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ON ISOMETRIES BETWEEN NON-COMMUTATIVE L^p-SPACES ASSOCIATED WITH ARBITRARY VON NEUMANN ALGEBRAS

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1. INTRODUCTION

It is interesting in non-commutative integration theory to consider the following problem; if there exists a linear isometry between non-commutative L^p-spaces associated with von Neumann algebras for $1 < p < \infty$ and $p \neq 2$, then what informations can we obtain about those von Neumann algebras and the structure of the isometry? This problem was originated by Banach [2], and Lamperti [18] completed the commutative cases. For semifinite von Neumann algebras, Broise [4], Russo [20], Katavolos [13], [14], [15], Tam [24] etc. had developed the corresponding theory, and there is a complete description of a general isometry for the case of *semifinite* von Neumann algebras in Yeadon [28]. On the other hand, after the development of the modular theory, Haagerup [10] (see also [25]), Araki-Masuda [1], Hilsum [11], Kosaki [17], Terp [26] etc. constructed non-commutative L^p-spaces associated with von Neumann algebras which are not necessarily semifinite. It is well-known that for a fixed von Neumann algebra those L^p-spaces are isometrically isomorphic each other.

In this article we consider for Haagerup's L^p-spaces the problem mentioned above. A difficulty to deal with Haagerup's L^p-spaces comes from the following facts. The intersection of two L^p-spaces with distinct values of p contains no non-zero element, particularly there is no non-zero bounded operator in the L^p-spaces. Each element in the L^p-space is not affiliated with the original von Neumann algebra in general. So it does not seem easy to obtain a common area between the L^p-spaces and the original von Neumann algebra, and it seems that many techniques used in semifinite cases are not effective. We decompose the problem to two parts as follows. At first, if there exists a linear isometry between non-commutative L^p-spaces, then are those von Neumann algebras Jordan $*$-isomorphic? Section 2 consists of some preliminaries. In Section 3, when the isometry is $*$-preserving, we will give an affirmative answer, using a well-known Dye's theorem. The result implies that non-commutative L^p-spaces associated with arbitrary von Neumann algebras do form a properly larger class of Banach spaces than those associated with semifinite von Neumann algebras. Secondly, can we describe the structure of the isometry in terms of the induced Jordan $*$-isomorphism? In Section 4, when the isometry is positive, we will give a necessary and sufficient condition for it to be the restriction of a Jordan $*$-isomorphism in the form of preserving an operator inequality, which implies a reduction to the answer.

2. PRELIMINARIES

First we recall some basic notions related to τ-measurable operators arising from semifinite von Neumann algebras. Full details are found in [19] and [25].

Let τ be a faithful normal semifinite trace on a semifinite von Neumann algebra N acting on a Hilbert space K. A densely defined closed operator a affiliated with N is said to be τ-measurable if there is, for each $\varepsilon > 0$, a projection e in N such that $eK \subset \mathcal{D}(a)$ and $\tau(1-e) \leqslant \varepsilon$. We denote by $\tilde{N}$ the set of all τ-measurable operators affiliated with N, which becomes a complete Hausdorff topological $*$-algebra in the measure topology. When it is necessary to indicate the trace under which we consider the measurability, we denote by $\tilde{N}^{\tau}$. For $\varepsilon, \delta > 0$, we set

$$N(\varepsilon, \delta) = \{a \in \tilde{N};\ \text{there exists a projection } e \text{ in } N \text{ with } \|ae\| \leqslant \varepsilon \text{ and } \tau(1-e) \leqslant \delta\}.$$

Then the family $\{N(\varepsilon, \delta);\ \varepsilon, \delta > 0\}$ gives a fundamental system of neighborhoods around 0 with respect to the measure topology. For each subset S of $\tilde{N}$, the set of all selfadjoint (resp. positive selfadjoint) operators in S is denoted by S_{sa} (resp. S_+). For $0 < p < \infty$, the non-commutative L^p-space arising from a semifinite von Neumann algebra is defined as follows:

$$L^p(N, \tau) = \{a \in \tilde{N};\ \tau(|a|^p) < \infty\} \quad \text{(see [5], [19], [21], [27])}.$$

Next, we collect some basic facts and definitions concerning (Haagerup's) non-commutative L^p-spaces associated with von Neumann algebras which are not necessarily semifinite. For details and proofs we refer to [10] and [25]. Let φ_0 be a fixed faithful normal semifinite weight on M acting on a Hilbert space H. Let $\{\sigma_t^{\varphi_0}\}_{t\in\mathbb{R}}$ be the modular automorphism group with respect to φ_0. We denote by N the crossed product $R(M, \sigma^{\varphi_0})$ which is the von Neumann algebra generated by $\pi(x), x \in M$ and $\lambda_s, s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t), \ \ \xi \in L^2(\mathbb{R}, H), \ t \in \mathbb{R},$$
$$(\lambda_s\xi)(t) = \xi(t-s), \qquad \xi \in L^2(\mathbb{R}, H), \ t \in \mathbb{R}.$$

The dual actions $\theta_s, s \in \mathbb{R}$, naturally extend to automorphisms on $\hat{N}_+$ which is the extended positive part of N ([8; Section 1]). Recall that the formula

$$\Phi x = \pi^{-1}\left(\int_{\mathbb{R}} \theta_s(x)\,\mathrm{d}s\right), x \in N_+,$$

defines a normal faithful semifinite operator valued weight Φ from N to M (cf. [9; Lemma 5.2]). For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ \Phi,$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ (cf. [8; Proposition 1.10]). Then $\tilde{\varphi}$ is a normal weight on N which is called the dual weight of φ. It is well-known that there exists a unique faithful normal semifinite trace τ on N characterized by the Connes' cocycle $(D\tilde{\varphi}_0 \colon D\tau)_t = \lambda_t, t \in \mathbb{R}$, and τ satisfies $\tau \circ \theta_s = \mathrm{e}^{-s}\tau, s \in \mathbb{R}$ (for the existence of τ, see [9; Lemma 5.2]). The dual actions $\theta_s, s \in \mathbb{R}$, are extended to continuous $*$-automorphisms of $\tilde{N}$. For $0 < p < \infty$, the Haagerup L^p-space $L^p(M) = L^p(M; \varphi_0)$ is defined by

$$L^p(M) = \left\{a \in \tilde{N}; \theta_s(a) = \mathrm{e}^{-s/p}a, \quad s \in \mathbb{R}\right\}.$$

For each normal weight φ on M, we simply denote by $h_\varphi = \dfrac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}$ the Radon-Nikodym derivative of $\tilde{\varphi}$ with respect to τ. It is well-known that $\varphi \in M_{*,+}$ if and only if h_φ is τ-measurable, and the mapping $\varphi \longrightarrow h_\varphi$ is extended to a linear order isomorphism from M_* onto $L^1(M)$, and so the positive linear functional tr on $L^1(M)$ is defined by $\mathrm{tr}(h_\varphi) = \varphi(1)$, $\varphi \in M_*$. For $0 < p < \infty$, the (quasi-)norm of $L^p(M)$ is defined by $\|a\|_p = \mathrm{tr}(|a|^p)^{1/p}$, $a \in L^p(M)$. When $1 \leqslant p < \infty, L^p(M)$ is a Banach space, and its dual Banach space is $L^q(M)$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ by the following duality:

$$\langle a, b\rangle = \mathrm{tr}(ab) \ (= \mathrm{tr}(ba)), \quad a \in L^p(M), b \in L^q(M).$$

One should note that for each $a = u|a| \in L^p(M)$ with its polar decomposition, u belongs to M and $|a|$ belongs to $L^p(M)_+$. Also for each $a = a_+ - a_- \in L^p(M)_{sa}$ with its Jordan decomposition, one has a_+ and a_- belong to $L^p(M)_+$.

Now we recall what happens if we change φ_0 to another faithful normal semifinite weight φ_1. We denote by $N(\varphi_1)$ the crossed product $R(M,\sigma^{\varphi_1})$, which is generated by $\pi_{\varphi_1}(x), x\in M$, and $\lambda_s, s\in\mathbb{R}$, where

$$(\pi_{\varphi_1}(x)\xi)(t)=\sigma^{\varphi_1}_{-t}(x)\xi(t),\ \xi\in L^2(\mathbb{R},H),\ t\in\mathbb{R}.$$

Define a unitary u on $L^2(\mathbb{R},H)$ by

$$(u\xi)(t)=(D\varphi_1:D\varphi_0)_{-t}\xi(t),\quad \xi\in L^2(\mathbb{R},H),\ t\in\mathbb{R}.$$

Let Φ_{φ_1} be the operator valued weight defined on $N(\varphi_1)$ as the preceding arguments. For each $\varphi\in M_{*,+}$, we denote by $\tilde{\varphi}^{\varphi_1}$ the dual weight on $N(\varphi_1)$ associated with Φ_{φ_1}. We also denote by τ_{φ_1} the canonical faithful normal semifinite trace on $N(\varphi_1)$.

LEMMA 2.1. ([25; Theorem 37, Corollary 38]). *Put* $\mathcal{K}(a)=uau^*, a\in N$. *Then* $\mathcal{K}$ *is an isomorphism from* N *onto* $N(\varphi_1)$ *which satisfies the following conditions*

$\pi_{\varphi_1}=\mathcal{K}\circ\pi,$

$\mathcal{K}\circ\theta_s=\theta_s\circ\mathcal{K}, s\in\mathbb{R},$

$\Phi_{\varphi_1}=\Phi\circ\mathcal{K}^{-1},$

$\tilde{\varphi}^{\varphi_1}=\tilde{\varphi}\circ\mathcal{K}^{-1}$, *for each normal weight* φ *on* M *and*

$\tau_{\varphi_1}=\tau\circ\mathcal{K}^{-1}.$

Moreover, the mapping $\mathcal{K}\colon N\longrightarrow N(\varphi_1)$ *extends to a topological* $*$*-isomorphism* $\tilde{\mathcal{K}}\colon\tilde{N}^{\tau}\longrightarrow N(\varphi_1)^{\sim\tau_{\varphi_1}}$.

Although the following lemma is well-known, we shall give it here for the convenience of readers.

LEMMA 2.2. *Let* tr_{φ_1} *be the linear functional on* $L^1(M;\varphi_1)$ *defined by* $\mathrm{tr}_{\varphi_1}\left(\frac{\mathrm{d}\tilde{\varphi}^{\varphi_1}}{\mathrm{d}\tau_{\varphi_1}}\right)=\varphi(1)$ *as in the above survey. Then we have*

$\mathrm{tr}_{\varphi_1}=\mathrm{tr}\circ\tilde{\mathcal{K}}^{-1}.$

Moreover the restriction of $\tilde{\mathcal{K}}$ *is a positive linear isometry from* $L^p(M;\varphi_0)$ *onto* $L^p(M;\varphi_1)$.

Proof. For each $\varphi\in M_{*,+}$, we have

$$\tau_{\varphi_1}\left(\tilde{\mathcal{K}}\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right)\cdot\right)=\tau_{\varphi_1}\circ\tilde{\mathcal{K}}\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\tilde{\mathcal{K}}^{-1}(\cdot)\right)=\tau\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\tilde{\mathcal{K}}^{-1}(\cdot)\right)=\tilde{\varphi}\circ\tilde{\mathcal{K}}^{-1}=\tilde{\varphi}^{\varphi_1},$$

which implies that $\tilde{\mathcal{K}}\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right)=\frac{\mathrm{d}\tilde{\varphi}^{\varphi_1}}{\mathrm{d}\tau_{\varphi_1}}$. It follows that

$$\mathrm{tr}\circ\tilde{\mathcal{K}}^{-1}\left(\frac{\mathrm{d}\tilde{\varphi}^{\varphi_1}}{\mathrm{d}\tau_{\varphi_1}}\right)=\mathrm{tr}\left(\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}\tau}\right)=\varphi(1)=\mathrm{tr}_{\varphi_1}\left(\frac{\mathrm{d}\tilde{\varphi}^{\varphi_1}}{\mathrm{d}\tau_{\varphi_1}}\right).$$

Hence $\mathrm{tr} \circ \tilde{\mathcal{K}}^{-1} = \mathrm{tr}_{\varphi_1}$. Secondly, it is obvious by the previous lemma that $\tilde{\mathcal{K}}$ maps $L^p(M;\varphi_0)$ onto $L^p(M;\varphi_1)$. For each $a \in L^p(M;\varphi_0)$, let $a = u|a|$ the polar decomposition of a. Since $\tilde{\mathcal{K}}$ is a $*$-isomorphism, we have $|\tilde{\mathcal{K}}(a)| = \tilde{\mathcal{K}}(|a|)$ and

$$\|\tilde{\mathcal{K}}(a)\|_p^p = \mathrm{tr}_{\varphi_1}(|\tilde{\mathcal{K}}(a)|^p) = \mathrm{tr}_{\varphi_1}(\tilde{\mathcal{K}}(|a|^p)) = \mathrm{tr}(|a|^p) = \|a\|_p^p.$$

This completes the proof. ∎

It is also well-known the following fact. Let M be semifinite with a faithful normal semifinite trace τ, and construct the Haagerup L^p-space $L^p(M;\varphi_0)$ taking τ as φ_0. Then $L^p(M;\varphi_0)$ is isometrically isomorphic to $L^p(M;\tau)$ defined by Dixmier or by Segal.

3. A CONSTRUCTION OF JORDAN $*$-ISOMORPHISM

In this section, we shall show that a $*$-preserving isometry between non-commutative L^p-spaces associated with two von Neumann algebras gives rise to a Jordan $*$-isomorphism between those von Neumann algebras.

LEMMA 3.1. *Let $0 < p < \infty$. Let M be a σ-finite von Neumann algebra. Then for any two equivalent projections e and f in M, we can take an element a in $L^p(M)$ such that the right support of a is e and left support of a is f.*

Proof. Take a partial isometry u in M such that $e = u^*u$ and $f = uu^*$. Since M is σ-finite, there is a faithful normal state φ_0 on M. Then $u\varphi_0u^*$ is a positive normal functional on M, where $u\varphi_0u^*(x) = \varphi_0(u^*xu)$, as usual. Moreover uu^* is the support projection of $u\varphi_0u^*$. In fact, at first we have

$$(u\varphi_0u^*)(1 - uu^*) = \varphi_0(u^*(1 - uu^*)u) = \varphi_0(u^*u - u^*uu^*u) = 0.$$

Secondly, for any projection q in M satisfying $(u\varphi_0u^*)(1-q) = 0$, we have $\varphi_0(u^*(1-q)u) = 0$ and $u^*(1-q)u = 0$. Hence $uu^* \leqslant q$. Thus $u^*(u\varphi_0u^*)$ is the form of the polar decomposition. We define an element a in $L^p(M)$ by $a = u^*(uh_{\varphi_0}u^*)^{1/p}$. It follows that this is also the form of the polar decomposition of a. Thus we can conclude that $r(a) = uu^* = f$ and $1 - n(a) = u^*u = e$, where we denote by $r(a)$ (resp. $n(a)$) the range projection (resp. the null projection) of a. This completes the proof. ∎

COROLLARY 3.2. *Let $0 < p < \infty$. Let M be a σ-finite von Neumann algebra. Then for any projection e, there is an element a in $L^p(M)_+$ such that the support projection $s(a)$ is e.*

LEMMA 3.3. *Let M_1 and M_2 be von Neumann algebras. Let T be a linear isometry from $L^p(M_1)$ onto $L^p(M_2)$.*

(1) *If T is $*$-preserving, then T^{-1} is also $*$-preserving. Therefore T maps $L^p(M_1)_{sa}$ onto $L^p(M_2)_{sa}$.*

(2) *If T is positivity preserving, then T^{-1} is also positivity preserving. Therefore T maps $L^p(M_1)_+$ onto $L^p(M_2)_+$.*

Proof. (1) Let y be an element in $L^p(M_2)_{sa}$. Since T is surjective, there is an element x in $L^p(M_1)$ such that $y = T(x)$. Then we have $T(x^*) = T(x)^* = y^* = y = T(x)$ due to the $*$-preserving property of T. Since T is injective, we have $x = x^*$. (2) Let y be an element in $L^p(M_2)_+$. From the previous argument, we can take an element x in $L^p(M_1)$ such that $T(x) = y$ and $x^* = x$. Let $x = x_+ - x_-$ be the Jordan decomposition of x. Then we have $T(x) = T(x_+) - T(x_-)$, $T(x_+) \geqslant 0$ and $T(x_-) \geqslant 0$ due to the positivity. Moreover, since T is isometric and since $x_+ x_- = 0$, we have the following equation

$$||T(x_+) + T(x_-)||_p^p = ||x_+ + x_-||_p^p = ||x_+||_p^p + ||x_-||_p^p = ||T(x_+)||_p^p + ||T(x_-)||_p^p.$$

Then we can conclude from [16; Corollary 6.5] that $T(x_+)T(x_-) = 0$. Thus $0 \leqslant y = T(x) = T(x_+) - T(x_-)$ is the Jordan decomposition. Hence we have $T(x_-) = 0$ and $x_- = 0$. This completes the proof. ■

Now we shall recall the notion of projection orthoisomorphisms between C^*-algebras, which is introduced by Dye.

DEFINITION 3.4. (cf.[6; p.75, Definition]). For a von Neumann algebra M, we denote by $\mathcal{P}(M)$ the set of all projections in M. A projection orthoisomorphism between von Neumann algebras M_1 and M_2 is a map $\theta: \mathcal{P}(M_1) \longrightarrow \mathcal{P}(M_2)$ which is one to one, onto, and such that $ef = 0$ if and only if $\theta(e)\theta(f) = 0$ for $e, f \in \mathcal{P}(M_1)$.

PROPOSITION 3.5. *Let $1 < p < \infty$ and $p \neq 2$. Let M_1 and M_2 be σ-finite von Neumann algebras. Let T be a $*$-preserving linear isometry from $L^p(M_1)$ onto $L^p(M_2)$. Then there exists an orthoisomorphism between M_1 and M_2.*

Proof. Supppose that $s(a) \leqslant s(b)$, $a, b \in L^p(M_1)_{sa}$. There exists from Corollary 3.2 an element x in $L^p(M_2)_{sa}$ such that $s(x) = 1 - s(T(b))$. Putting $c = T^{-1}(x)$, c belongs to $L^p(M_1)_{sa}$ by Lemma 3.3. Then we have $T(c)T(b) = 0$. Now appealing to the equality condition of the Clarkson's inequality (cf.[16; Theorem 6.6]), we have

$$||T(c) + T(b)||_p^p + ||T(c) - T(b)||_p^p = 2(||T(c)||_p^p + ||T(b)||_p^p).$$

Since T is isometric, it follows that

$$||c + b||_p^p + ||c - b||_p^p = 2(||c||_p^p + ||b||_p^p).$$

Therefore we have $cb = 0$ again by the equality condition. It follows that $s(c) \leqslant 1 - s(b)$. Multiplying $s(a)$, we have $s(a)s(c) = 0$. Hence $ac = 0$. Using the equality condition of the Clarkson's inequality quite similarly as above, we have $T(a)T(c) = 0$. Therefore we have $s(T(a))s(T(c)) = 0$, which implies that $s(T(a)) \leqslant 1 - s(T(c)) = s(T(b))$. Thus we can define a map J from $\mathcal{P}(M_1)$ to $\mathcal{P}(M_2)$ as follows:

$$J(s(a)) = s(T(a)), \ a \in L^p(M_1)_{sa}.$$

By Corollary 3.2, one has $\{s(a) : a \in L^p(M_1)_{sa}\} = \mathcal{P}(M_1)$. It follows that J is everywhere defined on $\mathcal{P}(M_1)$ and that J is surjective. Considering the inverse of T, we can conclude that J is one to one.

Suppose that e and f are orthogonal projections in M_1. There are elements a and b in $L^p(M_1)_{sa}$ such that $s(a) = e$ and $s(b) = f$. Then we have the orthogonality of $J(e)$ and $J(f)$ due to the equality condition of the Clarkson's inequality. Thus J is a projection orthoisomorphism between M_1 and M_2. This completes the proof. ■

THEOREM 3.6. *Let $1 < p < \infty$ and $p \neq 2$. Let M_1 and M_2 be σ-finite von Neumann algebras. Let T be a $*$-preserving linear isometry from $L^p(M_1)$ onto $L^p(M_2)$. Then there exists a Jordan $*$-isomorphism from M_1 onto M_2.*

Proof. When M_1 has no direct summands of type I_2, the result follows from the previous proposition and Dye's theorem ([6; p.83, Corollary]). In general case, take the central projection z in M_1 such that $M_1 z$ is continuous and $M_1(1-z)$ is discrete. Then we have a direct sum decomposition $L^p(M_1) = L^p(M_1 z) \oplus L^p(M_1(1-z))$. Put $z' = J(z)$. Since a projection orthoisomorphism preserves commutativity of two projections, z' commutes any $J(e), e \in \mathcal{P}(M_1)$. Thus z' is a central projection in M_2. Then we have $T(L^p(M_1)z) = L^p(M_2)z'$ and $T(L^p(M_1)(1-z)) = L^p(M_2)(1-z')$. In fact, if a is an element in $L^p(M_1)_+ z$, then we have $s(a) \leqslant z$. Therefore $s(T(a)) = J(s(a)) \leqslant J(z) = z'$. Hence we have $T(a) \in L^p(M_2)_+ z'$. The converse inclusion follows similarly.

At first, the restriction of J to $M_1 z$ is a Jordan $*$-isomorphism onto $M_2 z'$ due to Dye's theorem. We denote by T_1 the restriction of T^{-1} to $L^p(M_2(1-z'))$. Then T_1 induces a projection orthoisomorphism which is the restriction of J^{-1} to $M_2(1-z')$. We take the central projection $z'' \leqslant 1 - z'$ and decompose $M_2(1-z')$ to $M_2 z'' \oplus M_2(1 - z' - z'')$ where $M_2 z''$ is continuous and $M_2(1 - z' - z'')$ is discrete. Assume that $z'' \neq 0$. Then, by the Dye's theorem, the restriction of J^{-1} to $M_2 z''$ is a Jordan $*$-isomorphism onto $M_1 J^{-1}(z'')$, which is included in $M_1(1-z)$. However, since $M_1(1-z)$ is discrete, this is impossible. Hence $z'' = 0$ and $M_2(1-z')$ is a type I von Neumann algebra. Thus T_1^{-1} is a linear isometry between two type I von Neumann algebras. Therefore it induces a Jordan $*$-isomorphism ρ from $M_1(1-z)$

onto $M_2(1-z')$ by a well-known result (see [28; Theorem 2], for instance). The direct sum of the restriction of J to $M_1 z$ and ρ gives a desired Jordan $*$-isomorphism M_1 onto M_2. This completes the proof. ∎

The following corollary says that the class of non-commutative L^p-spaces associated with arbitrary von Neumann algebras defined by Haagerup (therefore by Hilsum, by Araki-Masuda, by Kosaki, by Terp) is actually properly wider than the class of non-commutative L^p-spaces associated with semifinite von Neumann algebras defined by Dixmier or by Segal.

COROLLARY 3.7. *Let $1 < p < \infty$ and $p \neq 2$. Let M_1 and M_2 be σ-finite von Neumann algebras. If M_1 is type III and if M_2 is semifinite with a faithful normal semifinite trace τ, then there exists no $*$-preserving surjective linear isometry between $L^p(M_1)$ and $L^p(M_2, \tau)$.*

We have another corollary for two type III factors.

COROLLARY 3.8. *Let $1 < p < \infty$ and $p \neq 2$. Let $0 \leqslant \lambda, \mu \leqslant 1$ and $\lambda \neq \mu$. Let M_1 (resp. M_2) be σ-finite type III_λ (resp. III_μ) factor . Then there exists no $*$-preserving surjective linear isometry between $L^p(M_1)$ and $L^p(M_2)$.*

4. ON THE STRUCTURE OF ISOMETRIES

In this section we will consider the structure of surjective positive linear isometries between non-commutative L^p-spaces. This problem is to describe an isometry in terms of Jordan $*$-isomorphism which is induced as in the previous section.

At first we assume the existence of a Jordan $*$-isomorphism and we shall canonically construct a surjective positive linear isometry between non-commutative L^p--spaces. Let M_1 and M_2 be σ-finite von Neumann algebras and let J be a Jordan $*$-isomorphism from M_1 onto M_2. We denote by z the unique central projection in M_1 such that J is a $*$-isomorphism on $M_1 z$ and a $*$-antiisomorphism on $M_1(1-z)$. Fix a faithful normal state φ_0 on M_1. It follows from the uniqueness of the modular automorphism group that

$$\sigma^{\varphi_0 \circ J^{-1}} = J \circ \sigma^{\varphi_0} \circ J^{-1}.$$

Thus the two W^*-dynamical system $(M_1, \mathbb{R}, \sigma^{\varphi_0})$ and $(M_2, \mathbb{R}, \sigma^{\varphi_0 \circ J^{-1}})$ are covariantly Jordan $*$-isomorphic. We denote by N_1 (resp. N_2) the crossed product of M_1 (resp. M_2) and $\mathbb{R}$ under the action of σ^{φ_0} (resp. $\sigma^{\varphi_0 \circ J^{-1}}$). Let π_i be the natural embedding of M_i into N_i and let θ^i be the dual actions on N_i, for $i = 1, 2$. Then we have a

Jordan $*$-isomorphism $\tilde{J}$ from N_1 onto N_2 which satisfies

$$\tilde{J} \circ \pi_1 = \pi_2 \circ J \quad \text{and} \quad \tilde{J} \circ \theta^1 = \theta^2 \circ \tilde{J}.$$

Moreover, $\pi_1(z)$ is the unique central projection in N_1 such that $\tilde{J}$ is a $*$-isomorphism on $N_1\pi_1(z)$ and a $*$-antiisomorphism on $N_1(1-\pi_1(z))$. Let τ_i be the unique faithful normal semifinite trace on N_i such that $\tau_i \circ \theta^i_s = \mathrm{e}^{-s}\tau_i$ for all $s \in \mathbb{R}$ and for $i = 1, 2$. We shall show that $\tau_1 \circ \tilde{J} = \tau_2$. Let Φ_i be the canonical operator valued weight from N_i to M_i, $i = 1, 2$. For each normal positive functional φ on M_i, we denote by $\tilde{\varphi}^i$ the dual weight of φ on N_i, $i = 1, 2$. We see that

$$J^{-1} \circ \Phi_2(x) = J^{-1}\left(\pi_2^{-1}\left(\int \theta^2_s(x)\,\mathrm{d}s\right)\right) = J^{-1}\left(\pi_2^{-1}\left(\int \tilde{J} \circ \theta^1_s \circ \tilde{J}^{-1}(x)\,\mathrm{d}s\right)\right) =$$

$$J^{-1} \circ \pi_2^{-1} \circ \tilde{J}\left(\int \theta^1_s\left(\tilde{J}^{-1}(x)\right)\,\mathrm{d}s\right) = \pi_1^{-1} \circ \tilde{J}\left(\int \theta^1_s\left(\tilde{J}^{-1}(x)\right)\,\mathrm{d}s\right) = \Phi_1 \circ \tilde{J}^{-1}(x)$$

for any x in the suitable domain. Thus $\Phi_2 \circ \tilde{J} = J \circ \Phi_1$. Therefore we have

$$(\varphi \circ J^{-1})^{\tilde{}2} = (\varphi \circ J^{-1})\hat{} \circ \Phi_2 = (\varphi \circ J^{-1})\hat{} \circ J \circ \Phi_1 \circ \tilde{J}^{-1} = \hat{\varphi} \circ \Phi_1 \circ \tilde{J}^{-1} = \tilde{\varphi}^1 \circ \tilde{J}^{-1},$$

for each element φ in $(M_1)_{*,+}$. Using the above fact, we compute the Connes' cocycle

$$\left(D\left(\varphi_0 \circ J^{-1}\right)^{\tilde{}2} : D\left(\tau_1 \circ \tilde{J}^{-1}\right)\right)_t = \left(D\left(\tilde{\varphi}^1_0 \circ \tilde{J}^{-1}\right) : D\left(\tau_1 \circ \tilde{J}^{-1}\right)\right)_t =$$

$$= \tilde{J}\left(\left(D\tilde{\varphi}^1_0 : D\tau_1\right)_t\right) = \tilde{J}\left(\lambda^1_t\right) = \lambda^2_t.$$

From the uniqueness of such trace (cf. [9; Lemma 5.2]), we can conclude that $\tau_1 \circ \tilde{J}^{-1} = \tau_2$. Let us recall that $x \in N(\varepsilon,\delta)_1$ if and only if $\tau_1(E_{(\varepsilon,\infty)}(|x|)) \leqslant \delta$ for any element in $\tilde{N}_1$, where we denote by $E_{(\varepsilon,\infty)}(|x|)$ the spectral projection of $|x|$ corresponding to the interval (ε, ∞). Since

$$\tau_2\left(E_{(\varepsilon,\infty)}\left(\tilde{J}(x)\right)\right) = \tau_2 \circ \tilde{J}\left(E_{(\varepsilon,\infty)}(x)\right) = \tau_1\left(E_{(\varepsilon,\infty)}(x)\right),$$

we have $\tilde{J}(N(\varepsilon,\delta)_1) = N(\varepsilon,\delta)_2$. This implies that we can extend $\tilde{J}$ to a Jordan $*$-isomorphism from $\tilde{N}_1$ to $\tilde{N}_2$ which is continuous under the measure topology. Also we extend $\theta^i_s, s \in \mathbb{R}$, to continuous $*$-automorphisms on $\tilde{N}_i, i = 1, 2$. We still use the same notation in both cases. For each element φ in $(M_1)_{*,+}$, we compute as follows, due to [19; Theorem 1.2] or [25; Chapter II, Lemma 5]:

$$\mathrm{tr}_1(h_\varphi) = \varphi(1) = \tau_1\left(E_{(1,\infty)}(h_\varphi)\right) = \tau_2 \circ \tilde{J}\left(E_{(1,\infty)}(h_\varphi)\right) =$$

$$= \tau_2\left(E_{(1,\infty)}\left(\tilde{J}(h_\varphi)\right)\right) = \mathrm{tr}_2\left(\tilde{J}(h_\varphi)\right).$$

Thus we have $\mathrm{tr}_1 = \mathrm{tr}_2 \circ \tilde{J}$ on $L^1(M_1)_+$. The linear extension shows the same relation on $L^1(M_1)$. Let a be an element in $L^p(M_1)$ and consider the polar decomposition $a = u|a|$. Then we have $|\tilde{J}(a)| = \tilde{J}(|a|z) + \tilde{J}(|a^*|(1-z))$ (More precisely, z shall be denoted by $\pi_1(z)$). Since $*$-operation is isometric on $L^p(M_1)$ and since z is a central projection in N_1, we have

$$\|\tilde{J}(a)\|_p^p = \mathrm{tr}_2\left(|\tilde{J}(a)|^p\right) = \mathrm{tr}_2\left(\tilde{J}\left(|a|^p z\right) + \tilde{J}\left(|a^*|^p(1-z)\right)\right) = \mathrm{tr}_1\left(|a|^p z\right) +$$

$$+\mathrm{tr}_1\left(|a^*|^p(1-z)\right) = \|\,|a|z\|_p^p + \|\,|a^*|(1-z)\|_p^p = \|\,|a|z\|_p^p + \|\,|a|(1-z)\|_p^p = \|a\|_p^p.$$

This shows that the restriction of $\tilde{J}$ to $L^p(M_1)$ gives a positivity preserving linear isometry onto $L^p(M_2)$.

THEOREM 4.1. *Let $1 < p < \infty$ and $p \neq 2$. Let M_1 and M_2 be σ-finite von Neumann algebras. Let φ_0 (resp. ψ_0) be a fixed faithful normal state on M_1 (resp. M_2). Let T be a positive linear isometry from $L^p(M_1;\varphi_0)$ onto $L^p(M_2;\psi_0)$. Let J be the Jordan $*$-isomorphism from M_1 onto M_2 induced by T due to Theorem 3.6. Let $\mathcal{K}$ be the canonical isomorphism associated with the change of ψ_0 and $\varphi_0 \circ J^{-1}$ which is mentioned as in preliminaries. Then the following conditions are equivalent:*

(1) $T = \tilde{\mathcal{K}}^{-1} \circ \tilde{J}$ on $L^p(M_1)$

(2) $T(a)^p + T(b)^p \leqslant T(c)^p$ if $a^p + b^p \leqslant c^p$
for any a, b and c in $L^p(M_1)_+$.

Proof. It is clear that the condition (1) yields the condition (2). We assume that the condition (2) holds. Let a, b and c be elements in $L^p(M_1)_+$ satisfying $a^p + b^p = c^p$. Then we easily have $T(c)^p = T(a)^p + T(b)^p$ by (2). We define a map β from $(M_1)_{*,+}$ onto $(M_2)_{*,+}$ by the following formula:

$$T\left(h_\varphi^{1/p}\right) = h_{\beta(\varphi)}^{1/p}.$$

For two elements φ and ψ in $(M_1)_{*,+}$, put $a = h_\varphi^{1/p}, b = h_\psi^{1/p}$ and $c = h_{\varphi+\psi}^{1/p}$. Then the above fact shows that $\beta(\varphi+\psi) = \beta(\varphi) + \beta(\psi)$. Moreover we have

$$\|\beta(\varphi)\| = \beta(\varphi)(1) = \mathrm{tr}_1(h_{\beta(\varphi)}) = \|h_{\beta(\varphi)}^{1/p}\|_p^p = \|T(h_\varphi^{1/p})\|_p^p = \|h_\varphi^{1/p}\|_p^p = \varphi(1) = \|\varphi\|$$

and

$$s\left(\beta\left(\varphi\right)\right) = s\left(h_{\beta(\varphi)}\right) = s\left(h_{\beta(\varphi)}^{1/p}\right) = s\left(T\left(a\right)\right) = J\left(s\left(a\right)\right) = J\left(s\left(\varphi\right)\right).$$

Suppose that $\varphi \leqslant \psi$. Putting $a = h_\varphi^{1/p}$ and $c = h_\psi^{1/p}$, we have $a^p \leqslant c^p$. It follows from the condition (2) that $h_{\beta(\varphi)} = T(a)^p \leqslant T(c)^p = h_{\beta(\psi)}$. Thus the map β is positive scalar homogeneous, additive, surjective, isometric, disjoint preserving and order preserving. We can naturally extend β to an $\mathbb{R}$-linear map from $(M_1)_{*,sa}$ to

$(M_2)_{*,sa}$. Then the transposed map ${}^t\beta$ of β from $(M_2)_{sa}$ to $(M_1)_{sa}$ is $\mathbb{R}$-linear, surjective, isometric and order preserving, where we identify the spaces $\big((M_i)_{*,sa}\big)^*$ and $(M_i)_{sa}$, $i = 1, 2$. Now we naturally extend ${}^t\beta$ from M_2 to M_1. It is one to one, onto and order preserving together with its inverse. We claim that ${}^t\beta(1) = 1$. Let B_i be the closed unit ball of $(M_i)_{sa}$, $i = 1, 2$. The $\mathbb{R}$-linear surjective isometry ${}^t\beta$ maps an extreme point of B_2 into that of B_1. Thus ${}^t\beta(1)$ is an extreme point of B_1 and ${}^t\beta(1) \geqslant 0$. The proof of [23; Lemma 10.1(ii)] shows that ${}^t\beta(1)$ is a projection in M_1. Considering $({}^t\beta)^{-1}$ which is also an $\mathbb{R}$-linear positive surjective isometry, we can conclude that ${}^t\beta(1) = 1$. Therefore ${}^t\beta$ is a Jordan $*$-isomorphism from M_2 to M_1 by [3; Theorem 3.2.3] and satisfies the following formula:

$$\beta(\varphi)(x) = x(\beta(\varphi)) = {}^t\beta(x)(\varphi) = \varphi \circ {}^t\beta(x), \quad x \in (M_2)_{sa}.$$

Especially we have $J(s(\varphi)) = s(\beta(\varphi)) = s(\varphi \circ {}^t\beta) = ({}^t\beta)^{-1}(s(\varphi))$, $\varphi \in (M_1)_{*,+}$. This implies by Lemma 3.1 that $J = ({}^t\beta)^{-1}$.

Now we simply write as $N_2 = N_2(\varphi_0 \circ J^{-1})$ and $\tau_2 = \tau_2(\varphi_0 \circ J^{-1})$. Let $N_2(\psi_0)$ be the crossed product $R(M_2, \sigma^{\psi_0})$ and let τ_{ψ_0} be the canonical trace on it. Let Φ_{ψ_0} be the canonical operator valued weight from $N_2(\psi_0)$ to M_2. For each normal weight ψ on M_2, we denote by $\tilde{\psi}^{\psi_0}$ the dual weight of ψ with respect to Φ_{ψ_0}. According to $(\varphi \circ J^{-1})^{\sim 2} = \tilde{\varphi}^1 \circ \tilde{J}^{-1}$, we see that

$$\tau_2\left(\tilde{J}\left(\frac{\mathrm{d}\tilde{\varphi}^1}{\mathrm{d}\tau_1}\right)\cdot\right) = \tau_1\left(\frac{\mathrm{d}\tilde{\varphi}^1}{\mathrm{d}\tau_1}\tilde{J}^{-1}(\cdot)\right) = \tilde{\varphi}^1 \circ \tilde{J}^{-1} = \left(\varphi \circ \tilde{J}^{-1}\right)^{\sim 2} =$$
$$= \tau_2\left(\frac{\mathrm{d}\left(\varphi \circ J^{-1}\right)^{\sim 2}}{\mathrm{d}\tau_2}\cdot\right),$$

i.e, $\tilde{J};\dfrac{\mathrm{d}\tilde{\varphi}^1}{\mathrm{d}\tau_1} \longrightarrow \dfrac{\mathrm{d}\left(\varphi \circ J^{-1}\right)^{\sim 2}}{\mathrm{d}\tau_2}$. On the other hand, by the definition of the map β, we have

$$T\left(\left(\frac{\mathrm{d}\tilde{\varphi}^1}{\mathrm{d}\tau_1}\right)^{\frac{1}{p}}\right) = \left(\frac{\mathrm{d}\left(\varphi \circ J^{-1}\right)^{\sim\psi_0}}{\mathrm{d}\tau_{\psi_0}}\right)^{\frac{1}{p}}.$$

It has been shown in the preliminaries that

$$\left(\varphi \circ J^{-1}\right)^{\sim\psi_0} \circ \tilde{\mathcal{K}}^{-1} = \left(\varphi \circ J^{-1}\right)^{\sim 2}.$$

Therefore we have the formula:

$$\tilde{\mathcal{K}}\left(\frac{\mathrm{d}\left(\varphi \circ J^{-1}\right)^{\sim\psi_0}}{\mathrm{d}\tau_{\psi_0}}\right) = \frac{\mathrm{d}\left(\left(\varphi \circ J^{-1}\right)^{\sim\psi_0} \circ \tilde{\mathcal{K}}^{-1}\right)}{\mathrm{d}\left(\tau_{\psi_0} \circ \tilde{\mathcal{K}}^{-1}\right)} = \frac{\mathrm{d}\left(\varphi \circ J^{-1}\right)^{\sim 2}}{\mathrm{d}\tau_2}.$$

It follows that

$$T\left(h_{\varphi}^{1/p}\right)^{p}=\frac{\mathrm{d}\left(\varphi\circ J^{-1}\right)^{\sim\psi}_{0}}{\mathrm{d}\tau_{\psi_0}}=\tilde{\mathcal{K}}^{-1}\left(\frac{\mathrm{d}\left(\varphi\circ J^{-1}\right)^{\sim 2}}{\mathrm{d}\tau_2}\right)=\tilde{\mathcal{K}}^{-1}\circ\tilde{J}\left(\frac{\mathrm{d}\tilde{\varphi}^1}{\mathrm{d}\tau_1}\right)=\tilde{\mathcal{K}}^{-1}\circ\tilde{J}(h_{\varphi}).$$

Since $\tilde{\mathcal{K}}^{-1}\circ\tilde{J}$ is a Jordan $*$-isomorphism, we have

$$T\left(h_{\varphi}^{1/p}\right)=\tilde{\mathcal{K}}^{-1}\circ\tilde{J}(h_{\varphi}^{1/p}),\quad \varphi\in(M_1)_{*,+},$$

which implies by the linearity that $T=\tilde{\mathcal{K}}^{-1}\circ\tilde{J}$ on $L^p(M_1;\varphi_0)$. This completes the proof. ■

REMARK 4.2. In the contrast to Theorem 3.6, the result stated in Theorem 4.1 is relatively unsatisfactory for the sake of the condition (2) which guarantees the aditivity and order preserving property of the map β. We believe that if T is a surjective positive linear isometry between non-commutative L^p-spaces then the condition (2) is satisfied automatically, so that T is the composition of two Jordan $*$-isomorphisms.

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REFERENCES

1. ARAKI, H.; MASUDA, T., *Positive cones and L^p-spaces for von Neumann algebras,* Publ. Res. Inst. Math. Sci. **18**(1982), 339–411.
2. BANACH, S., *Théorie des operations linearies,* Warsaw, 1932.
3. BRATTELI, O.; ROBINSON, D. W., *Operator Algebras and Quantum Statistical Mechanics I,* Springer, 1979.
4. BROISE, M. M., Sur les isomorphismes de cèrtaines algèbres de von Neumann, *Ann. Sci. Ecole Norm. Sup.*, **83**(1966), 91–111.
5. DIXMIER, J., Formes lineaires sur un anneau d'opérateurs, *Bull. Soc. Math. France*, **81**(1953), 9–39.
6. DYE, H. A., On the geometry of projections in certain operator algebras, *Ann. Math.*, **61**(1955), 73–88.
7. HAAGERUP, U., The standard form of von Neumann algebras, *Math. Scand.*, **37**(1975), 271–283.
8. HAAGERUP, U., Operator valued weight in von Neumann algebras I, *J. Funct. Anal.*, **32**(1979), 175–206.
9. HAAGERUP, U., Operator valued weight in von Neumann algebras II, *J. Funct. Anal.*, **33**(1979), 339–361.
10. HAAGERUP, U., *L^p-spaces associated with an arbitrary von Neumann algebra,* Colloq. Internat. CNRS, **274**(1979), 175–184.
11. HILSUM, M., Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale, *J. Funct. Anal.*, **40**(1981), 151–169.
12. KADISON, R. V., Isometries of operator algebras, *Ann. Math.*, **54**(1951), 325–338.

13. KATAVOLOS, A., Isometries of non-commutative L^p-spaces, *Can. J. Math.*, **28**(1976), 1180–1186.
14. KATAVOLOS, A., Are non-commutative L^p-spaces really non-commutative?, *Can. J. Math.*, **33**(1981), 1319–1327.
15. KATAVOLOS, A., Non-commutative L^p-spaces II, *Can. J. Math.*, **34**(1982), 1208–1214.
16. KOSAKI, H., Applications of uniform convexity of non-commutative L^p-spaces, *Trans. Amer. Math. Soc.*, **283**(1984), 265–282.
17. KOSAKI, H., Application of the complex interpolation method to a von Neumann algebra: Non-commutative L^p-spaces, *J. Funct. Anal.*, **56**(1984), 29–78.
18. LAMPERTI, J., On the isometries of certain function spaces, *Pac. J. Math.*, **8**(1958), 459–466.
19. NELSON, E., Note on non-commutative integration, *J. Funct. Anal.*, **15**(1974), 103–116.
20. RUSSO, B., Isometries of L^p-spaces associated with finite von Neumann algebras, *Bull. Amer. Math. Soc.*, **74**(1968), 228–232.
21. SEGAL, I. E., A non-commutative extension of abstract integration, *Ann. Math.*, **57** (1953), 401–457.
22. STRĂTILĂ, Ş., *Modular theory in operator algebras,* Abacus Press, 1981.
23. TAKESAKI, M., *Theory of operator algebra I,* Springer, 1979.
24. TAM, P. K., Isometries of L^p-spaces associated with semifinite von Neumann algebras, *Trans. Amer. Math. Soc.*, **254**(1979), 339–354.
25. TERP, M., *L^p-spaces associated with von Neumann algebras,* Notes, Copenhagen Univ., 1981.
26. TERP, M., Interpolation spaces between a von Neumann algebra and its predual, *J. Operator Theory,* **8**(1982), 327–360.
27. YEADON, F. J., Non-commutative L^p-spaces, *Math. Proc. Camb. Phil. Soc.*, **77**(1975), 91–102.
28. YEADON, F. J., Isometries of non-commutative L^p-spaces, *Math. Proc. Camb. Phil. Soc.*, **90** (1981), 41–50.

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