrated on some compact interval of the real line. If T is a well-bounded operator of type (B), then it is related to its spectral family E (concentrated on the compact interval J) via the formula

$$T = \int_J^{\oplus} \lambda \mathrm{d}E(\lambda).$$

This integration theory may be used to extend the functional calculus for T to $BV(J)$, the algebra of all functions of bounded variation on J, by the formula

$$g(T) = \int_J^{\oplus} g(\lambda) \mathrm{d}E(\lambda).$$

For well-bounded operators not of type (B) the situation is rather more complicated. In this case the spectral resolution is known as a *decomposition of the identity for* T. A decomposition of the identity for T is a naturally ordered family of projections $\{F(\lambda)\}_{\lambda \in \mathbb{R}}$ on X^* concentrated on $[a, b] \subset \mathbb{R}$, which satisfies certain natural properties and which is related to T by the formula

$$\langle Tx, x^* \rangle = b\langle x, x^* \rangle - \int_a^b \langle x, F(\lambda)x^* \rangle \mathrm{d}\lambda, \quad x \in X, \ x^* \in X^*.$$

Unfortunately, a well-bounded operator which is not of type (B) may possess many such decompositions of the identity. Full details of the spectral theory of well-bounded operators may be found in [10, Part 5].

Before we go further, it is important to show that if T is well-bounded of type (B) on two distinct L^p spaces, then the spectral families must agree, and conversely, that if $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ forms a spectral family on two L^p spaces, then it defines the same linear transformation on each space. These facts rest on the following well-known result.

LEMMA 3.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space and let $1 \leqslant p, \ r < \infty$. Suppose that $\{f_n\}$ is a sequence of functions in $L^p(\Omega, \mathcal{A}, \mu; X) \cap L^r(\Omega, \mathcal{A}, \mu; X)$ such that $f_n \to g_1$ in L^p and $f_n \to g_2$ in L^r. Then $g_1 = g_2$ almost everywhere with respect to μ.*

LEMMA 3.3. *Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$ and that $(\Omega, \mathcal{A}, \mu)$ is a positive measure space. Suppose also that T defines a well-bounded operator (of type* (B)*) on $L^p(\Omega, \mathcal{A}, \mu)$ for all $p \in (p_1, p_2)$ whith*

$$T = \int_{[a_p, b_p]}^{\oplus} \lambda \mathrm{d}E_p(\lambda).$$

If $p, \ r \in (p_1, p_2)$, then

$$E_p(\lambda)f = E_r(\lambda)f$$

(almost everywhere with respect to μ) for all $\lambda \in \mathbb{R}$ and all $f \in L^p \cap L^r$.

Proof. Without loss of generality we may assume that $a_p = a_r$ and that $b_p = b_r$. Let $J = [a_p, b_p]$ and suppose that $\lambda_0 \in J$. Define now the function $g = \chi_{[a_p,\lambda_0]}$ to be the characteristic function of the interval $[a_p, \lambda_0]$ and suppose that $f \in L^p \cap L^r$. Then

$$\int_J^{\oplus} g(\lambda)\mathrm{d}E_p(\lambda)f = E_p(\lambda_0)f$$

and

$$\int_J^{\oplus} g(\lambda)\mathrm{d}E_r(\lambda)f = E_r(\lambda_0)f.$$

Choose a uniformly bounded sequence g_n of polynomials in $AC(J)$ such that $g_n \to g$ pointwise on J. By [10, Proposition 17.5],

$$\int_J^{\oplus} g_n \mathrm{d}E_p \to \int_J^{\oplus} g\mathrm{d}E_p = E_p(\lambda_0)$$

in the strong operator topology. But $\int_J^{\oplus} g_n\mathrm{d}E_p f = g_n(T)f$, so $g_n(T)f \to E_p(\lambda_0)f$ in L^p norm.

Similarly,

$$g_n(T) = \int_J^{\oplus} g_n\mathrm{d}E_r f \to \int_J^{\oplus} g\mathrm{d}E_r f = E_r(\lambda_0)f$$

in L^r norm. The result now follows from Lemma 3.2. ∎

LEMMA 3.4. *Suppose that for each $\lambda \in \mathbb{R}$, $E(\lambda)$ is an idempotent linear transformation on $L^0(\Omega, \mathcal{A}, \mu)$. Assume that $1 \leqslant p < r < \infty$ and that $\{E(\lambda)\}$ defines spectral families $\{E_p(\lambda)\}$ on $L^p(\Omega, \mathcal{A}, \mu)$, and $\{E_r(\lambda)\}$ on $L^r(\Omega, \mathcal{A}, \mu)$, both concentrated on the compact set $J \subset \mathbb{R}$. Then for all $f \in L^p \cap L^r$,*

$$\int_J^{\oplus} \lambda\mathrm{d}E_p(\lambda)f = \int_J^{\oplus} \lambda\mathrm{d}E_r(\lambda)f \quad \text{(a.e.)}.$$

Proof. Let $\mathcal{P}$ denote the collection of partitions of J, partially ordered by inclusion. Define

$$T_p = \int_J^{\oplus} \lambda\mathrm{d}E_p(\lambda)$$

and similarly for T_r. Fix $f \in L^p \cap L^r$. For $\Lambda = \{a = \lambda_0, \lambda_1, \cdots, \lambda_n = b\} \in \mathcal{P}$, let

$$f_\Lambda = aE(a)f + \sum_{j=1}^{n} \lambda_j(E(\lambda_j) - E(\lambda_{j-1}))f.$$

We have that

$$T_p f = \lim_{\Lambda \in \mathcal{P}} f_\Lambda$$

in L^p norms whereas

$$T_r f = \lim_{\Lambda \in \mathcal{P}} f_\Lambda$$

in L^r norm. Choose $\Lambda_1 \in \mathcal{P}$ so that $\|f_{\Lambda_1} - T_p f\|_p \leqslant 1$. Now choose $\Lambda_2 \geqslant \Lambda_1$ so that $\|f_{\Lambda_2} - T_r f\|_r \leqslant 1/2$. The fact that $\Lambda_2 \geqslant \Lambda_1$ ensures that $\|f_{\Lambda_2} - T_p f\|_p \leqslant 1$. Continuing in this manner we can choose succesive refinements Λ_n so that

$$\|f_{\Lambda_n} - T_p f\|_p \to 0$$

and

$$\|f_{\Lambda_n} - T_r f\|_p \to 0.$$

It follows from Lemma 3.2 that $T_p f = T_r f$ (a.e.). ■

For $1 \leqslant p \leqslant \infty$, let p' denote the conjugate index to p, $1/p + 1/p' = 1$. It is trivial to show that if T is well-bounded on L^p, for some $1 \leqslant p < \infty$, then T^* is well-bounded on $L^{p'}$. One would expect that if T and T^* were both of type (B) that the natural relationship between their spectral families would hold and indeed this is the case. The proof of this requires some care however as the adjoint map is not continuous in the strong operator topology.

THEOREM 3.5. *Suppose that $1 < p < \infty$ and that T defines a well-bounded operator of type* (B) *on $L^p(\Omega, \mathcal{A}, \mu)$ with*

$$T = \int_J^{\oplus} \lambda \mathrm{d}E(\lambda).$$

Then T^* is well-bounded of type (B) *on $L^{p'}(\Omega, \mathcal{A}, \mu)$ and*

$$T^* = \int_J^{\oplus} \lambda \mathrm{d}(E(\lambda)^*).$$

Proof. Showing that T^* is well-bounded is easy, and since $L^{p'}(\Omega, \mathcal{A}, \mu)$ is reflexive, T^* must be of type (B). Let $\{F(\lambda)\}_{\lambda \in \mathbb{R}}$ denote the spectral family associated with T^*. Our aim then is to show that $F(\lambda) = E(\lambda)^*$ for all $\lambda \in \mathbb{R}$.

For $\lambda \in \mathbb{R}$ and $\delta > 0$, let $g_{\lambda,\delta}$ denote the absolutely continuous function which is 1 on $(-\infty, \lambda]$, 0 on $[\lambda + \delta, \infty)$ and linear on $[\lambda, \lambda + \delta]$. The proof of the spectral theorem for well-bounded operators (see [10, p. 348]) shows that

$$E(\lambda) = \text{wot-}\lim_{\delta \to 0} g_{\lambda,\delta}(T)$$

and

$$F(\lambda) = \text{wot-}\lim_{\delta\to 0} g_{\lambda,\delta}(T^*).$$

(Here wot means that the limits are taken in the weak operator topology). Fix functions $f \in L^p(\Omega, \mathcal{A}, \mu)$ and $\varphi \in L^{p'}(\Omega, \mathcal{A}, \mu)$. Then

$$\langle E(\lambda)f, \varphi\rangle = \lim_{\delta\to 0}\langle g_{\lambda,\delta}(T)f, \varphi\rangle = \lim_{\delta\to 0}\langle f, g_{\lambda,\delta}(T)^*\varphi\rangle =$$

$$= \lim_{\delta\to 0}\langle f, g_{\lambda,\delta}(T^*)\varphi\rangle = \langle f, F(\lambda)\varphi\rangle.$$

Thus $F(\lambda) = E(\lambda)^*$. ■

Example 5.4 later will show that it is possible for T to be well-bounded of type (B) on L^1 without T^* being of type (B) on L^∞. On the other hand Ricker [16] has shown that all the well-bounded operators of type (B) on L^∞ are *finite* combinations of projections. This allows an easy proof of the following result which we shall leave for the reader.

PROPOSITION 3.6. *Suppose that T is a bounded operator on $L^1(\Omega, \mathcal{A}, \mu)$ and that T^* is a well-bounded operator of type* (B) *on $L^\infty(\Omega, \mathcal{A}, \mu)$. Then T is well--bounded of type* (B) *on $L^1(\Omega, \mathcal{A}, \mu)$.*

COROLLARY 3.7. *Suppose that $1 < p < \infty$ and that $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ forms a spectral family on $L^p(\Omega, \mathcal{A}, \mu)$. Then $\{E(\lambda)^*\}_{\lambda\in\mathbb{R}}$ forms a spectral family on $L^{p'}(\Omega, \mathcal{A}, \mu)$.*

Proof. Suppose first that $\{E(\lambda)\}$ is concentrated on some compact interval of the real line. Then $\{E(\lambda)\}$ defines a well-bounded operator of type (B), say T, on L^p. By Theorem 3.5, $\{E(\lambda)^*\}$ is the spectral family associated with T^* acting on $L^{p'}$.

If $\{E(\lambda)\}$ is not concentrated on a compact interval, the required result may be obtained by considering truncated spectral families

$$E_N(\lambda) = \begin{cases} 0, & \text{for } \lambda < -N, \\ E(\lambda), & \text{for } \lambda \in [-N, N), \\ I, & \text{for } \lambda \geqslant N, \end{cases}$$

for larger and larger N. ■

We end this section with a consistency result that shows that if E is a linear transformation which defines a bounded operator on a range of L^p spaces, then the adjoints of the restriction of E to each of these spaces agree on the appropriate subsets of the dual spaces. In other words, we may unambiguously write E^*, without having to specify which space we are considering E to be acting on.

LEMMA 3.8. *Suppose that $1 \leqslant p$, $r < \infty$ and that E is a linear transformation which defines a bounded operator on $L^p(\Omega, \mathcal{A}, \mu)$ and $L^r(\Omega, \mathcal{A}, \mu)$. Then for all*

$f \in L^{p'}(\Omega, \mathcal{A}, \mu) \cap L^{r'}(\Omega, \mathcal{A}, \mu)$,

$$(E_p)^* f = (E_r)^* f$$

almost everywhere with respect to μ.

Proof. Let $M = L^p \cap L^r$ and let $M' = L^{p'} \cap L^{r'}$. Since M is dense in L^p and L^r, if $g \in M'$, then $E^* g$ is uniquely determined by the requirement that

$$\langle Ef, g \rangle = \langle f, E^* g \rangle$$

for all $f \in M$. ■

The results of this section carry through immediately to the spaces $L^p(\Omega, \mathcal{A}, \mu; X)$ for those Banach spaces X which make the corresponding Lebesgue-Bochner spaces reflexive for $1 < p < \infty$. For other types of interpolation space such as the spaces C_p, one has to prove an analogue of Lemma 3.2. For the C_p spaces this is easy and we leave it for the reader.

4. EXTRAPOLATION

We now turn to the more difficult question as to whether it is possible to deduce spectral information about an operator on L^{p_1} or L^{p_2} given information about the operator on L^p for $p \in (p_1, p_2)$. Lemma 4.1 is an easy consequence of the continuity of L^p norms with respect to p.

LEMMA 4.1. *Suppose that* $1 \leqslant p_1 < p_2 \leqslant \infty$ *and that* E *is a linear transformation which defines a bounded operator of norm at most* K *on* $L^p(\Omega, \mathcal{A}, \mu)$ *for all* $p \in$ $\in (p_1, p_2)$. *Then the domain of* E *can be extended uniquely so that* E *defines a bounded linear operator of norm at most* K *on* $L^{p_1}(\Omega, \mathcal{A}, \mu)$.

REMARK. This extension of course need not be proper. For example $\ell^1 = L^1(\mathbb{N})$ is strictly contained in ℓ^p for $p > 1$.

Proof. Let g be a non-zero simple integrable function. Then g lies in L^p for all p. Also since $\|g\|_p$ is a continuous function of p (see [11, VI.11.30]), $\|g\|_{p_1} =$ $= \lim_{p \to p_1^+} \|g\|_p$. Thus

$$\frac{\|Eg\|_{p_1}}{\|g\|_{p_1}} = \frac{\lim_{p \to p_1^+} \|Eg\|_p}{\lim_{p \to p_1^+} \|g\|_p} \leqslant \frac{\lim_{p \to p_1^+} K\|g\|_p}{\lim_{p \to p_1^+} \|g\|_p} = K.$$

It follows that E can be extended uniquely to an operator of norm at most K to all of L^{p_1}. ∎

Theorems 4.2 and 4.3 say that to extrapolate the property of being scalar-type spectral or of being well-bounded to L^{p_1}, one needs a uniform bound on the norms of the functional calculi. Theorem 4.2 is closely related to a result of Oberai [15, Theorem 2]. It should be noted however that Oberai incorrectly claims an extrapolation result to L^∞.

THEOREM 4.2. *Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$. Suppose also that T defines a real scalar-type spectral operator on $L^p(\Omega, \mathcal{A}, \mu)$ for all $p \in (p_1, p_2)$ and that T is bounded on L^{p_1}. Then a necessary and sufficient condition for T to define a real scalar-type spectral operator on L^{p_1} is that there exists $r \in (p_1, p_2)$ and $K < \infty$ such that*

$$\|g(T)\|_p \leqslant K \sup_{\lambda \in \sigma(T_p)} |g(\lambda)|$$

for all $g \in C(\sigma(T_p))$ and for all $p \in (p_1, r)$.

Proof. Sufficiency: It is well-known (see, for example [2, Proof of Theorem 3]) that if T is bounded on L^{p_1} and L^r, then the spectral radius of T_p is uniformly bounded for $p \in [p_1, r)$. Thus, for a suitably large closed interval $\Delta \subset \mathbb{R}$,

$$\sigma(T_p) \subset \Delta$$

for all $p \in (p_1, r)$. Thus T has a $C(\Delta)$ functional calculus on L^p for $p_1 < p < r$ of norm less than or equal to K. Again the fact that the polynomials are dense in $C(\Delta)$ ensures that for $g \in C(\Delta)$, the definition of $g(T)$ agrees on all the L^p spaces $(p_1 < p < r)$. By Lemma 4.1 then, we can extend $g(T)$ to an operator on L^{p_1} of norm less than or equal to $K \sup_{\lambda \in \Delta} |g(T)|$. Thus T has a $C(\Delta)$ functional calculus on L^{p_1} and, since $c_0 \not\subset L^{p_1}$, Lemma 2.1 implies that T is real scalar-type spectral.

Necessity: Suppose that T is real scalar-type spectral on L^p for all $p \in [p_1, p_2)$. Choose $r \in (p_1, p_2)$. Then as above we can choose $\Delta \subset \mathbb{R}$ such that $\sigma(T_p) \subset \Delta$ for all $p \in (p_1, r)$. Since T is scalar-type spectral on L^{p_1} and on L^r it must have a $C(\Delta)$ functional calculus on each of these spaces and hence, by the Riesz-Thorin Interpolation Theorem, on each of the intermediate spaces, with uniform bound (say K) on the norms of the functional calculi. Since $\sigma(T_p) \subset \Delta$ for each $p \in (p_1, r)$, the norm of the $C(\sigma(T_p))$ functional calculus must be bounded by K as well. ∎

In practice these same tehniques can often be used to handle scalar-type spectral operators whose spectrum is not contained in the real line. This is because one often has an explicit description of the $C(\sigma(T_p))$ functional calculi, and so the potential problems of deciding whether these agree is avoided. See Remark (iii) after Theorem 2.2.

THEOREM 4.3. *Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$. Suppose also that T defines a well-bounded operator (of type* (B)*) on $L^p(\Omega, \mathcal{A}, \mu)$ for all $p \in (p_1, p_2)$ with*

$$T = \int_{[a_p, b_p]}^{\oplus} \lambda \mathrm{d}E_p(\lambda).$$

A necessary and sufficient condition for T to define a well-bounded operator on L^{p_1} is that there exists $r \in (p_1, p_2)$ such that

$$M = \sup_{p_1 < p < r} \left\{ \sup_{\lambda \in \mathbb{R}} \|E_p(\lambda)\|_p \right\} < \infty.$$

REMARK. The condition that M be finite is equivalent to the condition that there is a uniform bound on the norms of the AC (or equivalently BV) functional calculi for T on each of the L^p spaces with $p_1 < p < r$. One consequence of Lemma 3.3 is that the spectral families for T on each of the L^p spaces are all concentrated on the same interval $[a, b] \subset \mathbb{R}$. Let $\psi_p : BV[a, b] \to B(L^p)$ denote the algebra homomorphism which maps g to $g(T)$, and let K_p denote the norm of this homomorphism when $BV[a, b]$ is equipped with the norm

$$|||g|||_{BV} = |g(b)| + \operatorname*{var}_{[a,b]} g.$$

That $K_p \leqslant M$ is a consequence of [4, Proposition 2.6] which shows that

$$\|g(T)\|_p \leqslant |||g|||_{BV} \sup_{\lambda \in \mathbb{R}} \|E_p(\lambda)\|.$$

The reverse inequality follows easily since $\psi_p(\chi_{[a,b]}) = E(\lambda)$.

Proof of Theorem 4.3. Sufficiency: Let g be a polynomial. Then, by the remark, $\|g(T)\|_p \leqslant M|||g|||_{BV}$. By Lemma 4.1 then, $\|g(T)\|_{p_1} \leqslant M|||g|||_{BV}$, and so T must be well-bounded on L^{p_1}.

Necessity: This follows easily from Riesz-Thorin. If

$$\|g(T)\|_{p_1} \leqslant K|||g|||_{BV}$$

and for some $r > p_1$

$$\|g(T)\|_r \leqslant L|||g|||_{BV}$$

then, by the Riesz-Thorin Interpolation Theorem,

$$\|g(T)\|_p \leqslant \max\{K, L\}\|\|g\|\|_{BV}$$

for all $p \in (p_1, r)$. Thus if T is well-bounded on L^{p_1} and L^r, then we have a uniform bound (say M) on the norm of its $AC[a,b]$ functional calculus on every L^p space with $p \in (p_1, r)$ and so for all such p, $\sup_{\lambda \in \mathbb{R}} \|E(\lambda)\|_p \leqslant M$. ■

If $p_1 > 1$ then L^{p_1} is reflexive, so that we may deduce that if T satisfies the conditions of Theorem 4.3, then it must be of type (B) on this space. If the underlying measure space is finite (a hypothesis that has *not* been required thus far) then the same conclusion holds for $p = 1$. As we shall see in the final section, this hypothesis is essential.

PROPOSITION 4.4. *Suppose that $1 \leqslant p$, $r < \infty$ and that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Suppose also that $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ is a family of linear transformations on $L^0(\Omega, \mathcal{A}, \mu)$ such that*

(i) *$\{E_p(\lambda)\}_{\lambda \in \mathbb{R}}$ forms a spectral family on $L^p(\Omega, \mathcal{A}, \mu)$, and*

(ii) *for each $\lambda \in \mathbb{R}$, $E_r(\lambda)$ is bounded on $L^r(\Omega, \mathcal{A}, \mu)$.*

Then a necessary and sufficient condition for $\{E_r(\lambda)\}_{\lambda \in \mathbb{R}}$ to be a spectral family on $L^r(\Omega, \mathcal{A}, \mu)$ is that there exists a constant M such that for all $\lambda \in \mathbb{R}$, $\|E_r(\lambda)\|_r \leqslant M$.

REMARK. It is clear that if $\{E_p(\lambda)\}_{\lambda \in \mathbb{R}}$ is concentrated on some interval, then $\{E_r(\lambda)\}_{\lambda \in \mathbb{R}}$ is concentrated on the same interval.

Proof. Necessity is obvious.

Suppose first then that $r \leqslant p$. Let $M = L^P \subset L^r$. For $f \in M$, the fact that $\{E_p(\lambda)\}$ forms a spectral family implies that

$$E(\mu)E(\lambda)f = E(\min\{\mu, \lambda\})f.$$

The density of M in L^r ensures that

$$E_r(\mu)E_r(\lambda) = E_r(\min\{\mu, \lambda\}).$$

Thus $\{E_r(\lambda)\}$ forms a naturally ordered family of projections on L^r.

It remains to show that $\{E_r(\lambda)\}$ is strong operator continuous on the right and has a strong operator left hand limit at each point in $\mathbb{R}$.

Suppose that $u \in L^r$. Fix $\varepsilon > 0$. Since L^p is dense in L^r we can choose $v \in L^p$ such that $\|u - v\|_r < \varepsilon$. Then (and it is here that is important that $\mu(\Omega) < \infty$),

$$\|(E(\lambda+\delta) - E(\lambda))u\|_r \leqslant \|(E(\lambda+\delta) - E(\lambda))v\|_r + \|(E(\lambda+\delta) - E(\lambda))(u - v)\|_r \leqslant$$

$$\leqslant \mu(\Omega)^{1/r-1/p}\|(E(\lambda+\delta)-E(\lambda))v\|_p+$$
$$+(\|E(\lambda+\delta)\|_r+\|E(\lambda)\|_r)\|u-v\|_r<\varepsilon+2M\varepsilon.$$

for $\delta > 0$ sufficiently close to zero, since $\{E(\lambda)\}$ is a spectral family on L^p. Thus $\{E_r(\lambda)\}$ is continuous on the right at λ in the strong operator topology. Showing that a strong left limit exists is similar.

If $p < r$, then $1 < r < \infty$. As above $\{E(\lambda)\}$ forms a naturally ordered family of projections on L^r. Since L^r is reflexive, a theorem of Lorch [14] implies that $\{E_r(\lambda)\}$ has strong operator topology limits on both the left and the right at each point of $\mathbb{R}$. We need to show that for all $\lambda \in \mathbb{R}$, the strong operator limit of $E_r(\lambda+\delta)$ as $\delta \to 0^+$ is actually $\{E_r(\lambda)\}$. Let $E \in B(L^r)$ denote this limit. Then for all $u \in L^r$,

$$\|(E(\lambda+\delta)-E)u\|_p \leqslant \mu(\Omega)^{1/p-1/r}\|(E(\lambda+\delta)-E)u\|_r \to 0$$

as $\delta \to 0^+$. As L^r is dense in L^p, this implies that $\{E(\lambda+\delta)\}$ has strong operator limit E as $\delta \to 0^+$. But since $\{E_r(\lambda)\}$ is a spectral family on L^p, E must equal $E_p(\lambda)$ on L^p and so result is proved. ■

THEOREM 4.5. *Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$ and that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Suppose also that T defines a well-bounded operator (of type* (B)*) on $L^p(\Omega, \mathcal{A}, \mu)$ for all $p \in (p_1, p_2)$ with*

$$T = \int_{[a_p,b_p]}^{\oplus} \lambda \mathrm{d}E_p(\lambda).$$

A necessary and sufficient condition for T to define a well-bounded operator on L^{p_1} is that there exists $r \in (p_1, p_2)$ such that

$$M = \sup_{p_1<p<r}\left\{\sup_{\lambda\in\mathbb{R}}\|E_p(\lambda)\|_p\right\} < \infty.$$

If this condition is satisfied, then T is well-bounded of type (B) *on L^{p_1}.*

Proof. We shall first prove sufficiency. As we have shown above, we may write unambiguously that

$$T = \int_{[a,b]}^{\oplus} \lambda \mathrm{d}E(\lambda)$$

on L^p for $p \in (p_1, p_2)$. We shall show that this also holds on L^{p_1}. By Lemma 4.1, $E(\lambda)$ defines a bounded operator on L^{p_1} for all $\lambda \in \mathbb{R}$. The hypothesis gives a uniform bound on the norms of these operators, so by Proposition 4.4, $\{E(\lambda)\}$ forms a spectral family on L^{p_1}.

Let $S = \int_J^{\oplus} \lambda \mathrm{d}E(\lambda)$ on L^{p_1}. It is not hard to see that S is a well-bounded operator which extends T_p on L^p. That this extension actually equals T on L^{p_1} follows since T_{p_1} is bounded and L^p is dense in L^{p_1}. That T_{p_1} is bounded is a consequence of Lemmas 3.4 and 4.1. Thus we have shown that T defines a well-bounded operator of type (B) on L^{p_1}.

Necessity is proved as in Theorem 4.3. ■

COROLLARY 4.6. *Suppose that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Suppose also that T defines a well-bounded operator on $L^1(\Omega, \mathcal{A}, \mu)$ and on $L^r(\Omega, \mathcal{A}, \mu)$ for some $r > 1$. Then T must be of type* (B) *on $L^1(\Omega, \mathcal{A}, \mu)$.*

Proof. Choose $a \leqslant b$ such that T has an $AC[a,b]$ functional calculus on L^1 and on L^r. Thus there exist $M_1,\ M_2 < \infty$ such that for all $g \in AC[a,b]$

$$\|g(T)\|_l \leqslant M_1 |||g|||_{BV}$$

$$\|g(T)\|_r \leqslant M_2 |||g|||_{BV}$$

Clearly then, if $M = \max\{M_1, M_2\}$, then

$$\|g(T)\|_p \leqslant M |||g|||_{BV}$$

for all $p \in (1, r)$. By Theorem 4.5 (and the remark after Theorem 4.3), T is well--bounded of type (B) on L^1. ■

In many situations it will be easier to apply this result by using the following special case, which is a consequence of the fact that all self-adjoint operators on a Hilbert space are well-bounded.

COROLLARY 4.7. *Suppose that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Suppose also that T defines a well-bounded operator on $L^1(\Omega, \mathcal{A}, \mu)$ and that T is self-adjoint on $L^2(\Omega, \mathcal{A}, \mu)$. Then T is of type* (B) *on $L^1(\Omega, \mathcal{A}, \mu)$.*

So far we have concentrated on the "left hand" endpoint of the interval (p_1, p_2). Similar results hold for extrapolation to the right hand endpoint, as long as $p_2 \neq \infty$. In the next section we shall give an example of a linear transformation which is scalar-type spectral, with uniform bound on the norms of its functional calculi on L^p for $p \in [1, \infty)$, but which is not scalar-type spectral on L^∞. For the property of well-boundedness however, we can get some information, even when $p_2 = \infty$.

COROLLARY 4.8. *Suppose that $1 \leqslant p_1 < p_2 \leqslant \infty$. Suppose also that T defines a well-bounded operator on $L^p(\Omega, \mathcal{A}, \mu)$ for all $p \in (p_1, p_2)$ with*

$$T = \int_{[a_p, b_p]}^{\oplus} \lambda \mathrm{d}E(\lambda).$$

A necessary and sufficient condition for T to define an well-bounded operator on L^{p_2} is that there exists $r \in (p_1, p_2)$ such that

$$M = \sup_{r<p<p_2} \left\{ \sup_{\lambda \in \mathbb{R}} \|E(\lambda)\|_p \right\} < \infty.$$

Proof. This is a standard duality argument. If $1 \leqslant p \leqslant \infty$, then we shall denote by p' its conjugate index, $1/p + 1/p' = 1$. Under the hypothesis, T^* is a well-bounded operator on L^p for all $p \in (p_2', p_1')$ with

$$T^* = \int_{[a,b]}^{\oplus} \lambda \mathrm{d}E(\lambda)^*.$$

By Theorem 4.3, T^* is well-bounded on $L^{p_2'}$ if and only if there exists $r' \in (p_2', p_1')$ such that

$$\sup_{p_2'<p<r'} \left\{ \sup_{\lambda \in \mathbb{R}} \|E(\lambda)^*\|_p \right\} = \sup_{r<p<p_2} \left\{ \sup_{\lambda \in \mathbb{R}} \|E(\lambda)\|_p \right\} < \infty.$$

Since T^* is well-bounded on $L^{p_2'}$ if and only if T is well-bounded on L^{p_2}, the result follows. ∎

Note that, as the examples show, there is no analogous result to Theorem 4.5 by which one may deduce that an operator is well-bounded of type (B) on L^∞.

5. SOME EXAMPLES

The first example shows that it may not be possible to extrapolate the property of being well-bounded of type (B) even if T is uniformly bounded on all the L^p spaces, and even if we assume that T has the stronger property of being scalar-type spectral on L^p for $p > 1$.

EXAMPLE 5.1. Define the projection E_n on $\ell^p(n)$ by

$$E_n(x_1, \ldots, x_n) = \left(x_1, \frac{x_1}{2}, \ldots, \frac{x_1}{n}\right).$$

It is easily seen that $\|E_n\|_p = \left(\sum_{k=1}^{n} \frac{1}{k^p}\right)^{\frac{1}{p}} = K_p$, say. Clearly $K_p < \infty$ for $p > 1$. Let $\lambda_n = 1/n$. By considering ℓ^p to be equal to $\bigoplus_{n=1}^{\infty} \ell^p(n)$, define the operator T by $T = \bigoplus_{n=1}^{\infty} \lambda_n E_n$. A simple calculation shows that $\|T\|_p = 1$ for $1 \leqslant p \leqslant \infty$ and that $\sigma(T) = \{0, 1, 1/2, 1/3, \ldots\}$ on each of these ℓ^p spaces.

Suppose that g is a polynomial. Then

$$g(T) = \bigoplus_{n=1}^{\infty} g(\lambda_n)E_n + g(0)\left(I - \bigoplus_{n=1}^{\infty} E_n\right)$$

and so

$$\|g(T)\|_p \leqslant \sup_n |g(\lambda_n) - g(0)|K_p + |g(0)| \leqslant (2K_p + 1) \sup_{\lambda\in\sigma(T)} |g(\lambda)|.$$

Thus, if $p > 1$, T has a continuous $C(\sigma(T))$ functional calculus, and so T is scalar-type spectral on ℓ^p for $1 < p < \infty$ and is well-bounded on ℓ^∞.

For $m = 1, 2, \ldots$, define the functions g_m as follows:

$$g_m(t) = \begin{cases} 1, & \text{if } t \geqslant 1/m; \\ mt, & \text{if } t \in [0, 1/m]; \\ 0, & \text{if } t \leqslant 0. \end{cases}$$

It is clear that g_m lies in $C(\sigma(T))$ and $AC[0,1]$ with $\|g_m\|_\infty = |||g_m|||_{BV} = 1$ (where have taken the left hand evaluation point in evaluating $|||\cdot|||_{BV}$). If T were to have a functional calculus for either of these algebras of functions then we would have to have that

$$g_m(T) = \left(\bigoplus_{n=1}^{m} E_n\right) \oplus \left(\bigoplus_{n=m+1}^{\infty} \frac{m}{n} E_n\right).$$

Thus

$$\|g_m(T)\|_1 \geqslant \sup_{1\leqslant n\leqslant m} \|E_n\|_1 = \sum_{k=1}^{m} \frac{1}{k}.$$

Since this last sum is unbounded as $m \to \infty$, it follows that T cannot have a bounded $C(\sigma(T))$ or $AC[0,1]$ functional calculus acting on ℓ^1 and so T is neither scalar-type spectral nor well-bounded on this space. ∎

The second example is of a transformation defined on $L^p(1,\infty)$ for all $p \geqslant 1$. This transformation defines a well-bounded operator of type (B) whose spectral family satisfies uniform bounds for $p > 1$ but which is not of type (B) when $p = 1$, thus proving that Theorem 4.5 can not be extended to cover arbitrary non-finite measure space.

EXAMPLE 5.2. Define projections $E(\lambda)$ for $\lambda \in (0,1)$, acting on measurable functions on $(1,\infty)$ by

$$(E(\lambda)f)(t) = \begin{cases} \dfrac{\lambda}{1-\lambda} \displaystyle\int_1^{1/\lambda} f(u)\,du, & t \leqslant 1/\lambda; \\ f(t), & t > 1/\lambda. \end{cases}$$

It is easy to see that each $E(\lambda)$ is in fact a conditional expectation operator, and so $\|E(\lambda)\|_p = 1$ for $1 \leqslant p \leqslant \infty$. For $\lambda \leqslant 0$, define $E(\lambda) = 0$ and for $\lambda \geqslant 1$, define $E(\lambda) = I$.

We shall show now that for $1 < p < \infty$, $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ forms a spectral family of projections concentrated on $[0,1]$, acting on the space $L^p(1,\infty)$. The main difficulty lies in showing that $E(\cdot)$ is right continuous in the strong operator topology and has a strong left hand limit at each point in $\mathbb{R}$. Actually, we shall show that $E(\cdot)$ is everywhere strongly continuous. Suppose then that $f \in L^p(1,\infty)$ and that $0 < \mu < < \lambda < 1$. Then

$$(E(\lambda)f - E(\mu)f)(t) = \begin{cases} \dfrac{\lambda}{1-\lambda}\displaystyle\int_1^{1/\lambda} f(u)\mathrm{d}u - \dfrac{\mu}{1-\mu}\displaystyle\int_1^{1/\mu} f(u)\mathrm{d}u, & t \leqslant 1/\lambda\ ; \\ f(t) - \dfrac{\mu}{1-\mu}\displaystyle\int_1^{1/\mu} f(u)\mathrm{d}u, & 1/\lambda < t \leqslant 1/\mu\ ; \\ 0, & t > 1/\mu. \end{cases}$$

Note that the function $\lambda \mapsto \dfrac{\lambda}{1-\lambda}\displaystyle\int_1^{1/\lambda} f(u)\mathrm{d}u$ is continuous on $(0,1)$. Now

$$\|E(\lambda)f - E(\mu)f\|_p \leqslant \left\{\int_1^{1/\lambda}\left|\frac{\lambda}{1-\lambda}\int_1^{1/\lambda} f(u)\mathrm{d}u - \frac{\mu}{1-\mu}\int_1^{1/\mu} f(u)\mathrm{d}u\right|^p \mathrm{d}v\right\}^{1/p} +$$

$$+\left\{\int_{1/\lambda}^{1/\mu}\left|f(v) - \frac{\mu}{1-\mu}\int_1^{1/\mu} f(u)\mathrm{d}u\right|^p \mathrm{d}v\right\}^{1/p} \leqslant$$

$$\leqslant \left(\frac{1-\lambda}{\lambda}\right)^{1/p}\left|\frac{\lambda}{1-\lambda}\int_1^{1/\lambda} f(u)\mathrm{d}u - \frac{\mu}{1-\mu}\int_1^{1/\mu} f(u)\mathrm{d}u\right| +$$

$$+\left\{\int_{1/\lambda}^{1/\mu}\left|f(v) - \frac{\mu}{1-\mu}\int_1^{1/\mu} f(u)\mathrm{d}u\right|^p \mathrm{d}v\right\}^{1/p} = \mathrm{I} + \mathrm{II},$$

say. Term I clearly vanishes as $|\lambda - \mu| \to 0$, whereas

$$\text{Term II} \leqslant \left\{\int_{1/\lambda}^{1/\mu} |f(v)|^p \mathrm{d}v\right\}^{1/p} + \frac{\mu}{1-\mu}\left(\frac{\lambda-\mu}{\lambda\mu}\right)^{1/p}\int_1^{1/\mu} |f(u)|\mathrm{d}u \to 0,$$

as $|\lambda - \mu| \to 0$. Thus $E(\cdot)$ is strongly continuous on $(0,1)$.

Clearly $E(\cdot)$ is strongly continuous on $(-\infty,0)$ and $(1,\infty)$, so it remains to check the behaviour at 0 and 1. This will require a lemma whose proof we omit.

LEMMA 5.3. *Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a uniformly bounded net of operators on a Banach space X. Suppose that $\mathcal{F}$ is a dense subset of X such that for all $y \in X$, $S_\lambda y$ converges. Then $\{S_\lambda\}_{\lambda\in\Lambda}$ converges in the strong operator topology on $B(X)$.*

Again let $1 < p < \infty$, and let $\lambda \in (0,1)$. For $\alpha > 1$, let

$$\mathcal{F}^\alpha = \{f \in L^p(1,\infty) : f \text{ is constant (a.e.) on } (1,\alpha)\}$$

and define $\mathcal{F} = \bigcup\limits_{\alpha>1} \mathcal{F}^\alpha$. Clearly $\mathcal{F}$ is dense in $L^p(1,\infty)$. Suppose that $g \in \mathcal{F}^\alpha$. Then, for $\lambda > 1/\alpha$,

$$\|E(1)g - E(\lambda)g\|_p = \left\{ \int_1^{1/\lambda} \left| g(v) - \frac{\lambda}{1-\lambda} \int_1^{1/\lambda} g(u)\mathrm{d}u \right|^p \mathrm{d}v \right\}^{1/p} = 0$$

since g is constant on $(1, 1/\lambda)$. By Lemma 5.3 then, $E(\lambda)$ converges to $E(1) = I$ in the strong operator topology as $\lambda \to 1^-$.

Define now

$$\mathcal{F}_\alpha = \{f \in L^p(1,\infty) : f(t) = 0 \text{ (a.e.) on } (\alpha,\infty)\}.$$

Again $\mathcal{F} = \bigcup\limits_{\alpha>1} \mathcal{F}_\alpha$ is dense in $L^p(1,\infty)$. For $g \in \mathcal{F}_\alpha$, and $\lambda < 1/\alpha$,

$$\|E(\lambda)g - E(0)g\|_p = \left\{ \int_1^{1/\lambda} \left| \frac{\lambda}{1-\lambda} \int_1^{1/\lambda} g(u)\mathrm{d}u \right|^p \mathrm{d}v \right\} \leqslant$$

$$\leqslant \left(\frac{\lambda}{1-\lambda}\right)^{(p-1)/p} \left| \int_1^{1/\lambda} g(u)\mathrm{d}u \right|^p \to 0 \quad \text{as } \lambda \to 0^+.$$

Using Lemma 5.3 again shows that $E(\cdot)$ is strongly right continuous at 0. Note that it is important here that $1 < p < \infty$. We shall omit the straightforward proof that $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ is naturally ordered.

It follows then that $\{E(\lambda)\}$ defines a well-bounded operator of type (B) on $L^p(1,\infty)$ for $1 < p < \infty$. Let the operator T be defined by

$$(Tf)(t) = \frac{1}{t} f(t) - \int_0^{1/t} \frac{\lambda}{1-\lambda} \int_1^{1/\lambda} f(u)\mathrm{d}u\mathrm{d}\lambda, \quad t \in (1,\infty).$$

Let $1 < p < \infty$ and as usual let p' denote the conjugate index to p. For $f \in L^p(1,\infty)$ and $\varphi \in L^{p'}(1,\infty)$ one can expand the right hand side of the following expression to show that

$$\langle Tf, \varphi\rangle = \langle f, \varphi\rangle - \int_0^1 \langle E(\lambda)f, \varphi\rangle \mathrm{d}\lambda$$

so that T is the well-bounded operator associated to $\{E(\lambda)\}$. Note that for $1 < p < \infty$ the norms of the projections $E(\lambda)$ are uniformly bounded by 1 on $L^p(1,\infty)$. By Theorem 4.3 then, T is well-bounded on $L^1(1,\infty)$. We shall show however, that T is not of type (B) on this space. Actually, [6, Theorem 4.1] (see also [8, Lemma 1]) implies that T is scalar-type spectral on $L^p(1,\infty)$ for $1 < p < \infty$.

By Lemma 3.3, if T were of type (B), then its spectral family would be given by $\{E(\lambda)\}$. Let

$$f(t) = \frac{1}{t^2}, \quad t \in (1,\infty).$$

It is easily checked that $f \in L^p(1,\infty)$ for all $p \geqslant 1$. Since f is positive, $\|E(\lambda)f\|_1 = 1$ for all $\lambda \in (0,1)$. Thus $E(\lambda)$ does not converge to $E(0)f = 0$ (in L^1 norm) as $\lambda \to 0^+$, and so $\{E(\lambda)\}$ can not be a spectral family on $L^1(1,\infty)$. It follows then that T is not of type (B) on $L^1(1,\infty)$.

It is easy to create analogous examples on spaces such as ℓ^p or C_p. For $n \geqslant 1$ define the linear transformations P_n, acting on sequences, by

$$P_n(x_1, x_2, \ldots,) = \Bigg(\underbrace{\frac{1}{n}\sum_{j=1}^{n} x_j, \ldots, \frac{1}{n}\sum_{j=1}^{n} x_j}_{n \text{ times}}, x_{n+1}, x_{n+2}, \ldots\Bigg).$$

Acting on ℓ^p with $1 < p < \infty$, $P_n \to 0$ in the strong operator topology as $n \to \infty$, so it is possible to construct a spectral family $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ by setting

$$E(\lambda) = \begin{cases} 0, & \text{for } \lambda \leqslant 0; \\ P_n, & \text{for } \lambda \in [1/(n+1), 1/n); \\ I, & \text{for } \lambda \geqslant 1. \end{cases}$$

Furthermore, $\|P_n\|_p \leqslant 1$ for all p. Note that $\{E(\lambda)\}$ does not form a spectral family on ℓ^1, so the same procedures may be used as in Example 5.2 to construct an operator which is well-bounded of type (B) on ℓ^p with uniformly bounded spectral family for $p > 1$, but which is not of type (B) on ℓ^1.

For the spaces C_p, one can construct projections Q_n on C_p by setting $Q_n A = = P_n A P_n$. Again one may construct a spectral family from these operators on C_p for $1 < p < \infty$, but not on C_1.

The final example shows that due to the paucity of well-bounded operators of type (B) on L^∞, there is no analogous result to Theorem 4.5 for extrapolation of this property to L^∞.

EXAMPLE 5.4. Define the linear transformation T on $L^0[0,1]$ by $(Tf)(t) = tf(t)$, $t \in [0,1]$. A simple calculation shows that T is well-bounded on $L^p[0,1]$ for $1 \leqslant p \leqslant \infty$ (and scalar-type spectral for $1 \leqslant p < \infty$). For $1 \leqslant p < \infty$ the spectral family for T is given by

$$E(\lambda)f = \begin{cases} 0, & \text{for } \lambda \leqslant 0; \\ \chi_{[0,\lambda]}f, & \text{for } \lambda \in (0,1); \\ I, & \text{for } \lambda \geqslant 1. \end{cases}$$

Clearly $\|E(\lambda)\|_p \leqslant 1$ for all p. However it is easily checked that $\{E(\lambda)\}$ does not form a spectral family on $L^\infty[0,1]$ so T can not be of type (B) on this space. Similarly, T is not scalar-type spectral on $L^\infty[0,1]$, even though there is a uniform bound on the norm of its $C[0,1]$ functional calculus on $L^p[0,1]$ for all p.

Acknowledgements. The author would like to express his thanks to Ernst Albrecht for his helpful comments on an earlier version of this paper.

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Received September 20, 1990; revised January 28, 1991.