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CHARACTERS AND FACTOR REPRESENTATIONS OF THE INFINITE DIMENSIONAL CLASSICAL GROUPS

ROBERT P. BOYER

1. INTRODUCTION

The starting point of this work was the announcement by S. V. Kerov and A. M. Vershik [11] that the finite characters of the inductive limit group $U(\infty)$ can all be obtained as limits of normalized characters of $U(N)$, which we call the Asymptotic Character Formula or the "ergodic method". In Section 2, we give a detailed proof of this Theorem for representations in the non-negative signatures. It is exactly this case which we need to establish the Asymptotic Formula for the other classical groups: the infinite symplectic, special orthogonal, and spin groups. In Proposition 2.2, we make a detailed study of the limiting behavior of the ratios of hooklengths which can be viewed as a refinement of the asymptotic methods of [10] and [26]. Essential use is made of the dimension formulas of [8]. Hence, the classification and asymptotic behavior of the characters of the infinite classical groups of compact type is complete.

In the final section, we use the techniques we developed in [6] and [7] to study factor representations of $SO(\infty)$ and $Sp(\infty)$ in the antisymmetric and symmetric tensors. These representations can be viewed as a generalization of the well-known quasi-free states of the algebras for the canonical commutation or anticommutation relations. By exploiting special combinatorial structures [23] and [24], we can make use of the dynamical system ideas of Strătilă and Voiculescu [21]. A much more general situation is studied in [2] and [3].

Our methods fall naturally in the theory of spherical functions for the infinite classical groups which have been studied by G. I. Ol'shankii [19] and D. Pickrell [20].

We would like to thank Professor D. Voiculescu for suggesting to us to complete the classification of finite characters which was given in [5, 6, 12] for the unitary and

symplectic groups.

Notation. For the unitary group $U(N)$, we parametrize its irreducible representations (corresponding to the ones with non-negative signatures) by Young diagrams $\{\lambda\}$ which have no more than N rows. We write: $\mathrm{Ch}\{\lambda\}$ or $\{\lambda\}$ for the corresponding character, and $\dim\{\lambda\}$ for the dimension of the representation. In addition, we let $|\lambda|$ denote the number of nodes in the diagram. We let $r_i(\lambda)$ (respectively, $c_i(\lambda)$) denote the length of the i-th row (respectively, column). The rank, $\mathrm{rk}(\lambda)$, of $\{\lambda\}$ is the length of the main diagonal. Next, we let $(a_1, a_2, \ldots, a_r | b_1, b_2, \ldots, b_r)$ denote the Frobenius coordinates of $\{\lambda\}$. For all this, see the book by I. Macdonald [16].

For the symplectic, special orthogonal, and spin groups, we adapt the above notation for the unitary group by replacing $\{\cdots\}$ with $\langle\cdots\rangle$ for the symplectic group $Sp(2N)$ while with $[\cdots]$ for the special orthogonal group $SO(2N+1)$. We let Δ denote the basic spin representation of $\mathrm{Spin}(2N+1)$. Then the proper spin representations of $\mathrm{Spin}(2N+1)$ can be parametrized $[\Delta, \lambda]$ where λ is a Young diagram with N or fewer rows.

We let f^λ denote the dimension of the irreducible representation associated to λ on the symmetric group on $|\lambda|$ letters.

2. ASYMPTOTIC FORMULAS FOR THE UNITARY GROUP

Let $\lambda_1 \prec \lambda_2 \prec \cdots$ with $\lambda_N \in \widehat{U(N)}$. In this section, we wish to give a detailed proof of the portions of the Asymptotic Character Formula that we will need in the next section. This result was announced by Kerov and Vershik in [12].

We record here the fundamental convergence conditions discovered by Kerov and Vershik. We require that the following limits all exist: $\alpha_i = \lim_{N\to\infty} r_i(\lambda_N)/N$, $\beta_i = \lim_{N\to\infty} c_i(\lambda_N)/N$, and $\theta = \lim_{N\to\infty} |\lambda_N|/N$. We call these limits the row, column, and occupancy frequencies, respectively. Further, we require a summation condition to hold: $\sum_{i=1}^{\infty}(\alpha_i + \beta_i) < \infty$. Note that the row and column numbers in the above limits can be replaced by the corresponding Frobenius coordinates.

By the Voiculescu Factorization Theorem [26], it suffices to consider a character of $U(N)$ restricted to one variable:

$$f_N(z) = \mathrm{Ch}\{\lambda_N\}(z, 1, \ldots, 1)/\dim\{\lambda_N\}_N = \sum_{k=0}^{|\lambda_N|} \frac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N} z^k.$$

Without loss of generality, we may assume that the length of the N-th row $r_N(\lambda_N)$ is zero. Also, we note that the irreducible components of $\{\lambda_N/k\}_{N-1}$ are simple [17]. To establish the Asymptotic Character Formula we shall show that the limit of $f_N(z)$ exists pointwise.

Since the coefficients of $f_N(z)$ are all non-negative and sum to one, we can treat f_N as a probability generating function on the non-negative integers. Hence, the pointwise limit of $f_N(z)$ exists if and only if each coefficient converges.

One further observation is that the row length $r_N(\lambda_N)$ gives the minimal power of z that occurs in the polynomial $f_N(z)$. So, the row lengths $r_N(\lambda_N)$ must be uniformly bounded in N, otherwise, the coefficients of $f_N(z)$ would all converge to 0.

To establish the limiting behavior of $\dim\{\lambda_N/k\}_{N-1}/\dim\{\lambda_N\}$, for fixed k, we need to exploit finer structures.

It will be necessary to separate out the numerator and the denumerator contributions to the dimension of $\lambda_N \in \widehat{U(N)}$:

$$\dim\{\lambda_N\} = \frac{\operatorname{num}\{\lambda_N\}}{H(\lambda_N)}$$

where

$$\operatorname{num}\{\lambda_N\} = \prod_{i=1}^{\operatorname{rk}\{\lambda_N\}} \frac{(N+a_i^{(N)})!}{(N-b_i^{(N)}-1)!}$$

$H(\lambda_N)$ is the usual product of the hook lengths and $(a_1^{(N)}, a_2^{(N)}, \cdots | b_1^{(N)}, b_2^{(N)}, \cdots)$ are the Frobenius coordinates of $\{\lambda_N\}$. Note that $H(\lambda_N)$ is independent of the number of variables of $U(N)$; in fact, it only depends of the shape of λ_N.

Let λ'_N be a summand of $\lambda_N|U(N-1)$, where λ'_N is obtained by deleting k nodes. It will be necessary to describe a sequence of λ'_N.

Let μ_N be a finite sequence $(x_1^{(N)}, \ldots, x_{s_N}^{(N)}, m_1^{(N)}, \ldots, m_{s_N}^{(N)}; y_1^{(N)}, \ldots, y_{t_N}^{(N)})$ where $x_i^{(N)}$ (respectively, $y_i^{(N)}$) denotes a row (respectively, a column) number and $m_i \in \mathbf{Z}^+$. We assume that $\sum m_i^{(N)} + t_N = k$. Then $(\lambda_N : \mu_N)$ denotes the diagram for which m_i nodes are deleted from row $x_i^{(N)}$ and 1 node is deleted from column $y_i^{(N)}$. We call μ_N the *deletion nodes.* When we use the notation $(\lambda : \mu)$, we assume that the result of deleting the nodes corresponding to μ results in another Young diagram. To emphasize this, we introducé further notation.

Let $\mathcal{D}_{k,N}$ denote the collection of all possible k nodes configurations that can be stripped from λ_N. Let $\mathcal{D}_k$ be the set of all sequences $\mu_N \in \mathcal{D}_{k,N}$. Let $\mathcal{C}_k$ denote the set of all sequences in $\mathcal{D}_k$ which are eventually constant. We simply write in this case that $\mu \in \mathcal{C}_k$.

For $\lambda = \lambda_N$, $\lambda' = (\lambda_N : \mu_N)$, $\mu_N \in \mathcal{D}_{k,N}$, the quotient $\dim\{\lambda'\}_{N-1}/\dim\{\lambda\}_N$ is the product of four factors:

(2.0.1) $$\frac{\operatorname{num}\{\lambda'\}_{N-1}}{\operatorname{num}\{\lambda'\}_N} \text{ -constant term analysis}$$

(2.0.2) $$\prod_{(i,j)\in\lambda\setminus\lambda'} \frac{N}{N+j-i}$$

-conversion factor between the constant term and higher order coefficients

(2.0.3) $$\frac{|\lambda|!}{|\lambda'|!}\Big/ N^k$$ -conversion factor between unitary and symmetric groups

(2.0.4) $$\frac{|\lambda'|!}{|\lambda|!}\Big/\frac{H(\lambda')}{H(\lambda)} = \frac{f^{\lambda'}}{f^{\lambda}}$$ -symmetric group asymptotics.

Comments. The first factor (2.0.1) involves the limiting behavior of the constant term of $f_N(z)$. We shall see that the middle two factors are elementary to analyze. The fourth factor (2.0.4) is connected with the Asymptotic Formula for the infinite symmetric group $S(\infty)$.

To state the following Lemma, it is necessary to introduce some notation and facts about $S(\infty)$.

We shall write $\tilde{\alpha_i} = \lim_{N\to\infty} r_i(\lambda_N)/|\lambda_N|$ and $\tilde{\beta_i} = \lim_{N\to\infty} c_i(\lambda_N)/|\lambda_N|$. So, $\alpha_i = \tilde{\alpha_i}\theta$ and $\beta_i = \tilde{\beta_i}\theta$. From the Asymptotic Character Formula for $S(\infty)$ [11] or [27], we know that

$$\lim_{N\to\infty} \frac{f^{\lambda_N/k}}{|\lambda_N|} = p_k$$

where

(2.0.5) $$G(t) = \sum_{k=0}^{\infty} p_k t^k = e^{\tilde{\lambda} t} \prod_{i=1}^{\infty} \frac{1+\tilde{\beta_i} t}{1-\tilde{\alpha_i} t},$$

and $0 \leqslant \tilde{\lambda} \leqslant 1$, $\tilde{\alpha_1} \geqslant \tilde{\alpha_2} \geqslant \cdots \geqslant 0$, $\tilde{\beta_1} \geqslant \tilde{\beta_2} \geqslant \cdots \geqslant 0$, and $\tilde{\lambda} + \sum \tilde{\alpha_i} + \tilde{\beta_i} = 1$. Moreover, we shall see in Proposition 2.2 that for a sequence with distinct row and column frequencies:

$$\tilde{\alpha_i} = \lim_{N\to\infty} \frac{f^{(\alpha_N:\mu_i)}}{f^{\lambda_N}}, \quad \tilde{\beta_i} = \lim_{N\to\infty} \frac{f^{(\lambda_N:\mu_i)}}{f^{\lambda_N}},$$

where μ_i (respectively, μ_i') denotes the deletion of one node in the i-th row (respectively, column).

Observation. In [11] or [27], shifted Frobenius coordinates are used in the sense that 1/2 is added to the usual coordinates. In those calculations, this has the effect of making the error term in the asymptotic formula be homogeneous in these coordinates and makes the sum of the normalized coordinates be $\leqslant 1$.

The first Lemma is immediate. We list it because of its frequent use.

LEMMA 2.1. *Suppose that μ and μ' are deletion nodes and that both limits*

$$f^{((\lambda_N:\mu):\mu')}/f^{(\lambda_N:\mu')} \text{ and } f^{(\lambda_N:\mu')}/f^{(\lambda_N:\mu)}$$

exist as $N \to \infty$, then the limit of $f^{((\lambda_N:\mu):\mu')}/f^{\lambda_N}$ exists as $N \to \infty$ and is equal to the product of the above limits.

The following Proposition is the version we need concerning the asymptotics of the symmetric group.

PROPOSITION 2.2. (i) *Let $\mu_n = \mu \in \mathcal{C}_k$, for all N, where*

$$\mu = (x_1, \ldots, x_s, m_1, \ldots, m_s; y_1 \ldots, y_t)$$

$\sum m_i + t = k$, $X = \{x_1, \ldots, x_s\}$ *and* $Y = \{y_1, \ldots, y_t\}$. *If all the row frequencies* $\tilde{\alpha}_{x_i}$, $1 \leqslant i \leqslant s$, *are distinct, then* $\lim\limits_{N\to\infty} \frac{f^{(\lambda_N:\mu)}}{f^{\lambda_N}}$ *exists and equals* $\prod\limits_{i\in Y} \tilde{\beta}_i \prod\limits_{i\in X} \tilde{\alpha}_i{}^{m_i}$

(ii) *Suppose that $\tilde{\alpha}_{x_1} = \cdots = \tilde{\alpha}_{x_s} = \tilde{\alpha}$, then*

$$\lim_{N\to\infty} \sum_{|\nu|=k} \frac{f^{(\lambda_N:\nu)}}{f^{\lambda_N}} = \binom{s+k-1}{s-1} \tilde{\alpha}^k,$$

where $(\lambda_N : \nu)$ denotes row deletion nodes chosen relative to the set X only.

(iii) *Suppose that $\tilde{\beta}_{y_1} = \cdots = \tilde{\beta}_{y_t} = \tilde{\beta}$, then*

$$\lim_{N\to\infty} \sum_{|\nu|=k} \frac{f^{(\lambda_N:\nu)}}{f^{\lambda_N}} = \binom{t}{k} \tilde{\alpha}^k,$$

where $(\lambda_N : \nu)$ denotes column deletion nodes chosen relative to the set Y only.

Proof. By Lemma 2.1, it is enough to consider the situation that p nodes are deleted from the ℓ-th row of λ_N. It is interesting to compare the calculations below with those of [11] and [27] in establishing the Asymptotic Character Formula for $S(\infty)$. As done there, we shall give a variation of an argument originally due to Frobenius (see [18, pp. 138–139]).

We first note that $\lim\limits_{N\to\infty} \mathrm{rk}(\lambda_N)/|\lambda_N| = 0$, since $[\mathrm{rk}(\lambda_N)]^2 \leqslant |\lambda_N|$. We write: $s_N = \mathrm{rk}(\lambda_N)$.

We define an auxiliary generating function:

$$F(y) = \prod_{j=1}^{s_N} \frac{y - a_i^{(N)}}{y + b_i^{(N)}},$$

where λ_N has Frobenius coordinates $(a_1^{(N)},\ldots,a_{s_N}^{(N)}\mid b_1^{(N)},\ldots,b_{s_N}^{(N)})$. Then:

$$\frac{f^{(\lambda_N:p)}}{f^{\lambda_N}}=\frac{-1}{p(|\lambda_N|)(|\lambda_N|-1)\cdots(|\lambda_N|-p+1)}\times$$
$$\times\mathrm{Res}\left(y(y-1)\cdots(y-p+1)\frac{F(y-p)}{F(y)},a_\ell^{(N)}\right).$$

Since this singularity is a simple pole, the desired residue equals:

$$\lim_{y\to a_\ell^{(N)}}(y-a_\ell^{(N)})y(y-1)\cdots(y-p+1)\frac{F(y-p)}{F(y)}=$$
$$=a_\ell^{(N)}(a_\ell^{(N)}-1)\cdots(a_\ell^{(N)}-p+1)\frac{-p}{a_\ell^{(N)}+b_\ell^{(N)}-p+1}\cdot\left(a_\ell^{(N)}+b_\ell^{(N)}+1\right)\times$$
$$\times\left(\prod_{j\neq\ell}^{s_N}\frac{a_\ell^{(N)}+b_j^{(N)}+1}{a_\ell^{(N)}+b_j^{(N)}-p+1}\,\frac{a_\ell^{(N)}-a_\ell^{(N)}-p+1}{a_\ell^{(N)}-a_\ell^{(N)}}\right)$$

(In [18], the sum of all finite residues is needed, in other words, the residue at ∞.) Now, the quotient of dimensions becomes:

$$\left(\frac{a_\ell^{(N)}}{|\lambda_N|}\right)\left(\frac{a_\ell^{(N)}-1}{|\lambda_N|-1}\right)\cdots\left(\frac{a_\ell^{(N)}-p+1}{|\lambda_N|-p+1}\right)\times\left(\frac{a_\ell^{(N)}+b_\ell^{(N)}+1}{a_\ell^{(N)}+b_\ell^{(N)}-p+1}\right)\times$$
$$\times\left(\prod_{j\neq\ell}^{s_N}1+\frac{p}{a_\ell^{(N)}+b_j^{(N)}-p+1}\right)\left(\prod_{j\neq\ell}^{s_N}1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right).$$

We claim that these last three products will converge to 1, as $N\to\infty$. Recall that p is a constant and that $s_N/|\lambda_N|\to\infty$, as $N\to\infty$. It is clear that the first product converges to 1. Next, we consider the product:

$$\text{(2.2.1)}\quad\prod_{j\neq\ell}\left(1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right)=\prod_{j=1}^{\ell-1}\left(1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right)\prod_{j=\ell+1}^{s_N}\left(1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right)$$

Again, it is immediate that $\prod_{j=1}^{\ell-1}\left(1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right)\to 1$. Moreover, we have the inequalities:

$$\left(1-\frac{p}{a_\ell^{(N)}-a_{\ell+1}^{(N)}}\right)^{s_n-\ell}\leqslant\prod_{j=\ell+1}^{s_N}\left(1-\frac{p}{a_\ell^{(N)}-a_j^{(N)}}\right)\leqslant 1.$$

In general, for a sequence $\{p_n\}$ of real numbers such that $0\leqslant p_n\leqslant 1$ and $p_n\to 0$, the sequence $(1-p_n)^n$ converges if and only if $np_n\to\lambda$, where $0\leqslant\lambda<\infty$. The limit, if it exists, equals $\mathrm{e}^{-\lambda}$. In our case, we have that

$$\lim_{N\to\infty}\frac{s_N}{a_\ell^{(N)}-a_{\ell+1}^{(N)}}=0,$$

since $\alpha_\ell^{(N)} > \alpha_{\ell+1}^{(N)}$. Hence, the entire product in (2.2.1) converges to 1.

By a similar argument, we see that

$$\lim_{N\to\infty} \prod_{j\neq\ell} \left(1 - \frac{p}{a_\ell^{(N)} + b_j^{(N)} - p + 1}\right) = 1.$$

So, we have the bounds

$$1 \leqslant \prod_{j\neq\ell} \left(1 + \frac{p}{a_\ell^{(N)} + b_j^{(N)} - p + 1}\right) \leqslant \left(1 - \frac{p}{a_\ell^{(N)} + b_j^{(N)} - p + 1}\right)^{-1},$$

since $\prod(1+p_n) \leqslant 1/\prod(1-p_n)$. Hence, we conclude that

$$\lim_{N\to\infty} \frac{(f^{\lambda_N:p})}{f^{\lambda_N}} = \tilde{\alpha}^p.$$

The case of column deletion follows from the row deletion argument with $p = 1$ since the dimensions f^λ and $f^{\lambda'}$ agree, where λ' is the conjugate of the partition λ.

The argument for the case of many identical row frequencies is very similar. To simplify notation, we assume that the first s row frequencies have a common value $\tilde{\alpha}$. All of the above calculations hold except that the factor $\prod_{j=1}^{s-1} \left(1 - \frac{p}{a_s^{(N)} - a_j^{(N)}}\right)$ in (2.2.1) converges to 1. In the base case in which the row lenghts are identical, so $a_1^{(N)} = a_2^{(N)} + 1 = \cdots = a_s^{(N)} + s - 1$, the sum in statement (ii) consists of a single term corresponding to deleting p nodes from row s. Then the factor in (2.2.1) becomes

$$\prod_{j=1}^{s-1} \left(1 + \frac{k}{j}\right) = \binom{k+s-1}{k}.$$

To handle the general case, we use induction on s. Suppose that $s = 2$ and that $a_1^{(N)} - a_2^{(N)} = m - 1$, with $m > 1$. Then the contribution of deleting j nodes from row 1 and $p - j$ nodes from row 2 is $(1 - j/m)(1 + (p-j)/m\tilde{\alpha})$. Hence, the total contribution is:

$$\sum_{j=0}^{p} \left(1 - \frac{j}{m}\right)\left(1 + \frac{p-j}{m-j}\right) = 2 + \sum_{j=1}^{p-1} \frac{m+p-2j}{m} = p + 1.$$

Note that the value of this sum is independent of the choice of m. Moreover, it is easy to see that if $a_1^{(N)} - a_2^{(N)} \to \infty$, then the total contribution is again $p + 1$. So, we now conclude that

$$\sum_{|\nu|=p} \frac{f^{(\lambda_N:\nu)}}{f^{\lambda_N}} = (p+1)\tilde{\alpha}^p,$$

where $(\lambda_N : \nu)$ denotes row node deletion from the first two rows only. Hence, statement (ii) holds if $s = 2$.

An interesting consequence of this argument is that $f^{(\lambda_N : p)}/f^{\lambda_N}$ converges if and only if $a_1^{(N)} - a_2^{(N)} \to \infty$ or $a_1^{(N)} - a_2^{(N)}$ is eventually constant.

Now, we assume the result for fewer than s rows or for s rows with fewer than p nodes deleted. Assume that $a_i^{(N)} - a_j^{(N)} = m_{i,j}$, with $1 \leqslant i < j \leqslant s$. We consider the case of p row nodes deletion over s rows. We break out the calculation into two cases. In the first case, we demand that every deletion contains at least 1 node in the first row. Hence, by induction, we have that the contribution is $\binom{s-p-2}{s-1}$. In the second case, we require that no deletions are made from the first row. Again, by induction, the contribution is: $\binom{s+p-2}{s-2}$. So, the total contribution is: $\binom{s+p-1}{s-1}$. We may now conclude that statement (ii) holds in general.

A similar argument can be given to deduce statement (iii). ∎

The following Corollary is easy to deduce from the proof of the Proposition.

COROLLARY 2.3. *Let $\varepsilon > 0$ be given. Choose m so that $\sum_{m+1}^{\infty}(\tilde{\alpha}_i + \tilde{\beta}_i) < \varepsilon$. Let $\mu_i^{(N)}$ be deletion nodes with coordinates $> m$, with $1 \leqslant i \leqslant k_N$. Then:*

(i) *Let $\mu \in C_1$ with coordinates $\leqslant m$ and distinct row or column frequencies. Then*

$$\lim_{N\to\infty} \sum_{i=1}^{k_N} \frac{f^{((\lambda_N : \mu_i^{(N)}) : \mu)}}{f^{\lambda_N}} = \lim_{N\to\infty} \left(\sum_{i=1}^{k_N} \frac{f^{(\lambda_N : \mu_i^{(N)})}}{f^{\lambda_N}} \right) \lim_{N\to\infty} \left(\frac{f^{(\lambda_N : \mu)}}{f^{\lambda}} \right).$$

(ii) *Suppose $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \cdots = \tilde{\alpha}_s = \tilde{\alpha}$ with coordinates $\leqslant m$. Then:*

$$\lim_{N\to\infty} \sum_{i=1}^{k_N} \sum_{|\nu|=k} \frac{f^{((\lambda_N : \mu_i^{(N)}) : \mu)}}{f^{\lambda_N}} = \binom{s+k-1}{s-1} \tilde{\alpha}^k \lim_{N\to\infty} \left(\sum_{i=1}^{k_N} \frac{f^{(\lambda_N : \mu_i^{(N)})}}{f^{\lambda_N}} \right),$$

where, $(\lambda_N : \nu)$ denotes row deletion from rows $x_1, \ldots, x_s$ only.

(iii) *Supppose $\tilde{\beta}_1 = \cdots = \tilde{\beta}_t = \tilde{\beta}$ with coordinates $\leqslant m$. Then:*

$$\lim_{N\to\infty} \sum_{i=1}^{k_N} \sum_{|\nu|=k} \frac{f^{((\lambda_N : \mu_i^{(N)}) : \mu)}}{f^{\lambda_N}} = \binom{t}{k} \tilde{\beta}^k \lim_{N\to\infty} \left(\sum_{i=1}^{k_N} \frac{f^{(\lambda_N : \mu_i^{(N)})}}{f^{\lambda_N}} \right),$$

where $(\lambda_N : \nu)$ denotes row deletion from columns $y_1, \ldots, y_t$ only.

The next Lemma establishes the connection between row and column frequencies normalized by $|\lambda_N|$ and N, the number of variables for $U(N)$.

LEMMA 2.4.

$$\lim_{N\to\infty} \frac{|\lambda_N|!}{|\lambda'_N|!} \Big/ N^k = \theta^k,$$

where $\theta = \lim_{N\to\infty} |\lambda_N|/N$. *In particular, the limit is independent of the choice of deleted nodes.*

Proof. We have:

$$\frac{|\lambda_N|!}{|\lambda'_N|!} \Big/ N^k = \prod_{j=0}^{k-1} \left(\frac{|\lambda_N| - j}{N} \right) \to \theta^k,$$

where $\theta = \lim_{N\to\infty} |\lambda_N|/N$. ∎

Notation. Let $\varepsilon > 0$ be given. Fix a positive integer k. Define $m_{\varepsilon,k} \in \mathbb{Z}^+$ so that $n \geqslant m_{\varepsilon,k} \in \mathbb{Z}^+$ implies that the t^j-th coefficients of $\prod_{i=1}^{n} \frac{1+\tilde{\beta}_i t}{1-\tilde{\alpha}_i t}$ and $\prod_{i=1}^{\infty} \frac{1+\tilde{\beta}_i t}{1-\tilde{\alpha}_i t}$ agree to within ε, $0 \leqslant j \leqslant k$, and $m_{\varepsilon,k-1} \leqslant m_{\varepsilon,k}$.

We shall list the $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots$ and $\tilde{\beta}_1, \tilde{\beta}_2, \ldots$ according to their multiplicity in the generating function $G(t)$, defined in (2.0.5).

Let X and Y be finite subsets of $\mathbb{Z}^+$ such that if $i \in X$ and $\tilde{\alpha}_i = \tilde{\alpha}_j$, then $j \in X$, while if $i \in Y$ and $\tilde{\beta}_i = \tilde{\beta}_j$, then $j \in Y$. We let X_0 be the subset of X consisting of indices of the distinct values of $\tilde{\alpha}_i$, $i \in X$. (We may choose the elements of X_0 be using the smallest choice of index, say.) Similarly, we define Y_0. Next, we give two sequences of non-negative integers. Let $m : m_1, m_2, \ldots$ be chosen so that $m_i = 0$ if $i > m_{\varepsilon,k}$ and $m_i = m_j$ whenever $\tilde{\alpha}_i = \tilde{\alpha}_j$. Similarly, we take $n : n_1, n_2, \ldots$ so $n_i = 0$ if $i > m_{\varepsilon,k}$ and $n_i = n_j$ if $\tilde{\beta}_i = \tilde{\beta}_j$. Moreover, we require that: $\sum\{m_i + n_j \; : \; i \in X_0, \; j \in Y_0\} = k$.

Now, (X, m, Y, n) determines a set $D = D(X, m, Y, n) \subset \mathcal{D}_N$ of deletion nodes when m_i nodes are deleted over the rows with row frequency $\tilde{\alpha}_i$, $i \in X_0$, and n_i nodes are deleted over the columns with column frequency $\tilde{\beta}_i$, $i \in Y_0$.

Note that if $\mu \in D$, then $|\mu| = k$. So, we can define $|D|$ as $|\mu|$, for any μ in D. Also, we define the multiplicity $m(D)$ of D by:

$$m(D) = \prod_{i\in X_0} \binom{s_i + m_i - 1}{s_i - 1} \prod_{i \in Y_0} \binom{t_i}{n_i},$$

where $\tilde{\alpha}_i, i \in X_0$ has multiplicity s_i and $\tilde{\beta}_i$, $i \in Y_0$ has multiplicity t_i.

Finally, we let $\mathcal{J}_{\varepsilon,N}$ be the collection of all such sets $D \subset \mathcal{D}_N$.

LEMMA 2.5. *For* $D = D(X, m, Y, n) \in \mathcal{J}_{\varepsilon,N} \subset \mathcal{D}_N$, $|D| = k - j$,

$$S(N, D) = \sum \left\{ \frac{f^{(\lambda_N : \mu\nu)}}{f^{\lambda_N}} \; : \; \mu \in D, \; \nu \in \mathcal{I}_{\varepsilon,N}, \; |\nu| = j \right\} \tag{2.5.1}$$

converges to within $m(D)(k+1)!\varepsilon$ *of* $m(D)\tilde{\lambda}^j/j! \times \prod_{i\in Y}\tilde{\beta}_i \prod_{i\in X}\tilde{\alpha}_i^{m_i}$.

Proof. We shall first assume that the row and column frequencies are distinct relative to the sets X and Y; that is $X = X_0$ and $Y = Y_0$, with $n_i = 1$.

Let $E_{k,j}$ be an error bound defined so that

$$\left| \lim_{N\to\infty} S(N,D) - \frac{\tilde{\lambda}^j}{j!} \prod_{i\in Y}\tilde{\beta}_i \prod_{i\in X}\tilde{\alpha}_i^{m_i} \right| < E_{k,j}\varepsilon.$$

We shall use the facts that $\lim_{N\to\infty} f^{\lambda_N/k}/f^{\lambda_N} = p_k$, the coefficients of the generating function $G(t)$ (2.0.5) are non-negative and bounded by 1, and that each p_k is a polynomial in $\tilde{\lambda}$ whose coefficients are functions in $\tilde{\alpha}_i, \tilde{\beta}_i$.

For $j = 0$, $E_{k,0} = 1$, by construction. For $k = 1$, $j = 1$, $E_{1,1} = 1$ since $p_1 = 1$ and $S(N,1)$ converges to within ε of $\sum_{i=1}^{\infty} \tilde{\alpha}_i + \tilde{\beta}_i = 1 - \tilde{\lambda}$. By the multiplicative properties of our limits, we have the relation: $E_{k,k-1} = 2E_{k-1,k-1} + 1$, $k \geqslant 2$. Moreover, $E_{k,k} = E_{k,0} + E_{k,1} + \cdots + E_{k,k-1}$, because of the limit expression for p_k. The desired upper bound $E_{k,j} \leqslant (k+1)!$ follows easily.

To obtain the general case, we can argue as in the proof of Proposition 2.2 by choosing convergent subsequences of $f^{(\lambda_N:\mu)}/f^{\lambda}$ with $\mu \in D$. Then the sum (2.5.1) always has the same limiting value which is independent of the choice of subsequence. ∎

The next Lemma will change constant term contributions in $f_N(z)$ to higher order contributions.

LEMMA 2.6. (i) *Let* $\mu_N = \mu \in C_k$. *Let* $\lambda'_N = (\lambda_N : \mu)$. *Then:*

$$\prod_{(i_N,j_N)\in\lambda_N\setminus\lambda'_N} \frac{N}{N + j_N - i_N} \to \prod_{i\in Y}\frac{1}{1-\beta_i}\prod_{i\in X}\left(\frac{1}{1+\alpha_i}\right)^{m_i}.$$

(ii) *Let* (i_N, j_N) *be the coordinates of a deleted node. If neither* i_N *nor* j_N *are eventually constant, then either* i_N^2 *or* j_N^2 *is* $O(N)$. *In particular,* $\frac{N}{(N + j_N - i_N)} \to 1$.

Proof. (i) This statement follows from two observations. The first is that $(i_N, j_N) \in \lambda_N \setminus \lambda'_N$ represent the coordinates of a deleted node. In particular, the limits of j_N/N and i_N/N both exist and given by a column and row frequency, respectively.

(ii) Now $0 \leqslant i_N$, $j_N \leqslant \mathrm{rk}(\lambda_N)$. But $[\mathrm{rk}(\lambda_N)]^2 \leqslant |\lambda_N|$ and $|\lambda_N|/N \to \lambda$ as $N \to \infty$. ∎

We now come to the Theorem announced by Kerov and Vershik [12].

THEOREM 2.7. *If the row, column, and occupancy frequencies exist and the summation conditions holds, then*

$$\lim_{N\to\infty} \frac{\dim\{\lambda_N/k\}_{N-1}}{\{\lambda_N\}_N}$$

exists and equals the z^k-coefficient of

$$F(z) = e^{\lambda(z-1)} \prod_{i=1}^{\infty} \frac{1-\beta_i+\beta_i z}{1+\alpha_i-\alpha_i z}.$$

Proof. We first write:

$$A_N = \frac{\operatorname{num}\{\lambda_N\}_{N-1}}{\operatorname{num}\{\lambda_N\}_N} = \prod_{i=1}^{\operatorname{rk}(\lambda_N)} \frac{1-b_i^{(N)}/N-1/N}{1+a_i^{(N)}/N}.$$

Since $\dfrac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N} \leqslant 1$, for all k and N, we can find a subsequence which converges to the value of $\limsup_{N\to\infty} A_N$. To simplify the notation, we shall surpress all subindices. By an application of Fatou's Lemma, we can further claim that:

$$\limsup_{N\to\infty} A_N = \mathrm{e}^{-a} \prod_{i=1}^{\infty} \frac{1-\beta_i}{1+\alpha_i}, \quad a \geqslant 0.$$

Further, we will inductively define a subsequence of $\{\lambda_N\}$ such that

$$\lim_{N\to\infty} \frac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N} \text{ exists for all } k.$$

Let $\varepsilon > 0$ be given. Fix $k \in \mathbb{Z}^+$. Let $\mu_N \in \mathcal{D}_{k,N}$, $\lambda'_N = (\lambda_N : \mu_N)$. Then:

$$\frac{\dim\{\lambda'_N\}_{N-1}}{\dim\{\lambda_N\}_N} = \left(\frac{\operatorname{num}\{\lambda'\}_{N-1}}{\operatorname{num}\{\lambda'\}_N}\right)\left(\prod_{(i,j)\in\lambda\setminus\lambda'} \frac{N}{N+j-i}\right)\left(\frac{|\lambda|!}{|\lambda'|!}\Big/ N^k\right)\left(\frac{f^{\lambda'}}{f^{\lambda}}\right).$$

It is easy to verify that

$$\lim_{N\to\infty} \frac{\operatorname{num}\{\lambda'_N\}_{N-1}}{\operatorname{num}\{\lambda'_N\}_N}$$

also converges to

$$\mathrm{e}^{-a} \prod_{i=1}^{\infty} \frac{1-\beta_i}{1+\alpha_i};$$

in other words, this limit is insensitive to the choice of the deletion nodes μ_N. We also have that:

$$\lim_{N\to\infty} \frac{|\lambda|!}{|\lambda'|!}\Big/ N^k = \theta^k.$$

For the remaining two products, their limiting values depend on the choice of deletion nodes μ_N. They are controlled by Proposition 2.2 and Lemma 2.6.

We fix $k \in \mathbb{Z}^+$. Let $\varepsilon > 0$ be given, and let $\varepsilon' = \varepsilon/(k+2)!2^k$. Then:

$$\lim_{N\to\infty} \frac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N} = \lim_{N\to\infty} \sum \left\{ \frac{\dim\{(\lambda_N : \mu)\}_{N-1}}{\dim\{\lambda_N\}_N} : \mu \in \mathcal{D}_{k,n} \right\} =$$

(2.7.1)

$$= \lim_{N\to\infty} \sum_{j=0}^{k} \sum \left\{ \frac{\dim\{(\lambda_N : \mu\nu)\}_{N-1}}{\dim\{\lambda_N\}_N} : \mu \in D \in \mathcal{J}_{\varepsilon',N},\ \nu \in \mathcal{I}_{\varepsilon',N},\ |D| = k-j,\ |\nu| = j \right\}.$$

By construction, for fixed $D \in \mathcal{J}_{\varepsilon',N}$ the sums

$$\sum \left\{ \frac{\dim\{(\lambda_N : \mu\nu)\}_{N-1}}{\dim\{\lambda_N\}_N} : \mu \in D,\ \nu \in \mathcal{I}_{\varepsilon',N},\ |\nu| = j \right\}$$

converge to within $\varepsilon/(k+2)2^k$ of

$$(2.7.2)\quad m(D)e^{-a} \left(\prod_{i \notin Y} 1 - \beta_i \right) \left(\prod_{i \in Y_0} \beta_i \right) \left(\prod_{i \notin X} \frac{1}{1+\alpha_i} \right) \left(\prod_{i \in X_0} \frac{\alpha_i^{m_i}}{(1+\alpha_i)^{m_i+1}} \right) \frac{\lambda^j}{j!}.$$

Hence, the limit of $\lim_{N\to\infty} \dim\{\lambda_N/k\}_{N-1}/\dim\{\lambda_N\}_N$ is within $\varepsilon/2^k$ of the z^k-coefficient of

$$g(z) = e^{-a} e^{\lambda z} \prod_{i=1}^{\infty} \frac{1 - \beta_i + \beta_i z}{1 + \alpha_i - \alpha_i z}.$$

But $f_N(1) = 1$, for all N. Hence,

$$\sum_{k=0}^{\infty} \left(\lim_{N\to\infty} \frac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N} \right) = 1.$$

So, $g(1)$ is within ε of 1, for all $\varepsilon > 0$. We conclude that $g(1) = 1$ or $a = \lambda$.

By the same method, we can show that the value of $\liminf_{N\to\infty} A_N$ agrees with the value of

$$\limsup_{N\to\infty} A_N.$$

In particular, the full limit of A_N exists. It follows at once that the desired limit

$$\lim_{N\to\infty} \frac{\dim\{\lambda_N/k\}_{N-1}}{\dim\{\lambda_N\}_N}$$

exists because of the factored expression (2.7.2) and the fact that the

$$\lim_{N\to\infty} \frac{\mathrm{num}\{\lambda_N'\}_{N-1}}{\mathrm{num}\{\lambda_N\}_N}$$

is independent of the deletion nodes. ■

COROLLARY 2.8. *If the row, column and occupancy frequencies exist and the summation conditions holds, then*

$$\lim_{N\to\infty} \frac{\mathrm{Ch}\{\lambda_N\}}{\dim\{\lambda_N\}}(V) = \prod_{i=1}^{M} F(z_i),$$

where $F(z)$ is given in the Theorem 2.7 and $z_1, z_2, \ldots, z_M$ are the eigenvalues of V different from 1.

We now turn to the converse of the Theorem. We focus on the case that we will need for the other classical groups, then make comments on the general case. Let $f_N(z)$ be as above. We assume that $f_N(z)$ converges pointwise to a function $f(z)$, where $f(1) = 1$.

The special assumption we make is that $\limsup_{N\to\infty} \beta_1 < 1$ and that $f(0) \neq 0$. Since $c_N(\lambda_N) \leqslant N$, for all N, we can find a subsequence of $\{\lambda_N\}$ such that all the column frequencies exists. Consider the constant term of $f_N(z)$:

$$A_N = \prod_{i=1}^{\mathrm{rk}(\lambda_N)} \frac{1 - b_i^{(N)}/N - 1/N}{1 + a_i^{(N)}/N}.$$

Since $A_N = f_N(0)$, the limit of A_N must exists and equals $A = f(0)$. But, $A_N \leqslant \prod_{i=1}^{\mathrm{rk}(\lambda_N)} 1 - b_i^{(N)}/N$, hence $A \leqslant \prod_{i=1}^{\infty} 1 - \beta_i$, by the definition of $b_i^{(N)}$. Since $0 \leqslant \beta_i < 1$ and $A > 0$, we must have $\sum \beta_i < \infty$. Moreover, we further have

$$A_N \leqslant \frac{1}{1 + a_1^{(N)}/N} \prod_{i=1}^{\mathrm{rk}(\lambda_N)} 1 - b_i^{(N)}/N,$$

so, $\limsup_{N\to\infty} a_1^{(N)}/N < \infty$ as well. Hence, we can find a subsequence of $\{\lambda_N\}$ so that the row frequencies exist as well. By the same method as for the $\beta's$, we can deduce that $\sum \alpha_i < \infty$ when $A = f(0) > 0$.

We now have shown:

LEMMA 2.9. *If $f_N(z) \to f(z)$ poinwise and $f(0) \neq 0$, then:*
(i) *For any subsequence for the all the column frequencies β_i exist, $\sum \beta_i < \infty$;*
(ii) $\limsup_{N\to\infty} a_i^{(N)}/N < \infty$;
(iii) *for any subsequence for which all the row frequencies α_i exists, $\sum \alpha_i < \infty$.*

We now show:

LEMMA 2.10.
$$\limsup_{N\to\infty} \frac{|\lambda_N|}{N} < \infty.$$

Proof. We first observe that
$$\sum\left\{\frac{1}{N+j-i} : (i,j)\in\lambda_N\right\} \leqslant \frac{1}{A_N}.$$
But $(N+j-i)^{-1}$ is minimized if $j = r_1(g_N)$ and $i = 1$. So,
$$\frac{1}{A_N} \geqslant \frac{|\lambda_N|}{N + r_1(\lambda_N) - 1}.$$
Since $\limsup_{N\to\infty} r_1(\lambda_N) < \infty$, $\limsup_{N\to\infty} |\lambda_N|/N = \infty$ implies that
$$\frac{|\lambda_N|}{N} + r_1(\lambda_N) - 1 \to \infty.$$
Hence, there is a subsequence of $\{\lambda_N\}$ such that
$$A_N = \prod_{(i,j)\in\lambda_N}\left(1 - \frac{1}{N+j-i}\right)$$
converges to 0. So, $f(0) = 0$. Contradiction. ■

COROLLARY 2.11.
$$\mathrm{rk}(\lambda_N) = O(N^{1/2}).$$

Proof. $[\mathrm{rk}(\lambda_N)]^2 \leqslant |\lambda_N|$. ■

So, summing up, we have:

PROPOSITION 2.12. *If $f_N(z)$ converges as $N \to \infty$ and if $\limsup_{N\to\infty} b_1^{(N)}/N < 1$, then the row, column, occupancy frequencies must exist and the summation conditon hold.*

3. ASYMPTOTIC FORMULAS FOR THE SYMPLECTIC, SPECIAL ORTHOGONAL, AND SPIN GROUPS

We shall now establish the Asymptotic Character Formula for the other classical groups. We must make extensive use of the dimension formulas of El Samra and King [9]. If $\langle\lambda\rangle \in \widehat{\mathrm{Sp}}(2N)$, then the dimension of this irreducible representation is:
$$\prod_{i=1}^{\mathrm{rk}(\lambda)}\left(\frac{(2N+a_i+1)!}{(2N-b_i)!}\prod_{j=i+1}^{\mathrm{rk}(\lambda)}\frac{(2N+a_i+a_j+2)}{(2N+a_i-b_j+1)}\frac{(2N-b_i-b_j)}{(2N-b_i+a_j+1)}\right)\Big/ H(\lambda),$$

where $(a_1, a_2, \dots \mid b_1, b_2, \dots)$ are Frobenius coordinates of λ.

As in Section 2, our standing assumption is that for the sequence of diagrams $\langle\lambda_N\rangle \in \widehat{\mathrm{Sp}}(2N)$ their row, column, and occupancy frequencies exist and the summation condition holds. Under this standing assumption, we shall show that the limit of the normalized characters exist.

LEMMA 3.1. *Let λ_N be an increasing sequence of signatures, $\lambda_N \in \mathrm{Sp}\widehat{(2N)}$. If the row, column, and occupancy frequencies exist for $\langle\lambda_N\rangle$ and the summation conditions holds, then*

$$\lim_{N\to\infty} \frac{\dim\langle\lambda_N\rangle_{N-1}}{\dim\langle\lambda_N\rangle_N}$$

exists and equals

$$\left(\mathrm{e}^{-\lambda} \prod_{i=1}^{\infty} \frac{1-\beta_i}{1+\alpha_i}\right)^2 .$$

Proof. We have that:

$$\frac{\dim\langle\lambda_N\rangle_{N-1}}{\dim\langle\lambda_N\rangle_N} = \prod_{i=1}^{\mathrm{rk}(\lambda_N)} \frac{(2N+a_i^{(N)}-1)!}{(2N+a_i^{(N)}+1)!} \frac{(2N-b_i^{(N)})!}{(2N-b_i^{(N)}-2)!} \times$$

$$\times \prod_{j=i+1}^{\mathrm{rk}(\lambda_N)} \left(\frac{2N+a_i^{(N)}+a_j^{(N)}}{2N+a_i^{(N)}-b_j^{(N)}-1}\right) \left(\frac{2N+a_i^{(N)}-b_j^{(N)}+1}{2N+a_i^{(N)}+a_j^{(N)}+2}\right) \times$$

$$\times \prod_{j=1}^{\mathrm{rk}(\lambda_N)} \left(\frac{2N-b_i^{(N)}-b_j^{(N)}-2}{2N-b_i^{(N)}+a_j^{(N)}-1}\right) \left(\frac{2N-b_i^{(N)}+a_j^{(N)}+1}{2N-b_i^{(N)}-b_j^{(N)}}\right) =$$

$$= \prod_{i=1}^{\mathrm{rk}(\lambda_N)} \left(\frac{(2N-b_i^{(N)})(2N-b_i^{(N)}-1)}{(2N-a_i^{(N)}+1)(2N+a_i^{(N)})}\right) =$$

$$= \prod_{j=i+1}^{\mathrm{rk}(\lambda_N)} \left(1-\frac{1}{2N+a_i^{(N)}+a_j^{(N)}+2}\right) \left(1+\frac{1}{2N+a_i^{(N)}-b_j^{(N)}}\right) \times$$

$$\times \prod_{j=i}^{\mathrm{rk}(\lambda_N)} \left(1-\frac{1}{2N+b_k^{(N)}-b_j^{(N)}}\right) \left(1+\frac{1}{2N-b_i^{(N)}+a_j^{(N)}}\right) =$$

$$= \prod_{i=1}^{\mathrm{rk}(\lambda_N)} \left(\frac{(2N-b_i^{(N)})(2N-b_i^{(N)}-1)}{(2N-a_i^{(N)}+1)(2N+a_i^{(N)})}\right) \times$$

$$\times \prod_{j=i+1}^{\mathrm{rk}(\lambda_N)} \left(1+\frac{a_i^{(N)}+b_i^{(N)}+1}{(2N-b_i^{(N)}+a_j^{(N)})(2N+a_i^{(N)}+a_j^{(N)}+2)}\right) \times$$

$$\times \prod_{j=i+1}^{\mathrm{rk}(\lambda_N)} \left(1-\frac{a_i^{(N)}+b_i^{(N)}+1}{(2N-b_i^{(N)}-b_j^{(N)})(2N+a_i^{(N)}-b_j^{(N)})}\right) \times$$

$$\times\left(1-\frac{1}{2N-2b_i^{(N)}}\right)\left(1+\frac{1}{2N-b_i^{(N)}+a_i^{(N)}}\right).$$

Under our assumptions, we have that

$$\lim_{N\to\infty}\prod_{i=1}^{\mathrm{rk}(\lambda_N)}\left(\frac{(2N-b_i^{(N)})(2N-b_i^{(N)}-1)}{(2N-a_i^{(N)}+1)(2N+a_i^{(N)})}\right)$$

converges to

$$\left(\mathrm{e}^{-\lambda}\prod_{i=1}^{\infty}\frac{1-\beta_i}{1+\alpha_i}\right)^2.$$

Moreover, since

$$\mathrm{rk}(\lambda_N)=O(N^{1/2}),$$

both

$$\left(1-\frac{1}{2N-2Nb_i^{(N)}}\right)^{\mathrm{rk}(\lambda_N)}\quad\text{and}\quad\left(1+\frac{1}{2N-b_i^{(N)}+a_i^{(N)}}\right)^{\mathrm{rk}(\lambda_N)}$$

converge to 1 as $N\to\infty$.

To see that the remaining factors converge to 1, let $\varepsilon>0$ be given. Choose m so large that for $N\geqslant m$, both $a_i^{(N)}/N$ and $b_i^{(N)}/N<\varepsilon$ for $i>m$. Clearly, these factors are bounded above by 1. Also, we observe that:

$$\prod_{i=m}^{\mathrm{rk}(\lambda_N)}\left(\prod_{j=m}^{\mathrm{rk}(\lambda_N)}1+\frac{a_i^{(N)}+b_i^{(N)}+1}{(2N-b_i^{(N)}+a_j^{(N)})(2N+a_i^{(N)}+a_j^{(N)}+2)}\right)\leqslant$$

$$\leqslant\prod_{i=m}^{\mathrm{rk}(\lambda_N)}\left(\prod_{j=m}^{\mathrm{rk}(\lambda_N)}1+\frac{2\varepsilon N+2}{(2N-\varepsilon N)(2N)}\right).$$

The limit of this last expression is bounded above by $\mathrm{e}^{k\varepsilon}$, where k is given by the inequality: $[\mathrm{rk}(\lambda_N)]^2\leqslant kN$. Since $\varepsilon>0$ is arbitrary, this last product must converge to 1. The remaining factors with indices in range $1\leqslant i\leqslant\mathrm{rk}(\lambda_N)$ and $i+1\leqslant j\leqslant m$ or in range $1\leqslant i\leqslant m$ and $i+1\leqslant j\leqslant\mathrm{rk}(\lambda_N)$ must converge to 1, since the number of factors involved has order of magnitute $O(N^{1/2})$. ∎

LEMMA 3.2.

$$\lim_{N\to\infty}\frac{\dim\langle\lambda_N/st\rangle_{N-1}}{\dim\langle\lambda_N\rangle_N}=\lim_{N\to\infty}\frac{\dim\{\lambda_N/st\}_{2N-1}}{\dim\{\lambda_N\}_{2N+1}}.$$

Proof. We will first assume that the row and column frequencies are distinct. We fix $s,t\geqslant 0$. It will suffice to consider the case of only one deletion process, that is, the deletion of s nodes, say. Let μ be a collection of deletion nodes, with $|\mu| = s$. We shall show that:

$$\lim_{N\to\infty} \frac{\dim\langle(\lambda_N:\mu)\rangle_{N-1}}{\dim\langle\lambda_N\rangle_N} = \lim_{N\to\infty} \frac{\dim\{(\lambda_N:\mu)\}_{2N-1}}{\dim\{\lambda_N\}_{2N+1}}.$$

To simplify notation, we will surpress the superscript N on the Frobenius coordinates, so we write a_i for $a_i^{(N)}$, for example. If the Frobenius coordinates of $(\lambda_N:\mu)$ differ from those of λ_N we place a prime $'$ on that coordinate. The Lemma will be proved if we can show:

$$\begin{aligned}\lim_{N\to\infty} \prod_{i=1}^{\mathrm{rk}(\lambda_N)} \prod_{j=i+1}^{\mathrm{rk}(\lambda_N)} \frac{2N+a_i'+a_j'}{2N+a_i'-b_j'-1} \Big/ \frac{2N+a_i+a_j+2}{2N+a_i-b_j+1} \times \\ \times \prod_{j=i}^{\mathrm{rk}(\lambda_N)} \frac{2N-b_i'-b_j'-2}{2N-b_i'+a_j'-1} \Big/ \frac{2N-b_i-b_j}{2N-b_i+a_j+1} = 1.\end{aligned} \tag{3.2.1}$$

We first deal with the case that s column nodes are deleted, with at most 2 per column.

Suppose nodes are deleted at the p_l-th columns, $1 \leqslant l \leqslant s$. Then there are at most $2\mathrm{rk}(\lambda_N)$ affected factors in the product (3.2.1). In particular, $2\mathrm{rk}(\lambda_N) = O(N)$. Write: $p = p_l$, $q = q_l$, then

$$\left(\frac{2N-b_p'-b_q'-1}{2N-b_p'+a_j'}\right)\left(\frac{2N-b_p+a_q+1}{2N-b_p-b_q}\right) =$$

$$= \left(1+\frac{(b_p-b_p')+(b_q-b_q')+1}{2N-b_p-b_q}\right)\left(1-\frac{(b_p-b_p')+1}{2N-b_p'+a_j}\right)$$

and

$$\left(\frac{2N+a_p+a_q+1}{2N+a_p-b_q'}\right)\left(\frac{2N+a_p-b_q+1}{2N+a_p+a_q+2}\right) =$$

$$= \left(1-\frac{1}{2N+a_p+a_q+2}\right)\left(1-\frac{b_p-b_p'+1}{2N+a_p-b_q'}\right).$$

Now, choose an index M such that if $i \geqslant M$ implies that a_i/N, $b_i/N < \varepsilon$. In the product (3.2.1), at most $2s$ factors differ from the constant term case. Moreover, the differing factors are bounded above by:

$$\left(1+\frac{5}{2N-2\varepsilon N}\right)^{\mathrm{rk}(\lambda_N)}, \quad \text{with } p \geqslant M. \tag{3.2.2}$$

But the limit sup of (3.2.2) is 1. Hence, the lim sup of the expression (3.2.1) is at most 1. Note: $0 \leqslant b_i - b_i' \leqslant 2$, for all i.

Similarly, for a lower bound, we replace the factor for $p, q \geqslant M$ with:

$$\left(1-\frac{3}{2N-\varepsilon N}\right)^{2\mathrm{rk}(\lambda_N)-2M}\left(1-\frac{1}{N+2}\right)^{\mathrm{rk}(\lambda_N)}. \tag{3.2.3}$$

But the lim inf of (3.2.3) is 1. Hence, the limit of (3.2.1) exists and 1 in this case.

We next turn to the case nodes are deleted only from rows.

Suppose that nodes are deleted from rows p and q. Then the affected factor in (3.2.1) is:

$$\left(\frac{2N+a'_p+a'_q+1}{2N+a'_p-b_q}\right)\left(\frac{2N+a_p-b_q+1}{2N+a_p+a_q+2}\right)=$$
$$=\left(1-\frac{(a_p-a'_p)+(a_q-a'_q)+1}{2N+a_p+a_q+2}\right)\left(1+\frac{a_p-a'_p+1}{2N+a'_p-b_q}\right)$$

and

$$\left(\frac{2N-b_p-b_q-1}{2N-b_p+a'_q}\right)\left(\frac{2N-b_q+a_q+1}{2N-b_p-b_q}\right)=$$
$$=\left(1-\frac{1}{2N-b_p-b_q}\right)\left(1+\frac{a_q-a'_q}{2N-b_p+a'_q}\right).$$

Since $0 \leqslant a_p - a'_p,\ a_q - a'_q \leqslant s$, we can argue as for column deletion. Moreover, the general deletion process can be decomposed into separate row and column deletion steps with the above estimates yielding the general result.

We drop our condition that the row and column frequencies are distinct by following the technique of Proposition 2.2, because the value of the hooklenghts have no effect on the product in (3.2.1). ■

Hence, we have established the Asymptotic Character Formula for the symplectic group $\mathrm{Sp}(\infty)$.

THEOREM 3.3. *If the row, column, and occupancy frequencies exist and the summation condition holds, then the sequence of symplectic characters* $\dfrac{\mathrm{Ch}\langle\lambda_N\rangle}{\dim\langle\lambda_N\rangle}$ *has a limit as* $N \to \infty$.

PROPOSITION 3.4. *Suppose that the limit of the normalized characters* $\dfrac{\mathrm{Ch}\langle\lambda_N\rangle}{\dim\langle\lambda_N\rangle}$ *exists, then the row, column, occupancy frequencies exist and the summation conditions holds.*

Proof. We consider the sequence $\dim\langle\lambda_N\rangle_{N-1}/\dim\langle\lambda_N\rangle_N$. Since these quotients are non-negative and bounded above by 1, there exists a convergent subsequence. To simplify notation, we omit subindices. Now, on the one hand, the value $\lim_{N\to\infty} \dim\langle\lambda_N\rangle_{N-1}/\dim\langle\lambda_N\rangle_N$ gives the measure of the cylinder set S of all paths that pass through the node $\langle 0\rangle_2$, $\langle 0\rangle_2 \in \widehat{\mathrm{Sp}(2)}$. On the other hand, by our classification of the characters of the infinite symplectic group, [6], the measure of S is given by an expression of the form:

$$\left(e^{-x}\prod_{i=1}^{\infty}\frac{1-y_i}{1+z_i}\right)^2,$$

where $0 \leqslant x,\ z_i < \infty,\ 0 \leqslant y_i \leqslant 1/2, \sum(y_i+z_i) < \infty$. In particular, the measure of the set S is 0 if and only if the measure itself is identically zero.

The key to the proof of the Proposition is the inequality:

$$\frac{\dim\langle\lambda_N\rangle_{N-1}}{\dim\langle\lambda_N\rangle_N} \leqslant \frac{\dim\{\lambda_N\}_{2N-1}}{\dim\{\lambda_N\}_{2N+1}}. \tag{3.4.1}$$

To establish the inequality (3.4.1), it will suffice to show that

$$\left[\left(\frac{N+a_i+a_j+1}{N+a_i-b_j}\right)\left(\frac{N+a_i+a_j+2}{N+a_i-b_j+1}\right)\right]\left[\left(\frac{N-b_i-b_j-1}{N-b_i+a_j}\right)\left(\frac{N-b_i-b_j}{N-b_i+a_j+1}\right)\right] \leqslant 1$$

To simplify notation, we have ommited the superscript N. Now, this product can be written as:

$$\left(1+\frac{a_j+b_j+1}{(N+a_i+a_j+2)(N+a_i-b_j)}\right)\left(1-\frac{a_j+b_j+1}{(N-b_i-b_j)(N-b_i+a_j)}\right) =$$
$$= 1+\frac{(a_j+b_j+1)(A-a_j-b_j-1)}{(N+a_i+a_j+2)(N+a_i-b_j)(N-b_i-b_j)(N-b_i+a_j)},$$

where

$$A = (N-b_i-b_j)(N-b_i+a_j)-(N+a_i+a_j+2)(N+a_j-b_j).$$

But A must be non-positive since $b_j \leqslant b_i \leqslant N$.

If a row frequency or occupancy of $\langle\lambda_N\rangle$ diverges to ∞ or the summation condition fails, then the ratio $\dim\{\lambda_N\}_{N-1}/\dim\{\lambda_N\}_N$ converges to 0 as $N \to \infty$. By the above inequality, this forces the probability measure that corresponds to the limit of the normalized characters to be identically zero. Contradiction. Moreover, the sequences $a_i^{(N)}/2N$, $b_i^{(N)}/2N$, and $|\lambda|/2N$ must converge, otherwise, the sequence of normalized characters would converge to more than one value because our assumptions are sufficient to guarantee convergence. ■

The same method can easily adapted to the special orthogonal group. We shall be satisfied to state the result. The classification portion of this Theorem can be derived using the method in our paper [6].

THEOREM 3.5. *If the row, column, and occupancy frequencies exist and the summation condition holds, then the sequence of characters $[\lambda_N]$ of the special orthogonal groups $SO(N)$ has a limit as $N \to \infty$. Moreover, every finite character χ of $SO(\infty)$ has the form:*

$$\chi(V) = \prod |f(z_j)|^2 = \det[f(V)],$$

where $f(z) = e^{\lambda(z-1)}\prod_{i=1}^{\infty}\frac{1-\beta_i+\beta_i z}{1+\alpha_i-\alpha_i z}$ *and* $V \in SO(2N+1) \subset SO(\infty)$ *has the eigenvalues* $z_1, \overline{z}_1, \ldots, z_N, \overline{z}_N, 1$.

The classification of the characters of the infinite spin group will follow from the next result.

THEOREM 3.6.

$$C^*(\mathrm{Spin}(\infty)) \cong C^*(SO(\infty)) \oplus C^*(\mathrm{Sp}(\infty)) \otimes \mathrm{UHF}(2^\infty).$$

Proof. By the branching laws for the spin groups [15], the representation $[\Delta;\lambda]_N$ decomposes into components again of the form $[\Delta;\lambda']_{N-1}$, so the group algebra $C^*(\mathrm{Spin}(\infty))$ is isomorphic to the direct sum $C^*(SO(\infty))$ with some algebra $\mathcal{A} = \varinjlim \mathcal{A}_n$. Our claim is that $\mathcal{A} \cong C^*(\mathrm{Sp}(\infty)) \otimes \mathrm{UHF}(2^\infty)$.

To complete the proof we now outline a general method to construct a Bratteli diagram for $B = A \otimes \mathrm{UHF}(2^\infty)$, where $A = \varinjlim A_n$ is an AF-algebra. Set $B = \varinjlim B_n$, $B_{2k} = (A_{2k} \oplus A_{2k}) \otimes M_{2^{k-1}}(\mathbb{C})$, and $B_{2k+1} = A_{2k+1} \otimes M_{2^k}(\mathbb{C})$. To describe the embedding $B_i \to B_{i+1}$, we let a' denote the image of $a \in A_i$ in A_{i+1}. Now, $B_{2k-1} \to B_{2k}$ or $A_{2k-1} \otimes M_{2^{k-1}}(\mathbb{C}) \to (A_{2k} \oplus A_{2k}) \otimes M_{2^{k-1}}$ by $a \otimes x \mapsto (a' \oplus a') \otimes x$. Also, $B_{2k} \to B_{2k+1}$ or $(A_{2k} \oplus A_{2k}) \otimes M_{2^{k-1}}(\mathbb{C})$ by $(a \oplus a_1) \otimes x \mapsto \begin{pmatrix} a \otimes x & 0 \\ 0 & a_1 \otimes x \end{pmatrix} \in A_{2k+1} \otimes M_{2^{k-1}}(\mathbb{C}) \otimes M_2(\mathbb{C})$.

Following many authors (for example [15]), we find it convenient to introduce the odd symplectic groups $\mathrm{Sp}(2N-1)$ whose characters are parametrized by signatures with N entries. The character and dimension formulas for the even symplectic groups still make sense if the even dimension $2N$ is replaced by an odd dimension.

It is straightforward to verify that the following diagrams are commutative.

$$\begin{array}{ccccc}
\mathcal{A}_{2k} & \longrightarrow & (A_{2k} \oplus A_{2k}) \otimes M_{2^{k-1}}(\mathbb{C}) & \longrightarrow & \mathcal{A}_{2k} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_{2k+1} & \longrightarrow & A_{2k+1} \otimes M_{2^k}(\mathbb{C}) & \longrightarrow & \mathcal{A}_{2k+1}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{A}_{2k-1} & \longrightarrow & A_{2k-1} \otimes M_{2^{k-1}}(\mathbb{C}) & \longrightarrow & \mathcal{A}_{2k-1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_{2k} & \longrightarrow & (A_{2k} \oplus A_{2k}) \otimes M_{2^{k-1}}(\mathbb{C}) & \longrightarrow & \mathcal{A}_{2k}
\end{array}$$

Here, the second mapping on each row is an isomorphism. The maps from A_i to A_{i+1} and $\mathcal{A}_i$ to $\mathcal{A}_{i+1}$ are the natural maps given via restriction of representations. The result now follows. ∎

To state the next result, we will let Δ denote the basic character of the infinite spin group. This and associated representations have been studied by R. Plymen [21]. By the result of Blackadar [4] on primitive ideals and traceable factor representations on tensor products of C^*-algebras, we have the following Corollary:

COROLLARY 3.7. (i) Prim(Spin(∞)) *is naturally isomorphic to the disjoint union of the primitive ideal spaces of* $SO(\infty)$ *and* $\mathrm{Sp}(\infty)$.

(ii) *Let* Δ *denote the character of the spin representation of* Spin(∞). *Then the finite characters* χ *of* Spin(∞) *have the form:*

$$\chi = \Delta \cdot \chi'$$

where χ' *is a finite character for* $SO(\infty)$.

4. FACTOR REPRESENTATIONS IN THE ANTISYMMETRIC AND SYMMETRIC TENSORS

In this final section, we extend the results of [7] and [8] to the special orthogonal and symplectic groups. In particular, we study the restriction of the positive-definite functions $p_A(V) = \det(I - A + AV)$, $0 \leqslant A \leqslant I$, which corresponds to representations in the antisymmetric tensors (exterior algebra) and $p_B(V) = \det((I - B)(I - BV)^{-1})$, $0 \leqslant B < I$, which corresponds to representations in the symmetric tensors. We study in detail the antisymmetric case. The arguments for the symmetric case follow in like manner.

We introduce the notation $\mathcal{A}_U$, $\mathcal{A}_{\mathrm{Sp}}$, and $\mathcal{A}_{SO}$, respectively for the quotients of the group algebras supporting the representations of the infinite unitary, symplectic, and special orthogonal groups, respectively, in the antisymmetric tensors.

We shall show:

THEOREM 4.1. *If* $0 \leqslant A \leqslant I$ *and* $\mathrm{Tr}[A(I - A)] = \infty$, *then the positive-definite function* $p_A = \det(I - A + AV)$ *is factorial on either* $\mathrm{Sp}(\infty)$ *or* $SO(\infty)$.

The proof rests on a close examination of the condition given in Chapter IV of [22]. We need to introduce some notation and ideas from there. First, we let $X_p = \{0, 1\}$, for all i. Following [14] or [27], we shall treat the spectrum of the MASA of an AF-algebra as a space of paths. So, $X_{(i,j)} = \prod_{p=1}^{j} X_p$ will be thought of as the set of all paths that begin in the set X_i and end in the set X_j. If $k \in X_{(i,j)}$, let $X_{(j+1,l)}(k)$ denote the set of all paths in $X_{(j+1,l)}$ such that $kk' \in X_{(i,l)}$ where $k' \in X_{(j+1,l)}$. Set $X = \varprojlim X_{(1,n)} = \prod_{i=1}^{\infty} X_i$, the space of infinite paths. Next, we let μ_p be a probability measure on X_p with $\mu_p(\{0\}) = p_i$, $\mu_p(\{1\}) = 1 - p_i$, $0 < p_i < 1$, and $t_i = (1-p_i)/p_i$. Let $\mu^{(p,q)} = \bigotimes_p^q \mu_i$ on the set $X_{(i,j)}$, and let $\mu = \prod_1^\infty \mu_i$ on X. Finally, we let G_r denote the group of path permutations on $X_{(1,r)}$.

Next, we let f be the characteristic function of the cylinder set based on the finite path α, $\alpha \in X_{(1,n)}$. We assume that $n < r < N$. Now as in [22], we assume that g is a G_r-invariant function of the form: $g(k,\gamma) = \varphi(k, \gamma_{(r+1,N)})$, where $k \in X_{(1,r)}$, $\gamma \in$ $\in X_{(r+1,\infty)}$. Here k_r denotes the r-th node on the path k. We now argue that we can drop the assumption that g is G_r-invariant. For ease of comparison with [22], we still write $g(k,\gamma) = \varphi(k, \gamma_{(r+1,N)})$.

The fundamental quantity to estimate in the factor condition is:

$$\left| \int_X fg \mathrm{d}\mu - \left(\int_X f \mathrm{d}\mu \right) \left(\int_X g \mathrm{d}\mu \right) \right| \leqslant$$

$$\leqslant \left| \sum_{k \in X_{(1,r)}} \sum_{\gamma \in X_{(r+1,N)}(k)} \varphi(k,\gamma) \mu^{(r+1,N)}(\gamma) [\mu^{(1,r)}(k) \mu^{(1,n)}(\alpha) - \mu^{(1,r)}(k_{(1,n)} = \alpha)] \right| .$$

Here, $\mu^{(1,r)}(k_{(1,n)} = \alpha)$ denotes the measure of the set of all paths k in $X_{(1,r)}$ such that their first n nodes agree with α. We next observe that:

$$\mu^{(1,r)}(k)\mu^{(1,n)}(\alpha) - \mu^{(1,r)}(k_{(1,n)} = \alpha) = \mu^{(1,n)}(\alpha)[\mu^{(1,r)}(k) - \mu^{(n+1,r)}(k_{(n+1,r)})] \leqslant 0,$$

for all $k \in X_{(1,r)}$. We now fix the choice of $\varphi(k,\gamma) = -1$, for all $(k,\gamma) \in X_{(1,N)}$, so the fundamental quantity becomes:

$$\sum_{k \in X_{(1,r)}} \sum_{\gamma \in X_{(r+1,N)}(k)} \mu^{(r+1,N)}(\gamma) \mu^{(1,r)}(\alpha) \left[\mu^{(n+1,r)}(k_{(n+1,r)}) - \mu^{(1,r)}(k) \right]$$

where each term is non-negative. This last quantity can be dominated by the sum:

$$\sum_{k \in X_{(1,r)}} [\mu^{(n+1,r)}(k_{(n+1,r)}) - \mu^{(1,r)}(k)] \mu^{(1,n)}(\alpha).$$

Since each term is non-negative, the terms can be grouped together with absolute values inserted at will.

We write the factor condition with the notation of [22]. For $x \in X_r$, let D_x denote the cylinder set of all paths in X that pass through the node x, and let $D'_x \subset D_x$ consist of all paths $\gamma \in D'_x$ with $\gamma_{(1,n)} = \alpha$. Our last estimate can now be written as:

$$\sum_{x \in X_r} \left| \mu(D'_x) - \mu(D_x) \mu^{(1,n)}(\alpha) \right| . \tag{4.1.1}$$

If $\sum p_i(1-p_i) = \infty$, then this last sum converges to 0 as $r \to \infty$ and the corresponding representation is a factor. [1] or [22]

We now describe the analogue of (4.1.1) for the dynamical system (Y, G_Y), associated to the AF-algebra $\mathcal{A}_{\mathrm{Sp}}$. (compare [22]). Y will be given as $\varprojlim Y(1, N)$.

Recall that the unitary character $\{1^j\}_{2N}$ restricts to $\mathrm{Sp}(2N)$ into the sum $\langle 1^j\rangle_{2N} + \langle 1^{j-2}\rangle_{2N} + \cdots + \langle 1^{j-2[j/2]}\rangle_{2N}$, $0 \leqslant j \leqslant N$. Moreover, a basis for the representation space for each $\langle 1^j\rangle_{2N}$ can be indexed by Young diagram consisting of a single column of length j whose entries are chosen from the alphabet: $1 < \overline{1} < 2 < \overline{2} < \cdots < N < \overline{N}$, are strictly increasing, and satisfy the condition that the entry in row i must be $\geqslant i$. We call these tableaux *symplectic* [25]. Such tableaux, though, can only index a portion of a basis for the j-th exterior power $\Lambda^j(\mathbb{C}^{2N})$. The basis elements for the other terms $\langle 1^{j-2}\rangle$, $\langle 1^{j-4}\rangle, \ldots$ are needed. From the representation theory of the unitary group, we know that a basis for $\Lambda^j(\mathbb{C}^{2N})$ is indexed by Young "unitary" tableaux consisting of a single column of length j whose entries are chosen to be strictly increasing from the alphabet: $1 < \overline{1} < \cdots < N < \overline{N}$. We now describe how to associate a column of length j with these shorter tableaux. Given a $(j-2)$-length column for $\langle 1^{j-2}\rangle_{2N}$, it can be imbedded into a j-length column by adding the entries i and $\overline{i}$ to it, where i and $\overline{i}$ are the first pair of letters that do not appear in the given tableau. This rule can be applied recursively to imbed the tableaux for $\langle 1^{j-4}\rangle_{2N}$, $\langle 1^{j-6}\rangle_{2N}, \ldots$ into single column tableaux of length j. Moreover, we can identify a basis for $\Lambda^j(\mathbb{C}^{2N})$ with $j > N$, by taking the $(2N-j)$-column T giving a basis element for $\Lambda^{2N-j}(\mathbb{C}^{2N})$, from a column of length j by choosing the entries to be exactly those not occuring in T. We shall call two unitary tableaux, not necessary symplectic, *sp-equivalent* if one can be obtained from the other using, perhaps repeatedly, the two procedures described in this paragraph.

We can now describe the infinite path space Y for $\mathcal{A}_{\mathrm{Sp}}$ in terms of symplectic tableaux. The space Y_N is parametrized by the irreducible representations $\langle 1^j\rangle_{2N}$, $0 \leqslant j \leqslant N$. A path in $Y_{(1,N)}$ can be identified with a symplectic tableau associated with some $\langle 1^j\rangle_{2N}$ with the map from $Y_{(1,N)}$ to $Y_{(1,N-1)}$ determined by deleting the entries N and $\overline{N}$ from the symplectic tableau.

The weight vectors for $\mathrm{Sp}(2N)$ that correspond to the symplectic tableaux do not, in general coincide with the weight vectors of $U(2N)$ on $\Lambda^j(\mathbb{C}^{2N})$. As a consequence, it is not clear which measure is determined by μ on the dynamical system (Y, G_Y) associated to $\mathrm{Sp}(\infty)$. Compare with [22, p. 80 and p. 120]. To simplify this process, we shall work with a quasi-equivalent representation of $U(\infty)$. Let Φ be the $*$-automorphism of $\mathcal{A}_U$ which will take a system of matrix units whose MASA is generated by the projections onto certain weight vectors of $\mathrm{Sp}(2N)$, $N \geqslant 1$, onto a system of matrix units whose MASA is generated by the projections onto the weight vectors of $U(2N)$. In particular, we choose the usual weight vectors for the standard copy of $\langle 1^j\rangle_{2N}$ contained in $\Lambda^j(\mathbb{C}^{2N})$. We construct further orthonormal vectors by choosing them consistently with the copies of $\langle 1^{j-2}\rangle_{2N}$, $\langle 1^{j-4}\rangle_{2N},\ldots$ contained in $\Lambda^j(\mathbb{C}^{2N})$. (This step is unnecessary if μ is a central measure, so p_A is a finite character). Then the probability measure ν on Y determined by this quasi-equivalent representation of $U(\infty)$ is given via the generating function:

$$\prod_{i=1}^{N}(1+t_{2i-1}z_i)(1+t_{2i}\overline{z_i}) \Big/ \prod_{i=1}^{N}(1+t_{2i-1})(1+t_{2i}).$$

As in the unitary case, we associate the coefficient of the monomial $w_{i_1}w_{i_2}\cdots w_{i_k}$ to a tableau with entries $i_1,i_2,\ldots i_k$. Here, $w_i=z_i$ if $i=1,2,\ldots,N$, while $w_i=\overline{z_i}$ if $i=\overline{1},\overline{2},\cdots,\overline{N}$. For $\mathrm{Sp}(\infty)$, the measure ν of the symplectic tableau T is the sum of the coefficients of the monomials associated to all unitary tableaux sp-equivalent to T.

Arguing as in [1] or [23], we have that the factor condition for $\mathrm{Sp}(\infty)$ is:

$$\lim_{r\to\infty}\sum_{y\in Y_r}\left|\nu(D'_y)-\nu(D_y)\nu^{(1,n)}(\alpha)\right|=0.$$

However, this sum is dominated by:

$$\sum_{x\in X_r}\left|\mu(D'_x)-\mu(D_x)\mu^{(1,n)}(\alpha)\right|.$$

This last sum converges to 0 as $r\to\infty$ when $\sum p_i(1-p_i)=\infty$. The measure ν corresponds to choosing A to be a diagonalizable operator in the formula: $\det(I-A+ +AV)$. By following the reasoning in [7] or [8] concerning quasi-equivalence, we can drop the condition that A is diagonalizable.

The situation for $SO(\infty)$ is even easier since the exterior powers $\Lambda^j(\mathbb{C}^{2N+1})$ are irreducible for $SO(2N+1)$, but the j-th exterior power is equivalent to the $(2N+1-j)$-th power. So the distinct irreducible representations occur for: $0\leqslant j\leqslant N$. So, in orthogonal case, the notion of equivalent tableaux is even simpler than in the symplectic situation. A tableau T of length j is equivalent to tableau T' if T' has length $2N+1-j$ and the entries of T and T' are disjoint. With this notion of equivalence, the argument for the symplectic group now extends to the special orthogonal group. This ends the reasoning for the antisymmetric tensors.

In the symmetric tensors, the unitary character $\{m\}_{2N}$ of $U(2N)$ when restricted to $\mathrm{Sp}(2N)$ remains irreducible. Hence, our factor condition [7] extends immediately. The above argument for the symplectic group in antisymmetric tensors will extend to the situation of the special orthogonal group in symmetric tensors because the measure

μ associated with the positive-definite function p_B is an infinite product measure on an infinite product space and there is a notion of *so-tableau* [24] analogous to symplectic tableau.

Recall that the unitary character $\{m\}_{2N+1}$ of $U(2N+1)$ restricts to the special orthogonal group $SO(2N+1)$ into the sum $[m]_{2N+1} + [m-2]_{2N+1} + \cdots + [m - -2[m/2]]_{2N+1}$, $m \geqslant 0$. Moreover, a basis for the representation space for each $[m]_{2N+1}$ can be indexed by a Young diagram consisting of a single row of length m whose entries chosen from the alphabet: $1 < \overline{1} < \cdots < N < \overline{N} < \infty$, such that the entries are chosen to be weakly increasing and ∞ occurs at most once. These tableaux are known as the *so-tableaux*, and are analogous to the symplectic tableaux. In order to obtain a basis for the j-th symmetric power of $\mathbb{C}^{2N+1}$, we must imbed the so-tableaux for $[m-2j]$ into an unitary Young tableaux consisting of a single row of length m whose entries chosen from the same alphabet as the so-tableaux except now all entries are chosen to be weakly increasing including ∞. To imbed a so-tableau T of length $m-2$ into an unitary tableau symply extend the tableau by repeating the entry ∞ twice. We can apply this rule recursively to imbed the so-tableaux for $[m-4], [m-6], \ldots$ into unitary tableaux of length m. We call two unitary tableaux *so-equivalent* if one can be obtained from the other using, perhaps repeatedly, this procedure.

Now the infinite path space $\mathcal{X}$ for the unitary group is given by: $\mathcal{X} = \prod_{n=1}^{\infty}(\mathbb{Z}^+ \cup \cup\{0\})$, with probability measure $\mu = \bigotimes_{n=1}^{\infty} \mu_n$ where μ_n is the probability measure on the non-negative integers with density $\{(1-c_n)c_n^k\}_{k=0}^{\infty}$ and $0 < c_n < 1$ for all n .

Let $\mathcal{Y}$ be the infinite path space associated to representations of $SO(\infty)$ in the symmetric tensors. Then the measure ν on $\mathcal{Y}$ will be determined by the generating function:

$$\prod_{i=1}^{N+1} \frac{1-c_{2i-1}}{1-c_{2i-1}z_{2i-1}} \prod_{i=1}^{N} \frac{1-c_{2i}}{1-c_{2i}\overline{z}_i},$$

as follows. To each so-tableau with entries $i_1, i_2, \ldots, i_k$, we associate the coefficient of the monomial $w_{i_1} w_{i_2} \cdots w_{i_k}$, where $w_i = z_i$ if $i = 1, 2, \ldots N; = \overline{z}_i$ if $i = \overline{1}, \overline{2}, \ldots \overline{N}$; however, if $i = \infty$, then the monomial contains the factor z_{2N+1}^j when ∞ is repeated j times. For $SO(\infty)$, the measure ν of the so-tableau T is the sum of the coefficients of the monomials associated to all unitary tableaux so-equivalent to T.

EXAMPLE. For the finite character $\chi(V) = \det((1-c)(1-cz)^{-1})$, we have, for $V \in SO(N)$,

$$\chi(V) = \chi(z_1, z_2, \ldots, z_N) = (1-c)^N \sum_{m=0}^{\infty} \{m\}_N c^m =$$

$$= (1-c)^N \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} c^{2n+m} \right) [m]_N = \frac{(1-c)^{N-1}}{1+c} \sum_{m=0}^{\infty} [m]_N c^m.$$

So, we are satisfied to state the following theorem.

THEOREM 4.2. *If* $\mathrm{Tr}(B) = \infty$, *then the positive-definite function* $p_B(V) = \det((I-B)(I-BV)^{-1})$ *is factorial on either* $\mathrm{Sp}(\infty)$ *or* $SO(\infty)$.

In [7], we state the factor condition as $\mathrm{Tr}[(I+B)(I-B)^{-1}] = \infty$. Since $0 \leqslant B < I$, this last condition implies that $\mathrm{Tr}(B) = \infty$.

We expect these techniques to extend to handle representations in various mixed classes of tensors.

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ROBERT P. BOYER
Department of Mathematical and Computer Science,
Drexel University,
Philadelphia, PA 19104,
U. S. A.

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